Acta Mechanica 168, 111–117 (2004)
DOI 10.1007/s00707-004-0081-6

Acta Mechanica
Printed in Austria

Note

Love waves in piezoelectromagnetic materials

J. S. Yang, Lincoln, Nebraska

Received July 31, 2003
Published online: March 11, 2004

Summary. The propagation of Love waves on a piezoelectric half space of polarized ceramics carrying an elastic layer is studied from the three-dimensional equations of linear piezoelectromagnetism with full electromagnetic coupling. Two cases when the elastic layer is a perfect conductor or a dielectric are analyzed.

1 Introduction

The theory of linear piezoelectricity is under the quasi-static approximation [1]. In this theory, although the mechanical equations are dynamic, the electromagnetic equations are static and the electric field and the magnetic field are not coupled. Therefore it does not describe the wave behavior of electromagnetic fields. Electromagnetic waves generated by mechanical fields [2] need to be studied in the calculation of radiated electromagnetic power from a vibrating piezoelectric device [3]–[6] and are also relevant in acoustic delay lines [7] and wireless acoustic wave sensors [8] where acoustic waves produce electromagnetic waves or vice versa. When electromagnetic waves are involved, the complete set of Maxwell's equations needs to be used, coupled to the mechanical equations of motion. Such a fully dynamic theory has been called piezoelectromagnetism by some researchers [9], [10]. Forced thickness vibration analysis of AT-cut and doubly rotated quartz plates and calculations of radiated electromagnetic power were performed in [3]–[5]. Variational formulations of the theory of piezoelectromagnetism were given in [9], [10]. The propagation of plane waves in a piezoelectromagnetic medium was studied in [11]–[13]. Vibrations of a piezoelectromagnetic body were studied in [14], [15]. Time-harmonic surface waves in a lithium niobate half space were calculated numerically in [16]. Transient surface waves in a ceramic half space under a surface load were analyzed in [17]. A fully dynamic piezoelectromagnetic surface wave solution representing a generalization of the well-known Bleustein-Gulyaev quasistatic piezoelectric surface waves [18], [19] was obtained in [20]. In this paper, we analyze the propagation of piezoelectromagnetic Love waves over a half space of polarized ceramics carrying an elastic layer of finite thickness. The equations of linear piezoelectromagnetism are summarized in Sect. 2. The case when the elastic layer is an ideal conductor is studied in Sect. 3, and the case of a dielectric layer in Sect. 4.

2 Equations of piezoelectromagnetism

For a piezoelectric but nonmagnetizable dielectric body the three-dimensional equations of linear piezoelectromagnetism [9], [10] consist of the equations of motion and Maxwell's equations

$$\begin{aligned} &T_{ji,j} + \rho f_i = \rho \ddot{u}_i, \\ &\varepsilon_{ijk} E_{k,j} = -\dot{B}_i, \quad \varepsilon_{ijk} H_{k,j} = \dot{D}_i, \quad B_{i,i} = 0, \quad D_{i,i} = 0, \end{aligned} \tag{1}$$

as well as the following constitutive relations:

$$T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad D_i = e_{ijk} S_{jk} + \varepsilon_{ij} E_j, \quad B_i = \mu_0 H_i, \tag{2}$$

where

$$S_{ij} = (u_{i,j} + u_{j,i})/2. \tag{3}$$

In Eqs. (1)–(3), u_i is the mechanical displacement vector, T_{ij} the stress tensor, S_{ij} the strain tensor, E_i the electric field, D_i the electric displacement vector, B_i the magnetic induction, and H_i the magnetic field. The coefficients c_{ijkl}, e_{kij} and ε_{ij} are the elastic, piezoelectric and dielectric constants, μ_0 the magnetic permeability of free space, ε_{ijk} the permutation tensor, ρ the mass density and f_i the body force. The summation convention for repeated tensor indices and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index are used. The indices i, j, k, l assume 1, 2, 3. A superimposed dot represents differentiation with respect to the time t.

3 Ceramic half space with an elastic metal layer

Consider a piezoelectromagnetic half space of polarized ceramics carrying an elastic layer (Fig. 1). The surface at $x_2 = -h$ is traction free. In this section, we consider the case that the elastic layer is a perfect conductor.

