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Incompressibility at large strains and finite-element implementation

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Summary. This contribution is concerned with the consideration of material incompressibility at large strains and proposes various methods for the enforcement of the corresponding constraint into finite-rotation shell models. The incompressibility condition can be expressed in terms of displacement as well as strain variables and is considered by means of three different procedures in the numerical implementation. As kinematic hypothesis a quadratic assumption with respect to the thickness coordinate is used in which the corresponding directors are decomposed into two stretch parameters and a common inextensible unit vector. Various constitutive laws holding for incompressible isotropic hyperelasticity are considered and directly coupled with shell equations through a numerical thickness integration. A 4-node isoparametric shell element is developed parameterizing the inextensible shell director in terms of rotation variables in the framework of an up-dated rotation formulation. Finally, several examples are analysed to identify the most effective procedure for modelling isochoric deformations in thin-walled structures.

1 Introduction

Many materials of modern technology can undergo very large strains in the elastic range (hyperelasticity) and are characterized by an incompressible (volume conserving) behavior. Materials with the cited characteristics are for example biological tissues such as those involved in the heart, veins or arteries. Accordingly the simulation of hyperelastic materials under the consideration of incompressibility is a challenging research topic particular in the field of Biomechanics. Constitutive models available for hyperelastic materials with isochoric material behaviour can be essentially classified into two groups: incompressible and compressible models. The first class of models requires a direct enforcement of the incompressibility condition in the corresponding model. On the contrary in compressible models the aforementioned condition can be fulfilled indirectly by a suitable selection of a material constant, e.g., Poisson's ratio $\nu \rightarrow 0.5$. The direct consideration of the incompressibility condition in hyperelastic material models requires in general lengthy algebra, while the indirect consideration of this constraint leads to computationally very expensive procedures as will be confirmed in this contribution. Starting from nonlinear continuum mechanics the formulation of the incompressibility condition (at large strains) in terms of displacement or strain variables is not a theoretically difficult task. In a 3D-formulation, such a condition renders a single displacement or strain variable as a dependent quantity. In case of a 2D-implementation, the incompressibility condition can be

†In memory of Y. Başar, who passed away on August 30, 2002.

expressed in an infinite power-series with respect to the thickness coordinate, which can be satisfied in accordance with the adopted kinematic assumption up to a certain order. In this way a set of displacement or strain variables is transformed into dependent quantities. In the present work emphasis is given to a 2D-realisation.

Various models have been proposed in the literature for the simulation of incompressible materials. The most popular formulation is the Mooney-Rivlin model including as a special case the Neo-Hookean material, where the strain energy function is expressed in terms of the right Cauchy-Green tensor. These models have been already adopted for the bending analysis of shells (Schiek et al. [1], Başar and Ding [2], [3]). A further possibility is the use of Ogden model, which has been mainly employed in membrane shell models (Wriggers and Taylor [4], Gruttmann and Taylor [5]). An extension to the bending analysis is accomplished by Eberlein [6] for axisymmetric deformations and by Başar and Itskov [7] for arbitrary bending deformations.

The question is: What is the most effective procedure to consider incompressible material behaviour? Can this be achieved only on the level of displacement variables or is it necessary to incorporate the strain quantities to achieve this purpose? Can a unified procedure work equally efficient for all types of constitutive models? Or conversely: Does the effectivity of a procedure depend upon the constitutive law to be considered? This contribution aims at the examination of the above mentioned questions in context with 2D finite-element models. The main purpose is to identify the best possible procedure for the consideration of the incompressibility constraint.

2 Notations

In this paper, index and absolute tensor notation will be employed. As usual, Latin indices represent the numbers 1, 2, 3 and the Greek ones the numbers 1, 2. For convenience we summarize the essential notations to be used in the present derivation:

Variable definitions

F_0, F	undeformed and deformed shell midsurface
V_0, V	undeformed and deformed shell continuum
dF_0, dV_0	surface and volume element
Θ^α, Θ^3	convective coordinates
$\overset{0}{\mathbf{X}}$	position vector of F_0
$\overset{1}{\mathbf{X}}$	general director of F_0
$\mathbf{N}$	unit vector $\perp$ to F_0
$\overset{0}{\mathbf{x}}$	position vector of F
$\overset{1}{\mathbf{x}}, \overset{2}{\mathbf{x}}$	general directors of F
$\overset{0}{\lambda}, \overset{1}{\lambda}$	stretch parameters
$\mathbf{d}$	inextensible director
$\mathbf{A}, \mathbf{a}$	metric tensor of F_0 and F
$\mathbf{G}, \mathbf{g}$	metric tensor of V_0 and V
A, G	determinant of $\mathbf{A}$ and $\mathbf{G}$
$\mathbf{A}_i, \mathbf{a}_i$	base vectors of F_0 and F
$\mathbf{G}_i, \mathbf{g}_i$	base vectors of V_0 and V
$\mathbf{F}$	deformation gradient
$\mathbf{C}$	right Cauchy-Green tensor

b	left Cauchy-Green tensor
E	Green-Lagrange strain tensor
S	2nd Piola-Kirchhoff stress tensor
$\boldsymbol{\sigma}$	Cauchy stress tensor

Tensor operations, symbols

$\mathbf{a}_{,i} = \dfrac{\partial \mathbf{a}}{\partial \Theta^i}$	partial derivative with respect to the convective coordinates
$\mathbf{a} \cdot \mathbf{b} = a_i b^i$	simple contraction of two vectors
$\mathbf{A} = \mathbf{a} \otimes \mathbf{b} = a_i b^j \mathbf{g}^i \otimes \mathbf{g}_j$	second-order tensor defined by dyadic product of two vectors
$\mathbb{D} = \mathbf{A} \otimes \mathbf{B} = A^{ij} B^{mn} \mathbf{G}_i \otimes \mathbf{G}_j \otimes \mathbf{G}_m \otimes \mathbf{G}_n$ $\mathbb{E} = \mathbf{A} \boxtimes \mathbf{B} = A^{ij} B^{mn} \mathbf{G}_i \otimes \mathbf{G}_m \otimes \mathbf{G}_j \otimes \mathbf{G}_n$	fourth-order tensors defined by tensor product of second-order ones
$\mathscr{S} = \mathbf{A} \otimes \mathbf{b} = A^{ij} b^m \mathbf{G}_i \otimes \mathbf{G}_j \otimes \mathbf{G}_m$	third-order tensor defined by tensor product of second- and first-order ones
$\mathbf{A}\mathscr{S} = A_{ik} S^k{}_{jm} \mathbf{G}^i \otimes \mathbf{G}^j \otimes \mathbf{G}^m$	simple contraction of two tensors
$\mathbf{A} : \mathbb{D} = A^{ij} D_{ijmn} \mathbf{G}^m \otimes \mathbf{G}^n$	double contraction of two tensors
$\det \mathbf{A} = \lvert A^i{}_j \rvert$	determinant of a tensor
$\delta \mathbf{A} = \delta A_{ij} \mathbf{G}^i \otimes \mathbf{G}^j$	variation of a material tensor
$\Delta \mathbf{B} = \Delta B_{ij} \mathbf{G}^i \otimes \mathbf{G}^j$	increment of a material tensor

3 Incompressibility

This section is concerned with the formulation of the incompressibility condition to be considered for modelling incompressible materials and its transformation into an incremental formulation needed for the subsequent finite-element implementation.

In this work we postulate a curvilinear coordinate system Θ^i having the property

$$G_{\alpha 3} = G^{\alpha 3} = 0, \quad G_{33} = G^{33}. \tag{3.1}$$

Furthermore, the coordinate system Θ^i is supposed to be subjected only to coordinate transformations of the form:

$$\bar{\Theta}^\alpha = \bar{\Theta}^\alpha(\Theta^1, \Theta^2), \quad \bar{\Theta}^3 = \Theta^3, \tag{3.2}$$

where $(.\bar{.}.)$ denotes a new set of coordinates. Accordingly, the position of the index "3" is irrelevant in component relations, and the base vector $\mathbf{G}_3 = \mathbf{G}^3$ is an invariant quantity. Note that such a coordinate system is useful for modelling shells (Başar and Krätzig [8]) as will be also observed in Sect. 5. If the conditions (3.1) and (3.2) hold, the variables

$$\hat{\mathbf{E}} = E_{\alpha\beta} \mathbf{G}^\alpha \otimes \mathbf{G}^\beta, \quad \mathbf{s} = E_{\alpha 3} \mathbf{G}^\alpha, \quad E_{33} = E^{33} \tag{3.3}$$

which represent the *tangential* components $E_{\alpha\beta}$ and the *transverse* shear strains $E_{\alpha 3}$ of the Green-Lagrange strain tensor $\mathbf{E} = E_{ij}\ \mathbf{G}^i \otimes \mathbf{G}^j$ form *surface* tensors of second and first order while $E_{33} = E^{33}$ is an invariant scalar-valued quantity. Note that surface tensors remain unchanged under coordinate transformations of the form (3.2) and tensors of the form (3.3) can actually be introduced on the basis of an arbitrary 3D second-order tensor.

The third invariant $I\!I\!I_{\mathbf{C}}$ of the right Cauchy-Green tensor $\mathbf{C}=\mathbf{F}^T\mathbf{F}$ plays a major role in dealing with *isochoric* deformations characterized by the *incompressibility condition*: $I\!I\!I_{\mathbf{C}}=1$. In terms of the Green-Lagrange strain tensor $\mathbf{E}=\frac{1}{2}(\mathbf{C}-\mathbf{G})$ the *incompressibility condition* reads (Green and Zerna [9], Başar and Weichert [10])

$$I\!I\!I_{\mathbf{C}}=\det\mathbf{C}=\left|C^i_j\right|=\det(2\mathbf{E}+\mathbf{G})=\left|2E^i_j+\delta^i_j\right|=1. \tag{3.4}$$

Written in full, the above relation can be solved for the transverse strains E_{33}. If, under consideration of (3.3), the *invariant* quantities

$$A=2E^\alpha_\alpha=2\mathrm{tr}\hat{\mathbf{E}}=2\hat{\mathbf{E}}:\mathbf{G}_\alpha\otimes\mathbf{G}^\alpha,$$

$$D=\frac{1}{2}\left(E^\alpha_\alpha E^\beta_\beta-E^\alpha_\beta E^\beta_\alpha\right)=\frac{1}{2}\left[(\mathrm{tr}\hat{\mathbf{E}})^2-\mathrm{tr}\hat{\mathbf{E}}^2\right],$$

$$Q=E^3_\alpha E^\alpha_3=\mathbf{s}\cdot\mathbf{s},$$

$$V=2\left(E^\alpha_3 E^\beta_\alpha E_{\beta 3}-E^\beta_\beta E^\alpha_3 E_{\alpha 3}\right)=2\left[\mathbf{s}\hat{\mathbf{E}}\mathbf{s}-(\mathrm{tr}\hat{\mathbf{E}})\mathbf{s}\cdot\mathbf{s}\right],$$

$$N=A+4D,\qquad S=-4(Q-V),\qquad R=\frac{S+N}{1+N} \tag{3.5}$$

are introduced as abbreviations, the corresponding result is expressible as

$$E^3_3=-\frac{1}{2}R=-\frac{1}{2}\frac{S+N}{1+N}. \tag{3.6}$$

For later use attention is now given to the first two invariants of the right Cauchy-Green tensor $I_{\mathbf{C}}$ and $I\!I_{\mathbf{C}}$, which are, using the definition $\mathbf{C}=2\mathbf{E}+\mathbf{G}$, expressible in terms of the invariant quantities given in (3.5) as

$$\begin{aligned}I_{\mathbf{C}}&=\mathrm{tr}\mathbf{C}=\mathrm{tr}(2\mathbf{E}+\mathbf{G})\\&=3+A+2E^3_3,\end{aligned} \tag{3.7}$$

$$\begin{aligned}I\!I_{\mathbf{C}}&=\frac{1}{2}\left[(\mathrm{tr}\mathbf{C})^2-\mathrm{tr}\mathbf{C}^2\right]=\frac{1}{2}\left[(\mathrm{tr}(2\mathbf{E}+\mathbf{G}))^2-\mathrm{tr}(2\mathbf{E}+\mathbf{G})^2\right]\\&=3+2A+4D+2E^3_3(2+A)-4Q,\end{aligned} \tag{3.8}$$

in accordance with the results presented in Başar and Ding [3].

