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Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid

T. Hayat, M. Khan and **S. Asghar,** Islamabad, Pakistan

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Summary. This paper deals with some steady unidirectional flows of an Oldroyd 8-constant magnetohydrodynamic (MHD) fluid in bounded domains. The fluid is electrically conducting in the presence of a uniform magnetic field. Three nonlinear flows are produced by the motion of a boundary or by sudden application of a constant pressure gradient or by the motion of a boundary and pressure gradient. The governing nonlinear differential equations are solved analytically using homotopy analysis method (HAM). Expressions for the velocity distribution are given. It is noted that for steady flow the solutions are strongly dependent on the non–Newtonian and magnetic parameters. The MHD solutions for a Newtonian fluid, as well as those corresponding to the Oldroyd 3 and 6-constant fluids, a Maxwell fluid and a second grade one, appear as limiting cases of our solutions. Finally, a physical interpretation of the results is given with the help of several graphs.

1 Introduction

In recent years, there has been a great deal of interest in understanding the behavior of non-Newtonian fluids as they are used in many engineering processes. Also, non–Newtonian fluids are intensively studied by mathematicians, essentially from the point of view of differential equations theory. On the other hand, in applied sciences such as rheology or physics of the atmosphere, the approach to fluid mechanics is in an experimental setup leading to the measurement of material coefficients. Moreover, in theoretically studying how to predict the weather, ordinary differential equations represent the main tool. Further, since the failures in the predictions are strictly related to a chaotic behavior, one may find it unessential to ask whether the fluids are really Newtonian.

Rheological properties of fluids are specified in general by their so-called constitutive equations. Amongst the many models which have been used to describe the non–Newtonian behavior exhibited by these fluids, the fluids of differential type [1] and those of rate type [2] have received special status. More recently, Baris et al. [3] considered an Oldroyd 8-constant model to discuss the steady flow in a convergent channel. In [3] a series method is used for the solutions involving linear ordinary differential equations. However, the solution in the more general context of the nonlinear equation for magnetohydrodynamic flow of an Oldroyd 8-constant fluid is not given.

To the authors' knowledge, no previous investigation has been reported to develope the governing equations for steady magneto-hydrodynamic (MHD) flow of an Oldroyd 8-constant fluid. In this work it is intended to construct the equations for an MHD flow of an Oldroyd

8-constant fluid. The motion of power law fluids in the presence of a magnetic field has been studied earlier by several authors [4]–[7]. Hayat et al. [8] examined some periodic MHD flows of an Oldroyd 3-constant fluid. Examples of non–Newtonian fluids which might be conductors of electricity were given by Sarpkaya [7], e.g., flow of nuclear slurries and of mercury amalgams, and lubrication with heavy oils and greases. The objective of the present study is therefore to discuss three fundamental flows (Couette, Poiseuille and generalized Couette) of an MHD Oldroyd 8-constant fluid. The governing equations are the conservation of mass and the conservation of linear momentum. The stress tensor from the constitutive equation is substituted in the momentum equation. The resulting nonlinear, ordinary differential equations are solved analytically using HAM [9]–[15].

2 Governing equations

Here, we consider the flow of an electrically conducting fluid. The steady motion of the conducting fluid in the Cartesian coordinate system is governed by the conservation laws of momentum and of mass which are

$$\rho(\mathbf{V} \cdot \nabla)\mathbf{V} = \nabla \cdot \mathbf{T} + \mathbf{J} \times \mathbf{B}, \tag{1}$$

$$div\mathbf{V} = 0, \tag{2}$$

in which $\mathbf{V} = (u, v, w)$ is the velocity vector, ρ the density, $\mathbf{J}$ the current density, $\mathbf{B}$ the total magnetic field so that $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, $\mathbf{b}$ is the induced magnetic field.

Neglecting the displacement currents, the Maxwell equations and the generalized Ohm's law are

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_m \mathbf{J}, \quad \nabla \times \mathbf{E} = 0, \tag{3}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}), \tag{4}$$

where μ_m is the magnetic permeability, $\mathbf{E}$ is the electric field and σ is the electric conductivity.

We make the following assumptions:

– The quantities ρ, μ_m and σ are all constant throughout the flow field.
– The magnetic field $\mathbf{B}$ is perpendicular to the velocity field $\mathbf{V}$ and the induced magnetic field is negligible compared with the imposed magnetic field so that the magnetic Reynolds number is small [16].
– The electric field is assumed to be zero.

In view of the above assumptions, the electromagnetic body force involved in Eq. (1) takes the following form:

$$(\mathbf{J} \times \mathbf{B}) = \sigma[\mathbf{B}_0(\mathbf{V} \cdot \mathbf{B}_0) - \mathbf{V}(\mathbf{B}_0 \cdot \mathbf{B}_0)] = -\sigma B_0^2 \mathbf{V}. \tag{5}$$

For an Oldroyd 8-constant fluid, the Cauchy stress tensor $\mathbf{T}$ [17], [18]

$$\mathbf{T} = -p_1 \mathbf{I} + \mathbf{S}, \tag{6}$$

in which p_1 is the pressure, $\mathbf{I}$ the identity tensor, and the extra stress $\mathbf{S}$ satisfies

$$\mathbf{S} + \lambda_1 \frac{D\mathbf{S}}{Dt} + \frac{\lambda_3}{2}(\mathbf{S}\mathbf{A}_1 + \mathbf{A}_1\mathbf{S}) + \frac{\lambda_5}{2}(tr\mathbf{S})\mathbf{A}_1 + \frac{\lambda_6}{2}[tr\,(\mathbf{S}\mathbf{A}_1)]\mathbf{I}$$

$$= \mu\left[\mathbf{A}_1 + \lambda_2 \frac{D\mathbf{A}_1}{Dt} + \lambda_4 \mathbf{A}_1^2 + \frac{\lambda_7}{2}\left[tr(\mathbf{A}_1^2)\right]\mathbf{I}\right], \tag{7}$$

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \quad \mathbf{L} = grad\mathbf{V}, \tag{8}$$

where $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor, $\mu, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ and λ_7 are the material constants, and the contravariant convected derivative D/Dt for steady flow is as follows:

$$\frac{D\mathbf{S}}{Dt} = (\mathbf{V} \cdot \nabla)\mathbf{S} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T. \tag{9}$$

We indicate the stress tensor and the velocity as

$$\mathbf{S}(y) = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}, \quad \mathbf{V}(y) = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}. \tag{10}$$

Using Eq. (10), Eq. (2) is satisfied identically and Eqs. (1), (5) and (6) to (10) yield the following scalar equations:

$$\frac{\partial p_1}{\partial x} = \frac{d}{dy} S_{xy} - \sigma B_0^2 u, \tag{11}$$

$$\frac{\partial p_1}{\partial y} = \frac{d}{dy} S_{yy}, \tag{12}$$

$$\frac{\partial p_1}{\partial z} = \frac{d}{dy} S_{zy}, \tag{13}$$

$$S_{xx} + (\lambda_3 + \lambda_6 - 2\lambda_1) S_{xy} \frac{du}{dy} = \mu(\lambda_4 + \lambda_7 - 2\lambda_2)\left(\frac{du}{dy}\right)^2, \tag{14}$$

$$S_{xy} - \lambda_1 S_{yy} \frac{du}{dy} + \left(\frac{\lambda_3 + \lambda_5}{2}\right)(S_{xx} + S_{yy})\frac{du}{dy} + \frac{\lambda_5}{2} S_{zz} \frac{du}{dy} = \mu \frac{du}{dy}, \tag{15}$$

$$S_{zx} + \left(\frac{\lambda_3 - 2\lambda_1}{2}\right) S_{zy} \frac{du}{dy} = 0, \tag{16}$$

$$S_{yy} + (\lambda_3 + \lambda_6) S_{xy} \frac{du}{dy} = \mu(\lambda_4 + \lambda_7)\left(\frac{du}{dy}\right)^2, \tag{17}$$

$$S_{zy} + \frac{\lambda_3}{2} S_{zx} \frac{du}{dy} = 0, \tag{18}$$

$$S_{zz} + \lambda_6 S_{xy} \frac{du}{dy} = \mu\lambda_7 \left(\frac{du}{dy}\right)^2, \tag{19}$$

$$S_{xx} + S_{yy} = 2\mu(\lambda_4 + \lambda_7 - \lambda_2)\left(\frac{du}{dy}\right)^2 - 2(\lambda_3 + \lambda_6 - \lambda_1) S_{xy} \frac{du}{dy}. \tag{20}$$

Making use of Eqs. (16) and (18) we have

$$S_{zx} = S_{zy} = 0. \tag{21}$$

From Eqs. (13), (15), (19) and (21), we get

$$\frac{\partial p_1}{\partial z} = 0, \tag{22}$$

$$S_{xy} - \lambda_1 S_{yy}\frac{du}{dy} + \left(\frac{\lambda_3+\lambda_5}{2}\right)(S_{xx}+S_{yy})\frac{du}{dy} - \frac{\lambda_5\lambda_6}{2}S_{xy}\left(\frac{du}{dy}\right)^2 + \frac{\mu\lambda_5\lambda_7}{2}\left(\frac{du}{dy}\right)^3 = \mu\frac{du}{dy}. \tag{23}$$

