Acta Mechanica 166, 103–118 (2003)
DOI 10.1007/s00707-003-0045-2

Acta Mechanica
Printed in Austria

Nonlinear dynamic behavior of a piezothermoelastic laminated plate with anisotropic material properties

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Received January 9, 2003; revised August 1, 2003
Published online: November 6, 2003

Summary. In this paper, we analyze the nonlinear dynamic behavior of a piezothermoelastic laminated plate with anisotropic material properties. The analytical model is a rectangular laminate composed of fiber-reinforced laminae and piezoelectric layers. The model is assumed to be a symmetric cross-ply laminate with all egdes simply-supported and to be subjected to mechanical, thermal and electrical loads as intended control procedures or as disturbances. The von Kármán strains are introduced to treat non-linear deformation. The behavior of the laminate is analyzed by using the Galerkin Method. We discuss the following quantities: (i) the buckling temperature due to in-plane thermal load; (ii) the large static deflection due to combined in-plane and anti-plane loads; (iii) the natural frequency of infinitesimal oscillation around the static equilibrium state; (iv) the natural frequency of the oscillation with finite amplitude around the static equilibrium state. Moreover, numerical examples are shown to investigate the methods to rise the buckling temperature, to linearize the thermal deflection and the natural frequencies by applying the electrical voltage to the piezoelectric actuators.

1 Introduction

Recently, smart structures have attracted much attention in engineering, medicine and other fields for the control of deformation. Smart structures consist of elements which serve as sensors and/or actuators. Among various materials as sensors and/or actuators in smart structures, piezoelectric materials have attracted much attention because of their superior coupling effect between the elastic and electric fields. Piezoelectric materials are often attached to structural laminates such as graphite/epoxy. The laminates composed of them, which are called *piezothermoelastic laminates*, have been used as devices for the deformation-control including shape-control, vibration-control and so forth in adaptive structures [1]. In aerospace applications, these structures are generally light and deform flexibly, and its flexibility can induce large deformation. Therefore, it is important to investigate large deformation of piezothermoelastic laminates. For example, Tzou and Zhou [2], [3] treated static and dynamic control of a nonlinear circular plate composed of two surface piezoelectric layers and one isotropic elastic layer with geometrical nonlinearity by introducing the von Kármán type non-linear deformation.

Piezothermoelastic laminates are often required to be of high specific strength, and their elastic layers are made of *several* layers composed of the matrices with low mass density and the reinforcing fibers with high strength. As a result, they often exhibit anisotropy and lamination properties. Therefore, it is important to study piezothermoelastic laminated plates with anisotropy.

In this paper, therefore, we analyze the nonlinear dynamic behavior of a piezothermoelastic laminated plate with anisotropic material properties. The analytical model is a rectangular laminate composed of fiber-reinforced laminae and piezoelectric layers which exhibit orthotropy. The model is assumed to be a symmetric cross-ply laminate with all egdes simply-supported and to be subjected to mechanical, thermal and electrical loads as intended control procedures or as disturbances. The von Kármán strains are introduced to treat non-linear deformation. The equations of motion of the laminate are solved using the Galerkin Method. Expressions of the following quantities are discussed: (i) the buckling temperature due to in-plane thermal load; (ii) the large static deflection due to combined in-plane and anti-plane loads; (iii) the natural frequency of infinitesimal oscillation around the static equilibrium state; (iv) the natural frequency of the oscillation with finite amplitude around the static equilibrium state. Moreover, numerical examples are shown to investigate the methods to rise the buckling temperature, to linearize the thermal deflection and the natural frequencies by applying the electrical voltage to the piezoelectric actuators.

2 Analysis

2.1 Problem

The analytical model is shown in Fig. 1. The model is a rectangular laminate with dimension $a \times b \times h$ composed of N layers: two of N layers ($z_{k-1} \le z \le z_k$, $z_{k'-1} \le z \le z_{k'}$) exhibit piezoelectricity while other layers do not. The laminate is a cross-ply laminate: all the layers exhibit orthotropy, and the principal axes of orthotropy coincide with the axes of the Cartesian coordinate system (x, y, z). The layers are laminated symmetrically with respect to the central plane $z = 0$: the i-th and $(N - i + 1)$-th layers are composed of the same material and have the same orthotropy with respect to the Cartesian coordinate system (x, y, z). All edges of the laminate are simply-supported.

The laminate is subjected to the following loads: transverse load q; temperature T_0 and T_N on the upper $(z = -h/2)$ and the lower $(z = h/2)$ surfaces of the laminate, respectively; electric potential V^k and $V^{k'}$ on the upper surface $(z = z_{k-1})$ of the k-th piezoelectric layer and the

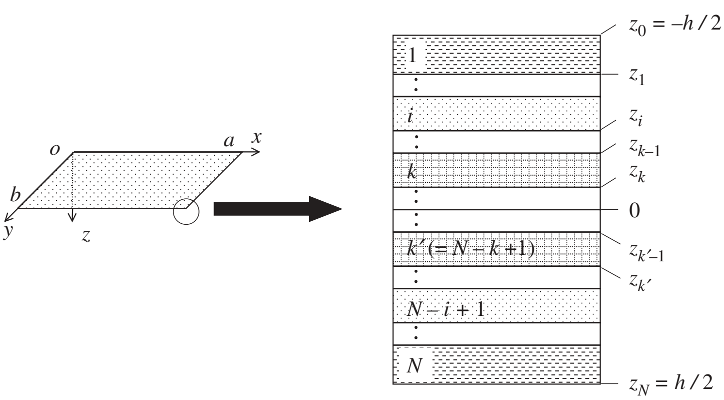

Fig. 1. Analytical model

lower surface ($z = z_{k'}$) of the k'-th piezoelectric layer, respectively. The lower surface ($z = z_k$) of the k-th layer and the upper surface ($z = z_{k'-1}$) of the k'-th layer are both the level surfaces of electric potential.

2.2 Governing equations

Based on the classical laminate theory, the displacement components in x-, y- and z-directions are taken to be, respectively,

$$u = u^0 - z\frac{\partial w^0}{\partial x}, \quad v = v^0 - z\frac{\partial w^0}{\partial y}, \qquad w = w^0, \tag{1}$$

where the superscript 0 denotes the quantities at the central plane. In order to treat non-linear deformation, the von Kármán strains ε_{ij} and γ_{ij} are introduced as

$$\begin{aligned} \varepsilon_{xx} &= \left[\frac{\partial u^0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] - z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{yy} = \left[\frac{\partial v^0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] - z\frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) - 2z\frac{\partial^2 w}{\partial x \partial y}. \end{aligned} \tag{2}$$

Electric fields E_i are expressed by an electric potential Φ as

$$E_x = -\frac{\partial \Phi}{\partial x}, \quad E_y = -\frac{\partial \Phi}{\partial y}, \quad E_z = -\frac{\partial \Phi}{\partial z}. \tag{3}$$