Our next goal is to construct on the basis of (3.6) the first variation δE^3_3 as well as its incremental form $\Delta\delta E^3_3$ to be used in the finite-element procedure. The first variation of an arbitrary tensor function $W=W(\mathbf{x})$ with respect to the independent position vector $\mathbf{x}$ and the corresponding incremental form are defined in terms of an arbitrary parameter ε by the following relations:

$$\delta W=\delta W(\mathbf{x},\delta\mathbf{x})=\frac{d}{d\varepsilon}W(\mathbf{x}+\varepsilon\delta\mathbf{x})|_{\varepsilon=0}=\frac{\partial W}{\partial\mathbf{x}}\delta\mathbf{x}, \tag{3.9}$$

$$\Delta\delta W=\Delta\delta W(\mathbf{x},\delta\mathbf{x},\Delta\mathbf{x})=\frac{d}{d\varepsilon}\delta W(\mathbf{x}+\varepsilon\Delta\mathbf{x},\delta\mathbf{x})|_{\varepsilon=0}=\Delta\mathbf{x}\frac{\partial^2 W}{\partial\mathbf{x}\partial\mathbf{x}}\delta\mathbf{x} \tag{3.10}$$

holding also if the notation δ is changed into Δ and vice versa. The application of the rules (3.9), (3.10) to (3.6) delivers, by considering (3.5), for the first variation δE^3_3:

$$\delta E_3^3 = -\frac{1}{2}\delta R = -\frac{1}{2}\frac{(1-S)}{(1+N)^2}\delta N - \frac{1}{2}\frac{\delta S}{(1+N)}, \tag{3.11}$$

and for its incrementation $\Delta\delta E_3^3$:

$$\Delta\delta E_3^3 = -\frac{1}{2}\Delta\delta R = \frac{1}{2}\left[\frac{2(1-S)}{(1+N)^3}\delta N\Delta N + \frac{1}{(1+N)^2}(\delta N\Delta S + \Delta N\delta S) - \frac{1-S}{(1+N)^2}\Delta\delta N - \frac{1}{(1+N)}\Delta\delta S\right], \tag{3.12}$$

where the expressions holding for $\delta N, \Delta\delta N, \ldots$ are summarized in Table 1 in index notation and in Table 2 in absolute notation. By using the corresponding results the variational and incremental terms occurring in (3.11) and (3.12) can be expressed as

Table 1. Abbreviations used for strain invariants and internal potential energy in index notation

$A = 2E_\alpha^\alpha$:

$\delta A = 2\delta E_\alpha^\alpha$

$\Delta\delta A = 2\Delta\delta E_\alpha^\alpha$

$D = \frac{1}{2}(E_\alpha^\alpha E_\beta^\beta - E_\beta^\alpha E_\alpha^\beta)$

$\delta D = \delta E_\alpha^\alpha E_\beta^\beta - \delta E_\beta^\alpha E_\alpha^\beta$

$\Delta\delta D = \delta E_\alpha^\alpha \Delta E_\beta^\beta - \delta E_\beta^\alpha \Delta E_\alpha^\beta + \Delta\delta E_\alpha^\alpha E_\beta^\beta - \Delta\delta E_\beta^\alpha E_\alpha^\beta$

$N = A + 4D$:

$\delta N = \delta A + 4\delta D$

$\Delta\delta N = \Delta\delta A + 4\Delta\delta D$

$Q = E_3^\alpha E_\alpha^3$:

$\delta Q = 2\delta E_3^\alpha E_\alpha^3$

$\Delta\delta Q = 2(\delta E_3^\alpha \Delta E_\alpha^3 + \Delta\delta E_3^\alpha E_\alpha^3)$

$V = 2E_3^\alpha(E_\beta^3 E_\alpha^\beta - E_\alpha^3 E_\beta^\beta)$:

$\delta V = 4\delta E_3^\alpha(E_\beta^3 E_\alpha^\beta - E_\alpha^3 E_\beta^\beta) + 2\delta E_\beta^\alpha(E_\alpha^3 E_3^\beta - \delta_\alpha^\beta E_\gamma^3 E_3^\gamma)$

$\Delta\delta V = 4\delta E_3^\alpha(E_\alpha^\beta - \delta_\alpha^\beta E_\gamma^\gamma)\Delta E_\beta^3 + 4\delta E_3^\alpha(\delta_\alpha^\gamma E_\beta^3 - \delta_\beta^\gamma E_\alpha^3)\Delta E_\gamma^\beta + 4\delta E_\beta^\alpha(\delta_\gamma^\beta E_\alpha^3 - \delta_\alpha^\beta E_\gamma^3)\Delta E_3^\gamma$
$\quad + 4\,\Delta\delta E_3^\alpha(E_\beta^3 E_\alpha^\beta - E_\alpha^3 E_\beta^\beta) + 2\Delta\delta E_\beta^\alpha(E_\alpha^3 E_3^\beta - \delta_\alpha^\beta E_\gamma^3 E_3^\gamma)$

$S = 4(V - Q)$:

$\delta S = 4(\delta V - \delta Q)$

$\Delta\delta S = 4(\Delta\delta V - \Delta\delta Q)$

$R = \frac{S+N}{1+N}$:

$\delta R = \delta N\frac{1-S}{(1+N)^2} + \delta S\frac{1}{1+N}$

$\Delta\delta R = -2\delta N\Delta N\frac{1-S}{(1+N)^3} - (\delta N\Delta S + \delta S\Delta N)\frac{1}{(1+N)^2} + \Delta\delta N\frac{1-S}{(1+N)^2} + \Delta\delta S\frac{1}{1+N}$

Table 2. Abbreviations used for strain invariants and internal potential energy in absolute notation

$$A = 2\,\mathrm{tr}\hat{\mathbf{E}}:$$
$$\delta A = 2\,\mathrm{tr}\delta\hat{\mathbf{E}} = 2\mathbf{G} : \delta\hat{\mathbf{E}}$$
$$\Delta\delta A = 2\,\mathrm{tr}\Delta\delta\hat{\mathbf{E}} = 2\mathbf{G} : \Delta\delta\hat{\mathbf{E}}$$
$$D = \frac{1}{2}\Big[(\mathrm{tr}\hat{\mathbf{E}})^2 - \mathrm{tr}\hat{\mathbf{E}}^2\Big]:$$
$$\delta D = \mathrm{tr}\hat{\mathbf{E}}\;\mathrm{tr}\delta\hat{\mathbf{E}} - \mathrm{tr}(\hat{\mathbf{E}}\delta\hat{\mathbf{E}}) = \Big[(\mathrm{tr}\hat{\mathbf{E}})\mathbf{G} - \hat{\mathbf{E}}\Big] : \delta\hat{\mathbf{E}}$$
$$\begin{aligned}\Delta\delta D &= \mathrm{tr}\Delta\hat{\mathbf{E}}\;\mathrm{tr}\delta\hat{\mathbf{E}} - \mathrm{tr}(\Delta\hat{\mathbf{E}}\delta\hat{\mathbf{E}}) + \mathrm{tr}\hat{\mathbf{E}}\;\mathrm{tr}\Delta\delta\hat{\mathbf{E}} - \mathrm{tr}(\hat{\mathbf{E}}\Delta\delta\hat{\mathbf{E}})\\ &= \Delta\hat{\mathbf{E}} : [\mathbf{G}\otimes\mathbf{G} - \mathbf{G}\boxtimes\mathbf{G}] : \delta\hat{\mathbf{E}} + \Big[(\mathrm{tr}\hat{\mathbf{E}})\mathbf{G} - \hat{\mathbf{E}}\Big] : \Delta\delta\hat{\mathbf{E}}\end{aligned}$$
$$N = A + 4D:$$
$$\delta N = \delta A + 4\delta D$$
$$\Delta\delta N = \Delta\delta A + 4\Delta\delta D$$
$$Q = \mathbf{s}\cdot\mathbf{s}:$$
$$\delta Q = 2\mathbf{s}\cdot\delta\mathbf{s}$$
$$\Delta\delta Q = 2\Delta\mathbf{s}\cdot\delta\mathbf{s} + 2\mathbf{s}\cdot\Delta\delta\mathbf{s}$$
$$V = 2\Big[\mathbf{s}\hat{\mathbf{E}}\mathbf{s} - (\mathbf{s}\cdot\mathbf{s})\mathrm{tr}\hat{\mathbf{E}}\Big]:$$
$$\delta V = 2\Big[\mathbf{s}\otimes\mathbf{s} - (\mathbf{s}\cdot\mathbf{s})\mathbf{G}\Big] : \delta\hat{\mathbf{E}} + 4\Big[\mathbf{s}\hat{\mathbf{E}} - (\mathrm{tr}\hat{\mathbf{E}})\mathbf{s}\Big]\delta\mathbf{s}$$
$$\begin{aligned}\Delta\delta V = \Delta\hat{\mathbf{E}} : &\Big[4\big(\mathbf{s}\otimes\mathbf{G} - \mathbf{G}\otimes\mathbf{s}\big)\Big]\delta\mathbf{s} + \Delta\mathbf{s}\Big[4\big(\mathbf{G}\otimes\mathbf{s} - \mathbf{s}\otimes\mathbf{G}\big)\Big] : \delta\hat{\mathbf{E}} + \Delta\mathbf{s}\Big[4\big(\hat{\mathbf{E}} - (\mathrm{tr}\hat{\mathbf{E}})\mathbf{G}\big)\Big]\delta\mathbf{s}\\ &+ 2\Big[\mathbf{s}\otimes\mathbf{s} - (\mathbf{s}\cdot\mathbf{s})\mathbf{G}\Big] : \Delta\delta\hat{\mathbf{E}} + 4\Big[\mathbf{s}\hat{\mathbf{E}} - (\mathrm{tr}\hat{\mathbf{E}})\mathbf{s}\Big]\Delta\delta\mathbf{s}\end{aligned}$$
$$S = 4(V - Q):$$
$$\delta S = 4(\delta V - \delta Q)$$
$$\Delta\delta S = 4(\Delta\delta V - \Delta\delta Q)$$
$$R = \frac{S+N}{1+N}:$$
$$\delta R = \delta N\frac{1-S}{(1+N)^2} + \delta S\frac{1}{1+N}$$
$$\Delta\delta R = -2\delta N\Delta N\frac{1-S}{(1+N)^3} - (\delta N\Delta S + \delta S\Delta N)\frac{1}{(1+N)^2} + \Delta\delta N\frac{1-S}{(1+N)^2} + \Delta\delta S\frac{1}{1+N}$$

$$\begin{aligned}\delta N &= \delta A + 4\delta D = \Big[(2 + 4\,\mathrm{tr}\hat{\mathbf{E}})\mathbf{G} - 4\hat{\mathbf{E}}\Big] : \delta\hat{\mathbf{E}} = \mathbf{A} : \delta\hat{\mathbf{E}},\\ \delta S &= -4\Big[(2 + 4\,\mathrm{tr}\hat{\mathbf{E}})\mathbf{s} - 4(\mathbf{s}\hat{\mathbf{E}})\Big]\cdot\delta\mathbf{s} + 8[\mathbf{s}\otimes\mathbf{s} - (\mathbf{s}\cdot\mathbf{s})\mathbf{G}] : \delta\hat{\mathbf{E}} = \mathbf{a}\cdot\delta\mathbf{s} + \mathbf{B} : \delta\hat{\mathbf{E}},\\ \Delta N\delta N &= \Delta\hat{\mathbf{E}} : \mathbf{A}\otimes\mathbf{A} : \delta\hat{\mathbf{E}} = \Delta\hat{\mathbf{E}} : \mathbb{E}_1 : \delta\hat{\mathbf{E}},\\ \Delta S\delta N &= \Delta\mathbf{s}(\mathbf{a}\otimes\mathbf{A}) : \delta\hat{\mathbf{E}} + \Delta\hat{\mathbf{E}} : \mathbf{B}\otimes\mathbf{A} : \delta\hat{\mathbf{E}} = \Delta\mathbf{s}\,\mathscr{S} : \delta\hat{\mathbf{E}} + \Delta\hat{\mathbf{E}} : \mathbb{E}_2 : \delta\hat{\mathbf{E}},\end{aligned} \tag{3.13}$$

where the last result holds also for $\delta S\Delta N$ if δ is replaced by Δ and vice versa. The fourth-order tensors $\mathbb{E}_1$ and $\mathbb{E}_2$ and the third-order tensor $\mathscr{S}$ defined in (3.13) are of significance to construct the so-called tangent moduli needed for the finite-element procedure.

By setting $\mathbf{x} = \mathbf{X}$ $(\mathbf{E} = \mathbf{0})$ the expressions summarized in Tables 1 and 2 can be specified for an infinitesimal neighborhood of the reference configuration. Thus we find the relations

$$\begin{aligned}
&\delta A = 2\mathbf{G} : \delta\hat{\mathbf{E}}, && \Delta\delta A = 2\mathbf{G} : \Delta\delta\hat{\mathbf{E}},\\
&\delta D = 0, && \Delta\delta D = \Delta\hat{\mathbf{E}} : [\mathbf{G}\otimes\mathbf{G} - \mathbf{G}\boxtimes\mathbf{G}] : \delta\hat{\mathbf{E}},\\
&\delta N = \delta A, && \delta\Delta N = \Delta\delta A + 4\Delta\delta D,\\
&\delta Q = 0, && \Delta\delta Q = 2\Delta\mathbf{s}\cdot\delta\mathbf{s},\\
&\delta V = 0, && \Delta\delta V = 0,\\
&\delta S = 0, && \Delta\delta S = -4\Delta\delta Q,\\
&\delta R = \delta N = \delta A, && \Delta\delta R = -2\Delta A\delta A + \Delta\delta A + 4\Delta\delta D - 4\Delta\delta Q
\end{aligned} \tag{3.14}$$

to be used later for the linearization of hyperelastic constitutive models at the point $\mathbf{x} = \mathbf{X}$. Consequently, the incompressibility condition (3.11) reduces $\mathrm{tr}\Delta\mathbf{E}$, in accordance with the well-known result of the linear theory at the point $\mathbf{x} = \mathbf{X}$, to

$$\mathrm{tr}\Delta\mathbf{E} = \Delta E_1^1 + \Delta E_2^2 + \Delta E_3^3 = 0, \tag{3.15}$$

where ΔE_i^i denote infinitesimal strains.