Taking

$$\hat{p} = p_1 - S_{yy} \tag{24}$$

we can write from Eqs. (11), (12) and (22):

$$\frac{\partial \hat{p}}{\partial x} = \frac{d}{dy}S_{xy} - \sigma B_0^2 u, \tag{25}$$

$$\frac{\partial \hat{p}}{\partial y} = \frac{\partial \hat{p}}{\partial z} = 0. \tag{26}$$

We note from Eq. (26) that $\hat{p}$ is independent of y and z and is a function of x only. Thus Eq. (25) becomes

$$\frac{d\hat{p}}{dx} = \frac{d}{dy}S_{xy} - \sigma B_0^2 u. \tag{27}$$

Using Eqs. (14) to (17), (20) and (23) we get:

$$S_{xx} = \frac{1}{M}\left\{\begin{array}{l} \mu[(\lambda_4+\lambda_7)-(\lambda_3+\lambda_6)+2(\lambda_1-\lambda_2)]\left(\frac{du}{dy}\right)^2 \\ \quad +\mu[(\lambda_4+\lambda_7)\alpha_2-\alpha_1(\lambda_3+\lambda_6)+2(\alpha_1\lambda_1-\alpha_2\lambda_2)]\left(\frac{du}{dy}\right)^4 \end{array}\right\}, \tag{28}$$

$$S_{xy} = \frac{1}{M}\left\{\mu\frac{du}{dy} + \mu\alpha_1\left(\frac{du}{dy}\right)^3\right\}, \tag{29}$$

$$S_{yy} = \frac{1}{M}\left\{\begin{array}{l} \mu[(\lambda_4+\lambda_7)-(\lambda_3+\lambda_6)]\left(\frac{du}{dy}\right)^2 \\ \quad +\mu[(\lambda_4+\lambda_7)\alpha_2-\alpha_1(\lambda_3+\lambda_6)](\frac{du}{dy})^4 \end{array}\right\}. \tag{30}$$

In the above equations

$$\alpha_1 = \lambda_1(\lambda_4+\lambda_7) - (\lambda_3+\lambda_5)(\lambda_4+\lambda_7-\lambda_2) - \frac{\lambda_5\lambda_7}{2}, \tag{31}$$

$$\alpha_2 = \lambda_1(\lambda_3+\lambda_6) - (\lambda_3+\lambda_5)(\lambda_3+\lambda_6-\lambda_1) - \frac{\lambda_5\lambda_6}{2}, \tag{32}$$

$$M = 1 + \alpha_2\left(\frac{du}{dy}\right)^2. \tag{33}$$

Making use of Eq. (29) in Eq. (27) we get the following nonlinear differential equation:

$$\frac{d^2u}{dy^2} + \left[(3\alpha_1-\alpha_2) + \alpha_1\alpha_2\left(\frac{du}{dy}\right)^2\right]\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} - \frac{1}{\mu}\left(\sigma B_0^2 u + \frac{d\hat{p}}{dx}\right)\left[1+\alpha_2\left(\frac{du}{dy}\right)^2\right]^2 = 0. \tag{34}$$

3 Plane Couette flow

Let us consider the steady flow of an incompressible Oldroyd 8-constant fluid between two parallel plates of infinite length at $y = 0$ and $y = d$. When the pressure $\hat{p}$ is constant (or there is zero pressure gradient in the x-direction) the velocity is zero everywhere for the given flow field. To maintain a velocity field, it is necessary to set one of the plates in motion. In this case, it is assumed that the top plate is moving at velocity U; the bottom plate is at rest. In the absence of a pressure gradient the governing equation (34) becomes

$$\frac{d^2u}{dy^2} + \left[(3\alpha_1 - \alpha_2) + \alpha_1\alpha_2\left(\frac{du}{dy}\right)^2\right]\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} - \frac{\sigma B_0^2}{\mu}u\left[1 + \alpha_2\left(\frac{du}{dy}\right)^2\right]^2 = 0 \tag{35}$$

with the boundary conditions

$$u = 0 \quad \text{for } y = 0,$$
$$u = U \quad \text{for } y = d. \tag{36}$$

Introducing the following dimensionless parameters:

$$u^* = \frac{u}{U}, \quad y^* = \frac{y}{d}, \quad \alpha_1^* = \frac{\alpha_1}{(d/U)^2}, \quad \alpha_2^* = \frac{\alpha_2}{(d/U)^2}, \quad m^{*^2} = \frac{\sigma B_0^2}{\mu/d^2} \tag{37}$$

the above governing boundary value problem after dropping "$*$" becomes

$$\frac{d^2u}{dy^2} + \left[(3\alpha_1 - \alpha_2) + \alpha_1\alpha_2\left(\frac{du}{dy}\right)^2\right]\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} - m^2u\left[1 + \alpha_2\left(\frac{du}{dy}\right)^2\right]^2 = 0, \tag{38}$$

$$u = 0 \quad \text{for } y = 0,$$
$$u = 1 \quad \text{for } y = 1. \tag{39}$$

We apply the homotopy analysis method to give an explicit, uniformly valid and totally analytic solution to the given problem. Using

$$\mathscr{L}[\bar{u}(y;p)] = \frac{\partial^2\bar{u}(y;p)}{\partial y^2} - m^2\bar{u}(y;p) \tag{40}$$

as an auxiliary linear operator, where $p \in [0, 1]$ is an embedding parameter, let us construct the zeroth-order deformation equation as

$$(1-p)\mathscr{L}[\bar{u}(y;p) - u_0(y)] = p\hbar\left[\frac{\partial^2\bar{u}(y;p)}{\partial y^2} + \left\{(3\alpha_1 - \alpha_2) + \alpha_1\alpha_2\left(\frac{\partial\bar{u}(y;p)}{\partial y}\right)^2\right\}\right.$$
$$\left.\times\left(\frac{\partial\bar{u}(y;p)}{\partial y}\right)^2\frac{\partial^2\bar{u}(y;p)}{\partial y^2} - m^2\bar{u}(y;p)\left\{1 + \alpha_2\left(\frac{\partial\bar{u}(y;p)}{\partial y}\right)^2\right\}^2\right] \tag{41}$$

with the boundary conditions

$$\bar{u}(0;p) = 0, \quad \bar{u}(1;p) = 1, \tag{42}$$

in which

$$u_0(y) = \beta_1 \sinh my \tag{43}$$

as the initial approximation of $u(y)$, where

$$\beta_1 = \frac{1}{\sinh m}$$

and $\hbar$ is a nonzero auxiliary parameter.

When $p = 0$, it is straightforward that the solution of Eqs. (41) and (42) is

$$\bar{u}(y;0) = u_0(y). \tag{44}$$

When $p = 1$, Eqs. (41) and (42) are the same as Eqs. (38) and (39), provided

$$\bar{u}(y;1) = u(y). \tag{45}$$

Therefore, according to Eqs. (44) and (45), the variation of p from 0 to 1 is just the continuous variation of $\bar{u}(y;p)$ from the initial approximation $u_0(y)$ to the unknown solution $u(y)$ of Eqs. (38) and (39).

Assume that the deformation $\bar{u}(y;p)$ governed by Eqs. (41) and (42) is smooth enough so that

$$u_0^{[k]}(y) = \left.\frac{\partial^k \bar{u}(y;p)}{\partial p^k}\right|_{p=0} \qquad (k \geq 1) \tag{46}$$

namely the kth-order deformation derivative exists.

Using Eq. (44), it is straightforward to expand $\bar{u}(y;p)$ in power series of the embedding parameter p as follows:

$$\bar{u}(y;p) = u_0(y) + \sum_{k=1}^{+\infty} u_k(y)p^k, \tag{47}$$

where

$$u_k(y) = \frac{1}{k!}\left.\frac{\partial^k \bar{u}(y;p)}{\partial p^k}\right|_{p=0} \qquad (k \geq 1). \tag{48}$$

Note that the zeroth-order deformation equation (41) contains a nonzero auxiliary parameter $\hbar$. Thus, $\bar{u}(y;p)$ and $u_k(y)$ are dependent upon the auxiliary parameter $\hbar$. Obviously, $\hbar$ also affects the convergence rate and region of the series (47). Assume that $\hbar$ is so properly chosen that series (47) is convergent at $p = 1$. Then, due to Eqs. (45) and (47), we have the relationship

$$u(y) = u_0(y) + \sum_{k=1}^{+\infty} u_k(y). \tag{49}$$

Differentiating k-times the zeroth-order deformation equations (41) and (42) with respect to p and then dividing them by $k!$ and finally setting $p = 0$, we have, due to definition (48), the kth-order deformation problem

$$\begin{aligned}\mathscr{L}[u_k(y) - \chi_k u_{k-1}(y)] = \hbar \Bigg[& u''_{k-1} + \sum_{n=0}^{k-1} u''_{k-n-1} \sum_{i=0}^{n} u'_{n-i} \left(\begin{array}{c} (3\alpha_1 - \alpha_2)u'_i \\ +\alpha_1\alpha_2 \sum_{j=0}^{i} u'_{i-j} \sum_{r=0}^{j} u'_{j-r}u'_r \end{array} \right) \\ & -m^2 \left\{ u_{k-1} + \alpha_2 \sum_{n=0}^{k-1} u_{k-n-1} \sum_{i=0}^{n} u'_{n-i} \left(2u'_i + \alpha_2 \sum_{j=0}^{i} u'_{i-j} \sum_{r=0}^{j} u'_{j-r}u'_r \right) \right\} \Bigg],\end{aligned} \tag{50}$$

$$u_k(0) = u_k(1) = 0, \tag{51}$$

where

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k \geq 2 \end{cases}$$

and prime denotes the derivative with respect to y.