The constitutive equations of piezothermoelasticity for the symmetrical cross-ply laminate are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & \bar{e}_{31} \\ 0 & 0 & \bar{e}_{32} \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix} - \begin{Bmatrix} \bar{\lambda}_1 \\ \bar{\lambda}_2 \\ 0 \end{Bmatrix} T, \tag{4}$$

where σ_{ij} denotes the stresses; T denotes the temperature; $\bar{Q}_{ij}$, $\bar{e}_{ij}$ and $\bar{\lambda}_i$ denote elastic stiffness coefficients, piezoelectric coefficients and stress-temperature coefficients, all of which are reduced and transformed [4]. The resultant forces N_x, N_y and N_{xy}, the resultant moments M_x, M_y and M_{xy} and the coefficient of translation inertia P are defined as

$$\{N_x, N_y, N_{xy}\} = \int_{-h/2}^{h/2} \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\} dz, \ \{M_x, M_y, M_{xy}\} = \int_{-h/2}^{h/2} \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\} z\, dz, \ P = \int_{-h/2}^{h/2} \rho\, dz, \tag{5}$$

where ρ denotes the mass density. Substitution of Eqs. (2) and (4) into Eqs. (5) gives

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u^0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v^0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{Bmatrix} - \begin{Bmatrix} N_x^T + N_x^E \\ N_y^T + N_y^E \\ 0 \end{Bmatrix}, \tag{6}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} - \begin{Bmatrix} M_x^T + M_x^E \\ M_y^T + M_y^E \\ 0 \end{Bmatrix}, \tag{7}$$

where the definitions of A_{ij}, D_{ij}, N_x^T, N_y^T, M_x^T, M_y^T, N_x^E, N_y^E, M_x^E and M_y^E are given as

$$\{A_{ij}, D_{ij}\} = \int_{-h/2}^{h/2} \bar{Q}_{ij}\{1, z^2\} dz \quad (i,j = 1,2,6), \tag{8}$$

$$\left\{N_x^T, N_y^T\right\} = \int_{-h/2}^{h/2} \{\bar{\lambda}_1, \bar{\lambda}_2\} T dz, \quad \left\{M_x^T, M_y^T\right\} = \int_{-h/2}^{h/2} \{\bar{\lambda}_1, \bar{\lambda}_2\} T z dz, \tag{9}$$

$$\left\{N_x^E, N_y^E\right\} = \int_{-h/2}^{h/2} \{\bar{e}_{31}, \bar{e}_{32}\} E_z dz, \quad \left\{M_x^E, M_y^E\right\} = \int_{-h/2}^{h/2} \{\bar{e}_{31}, \bar{e}_{32}\} E_z z dz. \tag{10}$$

The equations of motion which integrate the effect of in-plane resultant forces into anti-plane motion are given as follows:

$$\begin{aligned} &\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \\ &\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + q = P\frac{\partial^2 w}{\partial t^2}. \end{aligned} \tag{11}$$

By substituting Eqs. (6) and (7) into Eqs. (11), the equations of motion are expressed by the displacements as follows:

$$\begin{aligned} &L_1(u^0, v^0, w) = \frac{\partial}{\partial x}\left(N_x^T + N_x^E\right), \quad L_2(u^0, v^0, w) = \frac{\partial}{\partial y}\left(N_y^T + N_y^E\right), \\ &L_3(u^0, v^0, w) = P\frac{\partial^2 w}{\partial t^2} - q + \frac{\partial^2}{\partial x^2}\left(M_x^T + M_x^E\right) + \frac{\partial^2}{\partial y^2}\left(M_y^T + M_y^E\right), \end{aligned} \tag{12}$$

where the operators L_i are expressed as

$$\begin{aligned} L_1(u^0, v^0, w) = {} & A_{11}\frac{\partial^2 u^0}{\partial x^2} + A_{66}\frac{\partial^2 u^0}{\partial y^2} + (A_{12} + A_{66})\frac{\partial^2 v^0}{\partial x \partial y} \\ & + \left(A_{11}\frac{\partial^2 w}{\partial x^2} + A_{66}\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial w}{\partial x} + (A_{12} + A_{66})\frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial y}, \\ L_2(u^0, v^0, w) = {} & (A_{12} + A_{66})\frac{\partial^2 u^0}{\partial x \partial y} + A_{66}\frac{\partial^2 v^0}{\partial x^2} + A_{22}\frac{\partial^2 v^0}{\partial y^2} \\ & + \left(A_{66}\frac{\partial^2 w}{\partial x^2} + A_{22}\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial w}{\partial y} + (A_{12} + A_{66})\frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial x}, \\ L_3(u^0, v^0, w) = {} & -D_{11}\frac{\partial^4 w}{\partial x^4} - 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} - D_{22}\frac{\partial^4 w}{\partial y^4} - \left[\left(N_x^T + N_x^E\right)\frac{\partial^2 w}{\partial x^2} + \left(N_y^T + N_y^E\right)\frac{\partial^2 w}{\partial y^2}\right] \\ & + \left[\frac{\partial u^0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\left(A_{11}\frac{\partial^2 w}{\partial x^2} + A_{12}\frac{\partial^2 w}{\partial y^2}\right) + \left[\frac{\partial v^0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\left(A_{12}\frac{\partial^2 w}{\partial x^2} + A_{22}\frac{\partial^2 w}{\partial y^2}\right) \\ & + 2A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial x \partial y}. \end{aligned} \tag{13}$$

As the laminate is simply-supported at all edges, we have

$$\begin{aligned} x=0,a;\quad & u^0=v^0=0,\quad w=0,\quad M_x=-\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)-\left(M_x^T+M_x^E\right)=0,\\ y=0,b;\quad & u^0=v^0=0,\quad w=0,\quad M_y=-\left(D_{12}\frac{\partial^2 w}{\partial x^2}+D_{22}\frac{\partial^2 w}{\partial y^2}\right)-\left(M_y^T+M_y^E\right)=0. \end{aligned} \tag{14}$$

2.3 Galerkin method

We use the Galerkin method to solve the governing equations (12) under the boundary conditions (14). We choose trigonometric functions as the trial functions and consider that the displacements are expressed by series as

$$\{u^0, v^0, w\}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{u_{mn},v_{mn},w_{mn}\}\sin\alpha_m x\sin\beta_n y:\quad \alpha_m=\frac{m\pi}{a},\ \ \beta_n=\frac{n\pi}{b}, \tag{15}$$

so that the boundary conditions (14) may be satisfied. Hereafter, we consider that q, T_0, T_N, V^k and $V^{k'}$ are uniform with respect to the variables x and y. Moreover, we consider that the thickness of each layer is sufficiently small compared to its lengths a and b. Then, the heat flux in each layer is considered to occur only in the thickness direction and to be constant along the thickness direction. Therefore, the distribution of the temperature in each layer is uniform with respect to x and y and linear with respect to z and is obtained as

$$T=T_{i-1}+(T_i-T_{i-1})\frac{z-z_{i-1}}{z_i-z_{i-1}}:\quad z_{i-1}\le z\le z_i\ \ (i=1,\ldots,N), \tag{16}$$

where $T_i(i=1,\ldots,N-1)$ denotes the temperature at $z=z_i$ and is determined so as to satisfy the continuity conditions of the heat flux at $z=z_i(i=1,\ldots,N-1)$:

$$\lambda_{z,1}\frac{T_1-T_0}{z_1-z_0}=\lambda_{z,2}\frac{T_2-T_1}{z_2-z_1}=\cdots=\lambda_{z,N-1}\frac{T_{N-1}-T_{N-2}}{z_{N-1}-z_{N-2}}=\lambda_{z,N}\frac{T_N-T_{N-1}}{z_N-z_{N-1}}, \tag{17}$$

where $\lambda_{z,i}(i=1,\ldots,N)$ denotes the thermal conductivity in the z -direction in the i-th layer. Meanwhile, the electric field in each piezoelectric layer is considered to occur only in the thickness direction and to be constant along the thickness direction. Therefore, referring to Eqs. (3), the distribution of the electric potential in each piezoelectric layer is uniform with respect to x and y and linear with respect to z and is obtained as

$$\begin{aligned} \Phi&=V^k\frac{z_k-z}{z_k-z_{k-1}}:\quad z_{k-1}\le z\le z_k,\\ \Phi&=V^{k'}\frac{z-z_{k'-1}}{z_{k'}-z_{k'-1}}:\quad z_{k'-1}\le z\le z_{k'}. \end{aligned} \tag{18}$$

Then, from Eqs. (3), (9), (10), (16) and (18), it is found that the thermally-induced resultant forces N_x^T and N_y^T and the electrically-induced resultant forces N_x^E and N_y^E become uniform with respect to the variables x and y. On the other hand, in order to satisfy the boundary conditions (14), the thermally-induced resultant moments M_x^T and M_y^T and the electrically-induced resultant moments M_x^E and M_y^E are evaluated as

$$\begin{aligned} \left\{M_x^T,M_y^T,M_x^E,M_y^E\right\}&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{M_{x,mn}^T,M_{y,mn}^T,M_{x,mn}^E,M_{y,mn}^E\right\}\sin\alpha_m x\sin\beta_n y,\\ \left\{M_{x,mn}^T,M_{y,mn}^T,M_{x,mn}^E,M_{y,mn}^E\right\}&=\frac{4}{ab}\int_0^b\int_0^a\left\{M_x^T,M_y^T,M_x^E,M_y^E\right\}\sin\alpha_m x\sin\beta_n y\,dx\,dy, \end{aligned} \tag{19}$$

where, from Eqs. (3), (9), (10), (16), (18) and (19), the coefficients are calculated as

$$\left[M^T_{x,mn}, M^T_{y,mn}\right] = \begin{cases} \dfrac{16}{\pi^2 mn}\displaystyle\sum_{i=1}^{N}\left(\dfrac{T_{i-1}z_i - T_i z_{i-1}}{z_i - z_{i-1}}\dfrac{z_i^2 - z_{i-1}^2}{2} + \dfrac{T_i - T_{i-1}}{z_i - z_{i-1}}\dfrac{z_i^3 - z_{i-1}^3}{3}\right) \\ \left[\left(\bar{\lambda}_1\right)_i, \left(\bar{\lambda}_2\right)_i\right]: & m = \text{odd and } n = \text{odd} \\ [0,0]: & \text{otherwise} \end{cases}$$

$$\left[M^E_{x,mn}, M^E_{y,mn}\right] = \begin{cases} \left(V^k + V^{k'}\right)\frac{16}{\pi^2 mn}\frac{z_k + z_{k-1}}{2}[\bar{e}_{31}, \bar{e}_{32}]: & m = \text{odd and } n = \text{odd} \\ [0,0]: & \text{otherwise} \end{cases} \tag{20}$$

Then, applying the Galerkin method to Eqs. (12), we have

$$\begin{aligned} &\int_0^b\int_0^a L_1\left(u^0, v^0, w\right)\sin\alpha_{m'}x\sin\beta_{n'}y\,dx\,dy = 0, \\ &\int_0^b\int_0^a L_2\left(u^0, v^0, w\right)\sin\alpha_{m'}x\sin\beta_{n'}y\,dx\,dy = 0, \\ &\int_0^b\int_0^a \left[L_3\left(u^0, v^0, w\right) - P\frac{\partial^2 w}{\partial t^2} + q - \frac{\partial^2}{\partial x^2}\left(M_x^T + M_x^E\right) - \frac{\partial^2}{\partial y^2}\left(M_y^T + M_y^E\right)\right] \\ &\quad \times \sin\alpha_{m'}x\sin\beta_{n'}y\,dx\,dy = 0: \quad m', n' = 1, 2, 3, \ldots, \infty. \end{aligned} \tag{21}$$

Performing the integrations in Eqs. (21), we have the simultaneous non-linear equations with respect to u_{mn}, v_{mn} and w_{mn} as follows:

$$\begin{aligned} &a_{11,mn}u_{mn} + \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} a^{mn}_{12,m'n'}v_{m'n'} + \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b^{mn}_{11,m'n'ij}w_{m'n'}w_{ij} = 0, \\ &\sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} a^{mn}_{12,m'n'}u_{m'n'} + a_{22,mn}v_{mn} + \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b^{mn}_{22,m'n'ij}w_{m'n'}w_{ij} = 0, \\ &P\frac{d^2 w_{mn}}{dt^2} + \left(\frac{d_{c,33,mn}}{h^2} + a_{N,mn}\right)w_{mn} \\ &\quad + \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b^{mn}_{13,m'n'ij}w_{m'n'}u_{ij} + \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b^{mn}_{23,m'n'ij}w_{m'n'}v_{ij} \\ &\quad + \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} b^{mn}_{33,m'n'ijkl}w_{m'n'}w_{ij}w_{kl} = p_{c,mn}: \quad m, n = 1, 2, 3, \ldots, \infty, \end{aligned} \tag{22}$$

where $a_{N,mn}$ is the coefficient representing the effect of the thermally-induced and the electrically-induced resultant forces on anti-plane motion, defined as

$$a_{N,mn} = -\left[\alpha_m^2\left(N_x^T + N_x^E\right) + \beta_n^2\left(N_y^T + N_y^E\right)\right] \tag{23}$$

and

$$d_{c,33,mn} = h^2\left[D_{11}\alpha_m^4 + 2(D_{12} + 2D_{66})\alpha_m^2\beta_n^2 + D_{22}\beta_n^4\right], \tag{24}$$

$$p_{c,mn} = q_{mn} + \alpha_m^2 \left(M_{x,mn}^T + M_{x,mn}^E \right) + \beta_n^2 \left(M_{y,mn}^T + M_{y,mn}^E \right), \tag{25}$$

$$q_{mn} = \begin{cases} q \frac{16}{\pi^2 mn} : & m = \text{odd and } n = \text{odd} \\ 0 : & \text{otherwise} \end{cases} \tag{26}$$

and the definitions of $a_{11,mn}$, $a_{22,mn}$, $a_{12,m'n'}^{mn}$, $b_{11,m'n'ij}^{mn}$, $b_{13,m'n'ij}^{mn}$, $b_{22,m'n'ij}^{mn}$, $b_{23,m'n'ij}^{mn}$ and $b_{33,m'n'ijkl}^{mn}$ are given in the Appendix. It should be noted that $p_{c,mn}$ contributes to the bending of the laminate.