4 Material models

In this section constitutive models in the form of a Mooney-Rivlin model and a St. Venant-Kirchhoff model are introduced, which are considered in the finite-element implementation. The first model is capable to model *incompressible isotropic*, *rubber-like* materials at large strains, while the second one is applicable to *compressible isotropic* materials at the presence of *large rotations*, but comparatively small strains.

The Mooney-Rivlin model is described by an *energy density function* W (per unit undeformed volume) depending on the first two invariants $I_{\mathbf{C}}, I\!I_{\mathbf{C}}$ of $\mathbf{C}$ involving two experimentally determined material constants. By considering (3.7) and (3.8) it can be expressed as (Green and Zerna [9])

$$\begin{aligned}
W &= c_1(I_{\mathbf{C}} - 3) + c_2(I\!I_{\mathbf{C}} - 3)\\
&= c_1\big(A + 2E_3^3\big) + c_2\big[2A + 4(D - Q) + 2(2 + A)E_3^3\big],
\end{aligned} \tag{4.1}$$

where the incompressibility condition can be easily incorporated just replacing E_3^3 by (3.6). Note that, for $c_2 = 0$, this model reduces to a Neo-Hooke model involving a single material constant. Starting from (4.1), expressions for δW and $\Delta\delta W$ can be derived again by application of the operations (3.9) and (3.10), the results being, in view of the definitions (3.5), of the form:

$$\delta W = c_1\big(\delta A + 2\delta E_3^3\big) + c_2\big[2\big(1 + E_3^3\big)\delta A + 4(\delta D - \delta Q) + 2(2 + A)8E_3^3\big], \tag{4.2}$$

$$\begin{aligned}
\Delta\delta W = c_1\big(\Delta\delta A + 2\Delta\delta E_3^3\big) + c_2\big[2\big(\delta A\Delta E_3^3 + \Delta A\delta E_3^3\big)\\
+2\big(1 + E_3^3\big)\Delta\delta A + 4(\Delta\delta D - \Delta\delta Q) + 2(2 + A)\Delta\delta E_3^3\big].
\end{aligned} \tag{4.3}$$

The above relations hold for incompressibility if $E_3^3, \delta E_3^3$ and $\Delta\delta E_3^3$ are expressed according to (3.6), (3.11) and (3.12).

Attention is now confined to the St. Venant-Kirchhoff model where the energy density W is, by considering the terms given in (3.5), of the form:

$$W = \frac{\lambda}{2}(\mathrm{tr}\mathbf{E})^2 + \mu\, \mathrm{tr}\mathbf{E}^2$$
$$= \frac{\lambda}{2}\left(\frac{1}{4}A^2 + AE_3^3 + E_3^3E_3^3\right) + \mu\left(\frac{1}{4}A^2 - 2(D-Q) + E_3^3E_3^3\right), \tag{4.4}$$

λ and μ denoting the Lamé constants. By the usual procedure, the following expressions are then obtained for δW and $\Delta\delta W$:

$$\delta W = \frac{\lambda}{2}\left[\left(\frac{1}{2}A + E_3^3\right)\delta A + \left(A + 2E_3^3\right)\delta E_3^3\right] + \mu\left[\frac{1}{2}A\delta A - 2(\delta D - \delta Q) + 2E_3^3\delta E_3^3\right], \tag{4.5}$$

$$\Delta\delta W = \frac{\lambda}{2}\left[\left(\frac{1}{2}\Delta A + \Delta E_3^3\right)\delta A + \left(\Delta A + 2\Delta E_3^3\right)\delta E_3^3 + \left(\frac{1}{2}A + E_3^3\right)\Delta\delta A. + \left(A + 2E_3^3\right)\Delta\delta E_3^3\right]$$
$$+ \mu\left[\frac{1}{2}\Delta A\delta A + 2\Delta E_3^3\delta E_3^3 + \frac{1}{2}A\Delta\delta A - 2(\Delta\delta D - \Delta\delta Q) + 2E_3^3\Delta\delta E_3^3\right], \tag{4.6}$$

holding for compressible materials at an arbitrary point $\mathbf{x}$. Inserting the incompressibility conditions (3.6) and (3.11) into (4.2) and considering (3.14) it can be observed that $\delta W = 0$ at the point $\mathbf{x} = \mathbf{X}$. This is also true, according to (4.5), for the St. Venant-Kirchhoff model.

In order to establish a useful connection between the Mooney-Rivlin and the St. Venant-Kirchhoff material model in case of incompressibility we first recall that the linearization of a material model is defined by

$$L\delta W|_{\mathbf{x}=\mathbf{X}} = \delta W|_{\mathbf{x}=\mathbf{X}} + \Delta\delta W|_{\mathbf{x}=\mathbf{X}}. \tag{4.7}$$

If a strain-free reference configuration is postulated the first term on the right-hand side vanishes as has been already confirmed for the present models (4.1) and (4.4). Consequently, the final result is the value of $\Delta\delta W$ at $\mathbf{x} = \mathbf{X}$. Now we insert the incompressibility conditions (3.6), (3.11) and (3.12) into (4.3) and (4.6) in order to specify the corresponding results by using (3.14) at the point $\mathbf{x} = \mathbf{X}$. Thus we find for the Mooney-Rivlin model

$$\Delta\delta W|_{\mathbf{x}=\mathbf{X}} = 2(c_1 + c_2)(\Delta A\delta A - 2\Delta\delta D + 2\Delta\delta Q), \tag{4.8}$$

and for the St. Venant-Kirchhoff model

$$\Delta\delta W|_{\mathbf{x}=\mathbf{X}} = \mu(\Delta A\delta A - 2\Delta\delta D + 2\Delta\delta Q) = 2\mu\left(\Delta E_\alpha^\alpha \delta E_\beta^\beta + \Delta E_\beta^\alpha \delta E_\alpha^\beta + 2\Delta E_3^\alpha \delta E_\alpha^3\right). \tag{4.9}$$

Since both models must be identical at $\mathbf{x} = \mathbf{X}$, we find by comparison the condition

$$\mu = 2(c_1 + c_2), \quad \mu = \frac{1}{2}\sum_{p=1}^{N} \mu_p \alpha_p \tag{4.10}$$

to be satisfied by the material constants. In case of a Neo-Hookean model, we find $\mu = 2c_1$. The second equality in (4.10) can be derived, similarly, for the *incompressible* Ogden model (Başar and Weichert [10])

$$W = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left[(\lambda_1)^{\alpha_p} + (\lambda_2)^{\alpha_p} + (\lambda_3)^{\alpha_p} - 3\right], \tag{4.11}$$

where λ_i are the eigenvalues of the right stretch tensor $\mathbf{U} = \mathbf{C}^{1/2}$ and μ_p, α_p are material constants.

For later comparative studies we, finally, consider a compressible material model of Neo-Hookean type (Simo et al. [11], Ciarlet [12]):

$$W = \frac{1}{2}\kappa(\ln J)^2 + \frac{1}{2}\mu\Big(J^{-2/3} tr\mathbf{C} - 3\Big), \quad J = \sqrt{III_{\mathbf{C}}} \tag{4.12}$$

involving the bulk modulus κ and the shear modulus μ

$$\kappa = \frac{E}{3(1-2v)}, \quad \mu = \frac{E}{2(1+v)} \tag{4.13}$$

as material constants, yielding the following constitutive law for the second-kind Piola-Kirchhoff stress tensor (Başar and Weichert [10]):

$$\mathbf{S} = 2W,_{\mathbf{C}} = \kappa \ \ln J \ \mathbf{C}^{-1} + \mu \, J^{-2/3}\left[\mathbf{G} - \frac{1}{3}(\mathrm{tr}\mathbf{C})\mathbf{C}^{-1}\right]. \tag{4.14}$$

Since $\bar{\mathbf{C}} = (J^{-2/3}\mathbf{C})$ is associated with the incompressible part of the deformation gradient the second term on the right-hand side of (4.12) is concerned with the incompressible part of the deformation so that the compressible deformations affect solely the first term with the material constant κ. If the Poisson's ratio v tends to 0.5 the bulk modulus κ will approach infinity, which will assign for the compressible deformations associated with $\ln J$ an adequately large stiffness factor. Thus it is expected that, by using this penalty-method, the incompressibility condition can be indirectly enforced in the constitutive model. This model in the sequel denoted as com Neo-Hooke will be used for comparative studies in Sect. 9. Note that for $v = 0.5$ and under the conditions (4.10), the material model (4.12) is, in the linear case, identical with the Mooney-Rivlin model (4.1).

5 Shell kinematics

This section is concerned with the description of the reference and current configuration of the shell continuum. The deformed configuration of the shell is described by a *quadratic* kinematic hypothesis in the thickness coordinate Θ^3 with seven unknown parameters. In this context, a detailed discussion of the associated conditions, the inextensibility and incompressibility conditions, are given, which are to be satisfied by the *shell director* $\mathbf{d}$ and the *stretch parameters* $\overset{n}{\lambda}$ $(n = 0, 1)$, respectively.

5.1 Reference configuration

We first consider the *reference* (undeformed) configuration of a shell continuum. Let $\overset{0}{\mathbf{X}}$ be the position vector of the midsurface F_0 and let $\mathbf{N}$ be the *unit normal* vector of F_0. Thus, the position vector $\mathbf{X}$ of an arbitrary point P_0^* of the shell continuum can be expressed as (Fig. 1):

$$\mathbf{X} = \overset{0}{\mathbf{X}}(\Theta^\alpha) + \Theta^3 \overset{1}{\mathbf{X}}(\Theta^\alpha) = \overset{0}{\mathbf{X}}(\Theta^\alpha) + \Theta^3\mathbf{N}(\Theta^\alpha), \quad \Theta^3 \in \Big[-H/2, \ +H/2\Big], \tag{5.1}$$

where Θ^3 denotes the distance between P_0^* and F_0 measured in the $\mathbf{N}$-direction and H is the thickness of the shell. Starting from (5.1), the geometrical elements needed for the finite-element procedure can be derived by the standard procedure (Başar and Weichert [10]). The results read as:

Base vectors:

$$\begin{aligned} &\mathbf{G}_\alpha = \mathbf{X},_\alpha = \mathbf{A}_\alpha + \Theta^3\mathbf{N},_\alpha, \quad &&\mathbf{G}_3 = \mathbf{N}, \\ &\mathbf{G}^\alpha = G^{\alpha\beta}\mathbf{G}_\beta, &&\mathbf{G}^3 = \mathbf{G}_3. \end{aligned} \tag{5.2}$$

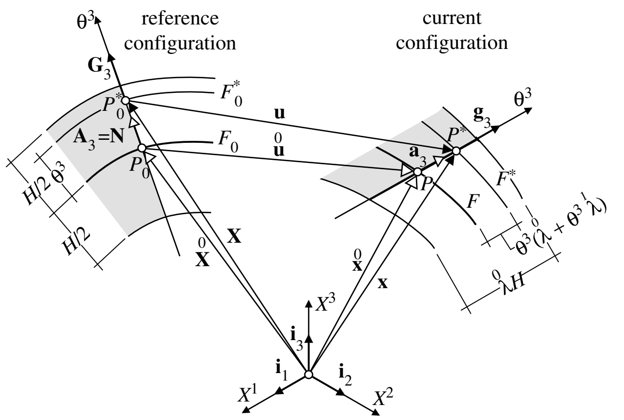

Fig. 1. Deformed and undeformed shell continuum, kinematic variables

Metric tensor components:

$$G_{\alpha\beta} = \mathbf{G}_\alpha \cdot \mathbf{G}_\beta, \quad G_{\alpha 3} = 0, \quad G_{33} = 1,$$

$$G^{\alpha\rho} G_{\rho\beta} = \delta^\alpha_\beta, \qquad G^{\alpha 3} = 0, \quad G^{33} = 1. \tag{5.3}$$

Determinant G and volume element dV_0:

$$\sqrt{G} = |\mathbf{G}_1 \mathbf{G}_2 \mathbf{G}_3| = \sqrt{|G_{ij}|},$$

$$dV_0 = \sqrt{G}\, d\Theta^1 d\Theta^2 d\Theta^3. \tag{5.4}$$

In (5.2), $\mathbf{A}_\alpha = \mathbf{G}_\alpha|_{\Theta^3=0}$ denotes the value of $\mathbf{G}_\alpha$ at $\Theta^3 = 0$ satisfying the usual relation $\mathbf{A}_\alpha \cdot \mathbf{A}^\beta = \delta^\beta_\alpha$ with the contravariant base vectors $\mathbf{A}^\alpha$. Note that, provided Θ^3 is kept unchanged, Eqs. (5.2) to (5.4) hold for arbitrary curvilinear coordinates Θ^α, particularly also for isoparametric ones $\xi^\alpha \in [-1, +1]$ to be used in the finite-element procedure. In the present development the exact relations (5.2) to (5.4) will be used for the determination of the geometrical elements, that is, the usual truncation of terms in Θ^3 e.g. for $G^{\alpha\beta}$ (Başar and Krätzig [8]) is omitted.