Now solving Eq. (50) subject to the boundary conditions (51) up to second-order of approximation, we obtain the three terms solution of the given problem (38) and (39) as follows:

$$u(y) = u_0(y) + u_1(y) + u_2(y), \tag{52}$$

where

$$u_1(y) = \hbar[f_1 \sinh 5my + f_2 \sinh 3my + f_3 my \cosh my + f_4 \sinh my], \tag{53}$$

$$u_2(y) = \frac{\hbar^2}{2!}\left[\begin{array}{c} f_5 \sinh 9my + f_6 \sinh 7my + f_7 \sinh 5my + f_8 \sinh 3my + (f_{13} + f_9 m^2 y^2) \\ \times \sinh my + my(f_{10} \cosh 5my + f_{11} \cosh 3my + f_{12} \cosh my) \end{array}\right] \tag{54}$$

and the involving constants are given in the Appendix.

4 Plane Poiseuille flow

Here the fluid is again bounded between two parallel plates of infinite length at $y = 0$ and $y = d$, which are at rest, but now a constant pressure gradient is applied in the x-direction to generate motion. For this flow our governing equation is (34) and the boundary conditions are given by

$$u = 0 \quad \text{for } y = 0,$$

$$u = 0 \quad \text{for } y = d. \tag{55}$$

The dimensionless boundary value problem is of the following form:

$$\frac{d^2u}{dy^2} + \left[(3\alpha_1 - \alpha_2) + \alpha_1\alpha_2\left(\frac{du}{dy}\right)^2\right]\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} - \left(m^2u + \frac{d\hat{p}}{dx}\right)\left[1 + \alpha_2\left(\frac{du}{dy}\right)^2\right]^2 = 0, \tag{56}$$

$$u = 0 \quad \text{for } y = 0,$$

$$u = 0 \quad \text{for } y = 1, \tag{57}$$

where

$$u^* = \frac{u}{U}, \quad x^* = \frac{x}{d}, \quad y^* = \frac{y}{d}, \quad \alpha_1^* = \frac{\alpha_1}{(d/U)^2},$$

$$\alpha_2^* = \frac{\alpha_2}{(d/U)^2}, \quad \hat{p}^* = \frac{\hat{p}}{\mu U/d}, \quad m^{*^2} = \frac{\sigma B_0^2}{\mu/d^2}. \tag{58}$$

From Eq. (56) and boundary conditions (57), we choose

$$u_0(y) = \frac{C}{m^2}[\cosh my + \beta_2 \sinh my - 1] \tag{59}$$

as the initial approximation of $u(y)$, where

$$\beta_2 = \frac{1-\cosh m}{\sinh m}, \quad C = \frac{d\hat{p}}{dx}.$$

Let us construct the zeroth-order deformation equation

$$(1-p)\mathscr{L}[\bar{u}(y;p) - u_0(y)] = p\hbar\left[\frac{\partial^2 \bar{u}(y;p)}{\partial y^2} + \left\{(3\alpha_1 - \alpha_2) + \alpha_1\alpha_2\left(\frac{\partial \bar{u}(y;p)}{\partial y}\right)^2\right\}\right.$$
$$\left.\times\left(\frac{\partial \bar{u}(y;p)}{\partial y}\right)^2 \frac{\partial^2 \bar{u}(y;p)}{\partial y^2} - (m^2\bar{u}(y;p) + C)\left\{1 + \alpha_2\left(\frac{\partial \bar{u}(y;p)}{\partial y}\right)^2\right\}^2\right] \tag{60}$$

with the boundary conditions

$$\bar{u}(0;p) = 0 = \bar{u}(1;p). \tag{61}$$

Differentiating k-times the zeroth-order deformation equations (60) and (61) with respect to p and then dividing them by $k!$ and finally setting $p = 0$, we have, due to definition (48), the following kth-order deformation problem:

$$\begin{aligned}\mathscr{L}[u_k(y) - \chi_k u_{k-1}(y)] = \hbar\Bigg[& u''_{k-1} + \sum_{n=0}^{k-1} u''_{k-n-1}\sum_{i=0}^{n} u'_{n-i}\left(\begin{array}{c}(3\alpha_1-\alpha_2)u'_i \\ +\ \alpha_1\alpha_2 \sum_{j=0}^{i} u'_{i-j}\sum_{r=0}^{j} u'_{j-r}u'_r\end{array}\right) \\ & -m^2\left\{u_{k-1} + \alpha_2\sum_{n=0}^{k-1} u_{k-n-1}\sum_{i=0}^{n} u'_{n-i}\left(2u'_i + \alpha_2\sum_{j=0}^{i} u'_{i-j}\sum_{r=0}^{j} u'_{j-r}u'_r\right)\right\} \\ & -\alpha_2 C\sum_{n=0}^{k-1} u'_{k-n-1}\left(2u'_n + \alpha_2\sum_{i=0}^{n} u'_{n-i}\sum_{j=0}^{i} u'_{i-j}u'_j\right)\Bigg],\end{aligned} \tag{62}$$

$$u_k(0) = u_k(1) = 0. \tag{63}$$

Now solving Eq. (62) subject to boundary conditions (63) up to second-order approximations, we obtain the three terms solution of the given problem (56) and (57) which is given by Eq. (52) in which

$$u_1(y) = \hbar\begin{bmatrix}(l_1 + l_9 + l_2 my)\cosh my + l_3\cosh 3my + l_4\cosh 5my \\ +(l_6 + l_{10} + l_5 my)\sinh my + l_7\sinh 3my + l_8\sinh 5my\end{bmatrix}, \tag{64}$$

$$u_2(y) = \frac{\hbar^2}{2!}\begin{bmatrix}(l_{11} + l_{27} + l_{12}my + l_{13}m^2y^2)\cosh my + (l_{14} + l_{15}my)\cosh 3my \\ +(l_{16} + l_{17}my)\cosh 5my + l_{18}\cosh 7my + l_{19}\cosh 9my \\ -\left(\frac{1}{2}l_{12} - l_{28} + 2l_{11}my - l_{20}m^2y^2\right)\sinh my + (l_{21} + l_{22}my)\sinh 3my \\ +(l_{23} + l_{24}my)\sinh 5my + l_{25}\sinh 7my + l_{26}\sinh 9my\end{bmatrix} \tag{65}$$

and the different constants are given in the Appendix.

5 Generalized Couette flow

In this case the velocity distribution is dependent on both the motion of the upper plate and the pressure gradient. For this flow the governing dimensionless problem consists of Eq. (56) and boundary condition (39). We choose

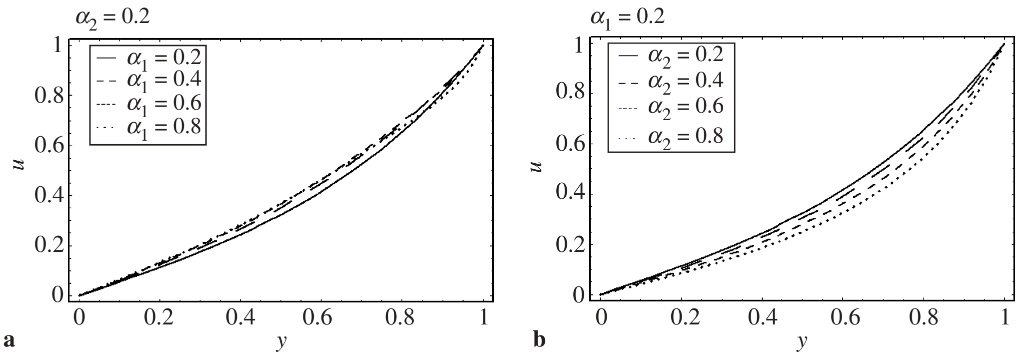

Fig. 1. Profiles of the dimensionless velocity $u(y)$ for plane Couette flow with various values of non–Newtonian parameters α_1 and α_2, respectively, for fixed values of $\hbar = -0.1$ and $m = 2$

