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On stresses conjugate to Eulerian strains

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Summary. A stress is considered conjugate to a strain if the product of the stress and an objective rate of the strain has a trace which is equal to the rate of work per unit volume. Using Kronecker product relations, apparently new expressions for stresses conjugate to the Finger strain $\mathbf{B}$, the Euler strain $\exists$, the Eulerian (right) stretch tensor $\mathbf{V}$, and $\log(\mathbf{V})$ are determined. In addition, a nonclassical strain $\mathfrak{z}$ is introduced which permits a constitutive equation expressing its Truesdell rate in terms of $\mathbf{B}$ and the Truesdell rate of the Cauchy stress.

1 Introduction

The stress arising in the current or deformed configuration is, of course, the Cauchy stress denoted by τ [1]. The rate of work done per unit *deformed* volume and referred to deformed coordinates is given by

$$\dot{w} = \mathrm{trace}(\tau\mathbf{D}) \tag{1}$$

in which $\mathbf{D}$ is the deformation rate tensor

$$\mathbf{D} = \frac{1}{2}(\dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}^{-T}\dot{\mathbf{F}}^{T}),$$

while $\mathbf{F}$ is the deformation gradient tensor. For the moment, let us consider

$$\boldsymbol{e} = \int \mathbf{D}dt$$

to be a strain, referred to the deformed configuration[1]. By virtue of Eq. (1) we also say that τ is *conjugate* to $\boldsymbol{e}$. $\mathbf{F}$ satisfies the Polar decomposition theorem by virtue of which $\mathbf{F} = \mathbf{U}\mathbf{Q}_1 = \mathbf{Q}_2\mathbf{V}$, where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are orthogonal tensors, $\mathbf{U}$ is the positive definite Lagrangian stretch tensor, and $\mathbf{V}$ is the positive definite Eulerian stretch tensor.

The rate of work per unit reference or *undeformed* volume is found by standard arguments to be given alternatively by

$$\dot{w}_o = \mathrm{trace}(\bar{\boldsymbol{\sigma}}\dot{\mathbf{F}}), \quad \dot{w}_o = \mathrm{trace}(\boldsymbol{\sigma}\dot{\boldsymbol{\varepsilon}}) \tag{2}$$

[1]We regard a tensor as a strain if (a) it is not affected by rigid body motion and (b) its current value, given suitable compatibility conditions, determines the current displacement field to within a rigid body translation and rotation. By these criteria $\boldsymbol{e}$ is not strictly a strain and instead we later refer to it as a pseudostrain.

in which $\bar{\boldsymbol{\sigma}}$ denotes the 1st Piola-Kirchhoff stress tensor, while $\boldsymbol{\sigma}$ denotes the 2nd Piola-Kirchhoff stress tensor. Also

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

is recognized as the Lagrangian or Green strain. Accordingly, $\bar{\boldsymbol{\sigma}}$ is said to be conjugate to $\mathbf{F}$[2], while $\boldsymbol{\sigma}$ is conjugate to $\boldsymbol{\varepsilon}$. The Right Cauchy-Green strain satisfies $\mathbf{C} = 2\boldsymbol{\varepsilon} + \mathbf{I}$, for which $\boldsymbol{\sigma}/2$ is the corresponding conjugate stress. Also $\mathbf{U}^2 = \mathbf{C}$.

We consider the objectivity of the rates of $\mathbf{C}$ and $\boldsymbol{\varepsilon}$. Consider two deformations differing only by a rigid body motion

$$\mathbf{x}_2 = \mathbb{Q}(t)\mathbf{x}_1 + \mathbf{b}(t). \tag{3}$$

A tensor $\mathbf{T}$ "observed" from $\mathbf{x}_1$ is denoted as $\mathbf{T}_1$ and from $\mathbf{x}_2$ as $\mathbf{T}_2$. It is elementary to show that $\mathbf{F}_2 = \mathbb{Q}(t)\mathbf{F}_1$.

A tensor $\mathbf{T}$ will be called objective if either (a) $\mathbf{T}_2 = \mathbf{T}_1$ or (b) $\mathbf{T}_2 = \mathbb{Q}(t)\mathbf{T}_1\mathbb{Q}^T(t)$. Now $\mathbf{C}_2 = \mathbf{C}_1$, $\boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}_1$, and hence the rates of $\mathbf{C}$ and $\boldsymbol{\varepsilon}$ are objective. Likewise $\mathbf{U}$ and $\dot{\mathbf{U}}$ are objective. Case (a) is characteristic of tensors referred to the undeformed configuration, while case (b) is relevant to tensors referred to the deformed configuration. Here, strains satisfying (b) are called *Eulerian*.