5.2 Current configuration

Let P^* be the deformed position of the point P^*_0 in the *current* configuration. In contrast to the geometrical elements of the reference state those ones associated with the deformed state will be denoted by lower-case letters. In the present development the position vector $\mathbf{x}$ of the point P^* is approximated by a *quadratic* polynomial in the thickness coordinate Θ^3 in the form (Fig. 1):

$$\mathbf{x} = \overset{0}{\mathbf{x}} + \Theta^3 \overset{1}{\mathbf{x}} + (\Theta^3)^2 \overset{2}{\mathbf{x}} = \overset{0}{\mathbf{x}} + \Theta^3 \left(\overset{0}{\lambda} + \Theta^3 \overset{1}{\lambda}\right) \mathbf{d}, \tag{5.5}$$

where $\mathbf{d}$ is supposed to be a unit vector subject therefore to the constraint

$$\mathbf{d} \cdot \mathbf{d} = 1 \quad \rightarrow \quad \mathbf{d}_{,\alpha} \cdot \mathbf{d} = 0. \tag{5.6}$$

Accordingly, the base vectors of the deformed state are given by

$$\mathbf{g}_\alpha = \mathbf{a}_\alpha + \Theta^3 \left[(\overset{0}{\lambda}\, \mathbf{d})_{,\alpha} + \Theta^3 (\overset{1}{\lambda}\, \mathbf{d})_{,\alpha} \right], \quad \mathbf{g}_3 = \left(\overset{0}{\lambda} + 2\Theta^3 \overset{1}{\lambda}\right) \mathbf{d} \tag{5.7}$$

as polynomials of second and first order in Θ^3, with $\mathbf{a}_\alpha = \overset{0}{\mathbf{x}}_{,\alpha}$ being $\mathbf{g}_\alpha$ at $\Theta^3 = 0$.

In (5.5), $\overset{n}{\mathbf{x}}\,(n = 0, 1, 2)$ or alternatively $\overset{0}{\mathbf{x}}, \mathbf{d}, \overset{n}{\lambda}\,(n = 0, 1)$ are 2D-kinematic quantities corresponding, in view of the constraint (5.6), to seven scalar-valued unknown variables. As will be confirmed in Sect. 6, the parameters $\overset{n}{\lambda}\,(n = 0, 1)$ describe through-the-thickness stretches of the shell continuum. Due to the separation of the numerically sensitive stretches $\overset{0}{\lambda}$ from the extensible director $\overset{1}{\mathbf{x}}$, the multiplicative decomposition $\overset{1}{\mathbf{x}} = \overset{0}{\lambda}\mathbf{d}$ of the first-order term in (5.5) is advantageously since the so-called curvature locking is practically eliminated. The inclusion of the quadratic term $\overset{1}{\lambda}$ is known (Başar and Ding [3]) as an efficient, kinematically consistent remedy against the Poisson locking in case of compressible materials. In this sense, it is not indispensable for incompressible materials since the *incompressibility condition* removes automatically the Poisson locking. But in the last cited case, the inclusion of $\overset{1}{\lambda}$ is in so far of significance as it offers the possibility to satisfy the incompressibility condition in a very effective manner only through the kinematic quantities $\overset{n}{\lambda}\,(n = 0, 1)$, that means, without eliminating the transverse strains E_{33} by using relation (3.6). A main purpose of this work is to show the effectivity or non-effectivity of the above mentioned approach.

Concerning incompressibility, the stretch variables $\overset{n}{\lambda}\,(n = 0, 1)$ occurring in (5.5) can be regarded as dependent kinematic quantities and can be evaluated through 2D-incompressibility conditions in terms of $\overset{0}{\mathbf{x}}$ and $\mathbf{d}$. To this end, we replace the incompressibility condition $III_{\mathbf{C}} = 1$, according to (5.4), by the following equivalent formulation:

$$\sqrt{g} = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 = \sqrt{G} = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3 \tag{5.8}$$

in terms of the base vectors $\mathbf{g}_i$ and $\mathbf{G}_i$ of the current and reference state, respectively. In (5.8) we express the cited base vectors according to (5.2) and (5.7) and then equate in the resulting polynomial in Θ^3 the first two coefficients to zero to obtain the following two constraints

$$\overset{0}{\lambda} = \frac{[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]}{[\mathbf{a}_1\mathbf{a}_2\mathbf{d}]} = \frac{\sqrt{A}}{[\mathbf{a}_1\mathbf{a}_2\mathbf{d}]}, \tag{5.9}$$

$$\overset{1}{\lambda} = \frac{1}{2}\frac{\overset{0}{\lambda}}{\sqrt{A}} e^{\alpha\beta}\Big[[\mathbf{A}_\alpha\mathbf{N}_{,\beta}\mathbf{N}] - (\overset{0}{\lambda})^2[\mathbf{a}_\alpha\mathbf{d}_{,\beta}\mathbf{d}]\Big], \quad e^{\alpha\beta} = e_{\alpha\beta} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{5.10}$$

Note that $e^{\alpha\beta}$ is the 2D-permutation tensor associated with orthogonal Cartesian coordinates. Table 3 summarizes expressions obtained from (5.9) and (5.10) for $\Delta\overset{0}{\lambda}, \Delta\delta\overset{0}{\lambda}, \ldots$ which will be used for the elimination of these quantities within an iteration step at the element level for the evaluation of the stretch parameters.

The nonlinear inextensibility condition does not ensure a unique determination of the director $\mathbf{d}$ when dealing with finite rotation phenomena (Başar [13]). An effective remedy against this deficiency is a suitable parametrization of the director $\mathbf{d}$. In this work, an *up-dated rotation* formulation is used for this purpose, where $\mathbf{d}$ is determined in each iteration step with respect to its foregoing position. Our aim is now to summarize the essential concepts of this procedure referring to Başar et al. [14], Başar and Kintzel [15] for a detailed description.

It can be easily confirmed that the conditions $\delta(\mathbf{d}\cdot\mathbf{d}) = \Delta(\mathbf{d}\cdot\mathbf{d}) = \Delta\delta(\mathbf{d}\cdot\mathbf{d}) = \delta\Delta(\mathbf{d}\cdot\mathbf{d}) = 0$ obtainable from the constraint (5.6) are automatically satisfied for arbitrary values of the vectors $\overset{V}{\boldsymbol{\omega}}$ and $\overset{L}{\boldsymbol{\omega}}$, if we set

$$\delta\mathbf{d} = \overset{V}{\boldsymbol{\omega}} \times \mathbf{d}, \quad \Delta\mathbf{d} = \overset{L}{\boldsymbol{\omega}} \times \mathbf{d}, \tag{5.11}$$

$$\Delta\delta\mathbf{d} = \delta\Delta\mathbf{d} = \frac{1}{2}\Big[\overset{V}{\boldsymbol{\omega}} \times (\overset{L}{\boldsymbol{\omega}} \times \mathbf{d}) + \overset{L}{\boldsymbol{\omega}} \times (\overset{V}{\boldsymbol{\omega}} \times \mathbf{d})\Big] \tag{5.12}$$

permitting to express $\delta\mathbf{d}, \Delta\mathbf{d}$ and $\Delta\delta\mathbf{d}$ in terms of $\overset{V}{\boldsymbol{\omega}}, \overset{L}{\boldsymbol{\omega}}$ during an iteration step. The relations in (5.11) can be regarded as ansatz while Eq. (5.12) is obtained from (5.11) under the condition

Table 3. Incompressibility conditions

$$\overset{0}{\lambda} = \frac{\sqrt{A}}{[\mathbf{a}_1\mathbf{a}_2\mathbf{d}]}:$$

$$\delta\overset{0}{\lambda} = -\frac{\overset{0}{\lambda}}{[\mathbf{a}_1\mathbf{a}_2\mathbf{d}]}\left\{e^{\alpha\beta}[\delta\mathbf{a}_\alpha\mathbf{a}_\beta\mathbf{d}] + [\mathbf{a}_1\mathbf{a}_2\delta\mathbf{d}]\right\}$$

$$\begin{aligned}\Delta\delta\overset{0}{\lambda} = -\frac{1}{[\mathbf{a}_1\mathbf{a}_2\mathbf{d}]}\Bigg\{ & \Delta\overset{0}{\lambda}\Big(e^{\alpha\beta}[\delta\mathbf{a}_\alpha\mathbf{a}_\beta\mathbf{d}] + [\mathbf{a}_1\mathbf{a}_2\delta\mathbf{d}]\Big) + \delta\overset{0}{\lambda}\Big(e^{\alpha\beta}[\Delta\mathbf{a}_\alpha\mathbf{a}_\beta\mathbf{d}] + [\mathbf{a}_1\mathbf{a}_2\Delta\mathbf{d}]\Big) \\ & + \overset{0}{\lambda}\Big(e^{\alpha\beta}\Big([\delta\mathbf{a}_\alpha\Delta\mathbf{a}_\beta\mathbf{d}] + [\delta\mathbf{a}_\alpha\mathbf{a}_\beta\Delta\mathbf{d}] + [\Delta\mathbf{a}_\alpha\mathbf{a}_\beta\delta\mathbf{d}]\Big) + [\mathbf{a}_1\mathbf{a}_2\Delta\delta\mathbf{d}]\Big)\Bigg\}\end{aligned}$$

$$\overset{1}{\lambda} = \frac{1}{2}\frac{\overset{0}{\lambda}}{\sqrt{A}}\,e^{\alpha\beta}\left\{[\mathbf{A}_\alpha\mathbf{N}_{,\beta}\mathbf{N}] - (\overset{0}{\lambda})^2[\mathbf{a}_\alpha\mathbf{d}_{,\beta}\mathbf{d}]\right\}:$$

$$\delta\overset{1}{\lambda} = \frac{1}{2\sqrt{A}}\,e^{\alpha\beta}\left\{\delta\overset{0}{\lambda}\Big([\mathbf{A}_\alpha\mathbf{N}_{,\beta}\mathbf{N}] - 3(\overset{0}{\lambda})^2[\mathbf{a}_\alpha\mathbf{d}_{,\beta}\mathbf{d}]\Big) - (\overset{0}{\lambda})^3\Big([\delta\mathbf{a}_\alpha\mathbf{d}_{,\beta}\mathbf{d}] + [\mathbf{a}_\alpha\delta\mathbf{d}_{,\beta}\mathbf{d}] + [\mathbf{a}_\alpha\mathbf{d}_{,\beta}\delta\mathbf{d}]\Big)\right\}$$

$$\begin{aligned}\Delta\delta\overset{1}{\lambda} = -\frac{1}{2\sqrt{A}}\,e^{\alpha\beta}\Bigg\{ & 6\,\overset{0}{\lambda}\,\Delta\overset{0}{\lambda}\,\delta\overset{0}{\lambda}\,[\mathbf{a}_\alpha\mathbf{d}_{,\beta}\mathbf{d}] + 3\Delta\overset{0}{\lambda}\,(\overset{0}{\lambda})^2\Big([\delta\mathbf{a}_\alpha\mathbf{d}_{,\beta}\mathbf{d}] + [\mathbf{a}_\alpha\delta\mathbf{d}_{,\beta}\mathbf{d}] + [\mathbf{a}_\alpha\mathbf{d}_{,\beta}\delta\mathbf{d}]\Big) \\ & + 3\delta\overset{0}{\lambda}\,(\overset{0}{\lambda})^2\Big([\Delta\mathbf{a}_\alpha\mathbf{d}_{,\beta}\mathbf{d}] + [\mathbf{a}_\alpha\Delta\mathbf{d}_{,\beta}\mathbf{d}] + [\mathbf{a}_\alpha\mathbf{d}_{,\beta}\Delta\mathbf{d}]\Big) + (\overset{0}{\lambda})^3\Big([\Delta\mathbf{a}_\alpha\delta\mathbf{d}_{,\beta}\mathbf{d}] + [\Delta\mathbf{a}_\alpha\mathbf{d}_{,\beta}\delta\mathbf{d}] + [\delta\mathbf{a}_\alpha\Delta\mathbf{d}_{,\beta}\mathbf{d}] \\ & + [\delta\mathbf{a}_\alpha\mathbf{d}_{,\beta}\Delta\mathbf{d}] + [\mathbf{a}_\alpha\Delta\mathbf{d}_{,\beta}\delta\mathbf{d}] + [\mathbf{a}_\alpha\delta\mathbf{d}_{,\beta}\Delta\mathbf{d}] + [\mathbf{a}_\alpha\Delta\delta\mathbf{d}_{,\beta}\mathbf{d}] + [\mathbf{a}_\alpha\mathbf{d}_{,\beta}\Delta\delta\mathbf{d}]\Big) \\ & - \Delta\delta\overset{0}{\lambda}\Big([\mathbf{A}_\alpha\mathbf{N}_{,\beta}\mathbf{N}] - 3(\overset{0}{\lambda})^2[\mathbf{a}_\alpha\mathbf{d}_{,\beta}\mathbf{d}]\Big)\Bigg\}\end{aligned}$$

that $\overset{V}{\boldsymbol{\omega}}$ and $\overset{L}{\boldsymbol{\omega}}$ behave like variation and linearization of an independent variable being not accessible to a further variation: $\delta\overset{V}{\boldsymbol{\omega}} = \Delta\overset{L}{\boldsymbol{\omega}} = 0$.