2.4 Buckling temperature

We consider the situation:

$$q = 0, \quad T_N = T_0, \quad V^k = -V^{k'} (\equiv V). \tag{27}$$

Referring to Eqs. (17), we have

$$T_i = T_0 \quad (i = 1, \ldots, N-1). \tag{28}$$

Then, from Eqs. (16), (20), (25) through (28), $p_{c,mn}$ is absent, and Eqs. (22) have a zero solution,

$$u_{mn} = v_{mn} = w_{mn} = 0 : \quad m, n = 1, 2, 3, \ldots, \infty. \tag{29}$$

Due to the non-linearity of Eqs. (22) with respect to w_{mn}, Eqs. (22) for the quasi-static case may have non-trivial solutions when

$$\frac{d_{c,33,mn}}{h^2} + a_{N,mn} = 0. \tag{30}$$

From Eqs. (3), (9), (10), (16), (18), (23), (24), (27) and (28), Eq. (30) gives the temperature as

$$T_0 = \frac{d_{c,33,mn} - 2Vh^2 \left(\alpha_m^2 \bar{e}_{31} + \beta_n^2 \bar{e}_{32} \right)}{h^2 \sum_{i=1}^{N} \left[\alpha_m^2 \left(\bar{\lambda}_1 \right)_i + \beta_n^2 \left(\bar{\lambda}_2 \right)_i \right] (z_i - z_{i-1})} \equiv T_{cr,mn}, \tag{31}$$

which is referred to as the buckling temperature. From Eq. (31), it is found that $T_{cr,mn}$ can be controlled by the electrical voltage V applied to the piezoelectric actuators.

2.5 Large deflection

On the other hand, when $p_{c,mn}$ is present, the laminate undergoes combined in-plane and anti-plane loads and deflects. By truncating infinite series in Eqs. (15), therefore in Eqs. (22), up to two terms, Eqs. (22) for $(m,n) = (1,1)$, $(m,n) = (1,2)$, $(m,n) = (2,1)$ and $(m,n) = (2,2)$ give the simultaneous non-linear equations with respect to u_{mn}, v_{mn} and w_{mn} for $(m,n) = (1,1)$, $(m,n) = (1,2)$, $(m,n) = (2,1)$ and $(m,n) = (2,2)$. These equations may have solutions which satisfy one of three exclusive conditions:

$$u_{21} \neq 0, \quad v_{12} \neq 0, \quad w_{11} \neq 0, \quad u_{11} = u_{12} = u_{22} = v_{11} = v_{21} = v_{22} = w_{12} = w_{21} = w_{22} = 0, \tag{32}$$

$$u_{21} \neq 0, \quad v_{12} \neq 0, \quad w_{12} \neq 0, \quad u_{11} = u_{12} = u_{22} = v_{11} = v_{21} = v_{22} = w_{11} = w_{21} = w_{22} = 0, \tag{33}$$

$$u_{21} \neq 0, \quad v_{12} \neq 0, \quad w_{21} \neq 0, \quad u_{11} = u_{12} = u_{22} = v_{11} = v_{21} = v_{22} = w_{11} = w_{12} = w_{22} = 0. \tag{34}$$

When $p_{c,11}$ is present, only the condition by Eqs. (32) is allowed among these conditions. Therefore, we consider the solution of Eqs. (22) which satisfies Eqs. (32). Equations (22) which satisfy Eqs. (32) give the simultaneous non-linear equations with respect to u_{21}, v_{12} and w_{11} as follows:

$$\begin{aligned}
&a_{11,21}u_{21} + a^{21}_{12,12}v_{12} + b^{21}_{11,1111}(w_{11})^2 = 0, \\
&a^{12}_{12,21}u_{21} + a_{22,12}v_{12} + b^{12}_{22,1111}(w_{11})^2 = 0, \\
&P\frac{d^2 w_{11}}{dt^2} + \left(\frac{d_{c,33,11}}{h^2} + a_{N,11}\right)w_{11} + b^{11}_{13,1121}w_{11}u_{21} + b^{11}_{23,1112}w_{11}v_{12} + b^{11}_{33,111111}(w_{11})^3 = p_{c,11}.
\end{aligned} \tag{35}$$

Furthermore, by eliminating u_{21} and v_{12} in Eqs. (35), we have the non-linear ordinary differential equation with respect to w_{11} as follows:

$$(Ph)\frac{d^2}{dt^2}\left(\frac{w_{11}}{h}\right) + k^L_{c,11}\left(\frac{w_{11}}{h}\right) + k^N_{11}\left(\frac{w_{11}}{h}\right)^3 = p_{c,11}, \tag{36}$$

where

$$\begin{aligned}
k^L_{c,11} &= \frac{d_{c,33,11}}{h} + h a_{N,11}, \\
k^N_{11} &= \frac{h^3}{a_{11,21}a_{22,12} - a^{12}_{12,21}a^{21}_{12,12}} \cdot \Big[b^{11}_{33,111111}\left(a_{11,21}a_{22,12} - a^{12}_{12,21}a^{21}_{12,12}\right) \\
&\quad - b^{11}_{13,1121}\left(a_{22,12}b^{21}_{11,1111} - a^{21}_{12,12}b^{12}_{22,1111}\right) - b^{11}_{23,1121}\left(a_{11,21}b^{12}_{22,1111} - a^{12}_{12,21}b^{21}_{11,1111}\right)\Big].
\end{aligned} \tag{37}$$

For the quasi-static case, Eq. (36) becomes an algebraic cubic equation, and the solution is obtained as follows:

$$\begin{aligned}
\frac{w_{11}}{h} &= \sqrt[3]{\frac{1}{2}\frac{p_{c,11}}{k^N_{11}} + \sqrt{\left(\frac{1}{2}\frac{p_{c,11}}{k^N_{11}}\right)^2 + \left(\frac{1}{3}\frac{k^L_{c,11}}{k^N_{11}}\right)^3}} \\
&\quad - \sqrt[3]{-\frac{1}{2}\frac{p_{c,11}}{k^N_{11}} + \sqrt{\left(\frac{1}{2}\frac{p_{c,11}}{k^N_{11}}\right)^2 + \left(\frac{1}{3}\frac{k^L_{c,11}}{k^N_{11}}\right)^3}} \quad : \quad \frac{k^L_{c,11}}{k^N_{11}} \geq 0, \\
\frac{w_{11}}{h} &= \sqrt[3]{\frac{1}{2}\frac{p_{c,11}}{k^N_{11}} + \sqrt{\left(\frac{1}{2}\frac{p_{c,11}}{k^N_{11}}\right)^2 + \left(\frac{1}{3}\frac{k^L_{c,11}}{k^N_{11}}\right)^3}} \\
&\quad + \sqrt[3]{\frac{1}{2}\frac{p_{c,11}}{k^N_{11}} + \sqrt{\left(\frac{1}{2}\frac{p_{c,11}}{k^N_{11}}\right)^2 + \left(\frac{1}{3}\frac{k^L_{c,11}}{k^N_{11}}\right)^3}} \quad : \quad -3\sqrt[3]{\left(\frac{1}{2}\frac{p_{c,11}}{k^N_{11}}\right)^2} \leq \frac{k^L_{c,11}}{k^N_{11}} \leq 0 \text{ and} \frac{p_{c,11}}{k^N_{11}} \geq 0,
\end{aligned}$$