There are other strain measures which are sometimes used, in particular the Left Cauchy-Green or Finger strain $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ in hyperelasticity [2], the Euler or Almansi strain

$$\exists = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}),$$

the Eulerian (spatial) stretch tensor $\mathbf{V}$ (which satisfies $\mathbf{V}^2 = \mathbf{B}$), and its logarithmic form

$$\log(\mathbf{V}) = \frac{1}{2}\log(\mathbf{B}).$$

These strains refer to the deformed configuration (i.e., are Eulerian) since they satisfy

$$\mathbf{B}_2 = \mathbb{Q}\mathbf{B}_1\mathbb{Q}^T, \quad \exists_2 = \mathbb{Q}\exists_1\mathbb{Q}^T, \quad \mathbf{V}_2 = \mathbb{Q}\mathbf{V}_1\mathbb{Q}^T, \quad \log(\mathbf{V}_2) = \mathbb{Q}\log(\mathbf{V}_1)\mathbb{Q}^T. \tag{4}$$

Note that the rates of these strains are not objective. Now $\mathbf{B}_2 = \mathbb{Q}(t)\mathbf{B}_1\mathbb{Q}^T(t)$ and $\exists_2 = \mathbf{Q}(t)\exists_1\mathbf{Q}^T(t)$. However, the rates satisfy e

$$\begin{aligned} \dot{\mathbf{B}}_2 &= \mathbb{Q}\dot{\mathbf{B}}_1\mathbb{Q}^T + \boldsymbol{\Omega}\mathbf{B}_2 - \mathbf{B}_2\boldsymbol{\Omega}, \\ \dot{\exists}_2 &= \mathbb{Q}\dot{\exists}_1\mathbb{Q}^T + \boldsymbol{\Omega}\exists_2 - \exists_2\boldsymbol{\Omega}, \\ \boldsymbol{\Omega} &= \dot{\mathbb{Q}}\mathbb{Q}^T. \end{aligned} \tag{5}$$

Of course, to circumvent non-objectivity a common practice is the replace the time rate with an objective rate such as the Jaumann rate

$$\overset{o}{\mathbf{B}} = \dot{\mathbf{B}} - \mathbf{W}\mathbf{B} + \mathbf{B}\mathbf{W}, \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T).$$

This prompts us to extend the notion of conjugacy as follows:

A stress Σ is *conjugate* to a strain $\mathbf{E}$ referred to the deformed configuration if, for an objective rate $\overset{o}{\mathbf{E}}$, the work per unit deformed volume is $\dot{w} = \mathrm{trace}(\boldsymbol{\Sigma}\overset{o}{\mathbf{E}})$.

In the sections below a derivation is presented of the stresses conjugate to $\mathbf{B}$, $\exists$, $\mathbf{V}$ and $\log(\mathbf{V})$. The derivation makes use of operations associated with Kronecker products of tensors, which

[2]Of course $\mathbf{F}$ is not a strain since it is affected by rigid body motion. We later refer to it as a pseudostrain.

are sketched in the next section. In later sections we consider a strain based on the Truesdell rate [3], and derive a corresponding (subordinate) objective rate for the stress conjugate to this strain.

2 Kronecker products on tensors

In the following sections all quantities are real. Let $\mathbf{A}$ be an $n \times n$ (second-order) tensor. Kronecker product notation [4] reduces $\mathbf{A}$ to a first-order $n^2 \times 1$ tensor (vector) as follows:

$$\mathrm{VEC}(\mathbf{A}) = \{ a_{11} \quad a_{21} \quad a_{31} \quad . \quad . \quad . \quad a_{n,n-1} \quad a_{nn} \}^T. \tag{6}$$

The inverse VEC operator, IVEC, is introduced by the obvious relation $\mathrm{IVEC}(\mathrm{VEC}(\mathbf{A})) = \mathbf{A}$. The Kronecker product of an $n \times m$ matrix $\mathbf{A}$ and an $r \times s$ matrix $\mathbf{B}$ generates an $nr \times ms$ matrix as follows:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & . & . & a_{1m}\mathbf{B} \\ a_{21}\mathbf{B} & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1}\mathbf{B} & . & . & . & a_{nm}\mathbf{B} \end{bmatrix}. \tag{7}$$

Five basic relations are introduced, followed by several subsidiary relations [4].