Furthermore, it can be proved that after evaluation of an iteration step the new value of the director $\overset{i+1}{\mathbf{d}}$ can be determined in an exact form if the incremental vector $\overset{L}{\boldsymbol{\omega}}$ occurring in (5.11) is identified with the Rodrigues rotation vector. The relation to be used for this purpose is of the form:

$$\overset{i+1}{\mathbf{d}} = \left(\mathbf{I} + \frac{\sin\omega}{\omega}\hat{\boldsymbol{\omega}} + \frac{1-\cos\omega}{\omega^2}\hat{\boldsymbol{\omega}}\hat{\boldsymbol{\omega}}\right)\overset{i}{\mathbf{d}} = \mathbf{R}\,\overset{i}{\mathbf{d}}, \quad \hat{\boldsymbol{\omega}} = \overset{L}{\boldsymbol{\omega}}\times, \quad \omega = ||\overset{L}{\boldsymbol{\omega}}||. \tag{5.13}$$

In the numerical implementation $\overset{V}{\boldsymbol{\omega}}$ and $\overset{L}{\boldsymbol{\omega}}$ are resolved with respect to a global reference frame $\boldsymbol{\omega} = \omega^i\mathbf{i}_i$ or with respect to a local coordinate system $\boldsymbol{\omega} = \tilde{\omega}^\alpha\mathbf{e}_\alpha + \tilde{\omega}^3\mathbf{d}$, respectively. The first decomposition is of significance to deal with compound shells, while the second one is suitable for single shells.

6 Strains

As strain measure we use the Green-Lagrange strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G})$ which is given in terms of the *reference* base vectors $\mathbf{G}_i$ and *current* base vectors $\mathbf{g}_i$ by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G}) = \frac{1}{2}\left[\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j\right]\mathbf{G}^i \otimes \mathbf{G}^j. \tag{6.1}$$

Inserting the series expansion (5.2) and (5.7) into (5.1) and neglecting cubic and higher-order terms in Θ^3 we finally obtain:

$$\mathbf{E} = \overset{0}{\mathbf{E}} + \Theta^3 \overset{1}{\mathbf{E}} + \left(\Theta^3\right)^2 \overset{2}{\mathbf{E}} \tag{6.2}$$

with n-th order strain tensors $\overset{n}{\mathbf{E}}$ $(n = 0, 1, 2)$ defined by the following kinematic relations:

$$\overset{0}{\mathbf{E}} = \overset{0}{E}_{ij}\, \mathbf{G}^i \otimes \mathbf{G}^j :$$

$$\overset{0}{E}_{\alpha\beta} = \frac{1}{2}\left[\,\mathbf{a}_\alpha \cdot \mathbf{a}_\beta - \mathbf{A}_\alpha \cdot \mathbf{A}_\beta\,\right], \tag{6.3}$$

$$\overset{0}{E}_{\alpha 3} = \frac{1}{2}\,\overset{0}{\lambda}\,\mathbf{a}_\alpha \cdot \mathbf{d}, \tag{6.4}$$

$$\overset{0}{E}_{33} = \frac{1}{2}\left[\,(\overset{0}{\lambda})^2 - 1\,\right], \tag{6.5}$$

$$\overset{1}{\mathbf{E}} = \overset{1}{E}_{ij}\, \mathbf{G}^i \otimes \mathbf{G}^j$$

$$\overset{1}{E}_{\alpha\beta} = \frac{1}{2}\left[\,\mathbf{a}_\alpha \cdot (\overset{0}{\lambda}\,\mathbf{d})_{,\beta} + \mathbf{a}_\beta \cdot (\overset{0}{\lambda}\,\mathbf{d})_{,\alpha} - \mathbf{A}_\alpha \cdot \mathbf{N}_{,\beta} - \mathbf{A}_\beta \cdot \mathbf{N}_{,\alpha}\,\right], \tag{6.6}$$

$$\overset{1}{E}_{\alpha 3} = \frac{1}{2}\,\overset{0}{\lambda}\overset{0}{\lambda}_{,\alpha} + \overset{1}{\lambda}\,\mathbf{a}_\alpha \cdot \mathbf{d}, \tag{6.7}$$

$$\overset{1}{E}_{33} = 2\,\overset{0}{\lambda}\overset{1}{\lambda}, \tag{6.8}$$

$$\overset{2}{\mathbf{E}} = \overset{2}{E}_{ij}\, \mathbf{G}^i \otimes \mathbf{G}^j$$

$$\overset{2}{E}_{\alpha\beta} = \frac{1}{2}\left[\overset{0}{\lambda}_{,\alpha}\overset{0}{\lambda}_{,\beta} + (\overset{0}{\lambda})^2\mathbf{d}_{,\alpha} \cdot \mathbf{d}_{,\beta} + \mathbf{a}_\alpha \cdot (\overset{1}{\lambda}\mathbf{d})_{,\beta} + \mathbf{a}_\beta \cdot (\overset{1}{\lambda}\mathbf{d})_{,\alpha} - \mathbf{N}_{,\alpha} \cdot \mathbf{N}_{,\beta}\,\right], \tag{6.9}$$

$$\overset{2}{E}_{\alpha 3} = \frac{1}{2}\left[\overset{0}{\lambda}\overset{1}{\lambda}_{,\alpha} + 2\,\overset{1}{\lambda}\overset{0}{\lambda}_{,\alpha}\right], \tag{6.10}$$

$$\overset{2}{E}_{33} = 2(\overset{1}{\lambda})^2. \tag{6.11}$$

It is observable that first- and second-order terms are only defined for the transverse strains E_{33} if the kinematic quantity $\overset{1}{\lambda}$ is included in the hypothesis (5.5). In this context, we note that the consideration of strain variables up to second-order is compatible with the quadratic kinematic assumption (5.5). Numerical investigations performed for compressible materials (Başar et al. [16]) have shown, that the truncation of terms in Θ^3 for $\mathbf{E}$ even after the linear term has practically no influence on the analysis accuracy. This fact is also well-known from classical shell theory formulations (Başar and Krätzig [8]).

The variation and linearization of the above cited relations (6.3) to (6.11) can be obtained by the standard procedure. Here, we solely give as example those holding for $\delta\overset{1}{E}_{\alpha\beta}$ and $\Delta\delta\overset{1}{E}_{\alpha\beta}$, thus

$$\delta \overset{1}{E}_{\alpha\beta} = \frac{1}{2}\Big[\,\mathbf{a}_\alpha\cdot(\mathbf{d}\delta\overset{0}{\lambda})_{,\beta} + \mathbf{a}_\alpha\cdot(\overset{0}{\lambda}\,\delta\mathbf{d})_{,\beta} + \delta\mathbf{a}_\alpha\cdot(\overset{0}{\lambda}\,\mathbf{d})_{,\beta}$$
$$+\,\mathbf{a}_\beta\cdot(\mathbf{d}\delta\overset{0}{\lambda})_{,\alpha} + \mathbf{a}_\beta\cdot(\overset{0}{\lambda}\,\delta\mathbf{d})_{,\alpha} + \delta\mathbf{a}_\beta\cdot(\overset{0}{\lambda}\,\mathbf{d})_{,\alpha}\Big], \tag{6.12}$$

$$\Delta\delta \overset{1}{E}_{\alpha\beta} = \frac{1}{2}\Big[\,\mathbf{a}_\alpha\cdot(\Delta\mathbf{d}\delta\overset{0}{\lambda})_{,\beta} + \Delta\mathbf{a}_\alpha\cdot(\mathbf{d}\delta\overset{0}{\lambda})_{,\beta} + \mathbf{a}_\alpha\cdot(\delta\mathbf{d}\Delta\overset{0}{\lambda})_{,\beta}$$
$$+\,\mathbf{a}_\alpha\cdot(\overset{0}{\lambda}\,\Delta\delta\mathbf{d})_{,\beta} + \Delta\mathbf{a}_\alpha\cdot(\overset{0}{\lambda}\,\delta\mathbf{d})_{,\beta} + \delta\mathbf{a}_\alpha\cdot(\mathbf{d}\Delta\overset{0}{\lambda})_{,\beta}$$
$$+\,\delta\mathbf{a}_\alpha\cdot(\overset{0}{\lambda}\,\Delta\mathbf{d})_{,\beta} + \mathbf{a}_\beta\cdot(\Delta\mathbf{d}\delta\overset{0}{\lambda})_{,\alpha} + \Delta\mathbf{a}_\beta\cdot(\mathbf{d}\delta\overset{0}{\lambda})_{,\alpha}$$
$$+\,\mathbf{a}_\beta\cdot(\delta\mathbf{d}\Delta\overset{0}{\lambda})_{,\alpha} + \mathbf{a}_\beta\cdot(\overset{0}{\lambda}\,\Delta\delta\mathbf{d})_{,\alpha} + \Delta\mathbf{a}_\beta\cdot(\overset{0}{\lambda}\,\delta\mathbf{d})_{,\alpha}$$
$$+\,\delta\mathbf{a}_\beta\cdot(\mathbf{d}\Delta\overset{0}{\lambda})_{,\alpha} + \delta\mathbf{a}_\beta\cdot(\overset{0}{\lambda}\,\Delta\mathbf{d})_{,\alpha} + \Delta\delta\mathbf{a}_\alpha\cdot(\overset{0}{\lambda}\,\mathbf{d})_{,\beta}$$
$$+\,\Delta\delta\mathbf{a}_\beta\cdot(\overset{0}{\lambda}\,\mathbf{d})_{,\alpha} + \mathbf{a}_\alpha\cdot(\mathbf{d}\Delta\delta\overset{0}{\lambda})_{,\beta} + \mathbf{a}_\beta\cdot(\mathbf{d}\Delta\delta\overset{0}{\lambda})_{,\alpha}\Big]\,. \tag{6.13}$$

After each iteration step the transverse strains $\overset{n}{E}_{33}\,(n = 0, 1, 2)$ are given directly by the incompressibility condition (3.6) or, alternatively, by considering the kinematic relations (6.5), (6.8) and (6.11). However, if the kinematic quantity $\overset{1}{\lambda}$ is omitted in (5.5) higher-order terms $\overset{n}{E}_{33}$ $(n = 1, 2)$ are not defined by means of the corresponding kinematic relations.

By means of Eqs. (6.3) to (6.11) and the definition of the stretches $\lambda_{<i>}$ related to Θ^i-curves it can be confirmed that the kinematic quantities $\overset{n}{\lambda}\,(n = 0, 1)$ correspond to constant and linear through-the-thickness stretches, thus (Malvern [17], Başar and Weichert [10]):

$$\lambda_{<3>} = \sqrt{\frac{2E_{33}}{G_{33}} + 1} \quad\rightarrow\quad \lambda_{<3>} = \overset{0}{\lambda} + 2\overset{1}{\lambda}\,\Theta^3. \tag{6.14}$$

For a further important observation attention is now given to the potential energy of an arbitrary structure. By using the Neo-Hookean material model (4.4) with $c_2 = 0$ the internal potential of the considered structure is, in view of (6.3) to (6.11), given by

$$\Pi_i = \int_{F_0}\int_{-H/2}^{H/2} W\; d\Theta^3 dF_0 = \int_{F_0} 2c_1\Big[H\,(\overset{0}{E}{}^{\alpha}_{\alpha} + \overset{0}{E}{}^{3}_{3}) + \frac{H^3}{12}(\overset{2}{E}{}^{\alpha}_{\alpha} + \overset{2}{E}{}^{3}_{3})\Big] dF_0. \tag{6.15}$$

Thus, we see that the first-order tangential strains $\overset{1}{E}_{\alpha\beta}$ responsible in classical shell theories for the consideration of bending effects as well as transverse shear strains $\overset{n}{E}_{\alpha 3}$ $(n = 0, 1, 2)$ are not present in the potential energy in case of using the Neo-Hookean material model (4.4).

Bending strains $\overset{1}{E}_{\alpha\beta}$ together with transverse shear strains $\overset{n}{E}_{\alpha 3}$ $(n = 0, 1, 2)$ are introduced in the formulation (15), if E_{33} is replaced by relation (3.6) which can be confirmed by a series expansion of $\overset{n}{E}_{33}$ with respect to Θ^3 (see, e.g., Başar and Ding [3]).

The incompressibility condition will be enforced in the sequel by three different procedures to be summarized in the following.

Procedure 1 presenting the most general and accurate approach is based on the satisfaction of the incompressibility constraint through the stretch variables $\overset{n}{\lambda}\,(n=0,1)$ as well as the transverse strains E_{33} by means of (5.9), (5.10) and (3.6), respectively, proceeding in the following steps:

(i) Application of the quadratic kinematic hypothesis (5.5) involving $\overset{1}{\lambda}$; consideration of the strains $\overset{n}{E}_{ij}\,(n=0,1,2)$ up to second-order terms together with the corresponding kinematic relations (6.3) to (6.11).