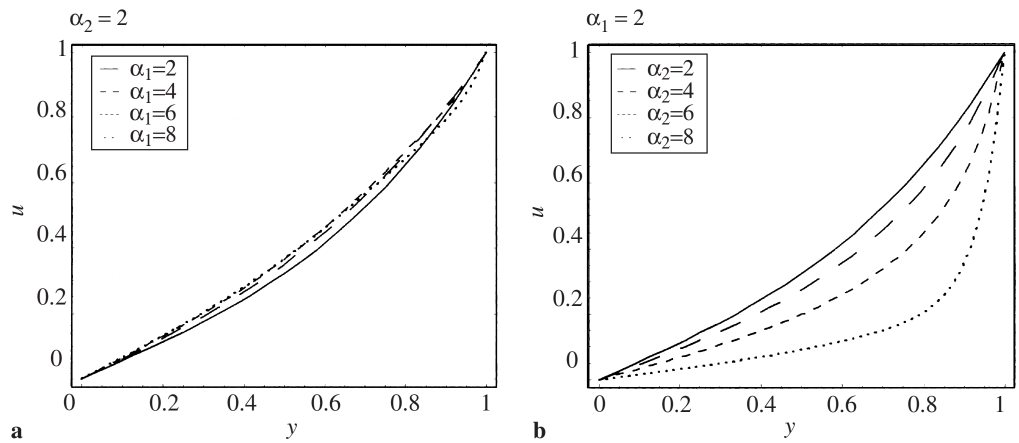

Fig. 2. Profiles of the dimensionless velocity $u(y)$ for plane Couette flow with various values of non–Newtonian parameters α_1 and α_2, respectively, for fixed values of $\hbar = -0.01$ and $m = 2$

$$u_0(y) = \frac{C}{m^2}[\cosh my + \beta_3 \sinh my - 1] \tag{66}$$

as the initial approximation of $u(y)$, where

$$\beta_3 = \frac{(1 - \cosh m) + \frac{m^2}{C}}{\sinh m}.$$

We observe that expressions (59) and (66) are the same except β_3 replaces β_2. Thus, the solutions in case of a generalized Couette flow can be obtained by replacing β_2 with β_3 in Sect. 4.

6 Discussion of results

6.1 Plane Couette flow

In Fig. 1, the velocity profiles are plotted for various values of the non–Newtonian parameters α_1 and α_2, respectively. From Eq. (34), we note that if $\alpha_1 = \alpha_2$ the Oldroyd fluid gives identical results to that of a Newtonian fluid. Therefore, the solid curves in Fig. 1 corresponding to

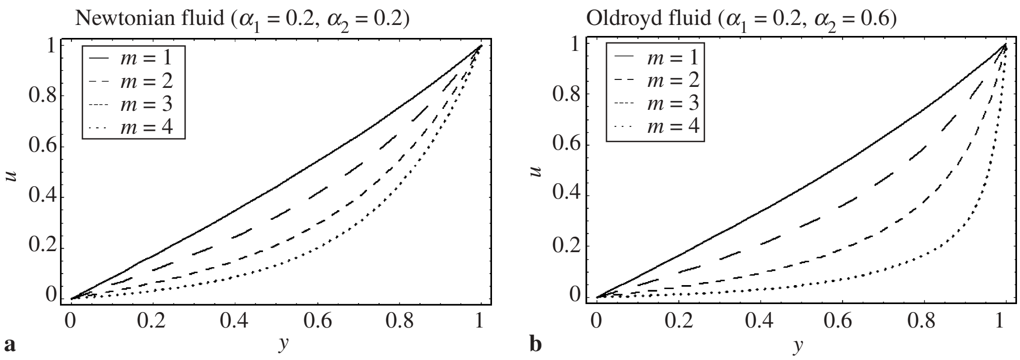

Fig. 3. Profiles of the dimensionless velocity $u(y)$ for plane Couette flow with various values of the magnetic parameter m for fixed values of $\hbar = -0.1$

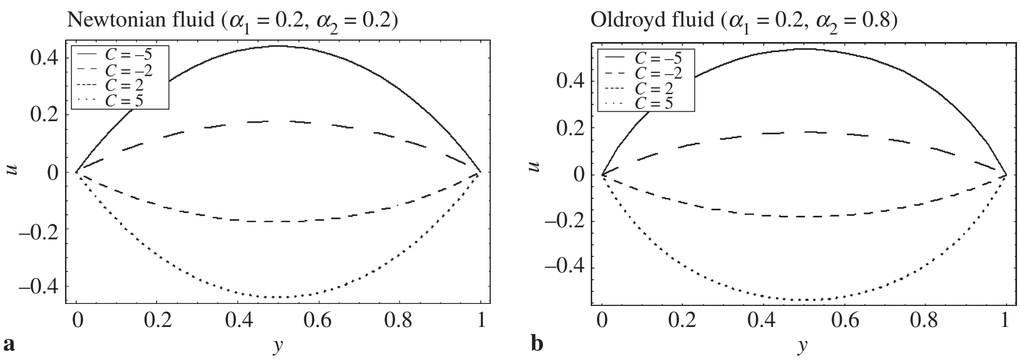

Fig. 4. Profiles of the dimensionless velocity $u(y)$ for plane Poiseuille flow with various values of the pressure gradient C for fixed values of $\hbar = -0.1$ and $m = 2$

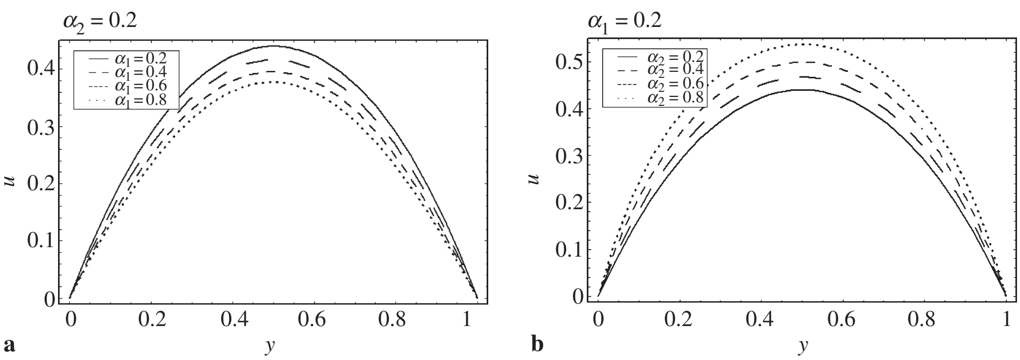

Fig. 5. Profiles of the dimensionless velocity $u(y)$ for plane Poiseuille flow with various values of the non–Newtonian material parameters α_1 and α_2, respectively, for fixed values of $\hbar = -0.1$, $C = -5$ and $m = 2$

$\alpha_1 = \alpha_2 = 0.2$ give the behavior of a Newtonian fluid. Furthermore, in the absence of a pressure gradient and magnetic parameter, Eq. (34) becomes $d^2u/dy^2 = 0$ for both the Newtonian and Oldroyd fluids, and the flow velocity is linear. It is obvious in Fig. 1a that for an Oldroyd 8-constant fluid, when the material parameter α_1 increases from $\alpha_1 = 0.2$ to 0.8 and fixed $\alpha_2 = 0.2$, the flow profiles tend to approach the linear distribution; thus, the shearing can

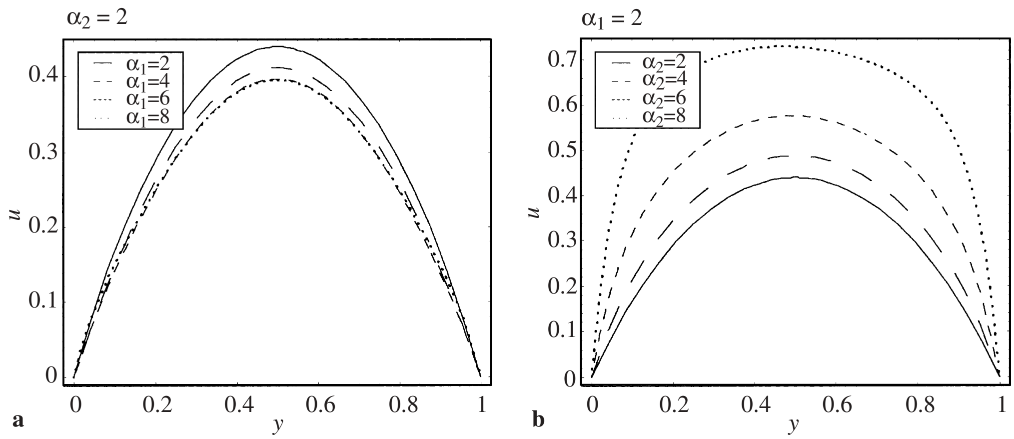

Fig. 6. Profiles of the dimensionless velocity $u(y)$ for plane Poiseuille flow with various values of the non–Newtonian material parameters α_1 and α_2, respectively, for fixed values of $\hbar = -0.1$, $C = -5$ and $m = 2$

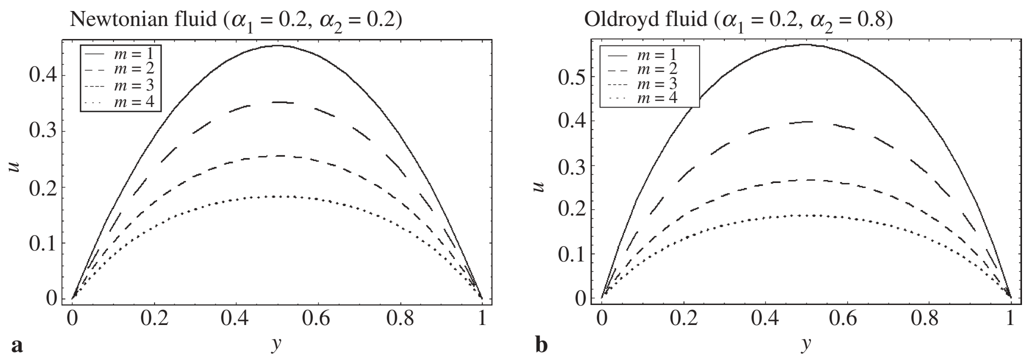

Fig. 7. Profiles of the dimensionless velocity $u(y)$ for Poiseuille flow with various values of the magnetic parameter m for fixed values of $\hbar = -0.1$ and $C = -4$

unattenuately extend to the whole domain from the boundaries, corresponding to a shear thickening phenomenon.