For ceramics poled in the x_3-direction the material tensors in Eq. (2) are represented by the following matrices under the compact notation [1]:

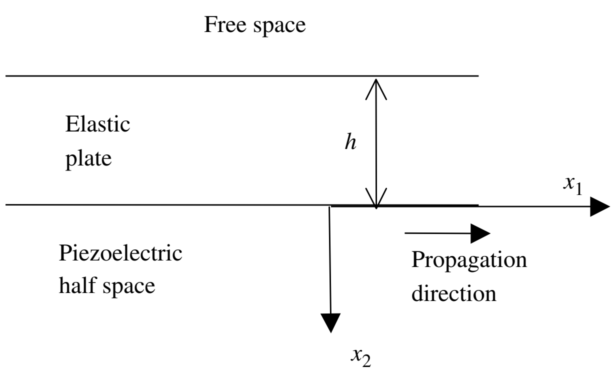

Fig. 1. A piezoelectric half space of polarized ceramics with an elastic layer

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}, \begin{pmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}, \tag{4}$$

where $c_{66} = (c_{11} - c_{12})/2$. We consider planar motions with

$$\begin{aligned} &u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2, t), \\ &E_1 = E_1(x_1, x_2, t), \quad E_2 = E_2(x_1, x_2, t), \quad E_3 = 0, \\ &H_1 = H_2 = 0, \quad H_3 = H_3(x_1, x_2, t). \end{aligned} \tag{5}$$

From Eq. (3), the non-vanishing strain components are

$$S_4 = u_{3,2}, \quad S_5 = u_{3,1}, \tag{6}$$

and then from Eq. (2) the non-vanishing components of T_{ij}, D_i and B_i are

$$T_4 = c_{44}u_{3,2} - e_{15}E_2, \quad T_5 = c_{44}u_{3,1} - e_{15}E_1, \tag{7.1, 2}$$

$$D_1 = e_{15}u_{3,1} + \varepsilon_{11}E_1, \quad D_2 = e_{15}u_{3,2} + \varepsilon_{11}E_2, \tag{7.3, 4}$$

$$B_3 = \mu_0 H_3. \tag{7.5}$$

The nontrivial ones of the equations of motion and Maxwell's equations in Eq. (1) take the forms:

$$c_{44}(u_{3,11} + u_{3,22}) - e_{15}(E_{1,1} + E_{2,2}) = \rho\ddot{u}_3, \tag{8.1}$$

$$e_{15}(u_{3,11} + u_{3,22}) + \varepsilon_{11}(E_{1,1} + E_{2,2}) = 0, \tag{8.2}$$

$$E_{2,1} - E_{1,2} = -\mu_0\dot{H}_3, \tag{8.3}$$

$$H_{3,2} = e_{15}\dot{u}_{3,1} + \varepsilon_{11}\dot{E}_1, \quad -H_{3,1} = e_{15}\dot{u}_{3,2} + \varepsilon_{11}\dot{E}_2. \tag{8.4}$$

Eliminating the electric field components from Eqs. (8.1,2) yields

$$\bar{c}_{44}(u_{3,11} + u_{3,22}) = \rho\ddot{u}_3, \tag{9}$$

where $\bar{c}_{44} = c_{44} + e_{15}^2/\varepsilon_{11}$ is a piezoelectrically stiffened elastic constant. Differentiating Eq. (8.3) with respect to time once and substituting from Eqs. (8.4,5), we have

$$H_{3,11} + H_{3,22} = \varepsilon_{11}\mu_0\ddot{H}_3. \tag{10}$$

Equations (9) and (10) govern the mechanical and magnetic fields. Once u_3 and H_3 are determined, E_1 and E_2 can be obtained from Eqs. (8.4,5). Consider the following waves propagating in the x_1-direction with

$$\begin{aligned} &u_3 = Ue^{-\xi_2 x_2}\cos(\xi_1 x_1 - \omega t), \\ &H_3 = He^{-\eta_2 x_2}\cos(\xi_1 x_1 - \omega t), \end{aligned} \tag{11}$$