(ii) Evaluation of the quantities $\delta\overset{n}{\lambda}, \Delta\overset{n}{\lambda}, \Delta\delta\overset{n}{\lambda}\,(n=0,1)$ according to Table 3 in terms of $\overset{0}{\mathbf{x}}, \mathbf{d}$ and the associated incremental quantities $\Delta\overset{0}{\mathbf{x}}, \ldots$. Determination of the fundamental state variables $\overset{n}{\lambda}\,(n=0,1)$ by means of (5.9), (5.10) after each iteration step.

(iii) Elimination of $\Delta E_{33}, \delta E_{33}, \Delta\delta E_{33}$ in δW (4.2) and $\Delta\delta W$ (4.3) by using (3.11), (3.12) within an iteration step; determination of E_{33} by means of (3.6) after accomplishment of an iteration step.

Procedure 2 presents a simplified version of Procedure 1. The simplification is due to the neglect of the kinematic quantity $\overset{1}{\lambda}$ in (5.5). Accordingly, the corresponding relation (5.10) together with the second-order strains $\overset{2}{E}_{ij}$ are not considered in this approach.

Procedure 3 is basically different from Procedure 1 and uses solely the relations (5.9), (5.10) holding for $\overset{n}{\lambda}\,(n=0,1)$ for the consideration of the incompressibility constraint. Accordingly, the last cited step of Procedure 1 is not considered in this approach resulting in a decisively simpler formulation. The fundamental state quantities $\overset{n}{E}_{33}\,(n=0,1)$ are determined in this approach by means of the kinematic relations (6.5), (6.8) and (6.11) indirectly coupled with the incompressibility constraint through $\overset{n}{\lambda}\,(n=0,1)$.

Note that neglecting (5.10) in Procedure 2 provides a significant simplification of the theoretical formulation coupled, correspondingly, with minor computational cost (see Table 3.) However, Procedure 2 is kinematically not fully consistent, since higher-order transverse strains $\overset{n}{E}_{33}\,(n=1,2)$ are introduced in the formulation trough (3.6) while for $\overset{1}{\lambda}=0$ the corresponding kinematic relations (6.8), (6.11) would prescribe vanishing values for them. This deficiency automatically disappears in Procedure 1 due to inclusion of the variable $\overset{1}{\lambda}$. We also note that Procedure 2 has been already considered in the numerical implementation by Başar and Ding [3], where in contrast to the present development all 3D-equations had been expanded in power series with respect to Θ^3 in order to carry out the thickness integration analytically. The numerical thickness integration favored in this contribution provides simplicity and the consideration of arbitrary constitutive models within a unified procedure.

7 Constitutive relation

The relation between stresses and strains will be established here for incompressible isotropic material models. The stress tensor energy conjugated to **C** is the second Piola-Kirchhoff tensor:

$$\mathbf{S} = 2W_{,\mathbf{C}} = 2\frac{\partial W}{\partial \mathbf{C}}, \tag{7.1}$$

which is given for an arbitrary incompressible isotropic material $W = W(I_{\mathbf{C}}, II_{\mathbf{C}})$ by

$$\mathbf{S} = 2\left[a_I \frac{\partial I_{\mathbf{C}}}{\partial \mathbf{C}} + a_{II}\frac{\partial II_{\mathbf{C}}}{\partial \mathbf{C}} + p\frac{\partial III_{\mathbf{C}}}{\partial \mathbf{C}}\right] = 2\left[(a_I + a_{II} I_{\mathbf{C}})\mathbf{G} - a_{II}\mathbf{C} + p\mathbf{C}^{-1}\right]. \tag{7.2}$$

Herein $a_I = \partial W/\partial I_{\mathbf{C}}$ and $a_{II} = \partial W/\partial II_{\mathbf{C}}$ are material constants, which depend on the invariants of $\mathbf{C}$, and p is a scalar function interpretable in the case of pure isochoric deformations $III_{\mathbf{C}} = 1$ as hydrostatic pressure. The corresponding expression for the Cauchy stress tensor

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^T, \quad J = \sqrt{III_{\mathbf{C}}} = 1 \tag{7.3}$$

reads accordingly

$$\boldsymbol{\sigma} = 2[(a_I + a_{II} I_{\mathbf{C}})\mathbf{b} - a_{II}\mathbf{b}^2 + p\mathbf{g}], \tag{7.4}$$

where $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ denotes the left Cauchy-Green tensor and $\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^i$ is the deformation gradient of the shell continuum. The unknown parameter p can be determined for sufficiently thin structures by the assumption that the component of the Cauchy stress vector $\mathbf{t}$ in direction of the unit normal vector $\mathbf{n} = \mathbf{g}^3/\sqrt{g^{33}}$ is of negligible magnitude in comparison to the remaining components. Thus

$$\mathbf{t} \cdot \mathbf{n} = 0, \tag{7.5}$$

wich can be expressed by means of the Cauchy theorem $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ as

$$\mathbf{g}^3\boldsymbol{\sigma}\mathbf{g}^3 = \sigma^{33} = 0 \tag{7.6}$$

and gives by virtue of (7.4) finally

$$p = -\frac{a_I + a_{II}(2 + 2E^{\alpha}_{\alpha})}{g^{33}}. \tag{7.7}$$

The stresses will be determined in the present FE-formulation directly from the 3D-relation (7.4) eliminating p by the above condition (7.7). Note that in case of Procedures 1 and 2 the relation (3.6) for the transversal strain components E^3_3 is to be considered to compute the deformed metric $\mathbf{g}$ and the strain invariants $I_{\mathbf{C}}$, $II_{\mathbf{C}}$.

8 Finite-element formulation

The theoretical fundamentals presented above are introduced into a 4-node isoparametric finite shell element. Geometrical and physical nonlinearities are treated by an incremental-iterative procedure according to the Newton-Raphson method. In the following we summarize the basic concepts of the development related to different aspects:

8.1 Material modelling

3D-constitutive laws as well as relation (3.6) in terms of the strain variables E_{ij} are directly introduced in the finite element formulation in combination with a numerical thickness integration.

8.2 Geometrical elements of the reference and current state

The geometrical elements $\overset{0}{\mathbf{X}}$ and $\mathbf{N}$ determining according to (5.1) the reference configuration as well as the independent kinematic quantity $\overset{0}{\mathbf{x}}$ occurring in the assumption (5.5) are approximated by using the standard bilinear polynomials, e.g.,

$$\overset{0}{\mathbf{x}} = \sum_{K=1}^{4} N_K \, \overset{0}{\mathbf{x}}{}^K \tag{8.1}$$

which are defined by

$$\begin{aligned} N_1 &= \frac{1}{4}\left(1-\xi^1\right)\left(1-\xi^2\right), \quad N_3 = \frac{1}{4}\left(1+\xi^1\right)\left(1+\xi^2\right), \\ N_2 &= \frac{1}{4}\left(1+\xi^1\right)\left(1-\xi^2\right), \quad N_4 = \frac{1}{4}\left(1-\xi^1\right)\left(1+\xi^2\right), \end{aligned} \tag{8.2}$$

where $\overset{0}{\mathbf{x}}{}^K$ denotes the nodal values of $\overset{0}{\mathbf{x}}$, and $\xi^\alpha \in [-1, +1]$ are isoparametric coordinates. Geometrical elements of the reference state are determined in an exact form by means of the continuum-based relations given in (5.2) and (5.4).

8.3 Stretch variables

The variational quantities $\Delta\overset{0}{\lambda}, \delta\overset{0}{\lambda}$ and $\Delta\delta\overset{0}{\lambda}$ related to the dependent stretch variable $\overset{0}{\lambda}$ are interpolated similar to $\overset{0}{\mathbf{x}}$ (8.1) by eliminating subsequently the nodal values $\Delta\overset{0}{\lambda}{}^K, \ldots$ occurring in the corresponding expressions at the element level by considering the constraints given in Table 3. After accomplishment of an iteration step the fundamental state quantity $\overset{0}{\lambda}$ is again interpolated according to (8.1) with the nodal values $\overset{0}{\lambda}{}^K$ evaluated by means of the exact relation (5.9). The dependent first-order stretch $\overset{1}{\lambda}$ is treated similarly. The direct interpolation of $\overset{n}{\lambda}$ $(n = 0, 1)$ and of all related quantities occurring in Table 3 particularly by considering the derivatives of the corresponding element in the form:

$$\overset{n}{\lambda}_{,\alpha} = \sum_{K=1}^{4} N_{K,\alpha} \, \overset{n}{\lambda}{}^K, \quad \delta\overset{n}{\lambda}_{,\alpha} = \sum_{K=1}^{4} N_{K,\alpha} \delta\overset{n}{\lambda}{}^K, \quad \Delta\delta\overset{n}{\lambda}_{,\alpha} = \sum_{K=1}^{4} N_{K,\alpha} \Delta\delta\overset{n}{\lambda}{}^K, \quad (n = 0, 1) \tag{8.3}$$

is variationally not consistent. But this approach provides a simple numerical implementation and is computationally efficient which will become evident in Sect. 9. The variationally consistent counterpart of this approach requires first the lengthy differentiation of the constraints from Table 3 which are then to be used for a point-wise determination of the shape functions of the derivatives $\overset{n}{\lambda}_{,\alpha}, \delta\overset{n}{\lambda}_{,\alpha}$ and $\Delta\delta\overset{n}{\lambda}_{,\alpha}$ $(n = 0, 1)$.

8.4 Transverse strains

The transverse strains E_{33} and its variational quantities δE_{33}, ΔE_{33}, $\Delta\delta E_{33}$ occurring in the incremental formulation (4.2), (4.3) are evaluated in Procedures 1 and 2 according to (3.6), (3.11), (3.12) in terms of $E_{\alpha i}, \delta E_{\alpha i}, \ldots$ and in Procedure 3 by using the corresponding kinematic relations (6.5), (6.8), (6.11), respectively.

8.5 Locking phenomena

The well-known *shear locking* is reduced by using the *natural assumed strain* concept (Bathe and Dvorkin [18]). To reduce *membrane* and *volume* locking the membrane strains $\overset{0}{E}_{\alpha\beta}$ are enhanced by incompatible modes according to the *enhanced-strain formulation* (Eckstein [19], Başar and Kintzel [15]) originally proposed by Simo and Rifai [20]. In case of compressible materials, Poisson locking is automatically omitted through the inclusion of linear stretches $\overset{1}{\lambda}$ in the kinematic assumption. For incompressible materials, $\overset{1}{\lambda}$ is not needed in this sense as the enforcement of the incompressibility condition is an efficient remedy against this deficiency.

9 Numerical examples

Extended numerical studies have been carried out to identify the computationally most effective procedure for modelling hyperelastic materials characterized by incompressibility. The three procedures introduced in Sect. 6 to be used for an explicit satisfaction of the incompressibility constraint have been examined particularly by considering only Mooney-Rivlin and Neo-Hookean material models. Some examples have also been analyzed by means of compressible material models of Neo-Hookean and St. Venant-Kirchhoff type for the sake of comparison. For this purpose the Poisson's ratio ν and Lame's constant μ have been selected to $\nu \rightarrow 0.5$ and $\mu = 2(c_1 + c_2)$. For clear presentation of the results the following abbreviations and symbols are used:

incom Neo-Hooke: incompressible Neo-Hooke model $(c_1, c_2 = 0)$

○ $\longrightarrow$ Procedure 1 (p1)
△ $\longrightarrow$ Procedure 2 (p2)
□ $\longrightarrow$ Procedure 3 (p3)

Mooney-Rivlin: incompressible Mooney-Rivlin model (c_1, c_2)

● $\longrightarrow$ Procedure 1 (p1)
▲ $\longrightarrow$ Procedure 2 (p2)
■ $\longrightarrow$ Procedure 3 (p3)

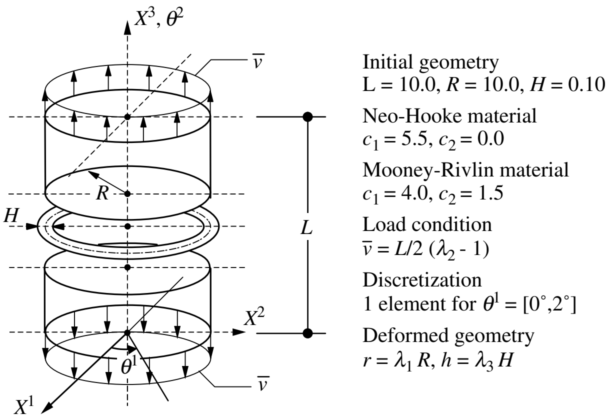

Fig. 2. Uniform extension of a circular cylindrical tube

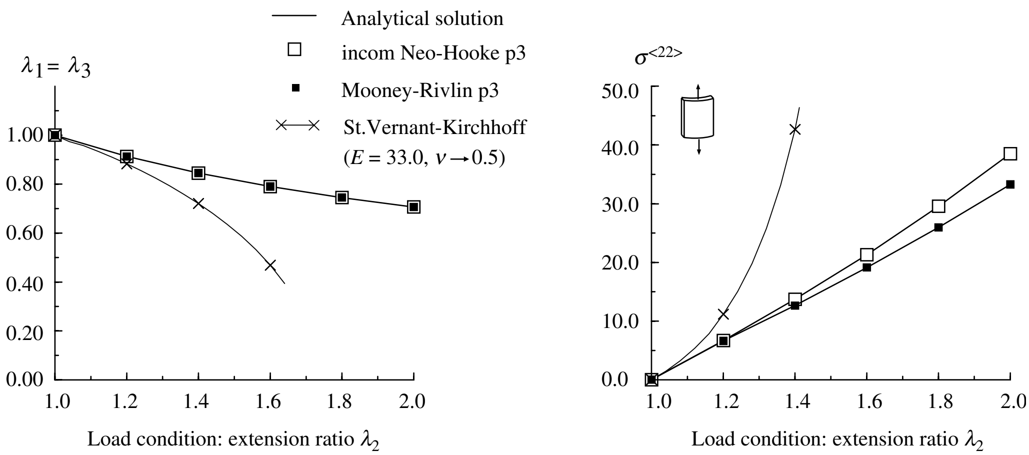

Fig. 3. Uniform extension of a cylindrical tube – load response diagrams

com Neo-Hooke: compressible Neo-Hooke model (μ, κ)

$+ \longrightarrow$ value of Poisson's ratio ν approaching 0.5

St. Venant-Kirchhoff: compressible St. Venant-Kirchhoff model (E, ν)

$\times \longrightarrow$ value of Poisson's ratio ν approaching 0.5.