(I) Let $\mathbf{A}$ denote an $n \times m$ matrix, with entry a_{ij} in the i-th row and j-th column. Let $I = (j-1)n + i$ and $J = (i-1)m + j$. Let $\mathscr{U}_{nm}$ denote the $nm \times nm$ matrix, independent of $\mathbf{A}$, satisfying

$$u_{JK} = \begin{cases} 1, & K = I \\ 0, & K \neq I \end{cases} \qquad u_{IK} = \begin{cases} 1, & K = J \\ 0, & K \neq J \end{cases}. \tag{8}$$

Then

$$\mathrm{VEC}(\mathbf{A}^T) = \mathscr{U}_{nm}\mathrm{VEC}(\mathbf{A}). \tag{9}$$

(II) If $\mathbf{A}$ and $\mathbf{B}$ are second-order $n \times n$ tensors, then

$$\mathrm{trace}(\mathbf{AB}) = \mathrm{VEC}^T(\mathbf{A}^T)\mathrm{VEC}(\mathbf{B}). \tag{10}$$

(III) If $\mathbf{I}_n$ denotes the $n \times n$ identity matrix and if $\mathbf{B}$ denotes an $n \times n$ tensor, then

$$\mathbf{I}_n \otimes \mathbf{B}^T = (\mathbf{I}_n \otimes \mathbf{B})^T. \tag{11}$$

(IV) Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ respectively denote $m \times n$, $r \times s$, $n \times p$ and $s \times q$ matrices, then

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}. \tag{12}$$

(V) If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $n \times m$, $m \times r$ and $r \times s$ matrices, then

$$\mathrm{VEC}(\mathbf{ACB}) = \mathbf{B}^T \otimes \mathbf{A}\mathrm{VEC}(\mathbf{C}). \tag{13}$$

Symmetry of $\mathscr{U}_{nn}$ was established in Relation (1).

Letting $\mathbf{I}_{n^2}$ denote the identity tensor in n^2-dimensional space,

$$\mathbf{I}_n \otimes \mathbf{I}_n = \mathbf{I}_{n^2}. \tag{14}$$

Note that $\mathrm{VEC}(\mathbf{A}) = \mathscr{U}_{nn}\mathrm{VEC}(\mathbf{A}^T) = \mathscr{U}_{nn}^2\mathrm{VEC}(\mathbf{A})$ if $\mathbf{A}$ is $n \times n$, and hence the matrix $\mathscr{U}_{nn}$ satisfies

$$\mathscr{U}_{nn}^2 = \mathbf{I}_{n^2} \quad \mathscr{U}_{nn} = \mathscr{U}_{nn}^T = \mathscr{U}_{nn}^{-1}. \tag{15}$$

$\mathscr{U}_{nn}$ is seen to be a permutation tensor for $n \times n$ matrices. If $\mathbf{A}$ is symmetric, $(\mathscr{U}_{nn} - \mathbf{I}_{n^2})\text{VEC}(\mathbf{A}) = \mathbf{0}$. If $\mathbf{A}$ is antisymmetric, $(\mathscr{U}_{nn} + \mathbf{I}_{n^2})\text{VEC}(\mathbf{A}) = \mathbf{0}$.

If $\mathbf{A}$ and $\mathbf{B}$ are second-order $n \times n$ tensors, then

$$\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA}). \tag{16}$$

If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ denote $n \times n$ tensors,

$$\begin{aligned} \text{VEC}(\mathbf{ACB}^T) &= \mathbf{I}_n \otimes \mathbf{A}\text{VEC}(\mathbf{CB}^T) \\ &= (\mathbf{B} \otimes \mathbf{I}_n)\text{VEC}(\mathbf{AC}) \\ &= \mathbf{B} \otimes \mathbf{A}\text{VEC}(\mathbf{C}). \end{aligned} \tag{17}$$

Another useful relation is

$$\mathbf{B} \otimes \mathbf{A} = \mathscr{U}_{n^2}\mathbf{A} \otimes \mathbf{B}\mathscr{U}_{n^2}, \mathscr{U}_{n^2} = \mathscr{U}_{nn} \tag{18}$$

If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ tensors,

$$\begin{aligned} (\mathbf{A} \otimes \mathbf{B})(\mathbf{A}^{-1} \otimes \mathbf{B}^{-1}) &= \mathbf{AA}^{-1} \otimes \mathbf{BB}^{-1} \\ &= \mathbf{I}_n \otimes \mathbf{I}_n \\ &= \mathbf{I}_{n^2}. \end{aligned} \tag{19}$$