$$\frac{w_{11}}{h} = -\sqrt[3]{-\frac{1}{2}\frac{p_{c,11}}{k_{11}^{N}} - \sqrt{\left(\frac{1}{2}\frac{p_{c,11}}{k_{11}^{N}}\right)^{2} + \left(\frac{1}{3}\frac{k_{c,11}^{L}}{k_{11}^{N}}\right)^{3}}}$$

$$- \sqrt[3]{-\frac{1}{2}\frac{p_{c,11}}{k_{11}^{N}} + \sqrt{\left(\frac{1}{2}\frac{p_{c,11}}{k_{11}^{N}}\right)^{2} + \left(\frac{1}{3}\frac{k_{c,11}^{L}}{k_{11}^{N}}\right)^{3}}} : \quad -3\sqrt[3]{\left(\frac{1}{2}\frac{p_{c,11}}{k_{11}^{N}}\right)^{2}} \le \frac{k_{c,11}^{L}}{k_{11}^{N}} \le 0 \text{ and } \frac{p_{c,11}}{k_{11}^{N}} \ge 0,$$

$$\frac{w_{11}}{h} = 2\sqrt{-\frac{1}{3}\frac{k_{c,11}^{L}}{k_{11}^{N}}}\cos\frac{\theta}{3} : \quad \frac{k_{c,11}^{L}}{k_{11}^{N}} \le -3\sqrt[3]{\left(\frac{1}{2}\frac{p_{c,11}}{k_{11}^{N}}\right)^{2}} \text{ and } \frac{p_{c,11}}{k_{11}^{N}} \ge 0,$$

$$\frac{w_{11}}{h} = 2\sqrt{-\frac{1}{3}\frac{k_{c,11}^{L}}{k_{11}^{N}}}\cos\left(\frac{\theta + 2\pi}{3}\right) : \quad \frac{k_{c,11}^{L}}{k_{11}^{N}} \le -3\sqrt[3]{\left(\frac{1}{2}\frac{p_{c,11}}{k_{11}^{N}}\right)^{2}} \text{ and } \frac{p_{c,11}}{k_{11}^{N}} \le 0, \tag{38}$$

where

$$\theta = \cos^{-1}\left[\frac{\left(\frac{1}{2}\frac{p_{c,11}}{k_{11}^{N}}\right)}{\sqrt{-\left(\frac{1}{3}\frac{k_{c,11}^{L}}{k_{11}^{N}}\right)^{3}}}\right] \quad (0 \le \theta \le \pi). \tag{39}$$

It should be noted that, when $p_{c,11}$ is absent, Eqs. (38) give the bifurcational solution as

$$\begin{aligned} \frac{w_{11}}{h} &= 0 : \quad \frac{k_{c,11}^{L}}{k_{11}^{N}} \ge 0, \\ \frac{w_{11}}{h} &= \pm\sqrt{-\frac{k_{c,11}^{L}}{k_{11}^{N}}} : \quad \frac{k_{c,11}^{L}}{k_{11}^{N}} \le 0. \end{aligned} \tag{40}$$

2.6 Free oscillation around static large deflection

We consider the free oscillation of the laminate around the static large deflection. Let $w_{11,s}$ denote the static large deflection due to static loads q, T_0, T_N, V^k and $V^{k'}$ and let $w_{11,d}(t)$ denote the free oscillation of the laminate around $w_{11,s}$. Then, from Eq. (36), the governing equations of $w_{11,s}$ and $w_{11,d}(t)$ are represented, respectively, by

$$k_{c,11}^{L}\left(\frac{w_{11,s}}{h}\right) + k_{11}^{N}\left(\frac{w_{11,s}}{h}\right)^{3} = p_{c,11}, \tag{41}$$

$$(Ph)\frac{d^2}{dt^2}\left[\frac{w_{11,d}(t)}{h}\right] + k_{c,11}^{L}\left[\frac{w_{11,s} + w_{11,d}(t)}{h}\right] + k_{11}^{N}\left[\frac{w_{11,s} + w_{11,d}(t)}{h}\right]^{3} = p_{c,11}. \tag{42}$$

By subtracting Eq. (41) from Eq. (42), we have

$$\frac{d^2}{dt^2}\left[\frac{w_{11,d}(t)}{h}\right] + \omega_{c,0}^{2}\left[\frac{w_{11,d}(t)}{h}\right] + v_2^2\left[\frac{w_{11,d}(t)}{h}\right]^{2} + v_3^2\left[\frac{w_{11,d}(t)}{h}\right]^{3} = 0, \tag{43}$$

where

$$\omega_{c,0} = \sqrt{\frac{k_{c,11}^{L} + 3k_{11}^{N}\left(\frac{w_{11,s}}{h}\right)^{2}}{Ph}}, \quad v_2 = \sqrt{\frac{3k_{11}^{N}\frac{w_{11,s}}{h}}{Ph}}, \quad v_3 = \sqrt{\frac{k_{11}^{N}}{Ph}}. \tag{44}$$

Since neglection of the nonlinear terms in Eq. (43) leads to

$$\frac{d^2}{dt^2}\left[\frac{w_{11,d}(t)}{h}\right]+\omega_{c,0}^2\left[\frac{w_{11,d}(t)}{h}\right]=0, \tag{45}$$

it is found that $\omega_{c,0}$ defined by Eqs. (44) denotes the natural frequency of infinitesimal oscillation around the static large deflection. When all terms in Eq. (43) are considered, the natural frequency of the oscillation with finite amplitude around the static large deflection, ω, is obtained by the Lindstedt-Poincaré method [5] as

$$\begin{aligned}\frac{\omega}{\omega_{c,0}}&=1+\frac{1}{24}\left[9\left(\frac{v_3}{\omega_{c,0}}\right)^2-10\left(\frac{v_2}{\omega_{c,0}}\right)^4\right]\widehat{W}^2\\&\quad+\frac{1}{6912}\left[6120\left(\frac{v_2^2v_3}{\omega_{c,0}^3}\right)^2-1940\left(\frac{v_2}{\omega_{c,0}}\right)^8-567\left(\frac{v_3}{\omega_{c,0}}\right)^4\right]\widehat{W}^4+\mathrm{O}\left(\widehat{W}^6\right),\end{aligned} \tag{46}$$

where $\widehat{W}$ denotes the amplitude of $w_{11,d}(t)/h$. From Eq. (46), it is found that the natural frequency is dependent on the amplitude when the oscillation with finite amplitude is considered. Moreover, by considering Eqs. (20), (25), (38) and (44), the natural frequency described by Eq. (46) is found to depend on the electrical voltage applied to the piezoelectric actuators and thus to be controlled by the voltage.

3 Numerical calculation

Some numerical calculation is carried out to investigate the methods to rise the buckling temperature, to linearize the thermal deflection and the natural frequencies with respect to the temperature by applying the electrical voltage to the piezoelectric actuators.