where U, H, ξ_1, ξ_2, η_2 and ω are undetermined constants. Substitution of Eq. (11) into Eq. (9) and Eq. (10) results in

$$\xi_2^2 = \xi_1^2 - \rho\omega^2/\bar{c}_{44} = \xi_1^2(1 - \frac{v^2}{v_T^2}) > 0,$$
$$\eta_2^2 = \xi_1^2 - \varepsilon_{11}\mu_0\omega^2 = \xi_1^2(1 - \frac{v^2}{c^2}) > 0, \tag{12}$$

where

$$v^2 = \frac{\omega^2}{\xi_1^2}, \quad v_T^2 = \frac{\bar{c}_{44}}{\rho}, \quad c^2 = \frac{1}{\varepsilon_{11}\mu_0}. \tag{13}$$

In Eq. (13), v is the surface wave speed, v_T is the speed of plane shear waves in the x_1-direction, and c is the speed of light in the x_1-direction. The inequalities are for decaying behavior from $x_2 = 0$. From Eq. (7.1), Eq. (8.4) and Eq. (11) we obtain:

$$T_4 = -\frac{1}{\varepsilon_{11}\omega}(\varepsilon_{11}\bar{c}_{44}\omega\xi_2 U e^{-\xi_2 x_2} + e_{15}\xi_1 H e^{-\eta_2 x_2})\cos(\xi_1 x_1 - \omega t),$$
$$E_1 = \frac{1}{\varepsilon_{11}\omega}(e_{15}\omega\xi_1 U e^{-\xi_2 x_2} + \eta_2 H e^{-\eta_2 x_2})\sin(\xi_1 x_1 - \omega t), \tag{14}$$

which will be needed for the interface continuity conditions.

For a metal layer of ideal conductor the electric field vanishes everywhere. We use a superimposed hat to represent the material parameters and undetermined constants. Similar to Eqs. (11)–(14) we have

$$u_3 = (\hat{U}\cos\hat{\xi}_2 x_2 + \hat{V}\sin\hat{\xi}_2 x_2)\cos(\xi_1 x_1 - \omega t), \tag{15}$$

$$\hat{\xi}_2^2 = \hat{\rho}\omega^2/\hat{c}_{44} - \xi_1^2 = \xi_1^2(\frac{v^2}{\hat{v}_T^2} - 1), \tag{16}$$

$$\hat{v}_T^2 = \frac{\hat{c}_{44}}{\hat{\rho}}, \tag{17}$$

$$T_4 = (-\hat{c}_{44}\hat{\xi}_2\hat{U}\sin\hat{\xi}_2 x_2 + \hat{c}_{44}\hat{\xi}_2\hat{V}\cos\hat{\xi}_2 x_2)\cos(\xi_1 x_1 - \omega t). \tag{18}$$

At the interface $x_2 = 0$ and the boundary $x_2 = -h$, we have the following continuity and boundary conditions [21] which represent four homogeneous linear algebraic equations for U, H, $\hat{U}$ and $\hat{V}$:

$$u_3(0^+) = U = \hat{U} = u_3(0^-),$$
$$T_4(0^+) = -\frac{1}{\varepsilon_{11}\omega}(\varepsilon_{11}\bar{c}_{44}\omega\xi_2 U + e_{15}\xi_1 H) = \hat{c}_{44}\hat{\xi}_2\hat{V} = T_4(0^-),$$
$$E_1(0^+) = \frac{1}{\varepsilon_{11}\omega}(e_{15}\omega\xi_1 U + \eta_2 H) = 0, \tag{19}$$
$$T_4(-h^+) = \hat{c}_{44}\hat{\xi}_2\hat{U}\sin\hat{\xi}_2 h + \hat{c}_{44}\hat{\xi}_2\hat{V}\cos\hat{\xi}_2 h = 0.$$