9.1 Uniform extension of a cylindrical tube

We consider first a cylindrical tube subjected to uniform extension in the longitudinal direction (Fig. 2). If the extension ratio λ_2 in this direction is given, the extension ratios in the other two directions can be obtained by means of the incompressibility condition as

$$\lambda_1 = \frac{r}{R} = \lambda_3 = \frac{h}{H} = \frac{1}{\sqrt{\lambda_2}}. \tag{9.1}$$

The analytical solution for the stress component $\sigma^{<22>}$ in the longitudinal direction according to Green and Zerna [9] is given by

$$\sigma^{<22>} = 2\left((\lambda_2)^2 - \frac{1}{\lambda_2}\right)\left(c_1 + \frac{c_2}{\lambda_2}\right). \tag{9.2}$$

Note that $\sigma^{<22>}$ represents the physical component of the Cauchy stress tensor $\boldsymbol{\sigma}$. The analysis has been performed for a $2°$ shell segment using a single element by considering the material models given in Fig. 2. The numerical results of all three procedures p1, p2, p3 are in full agreement with the analytical solution even for the most simple procedure p3 which can be confirmed in Fig. 3. It can be observed that the stresses $\sigma^{<22>}$ are, in contrast to the extension ratios $\lambda_1 = \lambda_3$, influenced by the material model. Numerical results obtained with the St. Venant-Kirchhoff material $(\times)$ are also plotted in Fig. 3 for comparison demonstrating clearly that the classical models (Başar et al. [21]) are not suitable for a large strain analysis.

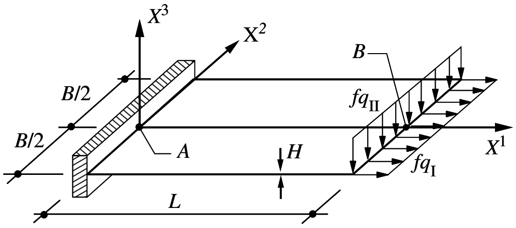

Fig. 4. Rectangular plate under in plane and transversal load

Table 4. Plate under in plane load – linear convergence study ($f = 1.0$)

Discretization	Δx_B^1						
	×	○	●	△	▲	□	■
10 × 4	3.2702	3.2660	3.2660	3.2660	3.2660	3.2593	3.2569
20 × 10	3.2756	3.2740	3.2740	3.2740	3.2740	3.2716	3.2707
40 × 20	3.2779	3.2768	3.2768	3.2768	3.2768	3.2759	3.2756
80 × 40	3.2788	3.2777	3.2777	3.2777	3.2777	3.2774	3.2773

Table 5. Plate under in plane load – nonlinear analysis for a 40 × 20 mesh

Load factor f	Δx_B^1					
	○	△	□	●	▲	■
1.0	3.8805	3.8805	3.8801	4.2186	4.2186	4.2175
2.0	9.1700	9.1700	9.1696	11.2389	11.2389	11.2367
3.0	15.9430	15.9430	15.9426	22.0455	22.0455	22.0410
5.5	37.2615	37.2616	37.2615	59.0678	59.0678	59.0520

9.2 Rectangular plate under in plane and transversal load

This example is selected to examine the different procedures p1, p2, p3, first, for membrane deformations and, subsequently, in a bending dominated situation. In this sense two load cases are considered for the cantilever rectangular plate (Fig. 4).

9.2.1 Rectangular plate under in plane load

The first load case q_I is suitable to analyse the different finite element formulations in case of membrane deformations. For a systematic comparison a linear convergence study has been performed concerning all three procedures in connection with the incompressible models incom Neo-Hooke and Mooney-Rivlin as well as the compressible St. Venant-Kirchhoff model. Some characteristic results for the deformation at point B are presented in Table 4. As expected the results obtained for the incompressible and compressible material models are in good agreement since the incom Neo-Hooke and Mooney-Rivlin models degenerate in the linear case to the St. Venant-Kirchhoff model. All three procedures (p1, p2, p3) produce similar results in the nonlinear case as can be seen in Table 5. However, the nonlinear deformation paths for the two

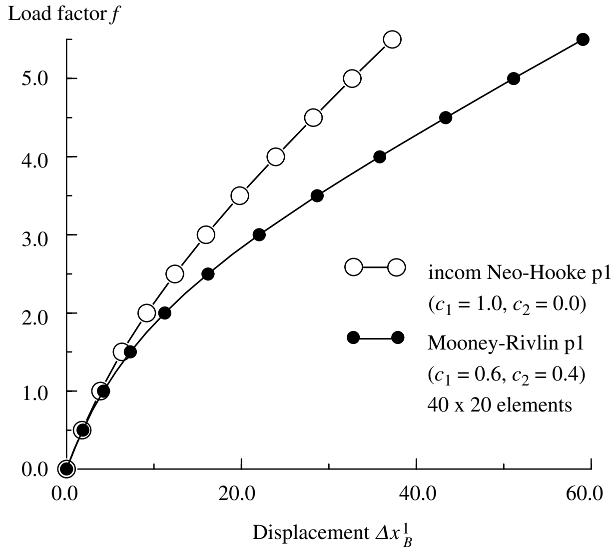

Fig. 5. Plate under in plane load – load-displacement diagram at point B

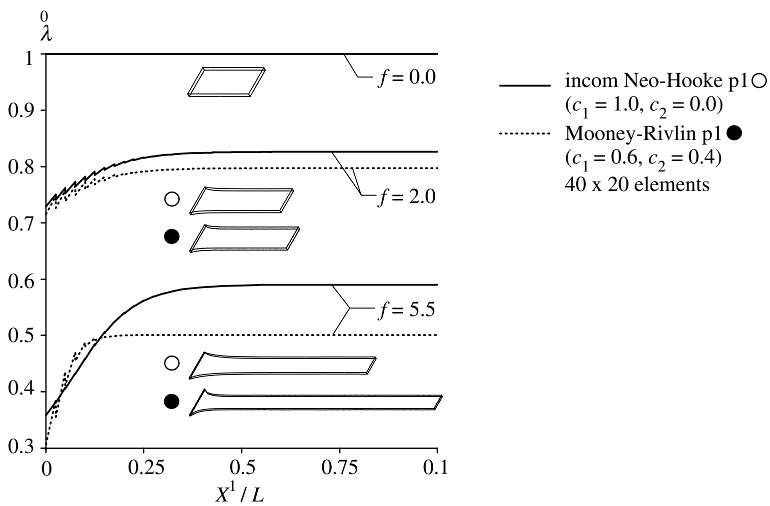

Fig. 6. Plate under in plane load – distribution of constant stretch parameter $\overset{0}{\lambda}$ for $X^2 = 0$

cited models, plotted in Fig. 5 for Procedure 1, are significantly different in the deep nonlinear range. Due to the fact that the stretch parameters $\overset{n}{\lambda}\,(n = 0, 1)$ are eliminated at the element level by means of the incompressibility condition our formulation does not provide an a-priori-satisfaction of C^0-continuity of $\overset{n}{\lambda}\,(n = 0, 1)$. The distribution of $\overset{0}{\lambda}$ for the coordinate line $X^2 = 0$ at several load levels is plotted in Fig. 6 where it can be easily observed that the condition in question is fulfilled with a sufficient accuracy.

9.2.2 Rectangular plate under transversal load

The second load case q_{II}, a transversal line load acting at the free edge, is selected to show the capability of the procedures in dealing with bending dominated situations. A linear

Table 6. Plate under transversal load – numerical results for the linear case with 10×4 elements ($f = 1.0$)

	×	○	△	●	▲
Δx_B^3	−0.4757	−0.4757	−0.4757	−0.4757	−0.4757
σ_A^{11}	$7.9743 \cdot 10^{-3}$	$7.9627 \cdot 10^{-3}$	$7.9662 \cdot 10^{-3}$	$7.9627 \cdot 10^{-3}$	$7.9662 \cdot 10^{-3}$
σ_A^{22}	$3.9044 \cdot 10^{-3}$	$3.8962 \cdot 10^{-3}$	$3.8964 \cdot 10^{-3}$	$3.8962 \cdot 10^{-3}$	$3.8964 \cdot 10^{-3}$

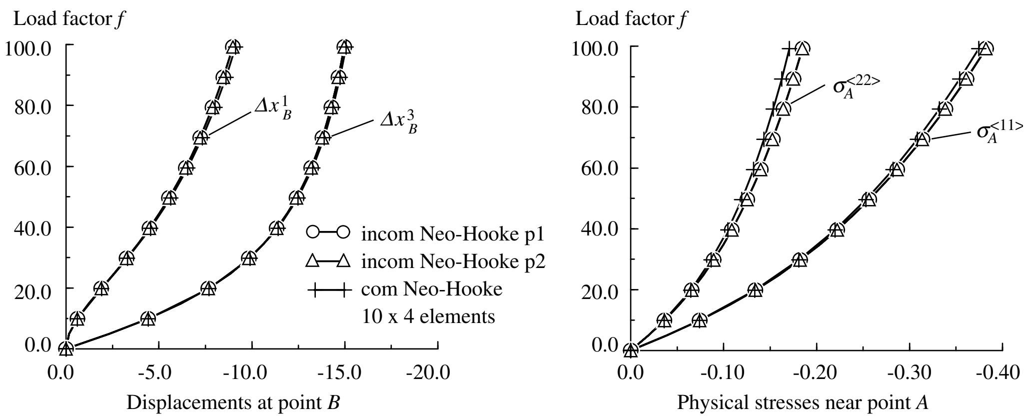

Fig. 7. Plate under transversal load – load-response diagrams

Table 7. Plate under transversal load – numerical results for the nonlinear case with 10×4 elements ($f = 100.0$)

	○	△	●	▲	+
Δx_B^3	−15.0961	−15.0893	−15.1033	−15.0964	−15.1913
Δx_B^1	−9.0498	−9.0498	−9.0569	−9.0564	−9.1982
$\sigma_A^{<11>}$	0.3857	0.3854	0.3871	0.3867	0.3777
$\sigma_A^{<22>}$	0.1865	0.1862	0.1939	0.1935	0.1723
max Δf	5.0	5.0	5.0	5.0	0.01
CPU-time [sec]	620	530	625	550	$144 \cdot 10^3$

convergence study had been performed at first, in which oscillating results could be observed for Procedure 3 for different discretizations. Moreover, Procedure 3 fails completely in the nonlinear analysis and is, therefore, not suitable for dealing with bending problems. However, Procedures 1 and 2 work very well and are for the linear case in excellent agreement with the compressible St. Venant-Kirchhoff model (Table 6). The numerical stress results presented for this load case are evaluated at the Gaussian point near to point A for $X^3 = H/2\sqrt{3}$. From the load response diagrams in Fig. 7 and from Table 7 it can easily be observed that the simplifications in p2 compared to p1 have no influence on the analysis accuracy and provide additionally a minor reduction in computational cost. This example was also analyzed by the com Neo-Hooke model considering the incompressibility condition by selecting the value of Poisson's ratio ν to 0.4999999. The results in Fig. 7 and Table 7