For $\alpha_2 = 0.2$ to 0.8 and fixed $\alpha_1 = 0.2$ Fig. 1b indicates an opposite phenomenon. This shows the shear thinning effects of the examined non–Newtonian fluid. Also from Figs. 1 and 2, it is observed that as $\hbar$ tends to zero from below, the convergence region enlarges.

Figure 3a and b provides the effects of a magnetic parameter for Newtonian and Oldroyd 8-constant fluids, respectively. From this figure, we observe that an increase in the magnetic parameter decreases the velocity profile.

6.2 Plane Poiseuille flow

The velocity profiles of a plane Poiseuille flow ($C \neq 0$) are shown in Figs. 4 to 7 for both a Newtonian fluid (Fig. 4a) and an Oldroyd 8-constant fluid (Fig. 4b). For both cases, symmetric parabolic flow profiles stretched between two zero boundary values are formed. The amplitudes of these parabolic profiles are strongly dependent on the magnitude of the pressure gradient, and the flow directions are against the direction of the pressure gradient. For $\alpha_1 = 0.2$ and $\alpha_2 = 0.8$ (Fig. 4b) the flow velocities are obviously much larger than those of the Newtonian fluid (Fig. 4a). However, this result cannot be generalized to an Oldroyd 8-constant fluid with other chosen material parameters.

From Fig. 5, we observe that in case of a Poiseuille flow the velocity profiles depend on the material parameters α_1 and α_2. It is found from Fig. 5a that for fixed $\alpha_2 = 0.2$ the velocity decreases by increasing $\alpha_1 = 0.2$ to 0.8. From Fig. 5b, we observe that the variation of α_2 from $\alpha_2 = 0.2$ to 0.8 gives the increase in the velocity. In case of Poiseuille flow, the variation of $\hbar$ regarding convergence can easily be seen from Figs. 5 and 6. It is clear that as $\hbar$ tends to zero from below, the convergence region enlarges. It is obvious, from Fig. 7, that the magnetic effects in this case are identical to that of the Couette flow case.

6.3 Generalized Couette flow

Figures 8 to 11 are prepared for a generalized Couette flow. For a favorable pressure gradient ($C < 0$), i.e., its direction is opposite to the velocity U of the top plate, the velocity is positive for both the Newtonian fluid (Fig. 8a) and Oldroyd fluid (Fig. 8b) across the entire cross section. But for an adverse pressure gradient ($C > 0$), i.e., its direction is the same as that of U, the velocity may either be all positive or a combination of a positive and negative regime for both the Newtonian fluid and Oldroyd fluid depending on the value of the adverse pressure gradient.

Figure 9 shows the effects of the material parameters α_1 and α_2 for the case of generalized Couette flow. From Fig. 9a, it is noted that the curvature of the velocity profile decreases by increasing α_1 from $\alpha_1 = 0.2$ to 0.8 for fixed $\alpha_2 = 0.2$ and approaches to a linear distribution. Further, the velocity profile becomes more parabolic and increases when α_2 varies from 0.2 to 0.8 and $\alpha_1 = 0.2$. From Figs. 9 and 10, we found identical behavior of $\hbar$ in the present case as for the cases of Couette and Poiseuille flows. Also from Fig. 11, it is revealed that the magnetic parameter has the same influence in this case as for Couette and Poiseuille flows.

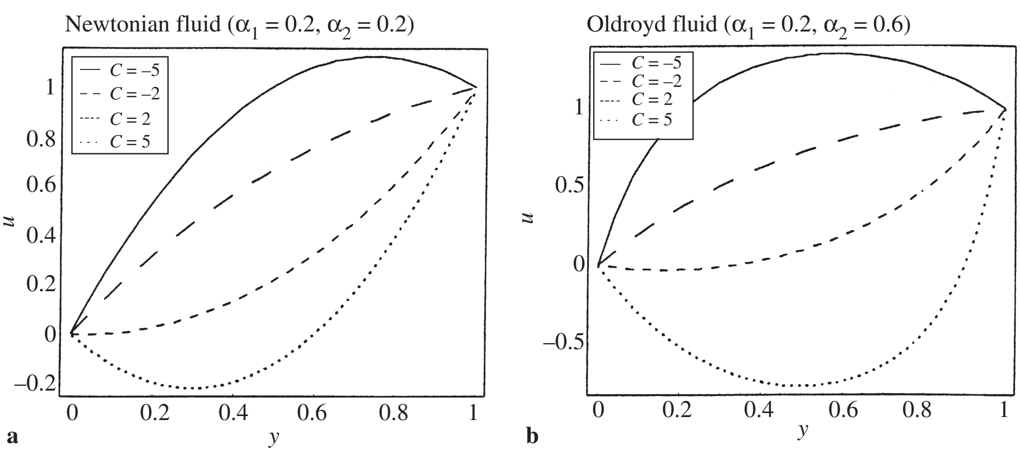

Fig. 8. Profiles of the dimensionless velocity $u(y)$ for generalized Couette flow with various values of the pressure gradient C for fixed values of $\hbar = -0.1$ and $m = 1$

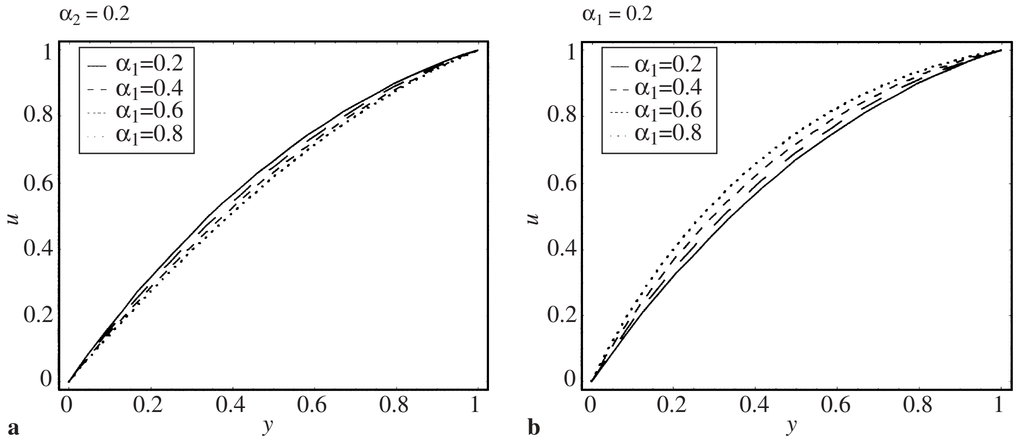

Fig. 9. Profiles of the dimensionless velocity $u(y)$ for generalized Couette flow with various values of the non–Newtonian material parameters α_1 and α_2, respectively, for fixed values of $\hbar = -0.1, C = -2$ and $m = 1$

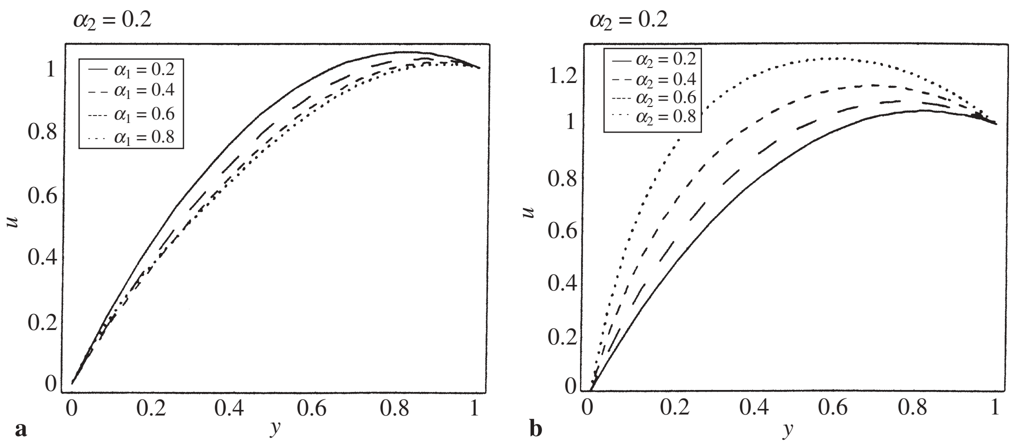

Fig. 10. Profiles of the dimensionless velocity $u(y)$ for generalized Couette flow with various values of the non–Newtonian material parameters α_1 and α_2, respectively, for fixed values of $\hbar = -0.01, C = -2$ and $m = 1$

7 Conclusions

In this paper, the modeling for MHD flow of an Oldroyd 8-constant fluid is given. Some analytic solutions of the governing nonlinear equations are discussed. These are the Couette flow, flow between two parallel plates one of which is suddenly moved, Poiseuille flow and generalized Couette flow which is a superposition of the Couette flow and the Poiseuille flow. The solutions of these flows are obtained using HAM.