The Kronecker sum and difference are defined as follows:

$$\mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{I}_n + \mathbf{I}_n \otimes \mathbf{B}; \quad \mathbf{A} \ominus \mathbf{B} = \mathbf{A} \otimes \mathbf{I}_n - \mathbf{I}_n \otimes \mathbf{B}. \tag{20}$$

Let α_j and β_k denote the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, and let $\mathbf{y}_j$ and $\mathbf{z}_k$ denote the corresponding eigenvectors. The Kronecker product, sum and difference have the following eigenstructures.

expression	*jk*-th eigenvalue	*jk*-th eigenvector
$\mathbf{A} \otimes \mathbf{B}$	$\alpha_j\beta_k$	$\mathbf{y}_j \otimes \mathbf{z}_k$
$\mathbf{A} \oplus \mathbf{B}$	$\alpha_j + \beta_k$	$\mathbf{y}_j \otimes \mathbf{z}_k$
$\mathbf{A} \ominus \mathbf{B}$	$\alpha_j - \beta_k$	$\mathbf{y}_j \otimes \mathbf{z}_k$

. (21)

Of particular importance to the following sections, if $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ tensors, and if $\mathbf{A}' = \mathbf{QAQ}^T, \mathbf{B}' = \mathbf{QBQ}^T$, then

$$\begin{aligned} \mathbf{A}' \otimes \mathbf{B}' &= (\mathbf{QAQ}^T) \otimes (\mathbf{QBQ}^T) \\ &= (\mathbf{Q} \otimes \mathbf{Q})(\mathbf{AQ}^T \otimes \mathbf{BQ}^T) \\ &= (\mathbf{Q} \otimes \mathbf{Q})(\mathbf{A} \otimes \mathbf{B})\mathbf{Q}^T \otimes \mathbf{Q}^T, \end{aligned} \tag{22}$$

from which

$$\begin{aligned} \mathbf{A}' \oplus \mathbf{B}' &= (\mathbf{Q} \otimes \mathbf{Q})(\mathbf{A} + \mathbf{B})\mathbf{Q}^T \otimes \mathbf{Q}^T, \\ \mathbf{A}' \ominus \mathbf{B}' &= (\mathbf{Q} \otimes \mathbf{Q})(\mathbf{A} \ominus \mathbf{B})\mathbf{Q}^T \otimes \mathbf{Q}^T. \end{aligned} \tag{23}$$

Note that $(\mathbf{Q} \otimes \mathbf{Q})^T = \mathbf{Q}^T \otimes \mathbf{Q}^T = \mathbf{Q}^{-1} \otimes \mathbf{Q}^{-1} = (\mathbf{Q} \otimes \mathbf{Q})^{-1}$. In fact, $\mathbf{Q} \otimes \mathbf{Q}$ represents a rotation in an n^2-dimensional vector space. Equation (23) is reminiscent of properties of tensors.

We now recapitulate recent extensions of Kronecker Product relations to fourth order tensors [5]. Let $\mathbf{A}$ and $\mathbf{B}$ be second-order $n \times n$ tensors and let $\mathscr{C}$ be a fourth-order $n \times n \times n \times n$ tensor with suitable symmetries. Suppose that $\mathbf{A} = \mathscr{C}\mathbf{B}$, which is equivalent to $a_{ij} = c_{ijkl}b_{kl}$ in which the range of i, j, k and l is $(1, n)$. The TEN22 operator is introduced implicitly using

$$\mathrm{VEC}(\mathbf{A}) = \mathrm{TEN22}(\mathscr{C})\mathrm{VEC}(\mathbf{B}). \tag{24}$$

Several useful properties are:

$$\mathrm{TEN22}(\mathbf{A}\mathscr{C}\mathbf{B}) = \mathbf{I}_n \otimes \mathbf{A}\mathrm{TEN22}(\mathscr{C})\mathbf{I}_n \otimes \mathbf{B}. \tag{25}$$

$$\mathrm{TEN22}(\mathscr{C}^{-1}) = \mathrm{TEN22}^{-1}(\mathscr{C}). \tag{26}$$

The inverse of the TEN22 operator is introduced using the obvious relation $\mathrm{ITEN22}(\mathrm{TEN22}(\mathscr{C})) = \mathscr{C}$.