For nontrivial solutions the determinant of the coefficient matrix has to vanish, which leads to the following dispersion relation that determines the wave speed:

$$\eta_2\left(\xi_2 - \frac{\hat{c}_{44}}{\bar{c}_{44}}\hat{\xi}_2\tan\hat{\xi}h\right) = k^2\xi_1^2, \tag{20}$$

or

$$\sqrt{1 - \frac{v^2}{c^2}}\left(\sqrt{1 - \frac{v^2}{v_T^2}} - \frac{\hat{c}_{44}}{\bar{c}_{44}}\sqrt{\frac{v^2}{\hat{v}_T^2} - 1}\ \tan\xi_1 h\sqrt{\frac{v^2}{\hat{v}_T^2} - 1}\right) = k^2, \tag{21}$$

where $k^2 = e_{15}^2/(\varepsilon_{11}\bar{c}_{44})$ is a dimensionless number called the electromechanical coupling factor which is usually smaller than one. We make the following observations from Eq. (21):

(i) The waves are dispersive.
(ii) When $k = 0$, i.e., the material is not piezoelectric, Eq. (21) reduces to

$$\sqrt{1-\frac{v^2}{v_T^2}}-\frac{\hat{c}_{44}}{\bar{c}_{44}}\sqrt{\frac{v^2}{\hat{v}_T^2}-1}\ \tan\xi_1 h\sqrt{\frac{v^2}{\hat{v}_T^2}-1}=0, \tag{22}$$

which is the well known equation that determines the speed of Love waves in elasticity [22].

(iii) When the speed of light goes to infinity Eq. (21) reduces to

$$\sqrt{1-\frac{v^2}{v_T^2}}-\frac{\hat{c}_{44}}{\bar{c}_{44}}\sqrt{\frac{v^2}{\hat{v}_T^2}-1}\ \tan\xi_1 h\sqrt{\frac{v^2}{\hat{v}_T^2}-1}=k^2, \tag{23}$$

which is the dispersion relation for the quasistatic piezoelectric Love waves in a ceramic half space carrying an elastic metal layer given in [23]. The existence of roots to Eq. (23) was discussed in [23]. As shown in the numerical examples in [20], the inclusion of full electromagnetic coupling modifies the elastic wave speeds by small amounts.

4 Ceramic half space with an elastic dielectric layer

We now consider the case when the elastic layer between $-h < x_2 < 0$ is a non-piezoelectric dielectric and carries at $x_2 = -h$ a thin electrode of ideal conductor with negligible mass and negligible elastic stiffness. The presence of the electrode confines the electromagnetic waves within the elastic later and the piezoelectric half space. In addition to the mechanical fields in Eqs. (15)–(18) there also exist the following electromagnetic fields in the elastic layer:

$$H_3 = (\hat{G}\cosh\hat{\eta}_2 x_2 + \hat{H}\sinh\hat{\eta}_2 x_2)\cos(\xi_1 x_1 - \omega t), \tag{24}$$

$$\hat{\eta}_2^2 = \xi_1^2 - \hat{\varepsilon}_{11}\mu_0\omega^2 = \xi_1^2(1-\frac{v^2}{\hat{c}^2}), \tag{25}$$

$$\hat{c}^2 = \frac{1}{\hat{\varepsilon}_{11}\mu_0}, \tag{26}$$

$$E_1 = \frac{1}{\hat{\varepsilon}_{11}\omega}(-\hat{G}\hat{\eta}_2\sinh\hat{\eta}_2 x_2 - \hat{H}\hat{\eta}_2\cosh\hat{\eta}_2 x_2)\sin(\xi_1 x_1 - \omega t). \tag{27}$$

At the interface $x_2 = 0$ and the boundary $x_2 = -h$, we have the following continuity and boundary conditions which represent six homogeneous linear algebraic equations for U, H, $\hat{U}$, $\hat{V}$, $\hat{G}$ and $\hat{H}$:

$$u_3(0^+) = U = \hat{U} = u_3(0^-),$$

$$T_4(0^+) = -\frac{1}{\varepsilon_{11}\omega}(\varepsilon_{11}\bar{c}_{44}\omega\xi_2 U + e_{15}\xi_1 H) = \hat{c}_{44}\hat{\xi}_2\hat{V} = T_4(0^-),$$