The major findings of the present study can be summarized as follows:

(i) The Oldroyd 8-constant fluid is the general case of the Newtonian, Maxwell, second grade, Oldroyd 3 and 6-constant fluid. When $\lambda_i = 0\,(i = 1 \text{ to } 7)$, it reduces to a Newtonian fluid. For $\lambda_1 \neq 0$, $\lambda_j = 0\,(j = 2 \text{ to } 7)$, it corresponds to a Maxwell fluid. When $\lambda_1 = 0$, $\mu\lambda_2 = \widetilde{\alpha}_1$,

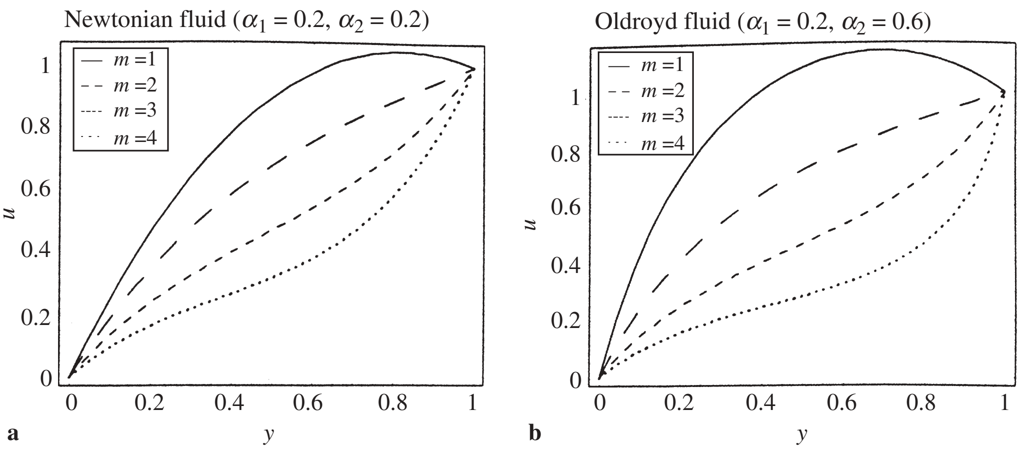

Fig. 11. Profiles of the dimensionless velocity $u(y)$ for generalized Couette flow with various values of the magnetic parameter m for fixed values of $\hbar = -0.1, C = -2$

$\lambda_k = 0\,(k = 3$ to $7)$ it reduces to a second grade fluid. For $\lambda_1 \neq 0 \neq \lambda_2$, $\lambda_k = 0\,(k = 3$ to $7)$, it becomes an Oldroyd 3-constant fluid and for $\lambda_6 = \lambda_7 = 0$ it reduces to an Oldroyd 6-constant fluid.

(ii) The solutions (52) obtained for Couette, Poiseuille and generalized Couette flows are valid for all values of the non–Newtonian parameters α_1 and α_2 (involving μ, λ_1 to λ_7) which is different from the case of an Oldroyd 3-constant fluid. It is further remarked here that solutions (52) for unidirectional steady flow of an Oldroyd 3-constant fluid (which involve μ, λ_1 and λ_2 only) are identical to that of the Newtonian fluid. It is also clear that Eq. (34) for Oldroyd 3-constants (i.e., μ, λ_1 and λ_2) corresponds to Newtonian fluid when λ_3 to λ_7 are zero in case of an Oldroyd 8-constant fluid. Further, the results obtained in the present analysis can easily be compared to the results of Nield [19] which are valid for Newtonian fluid by the appropriate choice of the involved parameters.

(iii) The increase of $\alpha_1\,(\alpha_2)$ decreases (increases) the velocity profile for the Poiseuille and generalized Couette flows.

(iv) The velocity in case of an Oldroyd 8-constant fluid is larger (smaller) than that of a Newtonian fluid case for $\alpha_1 < \alpha_2\,(\alpha_1 > \alpha_2)$.

(v) The convergence region of the obtained results is strongly dependent upon the choice of $\hbar$. Further, the convergence region is found to enlarge as $\hbar$ tends to zero from below.

(vi) The increase in the magnetic parameter decreases the velocity.

Appendix

Here, we provide the values of different constants appearing in Sects. 3 and 4:

$$f_1 = b_1 b_2, \quad f_2 = 3b_1(1 + 3b_2), \quad f_3 = 12b_1(1 + 2b_2),$$

$$f_4 = \frac{-1}{\sinh m}(f_1 \sinh 5m + f_2 \sinh 3m + f_3 m \cosh m),$$

$$f_5 = \frac{3}{640} f_1 m^4 \beta_1^4 (15\alpha_1 - 7\alpha_2)\alpha_2,$$

$$f_6 = \frac{1}{384} m^2\beta_1^2\left[12f_1(35\alpha_1 - 19\alpha_2) + m^2\beta_1^2\{(140f_1 + 21f_2)\alpha_1 - (44f_1 + 13f_2)\alpha_2\}\alpha_2\right],$$

$$f_7 = \frac{1}{2304}\left[\begin{array}{c} 4608\left(1+\frac{1}{\hbar}\right)f_1 + 48m^2\beta_1^2\{(150f_1 + 45f_2)\alpha_1 - (54f_1 + 29f_2)\alpha_2\} \\ +m^4\beta_1^4\left\{\begin{array}{c}(1800f_1 + 720f_2 + 47f_3 - 60f_4)\alpha_1 \\ -(72f_1 + 336f_2 + 23f_3 - 60f_4)\alpha_2\end{array}\right\}\alpha_2 \end{array}\right],$$

$$f_8 = \frac{1}{256}\left[\begin{array}{c} 512\left(1+\frac{1}{\hbar}\right)f_2 + 4m^2\beta_1^2\left\{\begin{array}{c}3(60f_1 + 72f_2 + 7f_3 - 12f_4)\alpha_1 \\ +(12f_1 - 88f_2 - 5f_3 + 36f_4)\alpha_2\end{array}\right\} \\ + 3m^4\beta_1^4\{(80f_1 + 72f_2 + 15f_3 - 20f_4)\alpha_1 + (48f_1 - 8f_2 - 7f_3 + 20f_4)\alpha_2\}\alpha_2 \end{array}\right],$$

$$f_9 = \frac{1}{16}\left[f_3 m^2\beta_1^2(\alpha_1 - \alpha_2)\{6 + \alpha_2 m^2\beta_1^2\}\right],$$

$$f_{10} = \frac{5}{192}\left[f_3 m^4\beta_1^4(\alpha_1 - \alpha_2)\alpha_2\right],$$

$$f_{11} = \frac{9}{64}\left[f_3 m^2\beta_1^2(\alpha_1 - \alpha_2)\{4 + \alpha_2 m^2\beta_1^2\}\right],$$

$$f_{12} = \frac{1}{16}\left[\begin{array}{c} 32\left(1+\frac{1}{\hbar}\right)f_3 + 2m^2\beta_1^2\{3(6f_2 + 7f_3 - 6f_4)\alpha_1 + (14f_2 - 13f_3 + 18f_4)\alpha_2\} \\ + m^4\beta_1^4\{(5f_1 + 15f_2 + 11f_3 - 10f_4)\alpha_1 + (19f_1 + 9f_2 - 7f_3 + 10f_4)\alpha_2\}\alpha_2 \end{array}\right],$$

$$f_{13} = \frac{-1}{\sinh m}\left[\begin{array}{c} f_5\sinh 9m + f_6\sinh 7m + f_7\sinh 5m + f_8\sinh 3m + f_9 m^2\sinh m \\ + m(f_{10}\cosh 5m + f_{11}\cosh 3m + f_{12}\cosh m) \end{array}\right],$$

$$b_1 = \frac{(\alpha_1 - \alpha_2)m^2\beta_1^3}{32}, \quad b_2 = \frac{\alpha_2 m^2\beta_1^2}{12}, \quad b_3 = \frac{(\alpha_1 - \alpha_2)C^3}{384m^6},$$

$$l_1 = -12b_3(-1 + \beta_2^2)\left[6m^2 + C^2(-1 + \beta_2^2)\alpha_2\right],$$

$$l_2 = 24b_3\beta_2(-1 + \beta_2^2)\left[6m^2 + C^2(-1 + \beta_2^2)\alpha_2\right],$$

$$l_3 = 9b_3(1 + 3\beta_2^2)\left[4m^2 + C^2(-1 + \beta_2^2)\alpha_2\right],$$

$$l_4 = b_3C^2\left[1 + 5\beta_2^2(2 + \beta_2^2)\right]\alpha_2,$$

$$l_5 = 24b_3(-1 + \beta_2^2)\left[6m^2 + C^2(-1 + \beta_2^2)\alpha_2\right],$$

$$l_6 = -12b_3\beta_2(-1 + \beta_2^2)\left[6m^2 + C^2(-1 + \beta_2^2)\alpha_2\right],$$