We assume that the piezoelectric layers are of $BaTiO_3$ and other layers are of graphite/epoxy (GE). Reduced material constants are given as follows:

for GE layer [6], [7]:

$$\begin{aligned}&Q_{11}^e=182[\mathrm{GPa}],\quad Q_{22}^e=10.3[\mathrm{GPa}],\quad Q_{12}^e=2.90[\mathrm{GPa}],\\&Q_{44}^e=2.87[\mathrm{GPa}],\quad Q_{55}^e=7.17[\mathrm{GPa}],\quad Q_{66}^e=7.17[\mathrm{GPa}],\\&\lambda_1^e=68.8\times10^3[\mathrm{PaK^{-1}}],\quad \lambda_2^e=233\times10^3[\mathrm{PaK^{-1}}],\\&\rho^e=1.580\times10^3[\mathrm{kgm^{-3}}];\end{aligned} \tag{47}$$

for $BaTiO_3$ (which exhibits 6mm symmetry) layer [8], [9]:

$$\begin{aligned}&Q_{11}^p=Q_{22}^p=120[\mathrm{GPa}],\ Q_{12}^p=36.2[\mathrm{GPa}],\ Q_{44}^p=Q_{55}^p=44.0[\mathrm{GPa}],\ Q_{66}^p=42.0[\mathrm{GPa}],\\&\lambda_1^p=\lambda_2^p=1.33\times10^6[\mathrm{PaK^{-1}}],\\&e_{31}=e_{32}=-12.3[\mathrm{Cm^{-2}}],\ e_{15}=e_{24}=11.4[\mathrm{Cm^{-2}}],\\&\eta_{11}^p=\eta_{22}^p=9.87\times10^{-9}[\mathrm{C^2N^{-1}m^{-2}}],\ \eta_{33}^p=13.2\times10^{-9}[\mathrm{C^2N^{-1}m^{-2}}],\\&\rho^p=5.700\times10^3[\mathrm{kgm^{-3}}],\end{aligned} \tag{48}$$

where η_{ij}^p denotes the permittivities of $BaTiO_3$. Reduced and transformed material properties can be obtained according to [4]. We assume that the square layers ($a=b$) are piled as $\left\{[\mathrm{BaTiO_3}:0°]/\left[\mathrm{GE}:(90°/0°)_2\right]\right\}_{\mathrm{sym}}$ ($N=10$, $k=1$, $k'=10$) and that each layer has the same thickness. Then, the thickness of each layer, t_i, is sufficiently small compared to its length

($t_i/a < 0.01$ for $a/h > 10$, for instance), which justifies the assumption introduced to derive Eqs. (16) and (18).

3.1 Buckling due to in-plane thermal load

Under the condition described by Eqs. (27), the laminate undergoes the buckling at the temperature given by Eq. (31). Figure 2 shows the variation of the buckling temperature with the length-to-thickness ratio a/h without the electric voltage applied. From Fig. 2, it is found that the buckling temperatures decreases as the length-to-thickness ratio decreases and that the buckling occurs for $(m, n) = (1, 1)$ because the buckling temperature for the case is the smallest among those for other combinations of (m, n).

Figure 3 shows the variation of the buckling temperature $T_{cr,11}$ normalized by $(h/a)^2$ with the length-to-thickness ratio for various values of the electric voltage applied to the piezoelectric actuators. From Fig. 3, it is found that the buckling temperature changes linearly with

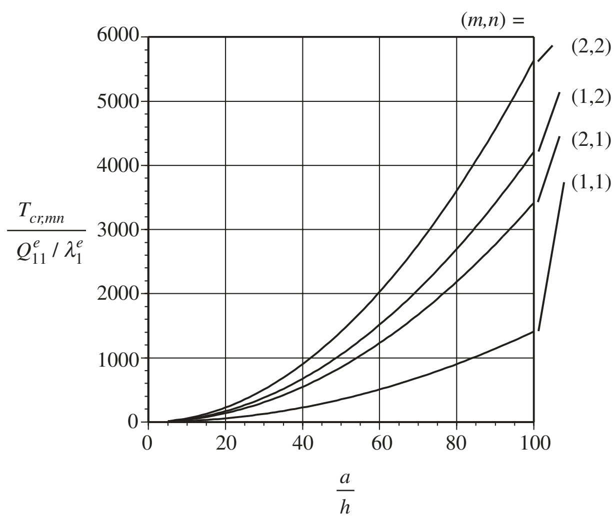

Fig. 2. Variation of the buckling temperature with the length-to-thickness ratio without the electric voltage applied

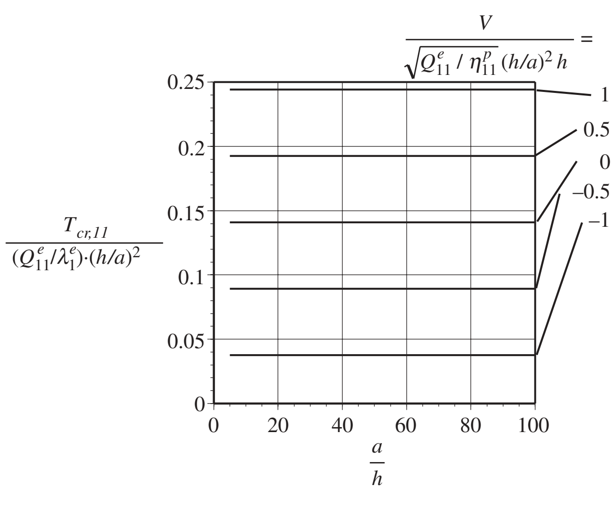

Fig. 3. Variation of the buckling temperature with the length-to-thickness ratio for various values of the electric voltage applied to the piezoelectric actuators

respect to the voltage and that appropriate voltage to the actuators can increase the buckling temperature.

3.2 Large deflection

We consider large deflection, which is described by Eqs. (38), of the laminate subjected to combined in-plane thermal load and anti-plane electrical load, letting

$$q = 0, \quad T_N = T_0, \quad V^k = V^{k'} \equiv V. \tag{49}$$

Figure 4 shows the variation of the deflection normalized by the thickness of the laminate with the normalized electrical voltage applied to the piezoelectric actuators. From Fig. 4, it is found that the deflection increases non-linearly with respect to the electrical voltage applied to the piezoelectric actuators and that the deflection increases as the temperature increases. Moreover, it is found that the deflection occurs even when the laminate is subjected only to the in-plane thermal load for two cases $T_0/\left[(Q_{11}^e/\lambda_1^e)\cdot(h/a)^2\right] = 0.15$ and $T_0/\left[(Q_{11}^e/\lambda_1^e)\cdot(h/a)^2\right] = 0.2$, which are larger than the buckling temperature $T_{cr,11}/\left[(Q_{11}^e/\lambda_1^e)\cdot(h/a)^2\right] \cong 0.14$ for $V = 0$ as shown in Fig. 3.

Figure 5 shows the variation of the deflection normalized by the thickness of the laminate with normalized temperature. From Fig. 5, it is found that the deflection increases non-linearly with respect to the temperature. For the case without the electrical voltage applied, the deflection is found to exhibit bifurcational behavior. An important aspect obtained from Fig. 5 is that the deflection can be linearized with respect to the temperature by applying an appropriate electrical voltage to the piezoelectric actuators.