$$H_3(0^+) = H = \hat{G} = H_3(0^-),$$

$$E_1(0^+) = \frac{1}{\varepsilon_{11}\omega}(e_{15}\omega\xi_1 U + \eta_2 H) = -\frac{1}{\hat{\varepsilon}_{11}\omega}\hat{H}\hat{\eta}_2 = E_1(0^-),$$

$$
\begin{aligned}
&T_4(-h^+) = \hat{c}_{44}\hat{\xi}_2\hat{U}\sin\hat{\xi}_2 h + \hat{c}_{44}\hat{\xi}_2\hat{V}\cos\hat{\xi}_2 h = 0,\\
&E_1(-h^+) = \frac{1}{\hat{\varepsilon}_{11}\omega}(\hat{G}\hat{\eta}_2\sinh\hat{\eta}_2 h - \hat{H}\hat{\eta}_2\cosh\hat{\eta}_2 h) = 0.
\end{aligned} \tag{28}
$$

For nontrivial solutions the determinant of the coefficient matrix has to vanish, which yields

$$
\left(\eta_2 + \frac{\varepsilon_{11}}{\hat{\varepsilon}_{11}}\hat{\eta}_2\tanh\hat{\eta}_2 h\right)\left(\xi_2 - \frac{\hat{c}_{44}}{\bar{c}_{44}}\hat{\xi}_2\tan\hat{\xi}h\right) = k^2\xi_1^2, \tag{29}
$$

or

$$
\begin{aligned}
&\left(\sqrt{1-\frac{v^2}{c^2}} + \frac{\varepsilon_{11}}{\hat{\varepsilon}_{11}}\sqrt{1-\frac{v^2}{\hat{c}^2}}\;\tanh\xi_1 h\sqrt{1-\frac{v^2}{\hat{c}^2}}\right)\\
&\qquad\left(\sqrt{1-\frac{v^2}{v_T^2}} - \frac{\hat{c}_{44}}{\bar{c}_{44}}\sqrt{\frac{v^2}{\hat{v}_T^2}-1}\;\tan\xi_1 h\sqrt{\frac{v^2}{\hat{v}_T^2}-1}\right) = k^2.
\end{aligned} \tag{30}
$$

We make the following observations from Eq. (30):

(i) The waves are dispersive.
(ii) When $k = 0$, i.e., the material of the half space is not piezoelectric, Eq. (30) reduces to

$$
\begin{aligned}
&\sqrt{1-\frac{v^2}{v_T^2}} - \frac{\hat{c}_{44}}{\bar{c}_{44}}\sqrt{\frac{v^2}{\hat{v}_T^2}-1}\;\tan\xi_1 h\sqrt{\frac{v^2}{\hat{v}_T^2}-1} = 0,\\
&\text{or}\quad \sqrt{1-\frac{v^2}{c^2}} + \frac{\varepsilon_{11}}{\hat{\varepsilon}_{11}}\sqrt{1-\frac{v^2}{\hat{c}^2}}\tanh\xi_1 h\sqrt{1-\frac{v^2}{\hat{c}^2}} = 0,
\end{aligned} \tag{31}
$$

which determines the speed of Love waves in elasticity and the speed of guided electromagnetic waves in the dielectric layer [24].

(iii) When the speed of light goes to infinity Eq. (30) reduces to

$$
\left(1 + \frac{\varepsilon_{11}}{\hat{\varepsilon}_{11}}\tanh\xi_1 h\right)\left(\sqrt{1-\frac{v^2}{v_T^2}} - \frac{\hat{c}_{44}}{\bar{c}_{44}}\sqrt{\frac{v^2}{\hat{v}_T^2}-1}\;\tan\xi_1 h\sqrt{\frac{v^2}{\hat{v}_T^2}-1}\right) = k^2, \tag{32}
$$

which is the dispersion relation for quasistatic piezoelectric Love waves in a ceramic half space carrying an elastic dielectric layer. This special result appears to be new.

5 Conclusion

Exact solutions are obtained from the three-dimensional equations of linear piezoelectromagnetism for Love waves in a ceramic half space carrying a metal or dielectric layer. The solutions obtained reduce to a few known elastic, electromagnetic, and quasistatic piezoelectric wave solutions in the literature as special cases.