$$l_7 = 9b_3\beta_2(3+\beta_2^2)\big[4m^2 + C^2(-1+\beta_2^2)\alpha_2\big],$$

$$l_8 = b_3\beta_2 C^2\big[5+\beta_2^2(10+\beta_2^2)\big]\alpha_2,$$

$$l_9 = -(l_1 + l_3 + l_4),$$

$$l_{10} = \frac{-1}{\sinh m}\begin{bmatrix} (l_1 + l_9 + l_2 m)\cosh m + l_3\cosh 3m + l_4\cosh 5m \\ +(l_6 + l_5 m)\sinh m + l_7\sinh 3m + l_8\sinh 5m \end{bmatrix},$$

$$\begin{aligned} l_{11} = \frac{C^2}{32m^4}\Bigg[& l_5\left\{\begin{array}{l} -32(1+\frac{1}{\hbar})\frac{m^4}{C^2} + 2m^2\big[21\alpha_1 - 13\alpha_2 - 3(5\alpha_1 + 3\alpha_2)\beta_2^2\big] \\ -C^2\alpha_2(-1+\beta_2^2)\big[-11\alpha_1 + 7\alpha_2 + (7\alpha_1 + 13\alpha_2)\beta_2^2\big] \end{array}\right\} \\ & + 36(-l_3 + l_9)m^2\alpha_1 - 4(7l_3 + 9l_9)m^2\alpha_2 + C^2\left\{\begin{array}{l} 5(3l_3 - l_4 - 2l_9)\alpha_1 \\ +(9l_3 - 19l_4 + 10l_9)\alpha_2 \end{array}\right\}\alpha_2 \\ & + 4\beta_2\left\{\begin{array}{l} 3(-2l_{10} - l_2 - 2l_6 + 6l_7)m^2\alpha_1 + C^2(-2l_{10} - 5l_2 - 2l_6 + l_7 + 19l_8)\alpha_2^2 \\ +\big[(6l_{10} + 11l_2 + 6l_6 + 14l_7)m^2 + C^2(2l_{10} + l_2 + 2l_6 - 9l_7 + 5l_8)\alpha_1\big]\alpha_2 \end{array}\right\} \\ & - 2\beta_2^2\left\{\begin{array}{l} 6(3l_3 + l_9)m^2\alpha_1 + \big[2(7l_3 - 3l_9)m^2 - 3C^2(3l_3 - 5l_4 + 2l_9)\alpha_1\big]\alpha_2 \\ + 3C^2(11l_3 + 19l_4 + 2l_9)\alpha_2^2 \end{array}\right\} \\ & + 4C^2\alpha_2\beta_2^3\{(-2l_{10} - l_2 - 2l_6 + 3l_7 + 5l_8)\alpha_1 + (2l_{10} + 5l_2 + 2l_6 + 21l_7 + 19l_8)\alpha_2\} \\ & - C^2\alpha_2\beta_2^4\{(9l_3 + 5l_4 + 2l_9)\alpha_1 + (31l_3 + 19l_4 - 2l_9)\alpha_2\} \\ & -2l_1(\alpha_1 - \alpha_2)\big\{6m^2(-3+\beta_2^2) + C^2\alpha_2(5 - 6\beta_2^2 + \beta_2^4)\big\}\Bigg], \end{aligned}$$

$$\begin{aligned} l_{12} = \frac{C^2}{16m^4}\Bigg[& l_2\left\{\begin{array}{l} 32(1+\frac{1}{\hbar})\frac{m^4}{C^2} + C^2\alpha_2(-1+\beta_2^2)\big[-7\alpha_1 - 13\alpha_2 + (11\alpha_1 - 7\alpha_2)\beta_2^2\big] \\ + 2m^2\big[3\alpha_1(-5+7\beta_2^2) - \alpha_2(9+13\beta_2^2)\big] \end{array}\right\} \\ & + 4l_7(9\alpha_1 + 7\alpha_2)m^2 + C^2\{(-9l_7 + 5l_8)\alpha_1 + (-31l_7 + 19l_8)\alpha_2\}\alpha_2 \\ & - 4\beta_2\left\{\begin{array}{l} 3(2l_1 + 6l_3 + l_5 + 2l_9)m^2\alpha_1 + C^2(2l_1 - 21l_3 + 19l_4 + 5l_5 + 2l_9)\alpha_2^2 \\ -\big[(6l_1 - 14l_3 + 11l_5 + 6l_9)m^2 + C^2(2l_1 + 3l_3 - 5l_4 + l_5 + 2l_9)\alpha_1\big]\alpha_2 \end{array}\right\} \end{aligned}$$

$$- 4C^2\alpha_2\beta_2^3\{(2l_1+9l_3+5l_4+l_5+2l_9)\alpha_1-(2l_1+l_3-19l_4+5l_5+2l_9)\alpha_2\}$$

$$+ C^2\alpha_2\beta_2^4\{5(3l_7+l_8)\alpha_1+(9l_7+19l_8)\alpha_2\}$$

$$\left. + 2(\alpha_1-\alpha_2)(l_6+l_{10})\left\{6m^2(-1+3\beta_2^2)+C^2\alpha_2(1-6\beta_2^2+5\beta_2^4)\right\}\right],$$

$$l_{13}=\frac{(\alpha_1-\alpha_2)C^2}{16m^4}\left[-4l_2\beta_2\left\{3m^2+C^2\alpha_2(-1+\beta_2^2)\right\}+l_5\left\{\begin{array}{c}6m^2(-1+3\beta_2^2)\\ +C^2\alpha_2(1-6\beta_2^2+5\beta_2^4)\end{array}\right\}\right],$$

$$l_{14}=\frac{C^2}{256m^4}\left[8l_3\left\{\begin{array}{l}64(1+\frac{1}{\hbar})\frac{m^4}{C^2}+4m^2(27\alpha_1-11\alpha_2)(-1+\beta_2^2)\\ \quad +3C^2(9\alpha_1-\alpha_2)(-1+\beta_2^2)^2\alpha_2\end{array}\right\}\right.$$

$$+ (84l_5+144l_9)m^2\alpha_1-(20l_5+144l_9)m^2\alpha_2+C^2\left\{\begin{array}{l}(-45l_5-60l_9)\alpha_1\\ +(21l_5+60l_9)\alpha_2\end{array}\right\}\alpha_2$$

$$- 4\beta_2\left\{\begin{array}{l}6(-12l_{10}-7l_2-12l_6+60l_8)m^2\alpha_1+3C^2(-4l_{10}+9l_2-4(l_6+6l_8))\alpha_2^2\\ \quad +\left[2(36l_{10}+5l_2+12(3l_6+l_8))m^2-3C^2(-4l_{10}+l_2-4l_6+40l_8)\alpha_1\right]\alpha_2\end{array}\right\}$$

$$+ 2\beta_2^2\left\{-36l_9(\alpha_1-\alpha_2)(-2m^2+C^2\alpha_2)+l_5\left[\begin{array}{c}2m^2(21\alpha_1-5\alpha_2)\\ +3C^2\alpha_2(-17\alpha_1+25\alpha_2)\end{array}\right]\right\}$$

$$- 4C^2\alpha_2\beta_2^3\{(-31l_2-36l_6+120l_8)\alpha_1-36l_{10}(\alpha_1-\alpha_2)+(23l_2+36l_6+72l_8)\alpha_2\}$$

$$+ C^2\alpha_2\beta_2^4\{36l_9(\alpha_1-\alpha_2)+l_5(11\alpha_1+29\alpha_2)\}$$

$$+ 48l_4(1+\beta_2^2)\left\{m^2(15\alpha_1+\alpha_2)+C^2\alpha_2(5\alpha_1+3\alpha_2)(-1+\beta_2^2)\right\}$$

$$\left. +12l_1(\alpha_1-\alpha_2)\left\{12m^2(1+\beta_2^2)+C^2\alpha_2(-5-6\beta_2^2+3\beta_2^4)\right\}\right],$$

$$l_{15}=\frac{3(\alpha_1-\alpha_2)C^2}{64m^4}\left[\begin{array}{c}4l_5\beta_2\left\{6m^2+C^2\alpha_2(-1+3\beta_2^2)\right\}\\ +l_2\left\{12m^2(1+\beta_2^2)+C^2\alpha_2(-5-6\beta_2^2+3\beta_2^4)\right\}\end{array}\right],$$

$$l_{16} = \frac{C^2}{2304m^4}\left[72l_4\left\{\begin{array}{c} 64(1+\frac{1}{\hbar})\frac{m^4}{C^2} + 4m^2(25\alpha_1 - 9\alpha_2)(-1+\beta_2^2) \\ + C^2\alpha_2(25\alpha_1 - \alpha_2)(1+\beta_2^2)^2 \end{array}\right\}\right.$$

$$+ 96l_7m^2(45\alpha_1 - 29\alpha_2)\beta_2 + 48l_3(1+\beta_2^2)\left\{\begin{array}{c} m^2(45\alpha_1 - 29\alpha_2) \\ +C^2\alpha_2(15\alpha_1 - 7\alpha_2)(-1+\beta_2^2) \end{array}\right\}$$