3.3 Free oscillation around static large deflection

Numerical calculations for the free oscillation of the laminate around the static large deflection, which is treated in Sect. 2.6, are performed. Figure 6 shows the variation of the normalized

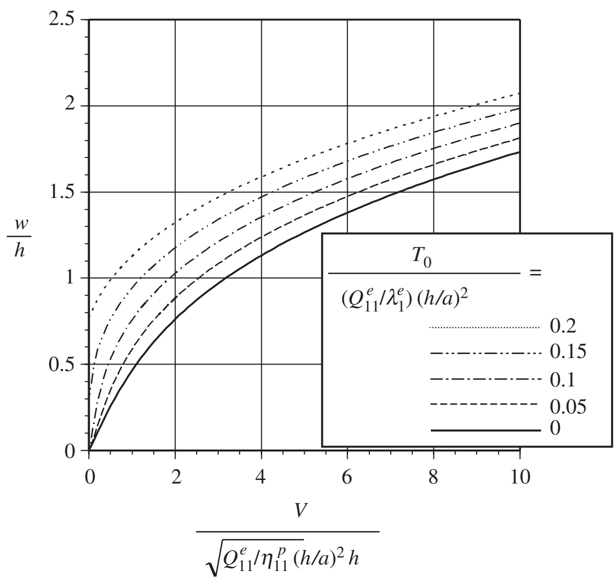

Fig. 4. Variation of the deflection with the electrical voltage applied to the piezoelectric actuators $(a/h = 50)$

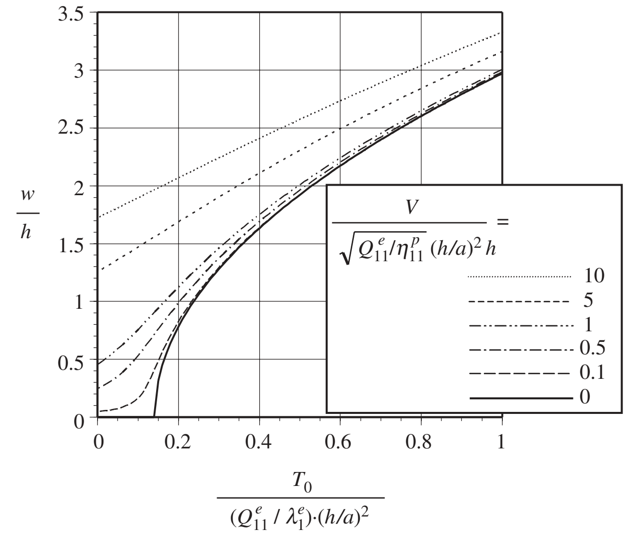

Fig. 5. Variation of the deflection with the temperature $(a/h = 50)$

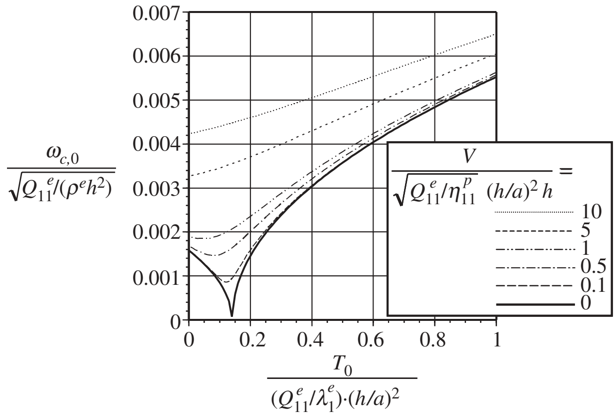

Fig. 6. Variation of the natural frequency of infinitesimal oscillation with the temperature $(a/h = 50)$

natural frequency of infinitesimal oscillation with the normalized temperature for various values of the electrical voltage applied to the piezoelectric actuators. From Fig. 6, it is found that the natural frequency of infinitesimal oscillation is decreased by applying the temperature around the buckling temperature $T_{cr,11}/\left[\left(Q^e_{11}/\lambda^e_1\right) \cdot (h/a)^2\right] \cong 0.14$ and that the natural frequency can be increased by applying an appropriate electrical voltage to the piezoelectric actuators. Moreover, it is found that the natural frequency of infinitesimal oscillation can be linearized with respect to the temperature by applying an appropriate electrical voltage to the piezoelectric actuators.

Figures 7 and 8 show the relation between the natural frequency and the amplitude of the oscillation with finite amplitude for $T_0 < T_{cr,11}$ and $T_0 > T_{cr,11}$, respectively. From Fig. 7, it is found that, for the case $T_0 < T_{cr,11}$, the natural frequency increases as the amplitude increases for relatively small magnitude of the electrical voltage and that it decreases as the amplitude increases for relatively large magnitude of the electrical voltage. By choosing an appropriate magnitude of the electric voltage, the natural frequency is found to be linearized with respect to the amplitude. From Fig. 8, it is found that, for the case $T_0 > T_{cr,11}$, the natural frequency decreases as the amplitude increases. Also for this case, the natural frequency is found to be

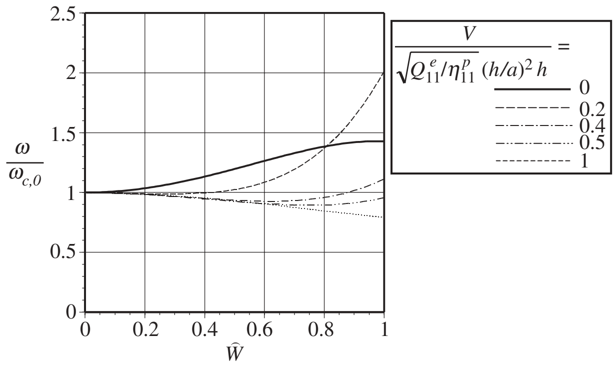

Fig. 7. Relation between the natural frequency and the amplitude of the oscillation with finite amplitude $\left(T_0/\left[(Q^e_{11}/\lambda^e_1)(h/a)^2\right] = 0.1; a/h = 50\right)$

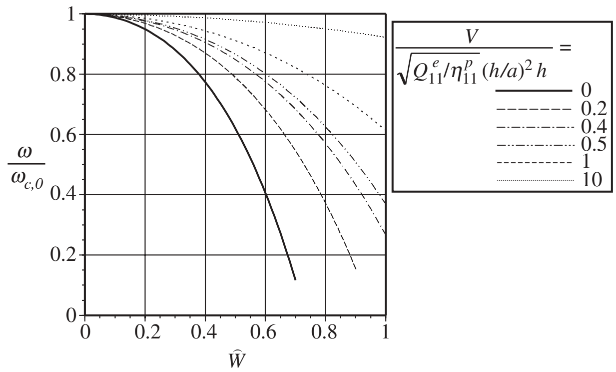

Fig. 8. Relation between the natural frequency and the amplitude of the oscillation with finite amplitude $(T_0/\left[\left(Q^e_{11}/\lambda^e_1\right)(h/a)^2\right] = 0.2; a/h = 50)$

linearized with respect to the amplitude by appropriate application of the electric voltage to the piezoelectric actuators.