Suppose $\mathbf{A}$ and $\mathbf{B}$ are second order $n \times n$ tensors and $\mathscr{C}$ is a fourth order $n \times n \times n \times n$ tensor such that $\mathbf{A} = \mathscr{C}\mathbf{B}$. All are referred to a coordinate system denoted as Y. Let the orthogonal tensor $\mathbf{Q}$ represent a rotation which gives rise to a coordinate system Y', and let $\mathbf{A}'$, $\mathbf{B}'$ and $\mathscr{C}'$ denote the counterparts of $\mathbf{A}$, $\mathbf{B}$ and $\mathscr{C}$, referred to Y'. Now, since $\mathbf{A}' = \mathbf{Q}\mathbf{A}\mathbf{Q}^T$,

$$\mathrm{VEC}(\mathbf{A}') = \mathbf{Q} \otimes \mathbf{Q}\mathrm{VEC}(\mathbf{A}). \tag{27}$$

Now write $\mathbf{A}' = \mathscr{C}'\mathbf{B}'$ from which

$$\mathbf{Q} \otimes \mathbf{Q}\mathrm{VEC}(\mathbf{A}) = \mathrm{TEN22}(\mathscr{C}')\mathbf{Q} \otimes \mathbf{Q}\mathrm{VEC}(\mathbf{B}). \tag{28}$$

The transformation properties of TEN22 follow as

$$\mathrm{TEN22}(\mathscr{C}') = \mathbf{Q} \otimes \mathbf{Q}\mathrm{TEN22}(\mathscr{C})(\mathbf{Q} \otimes \mathbf{Q})^T. \tag{29}$$

Recall that $\mathbf{Q} \otimes \mathbf{Q}$ represents a rotation in n^2-dimensional space, for which reason Eq. (29) establishes tensor-like properties for TEN22.

2 Stress conjugate to the left Cauchy-Green strain B

Elementary manipulations serve to derive that

$$\begin{aligned}\dot{\mathbf{B}} &= \dot{\mathbf{F}}\mathbf{F}^T + \mathbf{F}\dot{\mathbf{F}}^T \\ &= \dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{F}\mathbf{F}^T + \mathbf{F}\mathbf{F}^T\mathbf{F}^{-T}\dot{\mathbf{F}}^T \\ &= \mathbf{L}\mathbf{B} + \mathbf{B}\mathbf{L}^T \\ &= \mathbf{D}\mathbf{B} + \mathbf{B}\mathbf{D} + \mathbf{W}\mathbf{B} - \mathbf{B}\mathbf{W}, \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T)\end{aligned} \tag{30}$$

and hence

$$\mathbf{D}\mathbf{B} + \mathbf{B}\mathbf{D} = \dot{\mathbf{B}} - \mathbf{W}\mathbf{B} + \mathbf{B}\mathbf{W}.$$

The right-hand side is recognized as the objective Jaumann rate $\overset{o}{\mathbf{B}}$ of $\mathbf{B}$.

Let $\boldsymbol{\Upsilon}$ denote the stress conjugate to $\mathbf{B}$. Using Kronecker product notation, we find that, by virtue of the symmetry of $\mathbf{D}$,

$$\begin{aligned}\dot{w} &= \mathrm{VEC}^T(\boldsymbol{\Upsilon})[\mathbf{B} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{B}]\mathrm{VEC}(\mathbf{D}) \\ &= \mathrm{VEC}^T(\boldsymbol{\Upsilon})[\mathbf{B} \otimes \mathbf{I} + \mathscr{U}_9\mathbf{B} \otimes \mathbf{I}\mathscr{U}_9]\mathrm{VEC}(\mathbf{D}) \\ &= \mathrm{VEC}^T(\boldsymbol{\Upsilon})[(\mathbf{I}_9 + \mathscr{U}_9)\mathbf{B} \otimes \mathbf{I}]\mathrm{VEC}(\mathbf{D})\end{aligned} \tag{31}$$

$$\mathscr{U}_9 = \mathscr{U}_{33}, \quad \mathbf{I}_9 = \mathbf{I} \otimes \mathbf{I}.$$

Since $\boldsymbol{\tau}$ is conjugate to $\mathbf{D}$,

$$
\begin{aligned}
\mathrm{VEC}^T(\boldsymbol{\tau}) &= \mathrm{VEC}^T(\boldsymbol{\Upsilon})(\mathbf{I}_9 + \mathscr{U}_9)\mathbf{B}\otimes\mathbf{I} \\
\mathrm{VEC}^T(\boldsymbol{\tau})\mathbf{B}^{-1}\otimes\mathbf{I} &= \mathrm{VEC}^T(\boldsymbol{\Upsilon})(\mathbf{I}_9 + \mathscr{U}_9) \\
\mathbf{B}^{-1}\otimes\mathbf{I}\mathrm{VEC}(\boldsymbol{\tau}) &= (\mathbf{I}_9 + \mathscr{U}_9)\mathrm{VEC}(\boldsymbol{\Upsilon}) \\
\mathrm{VEC}(\boldsymbol{\tau}\mathbf{B}^{-1}) &= (\mathbf{I}_9 + \mathscr{U}_9)\mathrm{VEC}(\boldsymbol{\Upsilon}).
\end{aligned} \tag{32}
$$