$$+ C^2\alpha_2\{60(\alpha_1 - \alpha_2)(l_1 + l_9) + l_5(47\alpha_1 - 23\alpha_2)\}(1 + 6\beta_2 + \beta_2^4)$$

$$+ 4\alpha_2\beta_2C^2\{(60l_{10} + 47l_2 + 60(l_6 - 6l_7))\alpha_1 - (60l_{10} + 23l_2 + 60l_6 - 168l_7)\alpha_2\}$$

$$\left. + 4\alpha_2\beta_2^3C^2\{(60l_{10} + 47l_2 + 60(l_6 + 6l_7))\alpha_1 - (60l_{10} + 23l_2 + 60l_6 + 168l_7)\alpha_2\}\right],$$

$$l_{17} = \frac{5(\alpha_1 - \alpha_2)\alpha_2C^4}{192m^4}\left[4l_5\beta_2(1+\beta_2^2) + l_2(1 + 6\beta_2^2 + \beta_2^4)\right],$$

$$l_{18} = \frac{C^2}{384m^4}\left[24l_8m^2(35\alpha_1 - 19\alpha_2)\beta_2 + 4l_4(1+\beta_2^2)\left\{\begin{array}{c} 3m^2(35\alpha_1 - 19\alpha_2) \\ + C^2(35\alpha_1 - 11\alpha_2)\alpha_2(-1+\beta_2^2) \end{array}\right\}\right.$$

$$\left. +C^2\alpha_2\left\{l_3(21\alpha_1 - 13\alpha_2)(1 + 6\beta_2^2 + \beta_2^4) + 4\beta_2\left(\begin{array}{c} 2l_8(35\alpha_1 - 11\alpha_2)(-1+\beta_2^2) \\ + l_7(21\alpha_1 - 13\alpha_2)(1+\beta_2^2) \end{array}\right)\right\}\right],$$

$$l_{19} = \frac{3(15\alpha_1 - 7\alpha_2)\alpha_2C^4}{640m^4}\left[4l_8\beta_2(1+\beta_2^2) + l_4(1 + 6\beta_2^2 + \beta_2^4)\right],$$

$$l_{20} = \frac{(\alpha_1 - \alpha_2)C^2}{16m^4}\left[\begin{array}{c} 4l_5\beta_2\{3m^2 + C^2\alpha_2(-1+\beta_2^2)\} \\ + l_2\{6m^2(-3+\beta_2^2) + C^2\alpha_2(5 - 6\beta_2^2 + \beta_2^4)\} \end{array}\right],$$

$$l_{21} = \frac{C^2}{256m^4}\left[8l_7\left\{\begin{array}{c} 64(1+\frac{1}{\hbar})\frac{m^4}{C^2} + 4m^2(27\alpha_1 - 11\alpha_2)(-1+\beta_2^2) \\ + 3C^2(9\alpha_1 - \alpha_2)\alpha_2(-1+\beta_2^2)^2 \end{array}\right\}\right.$$

$$+ 12l_{10}(\alpha_1 - \alpha_2)\{12m^2(1+\beta_2^2) + C^2\alpha_2(-3 + 6\beta_2^2 + 5\beta_2^4)\}$$

$$+ 48l_8\left[(15\alpha_1 + \alpha_2)m^2 - C^2\alpha_2(5\alpha_1 + 3\alpha_2)\right] + 48l_8m^2(15\alpha_1 + \alpha_2)\beta_2^2$$

$$+ l_2\left\{4m^2(21\alpha_1 - 5\alpha_2)(1+\beta_2^2) - C^2\alpha_2\begin{bmatrix}11\alpha_1 + 29\alpha_2 + 6(-17\alpha_1 + 25\alpha_2)\beta_2^2\\ \\ + 3(-15\alpha_1 + 7\alpha_2)\beta_2^4\end{bmatrix}\right\}$$

$$+ 4\beta_2\left\{\begin{matrix}6(12l_1 - 60l_4 + 7l_5 + 12l_9)m^2\alpha_1 + C^2(36l_1 + 72l_4 + 23l_5 + 36l_9)\alpha_2^2\\ \\ -\left[2(36l_1 + 12l_4 + 5l_5 + 36l_9)m^2 + C^2(36l_1 - 120l_4 + 31l_5 + 36l_9)\alpha_1\right]\alpha_2\end{matrix}\right\}$$

$$- 12C^2\alpha_2\{(-4l_1 + 40l_4 + l_5 - 4l_9)\alpha_1 + (4l_1 + 24l_4 - 9l_5 + 4l_9)\alpha_2\}\beta_2^3$$

$$\left. + 48C^2 l_8\alpha_2(5\alpha_1 + 3\alpha_2)\beta_2^4 + 12l_6(\alpha_1 - \alpha_2)\left\{\begin{matrix}12m^2(1+\beta_2^2)\\ \\ + C^2\alpha_2(-3 + 6\beta_2^2 + 5\beta_2^4)\end{matrix}\right\}\right],$$

$$l_{22} = \frac{3(\alpha_1 - \alpha_2)C^2}{16m^4}\begin{bmatrix}4l_2\beta_2\{6m^2 + C^2\alpha_2(-3+\beta_2^2)\}\\ + l_5\{12m^2(1+\beta_2^2) + C^2\alpha_2(-3 + 6\beta_2^2 + 5\beta_2^4)\}\end{bmatrix},$$

$$l_{23} = \frac{C^2}{2304m^4}\left[72l_8\left\{\begin{matrix}64(1+\frac{1}{\hbar})\frac{m^4}{C^2} + 4m^2(25\alpha_1 - 9\alpha_2)(-1+\beta_2^2)\\ \\ + C^2(25\alpha_1 - \alpha_2)\alpha_2(-1+\beta_2^2)^2\end{matrix}\right\}\right.$$

$$+ 96l_3m^2(45\alpha_1 - 29\alpha_2)\beta_2 + 48l_7(1+\beta_2^2)\left\{\begin{matrix}m^2(45\alpha_1 - 29\alpha_2)\\ \\ + C^2\alpha_2(15\alpha_1 - 7\alpha_2)(-1+\beta_2^2)\end{matrix}\right\}$$

$$+ C^2\alpha_2\{60(\alpha_1 - \alpha_2)(l_6 + l_{10}) + l_2(47\alpha_1 - 23\alpha_2)\}(1 + 6\beta_2 + \beta_2^4)$$

$$+ 4\alpha_2\beta_2C^2\{(60l_1 - 360l_3 + 47l_5 + 60l_9))\alpha_1 - (60l_1 - 168l_3 + 23l_5 + 60l_9)\alpha_2\}$$

$$\left. + 4\alpha_2\beta_2^3C^2\{(60l_1 + 360l_3 + 47l_5 + 60l_9))\alpha_1 - (60l_1 + 168l_3 + 23l_5 + 60l_9)\alpha_2\}\right],$$

$$l_{24} = \frac{5(\alpha_1 - \alpha_2)\alpha_2C^4}{192m^4}\left[4l_2\beta_2(1+\beta_2^2) + l_5(1 + 6\beta_2^2 + \beta_2^4)\right],$$

$$l_{25} = \frac{C^2}{384m^4}\left[24l_4m^2(35\alpha_1 - 19\alpha_2)\beta_2 + 4l_8(1+\beta_2^2)\left\{\begin{matrix}3m^2(35\alpha_1 - 19\alpha_2)\\ + C^2(35\alpha_1 - 11\alpha_2)\alpha_2(-1+\beta_2^2)\end{matrix}\right\}\right.$$

$$\left. + C^2\alpha_2\left\{l_7(21\alpha_1 - 13\alpha_2)(1 + 6\beta_2^2 + \beta_2^4) + 4\beta_2\begin{pmatrix}2l_4(35\alpha_1 - 11\alpha_2)(-1+\beta_2^2)\\ \\ + l_3(21\alpha_1 - 13\alpha_2)(1+\beta_2^2)\end{pmatrix}\right\}\right],$$

$$l_{26} = \frac{3(15\alpha_1 - 7\alpha_2)\alpha_2C^4}{640m^4}\left[4l_4\beta_2(1+\beta_2^2) + l_8(1 + 6\beta_2^2 + \beta_2^4)\right],$$

$$l_{27} = -(l_{11} + l_{14} + l_{16} + l_{18} + l_{19}),$$

$$l_{28} = \frac{-1}{\sinh m}\begin{bmatrix} (l_{11} + l_{27} + l_{12}m + l_{13}m^2)\cosh m + (l_{14} + l_{15}m)\cosh 3m \\ +(l_{16} + l_{17}m)\cosh 5m + l_{18}\cosh 7m + l_{19}\cosh 9m \\ -(\frac{1}{2}l_{12} + 2l_{11}m - l_{20}m^2)\sinh m + (l_{21} + l_{22}m)\sinh 3m \\ +(l_{23} + l_{24}m)\sinh 5m + l_{25}\sinh 7m + l_{26}\sinh 9m \end{bmatrix}.$$

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Authors' address: T. Hayat, M. Khan and S. Asghar, Department of Mathematics, Quaid-i-Azam University 45320, Islamabad, Pakistan (E-mail: t_pensy@hotmail.com)