4 Concluding remarks

We analyze the non-linear dynamic behavior of a piezothermoelastic laminated plate with anisotropic material properties. We consider that the laminate is a symmetric cross-ply laminate with all egdes simply-supported and is subjected to mechanical, thermal and electrical loads as intended control procedures or as disturbances. By using the von Kármán strains and the Galerkin Method, expressions of the following quantities are obtained: (i) the buckling temperature for in-plane thermal load; (ii) the large static deflection due to combined in-plane and anti-plane loads; (iii) the natural frequency of infinitesimal oscillations around the static large deflection; (iv) the natural frequency of the oscillation with finite amplitude around the static large deflection. Moreover, by performing numerical calculations, the non-linearity of the deflection with respect to thermal and electrical loads are shown qualitatively, and it is found that the application of an appropriate electrical voltage to the piezoelectric actuators can rise the buckling temperature and the natural frequency of infinitesimal oscillations and linearize the large deflection with respect to temperature and the natural frequencies with respect to the finite amplitude.

Appendix

The definitions of $a_{11,mn}$, $a_{22,mn}$, $a^{mn}_{12,m'n'}$, $b^{mn}_{11,m'n'ij}$, $b^{mn}_{13,m'n'ij}$, $b^{mn}_{22,m'n'ij}$, $b^{mn}_{23,m'n'ij}$ and $b^{mn}_{33,m'n'ijkl}$ are given as follows:

$$a_{11,mn} = A_{11}\alpha_m^2 + A_{66}\beta_n^2 \ , \ a_{22,mn} = A_{66}\alpha_m^2 + A_{22}\beta_n^2 \ , \ a_{12,mn} = (A_{12} + A_{66})\alpha_m\beta_n \ ,$$

$$a_{13,mn} = A_{11}\alpha_m^2 + A_{12}\beta_n^2 \ , \ a_{23,mn} = A_{12}\alpha_m^2 + A_{22}\beta_n^2 \ , \ a_{33,mn} = 2A_{66}\alpha_m\beta_n \ ,$$

$$a^{mn}_{12,m'n'} = -\frac{16}{\pi^2}a_{12,m'n'}\delta_{mm'}\delta_{nn'},$$

$$\begin{Bmatrix} b^{mn}_{11,m'n'ij} \\ b^{mn}_{13,m'n'ij} \end{Bmatrix} = -\frac{2}{\pi}\left[\begin{Bmatrix} a_{11,m'n'} \\ a_{13,m'n'} \end{Bmatrix}\alpha_i\delta_{m'im}\delta_{s,n'jn} - \begin{Bmatrix} a_{12,m'n'} \\ a_{33,m'n'} \end{Bmatrix}\beta_j\delta_{mm'i}\delta_{c,n'jn}\right],$$
$$\begin{Bmatrix} b^{mn}_{22,m'n'ij} \\ b^{mn}_{23,m'n'ij} \end{Bmatrix} = -\frac{2}{\pi}\left[\begin{Bmatrix} a_{22,m'n'} \\ a_{23,m'n'} \end{Bmatrix}\beta_j\delta_{s,m'im}\delta_{n'jn} - \begin{Bmatrix} a_{12,m'n'} \\ a_{33,m'n'} \end{Bmatrix}\alpha_i\delta_{c,m'im}\delta_{nn'j}\right], \tag{A.1}$$

$$b^{mn}_{33m'n'ijkl} = \frac{1}{32}\big(a_{13,m'n'}\alpha_i\alpha_k\Delta_{c,ikm'm}\Delta_{s,jln'n} + a_{23,m'n'}\beta_j\beta_l\Delta_{s,ikm'm}\Delta_{c,jln'n}$$
$$-2a_{33,m'n'}\alpha_i\beta_l\Delta_{c,m'ikm}\Delta_{c,n'ljn}\big),$$

where

$$\delta_{ij} = \begin{cases} \frac{i}{i^2-j^2} & : \mathrm{mod}(i,2) \neq \mathrm{mod}(j,2) \\ 0 & : \mathrm{mod}(i,2) = \mathrm{mod}(j,2) \end{cases}, \tag{A.2}$$

$$\delta_{ijk} = \begin{cases} +1 & : k = i + j \text{ or } i = j + k \\ -1 & : j = k + i \\ 0 & : \text{otherwise} \end{cases}, \tag{A.3}$$

$$\delta_{s,ijk} = \begin{cases} \frac{2ijk}{(i+j+k)(i+j-k)(i-j+k)(i-j-k)} & : \begin{pmatrix} \mathrm{mod}\,(i,2)\cdot\mathrm{mod}(j,2)\cdot\mathrm{mod}(k,2) = 1 \\ \text{or } \mathrm{mod}\,(i,2)\cdot[\mathrm{mod}(j,2)+1]\cdot[\mathrm{mod}(k,2)+1] = 1 \\ \text{or } \mathrm{mod}\,(j,2)\cdot[\mathrm{mod}(k,2)+1]\cdot[\mathrm{mod}(i,2)+1] = 1 \\ \text{or } \mathrm{mod}\,(k,2)\cdot[\mathrm{mod}(i,2)+1]\cdot[\mathrm{mod}(j,2)+1] = 1 \end{pmatrix}, \\ 0 & : \text{otherwise} \end{cases} \tag{A.4}$$

$$\delta_{c,ijk} = \begin{cases} \frac{k(i^2+j^2-k^2)}{(i+j+k)(i+j-k)(i-j+k)(i-j-k)} & : \begin{pmatrix} \mathrm{mod}\,(i,2)\cdot\mathrm{mod}(j,2)\cdot\mathrm{mod}(k,2) = 1 \\ \text{or } \mathrm{mod}\,(i,2)\cdot[\mathrm{mod}(j,2)+1]\cdot[\mathrm{mod}(k,2)+1] = 1 \\ \text{or } \mathrm{mod}\,(j,2)\cdot[\mathrm{mod}(k,2)+1]\cdot[\mathrm{mod}(i,2)+1] = 1 \\ \text{or } \mathrm{mod}\,(k,2)\cdot[\mathrm{mod}(i,2)+1]\cdot[\mathrm{mod}(j,2)+1] = 1 \end{pmatrix}, \\ 0 & : \text{otherwise} \end{cases} \tag{A.5}$$

$$\left(\Delta_{s,ijkl}\ ,\Delta_{c,ijkl}\right)=\begin{cases}(3,1) & : i=j=k=l\\ (2,2) & : i=j\neq k=l\\ (2,0) & : i=k\neq j=l \text{ or } i=l\neq j=k\\ (-1,-1) & : i=j+k+l \text{ or } j=k+l+i\\ (-1,1) & : k=l+i+j \text{ or } l=i+j+k\\ (1,-1) & : i+j=k+l \text{ and } i\neq k \text{ and } i\neq l \text{ and } j\neq k \text{ and } j\neq l\\ (1,1) & : \left[\begin{array}{l}(i+k=j+l \text{ and } i\neq j \text{ and } i\neq l \text{ and } k\neq j \text{ and } k\neq l)\\ \text{or } (i+l=j+k \text{ and } i\neq j \text{ and } i\neq k \text{ and } l\neq j \text{ and } l\neq k)\end{array}\right]\\ (0,0) & : \text{otherwise}\end{cases}. \tag{A.6}$$

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