Note that

$$
(\mathbf{I}_9 + \mathscr{U}_9)^2 = \mathbf{I}_9 + 2\mathscr{U}_9 + \mathscr{U}_9^2 = 2(\mathbf{I}_9 + \mathscr{U}_9) \tag{33}
$$

from which we find

$$
(\mathbf{I}_9 + \mathscr{U}_9)\mathrm{VEC}(\boldsymbol{\tau}\mathbf{B}^{-1}) = 2(\mathbf{I}_9 + \mathscr{U}_9)\mathrm{VEC}(\boldsymbol{\Upsilon}). \tag{34}
$$

Exploiting the symmetry of $\boldsymbol{\Upsilon}$, we find the desired conjugate stress as

$$
\begin{aligned}
4\mathrm{VEC}(\boldsymbol{\Upsilon}) &= \mathrm{VEC}(\boldsymbol{\tau}\mathbf{B}^{-1} + \mathbf{B}^{-1}\boldsymbol{\tau}) \\
\boldsymbol{\Upsilon} &= \frac{\boldsymbol{\tau}\mathbf{B}^{-1} + \mathbf{B}^{-1}\boldsymbol{\tau}}{4}.
\end{aligned} \tag{35}
$$

3 The Euler strain $\exists$

The Euler strain refers to the deformed configuration. Its rate satisfies

$$
\begin{aligned}
\dot{\exists} &= \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})^{\cdot} \\
&= -\frac{1}{2}(\mathbf{B}^{-1})^{\cdot} \\
&= \frac{1}{2}\mathbf{B}^{-1}\dot{\mathbf{B}}\mathbf{B}^{-1}.
\end{aligned} \tag{36}
$$

It likewise is not an objective strain rate. We introduce

$$
\overset{o}{\exists} = \frac{1}{2}\mathbf{B}^{-1}\overset{o}{\mathbf{B}}\mathbf{B}^{-1} \tag{37}
$$

which is objective since

$$
\begin{aligned}
\overset{o}{\exists}_2 &= \frac{1}{2}\mathbf{B}_2^{-1}\overset{o}{\mathbf{B}}_2\mathbf{B}_2^{-1} \\
&= \frac{1}{2}(\mathbb{Q}\mathbf{B}_1^{-1}\mathbb{Q}^T)(\mathbb{Q}\overset{o}{\mathbf{B}}_1\mathbb{Q}^T)(\mathbb{Q}\mathbf{B}_1^{-1}\mathbb{Q}^T) \\
&= \frac{1}{2}\mathbb{Q}(\mathbf{B}_1^{-1}\overset{o}{\mathbf{B}}_1\mathbf{B}_1^{-1})\mathbb{Q}^T \\
&= \mathbb{Q}\overset{o}{\exists}_1\mathbb{Q}^T.
\end{aligned} \tag{38}
$$

Let $\boldsymbol{\Psi}$ denote the stress conjugate to $\exists$. The rate of work per unit *deformed* volume satisfies

$$
\begin{aligned}
\dot{w} &= \mathrm{trace}(\boldsymbol{\Psi}\overset{o}{\exists}) \\
&= \mathrm{trace}\left(\boldsymbol{\Psi}\frac{1}{2}\mathbf{B}^{-1}\overset{o}{\mathbf{B}}\mathbf{B}^{-1}\right) \\
&= \mathrm{trace}\left(\left(\frac{1}{2}\mathbf{B}^{-1}\boldsymbol{\Psi}\mathbf{B}^{-1}\right)\overset{o}{\mathbf{B}}\right)
\end{aligned} \tag{39}
$$

and hence the desired conjugate stress is

$$\begin{aligned}\boldsymbol{\Psi} &= 2\mathbf{B}\boldsymbol{\Upsilon}\mathbf{B}\\ &= \frac{1}{2}(\boldsymbol{\tau}\mathbf{B} + \mathbf{B}\boldsymbol{\tau}).\end{aligned} \tag{40}$$