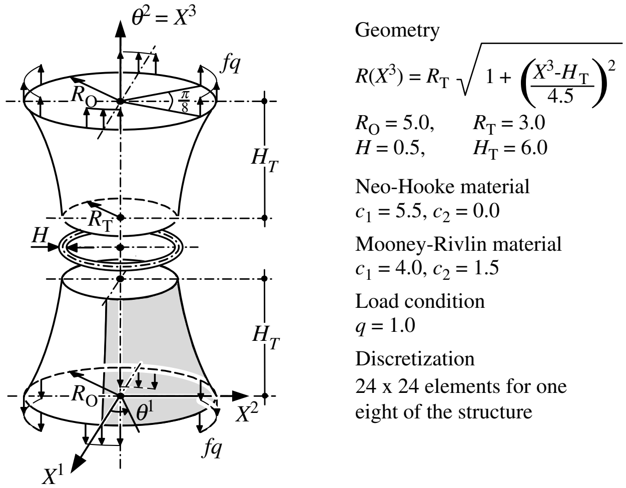

Fig. 8. Hyperbolic shell under four pairs of locally distributed vertical loads

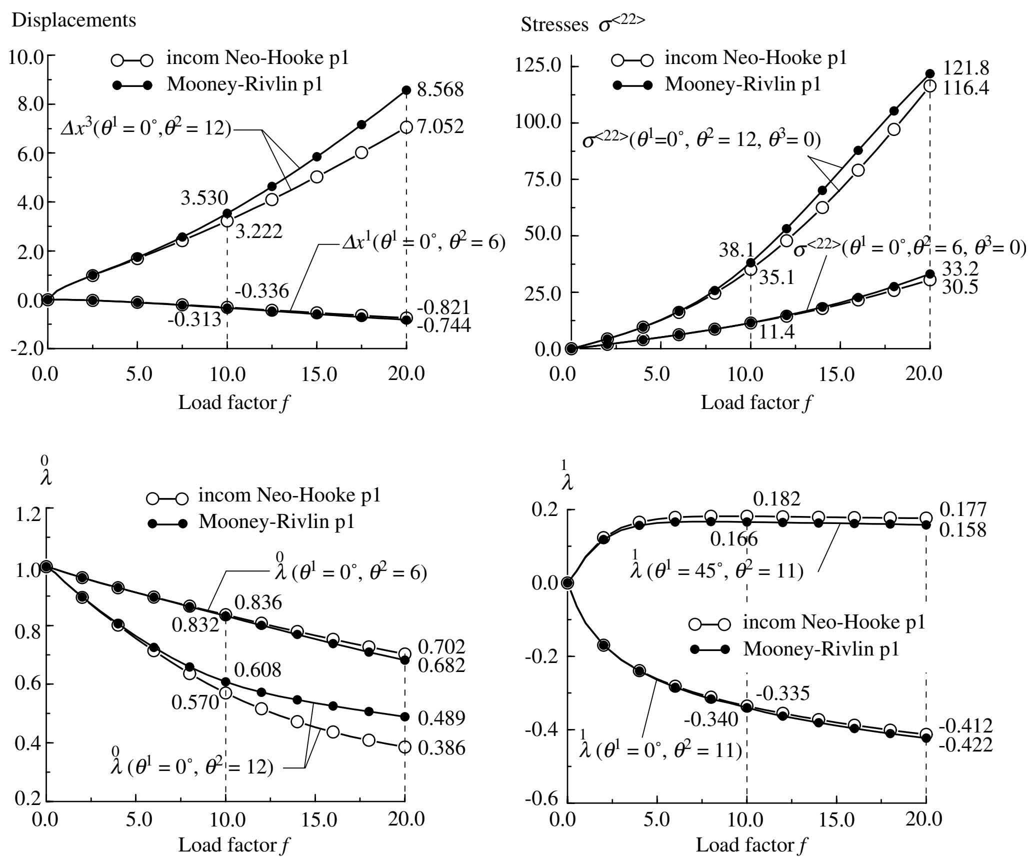

Fig. 9. Hyperbolic shell – load-response diagrams

demonstrate that in this way the incompressibility condition can be considered very accurately even in the deep nonlinear range, but with a large number of load steps needed to ensure that the Newton-Raphson iterative process reaches convergence. In the present case

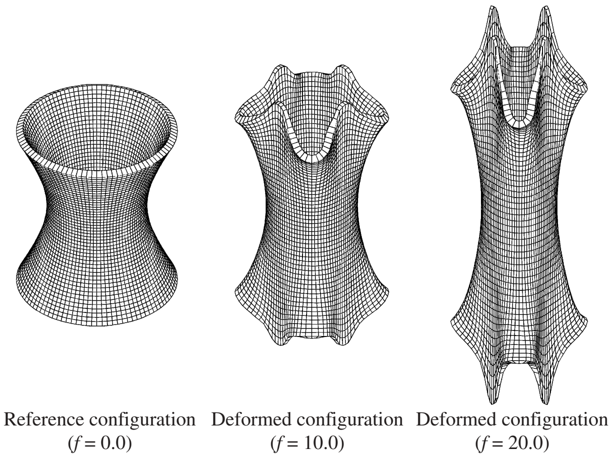

Fig. 10. Hyperbolic shell – undeformed and deformed configuration (Mooney-Rivlin material)

the incompressible models (○, △, ●, ▲) have been computed by using a maximum load increment of $\Delta f = 5.0$, while the compressible one (+) needs 500 times more increments ($\Delta f = 0.01$). Unfortunately, using large increments for compressible models leads to large volume changes especially for those elements containing nodes with prescribed displacements, which, because of the relatively large bulk modulus κ, give extremely large internal pressures at those elements. These unrealistic pressure values lead in turn to very large residual forces such that the Newton-Raphson process fails.

9.3 Hyperbolic shell subjected to nearly concentrated loads

A hyperbolic shell made of hyperelastic material considered as next example is subjected to four pairs of locally distributed vertical loads (Fig. 8). The deformation of this shell is characterized by combined membrane and bending deformations. Due to the symmetry of the structure and loads, only one eighth of the shell is analysed by using 24×24 elements. In this example solely Procedure 1 has been applied. Some characteristic results are given in Fig. 9 in the form of load-response diagrams. Due to the very good agreement of the numerical results with the solution of Başar and Ding [3] it can be concluded that the interpolation of the dependent stretch variables $\overset{n}{\lambda}$ $(n = 0, 1)$ and the related variational quantities does not influence the analysis accuracy. The undeformed and deformed configurations given in Fig. 10 show clearly the large displacements and rotations of the structure during the deformation. The distribution of the Cauchy stresses $\sigma^{<22>}$ $(\Theta^3 = 0)$ and the stretch parameters $\overset{n}{\lambda}$ $(n = 0, 1)$ at the load levels $f = 10.0$ and $f = 20.0$ are plotted in Fig. 11. For the part of the structure far away from the locally distributed loads, the stresses $\sigma^{<22>}$ and the thickness stretching $\overset{0}{\lambda} = h/H$ show an almost uniform distribution along the circumferential direction. On the contrary, stress concentrations due to the locally distributed loads can be observed along the coordinate line $\Theta^2 = 0$ while the zero stress boundary condition is approximately satisfied along the unloaded free end. The distribution of the constant stretch parameter $\overset{0}{\lambda}$ plotted in Fig. 11 exhibits strong thickness changes of the shell near the loaded boundary especially for higher load levels. The distribution of the linear stretch parameter $\overset{1}{\lambda}$ identifies the parts of the structure wich are affected by bending effects.

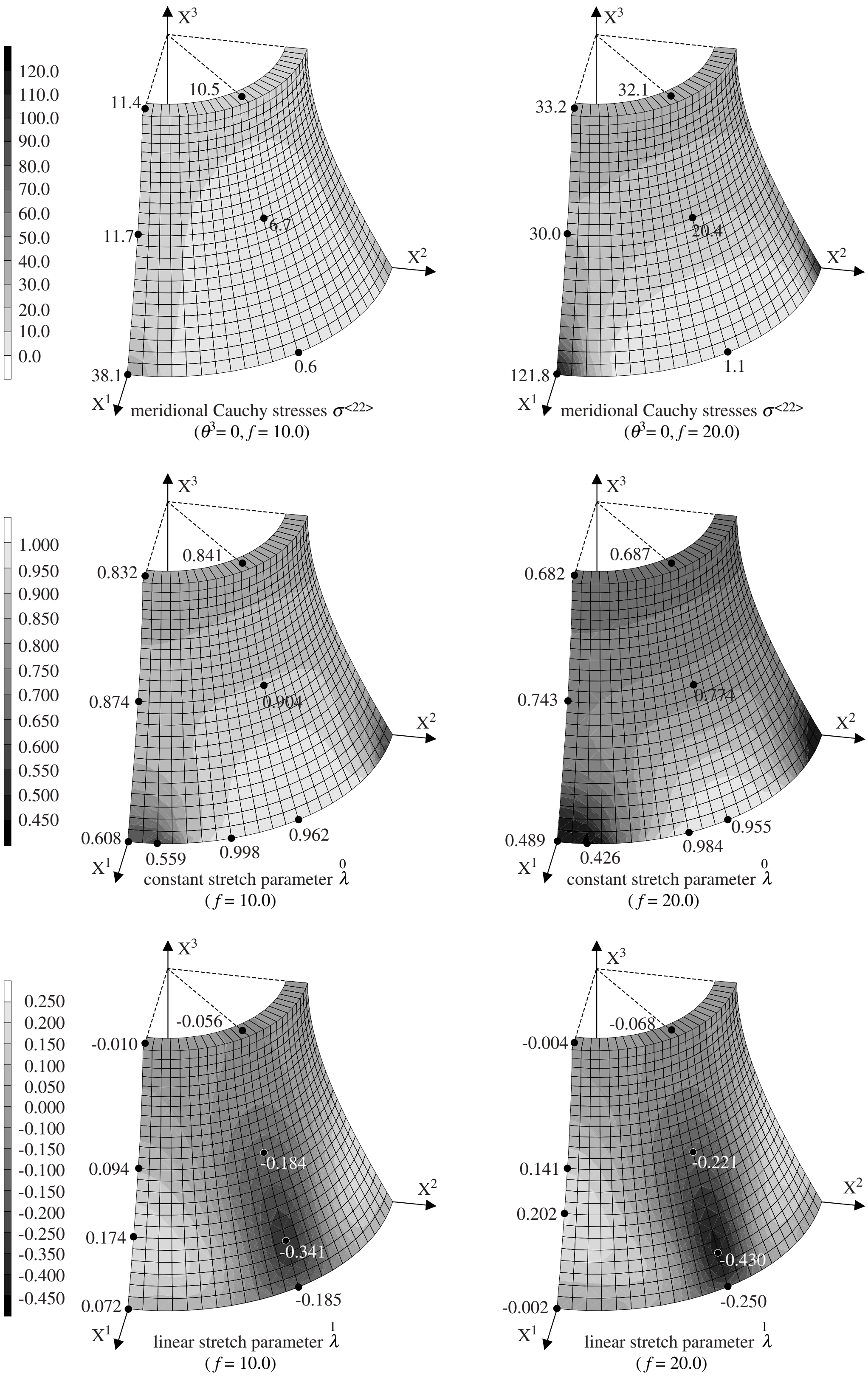

Fig. 11. Hyperbolic shell – distribution of the stresses $\sigma^{<22>}(\Theta^3 = 0)$ and the stretch parameters $\overset{0}{\lambda}$, $\overset{1}{\lambda}$ (Mooney-Rivlin material)

10 Conclusions

For the modelling of hyperelastic materials characterized by incompressibility three different procedures have been proposed for the enforcement of the corresponding condition in shell kinematics. Furthermore extensive numerical studies have been performed to find out the computationally most effective procedure. Procedure 1 is based on the consideration of the incompressibility condition both by means of displacement quantities appearing in the kinematic hypothesis and strain variables. Procedure 3 uses for this purpose only the displacement variables and is, therefore, by nature simpler in its formulation than Procedure 1. Procedure 2 presents a simplified, kinematically not fully consistent variant of Procedure 1 where the first-order stretch $\overset{1}{\lambda}$ is neglected. Concerning the effectivity of the above mentioned procedures the following conclusions are drawn:

Procedure 1 works very well for all types of deformations governed particularly by bending effects and presents, in this sense, a reliable procedure. Procedure 2 is also generally applicable and provides additionally a minor reduction of computational cost connected with neglecting the stretch variable $\overset{1}{\lambda}$. Disregarding $\overset{1}{\lambda}$ has no influence on the accuracy of the numerical analysis of sufficiently thin structures. Procedure 3 is only effective when dealing with membrane deformations but fails completely if bending dominated problems are considered and the material behaviour is described by a Mooney-Rivlin material model involving linear strains as significant terms. Consequently, the applicability of this procedure, despite the simplicity of its formulation, is restricted to large strain analysis of membrane shells only. This deficiency seems to be caused primarily by the considered material model whose significant strain terms are linear, since the tangential strains of first-order $\overset{1}{E}_{\alpha\beta}$, which are known as responsible for bending effects, disappear in the variational principle.

In contrast to material models of Mooney-Rivlin type which require an explicit enforcement of the incompressibility condition, this condition can be achieved in the compressible material model of Neo-Hookean type (Simo et al. [11]) implicitly by selecting the value of Poisson's ratio ν near to 0.5 which in turn yields large values for the bulk modulus $\kappa \to \infty$. Numerical studies have demonstrated that this allows for an accurate consideration of the incompressibility condition even for highly nonlinear problems. However, in this case the computations show a poor convergence behavior.

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