References

[1] Tiersten, H. F.: Linear piezoelectric plate vibrations. New York: Plenum 1969.
[2] Eringen, A. C., Maugin, G. A.: Electrodynamics of continua, vol. I. New York: Springer 1990.

[3] Mindlin, R. D.: Electromagnetic radiation from a vibrating quartz plate. Int. J. Solids Struct. **9,** 697–702 (1972).
[4] Lee, P. C. Y.: Electromagnetic radiation from an AT-cut quartz plate under lateral-field excitation. J. Appl. Phys. **65,** 1395–1399 (1989).
[5] Lee, P. C. Y., Kim, Y.-G., Prevost, J. H.: Electromagnetic radiation from doubly rotated piezoelectric crystal plates vibrating at thickness frequencies. J. Appl. Phys. **67,** 6633–6642 (1990).
[6] Campbell, C. F., Weber, R. J.: Calculation of radiated electromagnetic power from bulk acoustic wave resonators. In: Proc. IEEE Int. Frequency Control Symp., pp. 472–475, 1993.
[7] Oliner, A. A.: Acoustic surface waves. New York: Springer 1978.
[8] Scholl, G., Schmidt, F., Ostertag, T., Reindl, L., Scherr, H., Wolff, U.: Wireless passive SAW sensor systems for industrial and domestic applications. In: Proc. IEEE Int. Frequency Control Symp., pp. 595–601, 1998.
[9] Mindlin, R. D.: A variational principle for the equations of piezoelectromagnetism in a compound medium. In: Complex variable analysis and its applications (I. N. Vekua 70th Birthday Volume), pp. 397–400. Moscow: Nauka, Academy of Sciences USSR 1978.
[10] Lee, P. C. Y.: A variational principle for the equations of piezoelectromagnetism in elastic dielectric crystals. J. Appl. Phys. **69,** 7470–7473 (1991).
[11] Kyame, J. J.: Wave propagation in piezoelectric crystals. J. Acoust. Soc. Am. **21,** 159–167 (1949).
[12] Pailloux, P. M. H.: Piezoelectricite calcul des vitesses de propagation. Le Journal de Physique et le Radium **19,** 523–526 (1958).
[13] Hruska, H.: The rate of propagation of ultrasonic waves in ADP and in Voigt's theory. Czech. J. Phys. **B16,** 446–453 (1966).
[14] Yang, J. S., Wu, X. Y.: The vibration of an elastic dielectric with piezoelectromagnetism. Q. Appl. Math. **53,** 753–760 (1995).
[15] Yang, J. S.: Variational principles for the vibration of an elastic dielectric. Arch. Mech. **45,** 279–284 (1993).
[16] Spaight, R. N., Koerber, G. G.: Piezoelectric surface waves on $LiNbO_3$. IEEE Trans. on Sonics and Ultrasonics **SU-18,** 237–238 (1971).
[17] Sedov, A., Schmerr, Jr., L. W.: Some exact solutions for the propagation of transient electroacoustic waves I: piezoelectric half-space. Int. J. Engng Sci. **24,** 557–568 (1986).
[18] Bleustein, J. L.: A new surface wave in piezoelectric materials. Appl. Phys. Lett. **13,** 412–413 (1968).
[19] Gulyaev, Yu. V.: Electroacoustic surface waves in solids. Sov. Phys. JETP Lett. **9,** 37–38 (1969).
[20] Yang, J. S.: Bleustein-Gulyaev waves in piezoelectromagnetic materials. Int. J. Appl. Electromagnetics Mech. **12,** 235–240 (2000).
[21] Jackson, J. D.: Classical electrodynamics. New York: Wiley 1990.
[22] Graff, K. F.: Wave motion in elastic solids. New York: Dover 1991.
[23] Curtis, R. G., Redwood, M.: Transverse surface waves on a piezoelectric material carrying a metal layer of finite thickness. J. Appl. Phys. **44,** 2002–2007 (1973).
[24] Marcuse, D.: Light transmission optics. Florida: Krieger 1989.

Author's address: J. S. Yang, Department of Engineering Mechanics, University of Nebraska, Lincoln, NE 68 588, U.S.A. (E-mail: jyang1@unl.edu)