4 The Lagrangian and Eulerian stretch tensors

For the previously defined Lagrangian stretch tensor, the work per unit *undeformed* volume is given by

$$\begin{aligned}\dot{w}_o &= \text{trace}\left(\frac{1}{2}\boldsymbol{\sigma}\dot{\mathbf{C}}\right)\\ &= \text{trace}\left(\frac{1}{2}\boldsymbol{\sigma}(\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U})\right)\\ &= \text{trace}\left(\frac{1}{2}\boldsymbol{\sigma}\mathbf{U}\dot{\mathbf{U}}\right) + \text{trace}\left(\frac{1}{2}\boldsymbol{\sigma}\dot{\mathbf{U}}\mathbf{U}\right)\\ &= \text{trace}\left(\frac{1}{2}(\boldsymbol{\sigma}\mathbf{U} + \mathbf{U}\boldsymbol{\sigma})\dot{\mathbf{U}}\right).\end{aligned} \tag{41}$$

Consequently

$$\frac{1}{2}(\boldsymbol{\sigma}\mathbf{U} + \mathbf{U}\boldsymbol{\sigma})$$

is the stress conjugate to $\mathbf{U}$.

The Eulerian stretch tensor satisfies $\mathbf{V}^2 = \mathbf{B}$. The Jaumann rate of $\mathbf{B}$ has been previously introduced as

$$\overset{o}{\mathbf{B}} = \dot{\mathbf{B}} - \mathbf{W}\mathbf{B} + \mathbf{B}\mathbf{W}. \tag{42}$$

We now prove that $\overset{o}{\mathbf{B}}$ is objective. Now $\mathbf{D}$ and $\mathbf{B}$ are objective. Using Eq. (22) gives

$$\begin{aligned}\overset{o}{\mathbf{B}}_2 &= \mathbf{D}_2\mathbf{B}_2 + \mathbf{B}_2\mathbf{D}_2\\ &= (\mathbb{Q}\mathbf{D}_1\mathbb{Q}^T)(\mathbb{Q}\mathbf{B}_1\mathbb{Q}^T) + (\mathbb{Q}\mathbf{B}_1\mathbb{Q}^T)(\mathbb{Q}\mathbf{D}_1\mathbb{Q}^T)\\ &= \mathbb{Q}(\mathbf{D}_1\mathbf{B}_1 + \mathbf{B}_1\mathbf{D}_1)\mathbb{Q}^T\\ &= \mathbb{Q}\overset{o}{\mathbf{B}}_1\mathbb{Q}^T.\end{aligned} \tag{43}$$

Note that $\mathbf{V}\dot{\mathbf{V}} + \dot{\mathbf{V}}\mathbf{V} = \dot{\mathbf{B}}$. We introduce the Jaumann rate $\overset{o}{\mathbf{V}}$ for $\mathbf{V}$ and note that

$$\begin{aligned}\mathbf{V}\overset{o}{\mathbf{V}} + \overset{o}{\mathbf{V}}\mathbf{V} &= \mathbf{V}(\dot{\mathbf{V}} - \mathbf{W}\mathbf{V} + \mathbf{V}\mathbf{W}) + (\dot{\mathbf{V}} - \mathbf{W}\mathbf{V} + \mathbf{V}\mathbf{W})\mathbf{V}\\ &= \mathbf{V}\dot{\mathbf{V}} + \dot{\mathbf{V}}\mathbf{V} - \mathbf{W}\mathbf{V}^{\mathbf{2}} + \mathbf{V}\mathbf{W}^{\mathbf{2}}\\ &= \overset{o}{\mathbf{B}}.\end{aligned} \tag{44}$$

Let Д denote the stress conjugate to $\mathbf{V}$. Now

$$\begin{aligned}\dot{w} &= \text{trace}(\boldsymbol{\Upsilon}\overset{o}{\mathbf{B}})\\ &= \text{trace}(\boldsymbol{\Upsilon}(\mathbf{V}\overset{o}{\mathbf{V}} + \overset{o}{\mathbf{V}}\mathbf{V}))\\ &= \text{trace}((\boldsymbol{\Upsilon}\mathbf{V} + \mathbf{V}\boldsymbol{\Upsilon})\overset{o}{\mathbf{V}}) \qquad (45)\end{aligned}$$

so that the desired conjugate stress is

$$\text{Д} = \boldsymbol{\Upsilon}\mathbf{V} + \mathbf{V}\boldsymbol{\Upsilon}. \qquad (46)$$

5 Logarithmic strain

The strain

$$\log(\mathbf{V}) = \frac{1}{2}\log(\mathbf{B})$$

is thought to have interesting properties and has been the object of a number of recent research studies [6]. Here, to find a stress conjugate to an objective rate of this strain we invoke a relation for the differential of an isotropic tensor valued function of a tensor. But first we introduce several useful relations in terms of Kronecker Product notation.

Let $\overset{o}{\mathbf{b}} = \text{VEC}(\overset{o}{\mathbf{B}})$. Objectivity of $\overset{o}{\mathbf{B}}$ is equivalent to

$$\begin{aligned}\overset{o}{\mathbf{b}}_2 &= \text{VEC}(\mathbb{Q}(t)\mathbf{B}_1\mathbb{Q}(t)^T)\\ &= \mathbb{Q}(t) \otimes (t)\overset{o}{\mathbf{b}}_1. \qquad (47)\end{aligned}$$

Now suppose that there is another rate given by $\overset{\square}{\mathbf{A}} = \mathscr{G}\overset{o}{\mathbf{B}}$ where $\mathscr{G}$ is a fourth-order tensor with suitable symmetry properties. We now will see that if $\mathscr{G}$ satisfies

$$\text{TEN22}(\mathscr{G}_2) = \mathbb{Q}(t) \otimes \mathbb{Q}(t)\text{TEN22}(\mathscr{G}_1)\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t) \qquad (48)$$

then $\overset{\square}{\mathbf{A}}$ is objective. As proof, observe that

$$\begin{aligned}\text{VEC}(\overset{\square}{\mathbf{A}}_2) = {} & \text{TEN22}(\mathscr{G}_2)\text{VEC}(\overset{\square}{\mathbf{B}}_2)\\ &= [\mathbb{Q}(t) \times \mathbb{Q}(t)\ \text{TEN22}(\mathscr{G}_1)\mathbb{Q}^T(t) \times \mathbb{Q}^T(t)]\mathbb{Q}(t) \times \mathbb{Q}(t)\text{VEC}(\overset{\square}{\mathbf{B}}_1)\\ &= \mathbb{Q}(t) \times \mathbb{Q}(t)\ \text{TEN22}(\mathscr{G}_1)[[\mathbb{Q}^T(t) \times \mathbb{Q}^T(t)]\mathbb{Q}(t) \times \mathbb{Q}(t)]\text{VEC}(\overset{\square}{\mathbf{B}}_1)\\ &= \mathbb{Q}(t) \times \mathbb{Q}(t)\ \text{TEN22}(\mathscr{G}_1)\text{VEC}(\overset{\square}{\mathbf{B}}_1) \qquad (49)\\ &= \mathbb{Q}(t) \times \mathbb{Q}(t)\text{VEC}(\overset{\square}{\mathbf{A}}_1).\end{aligned}$$

We say that the flux $\overset{\square}{(.)}$ is subordinate to the flux $\overset{o}{(.)}$.

Another useful relation is a general expression for the differential of an isotropic tensor-valued function of a tensor presented in Nicholson and Lin [5] under the assumption that the eigenvalues of $\mathbf{B}$ are distinct. It is applied here to the derivative of $\log(\mathbf{B})$ with respect to $\mathbf{B}$. Let $\ell = \text{VEC}(\log(\mathbf{B}))$ and $\mathbf{b} = \text{VEC}(\mathbf{B})$. Then, from Nicholson and Lin [5]

$$\frac{d\ell}{d\mathbf{b}} = \frac{1}{2}\mathbf{B}^{-1} \oplus \mathbf{B}^{-1} - [\mathbf{B} \ominus \mathbf{B}]^{\mathrm{I}}\mathscr{W} \qquad (50)$$

in which $[\mathbf{B} \ominus \mathbf{B}]^{\mathrm{I}}$ denotes the Morse-Penrose inverse (Dahlqvist and Bjork [7]) of $\mathbf{B} \ominus \mathbf{B}$, and

$$\mathscr{W} = \left[\log(\mathbf{B}) - \frac{1}{2}\mathbf{I}\right] \ominus \left[\log(\mathbf{B}) - \frac{1}{2}\mathbf{I}\right]. \tag{51}$$

Equations (22), (23) serve to prove that

$$\begin{aligned} &\mathbf{B}_2^{-1} \oplus \mathbf{B}_2^{-1} = \mathbb{Q}(t) \otimes \mathbb{Q}(t)\mathbf{B}_1^{-1} \oplus \mathbf{B}_1^{-1}\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t), \\ &\mathscr{W}_2 = \mathbb{Q}(t) \otimes \mathbb{Q}(t)\mathscr{W}_1\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t). \end{aligned} \tag{52}$$

We now consider the transformation properties of the Morse-Penrose inverse of $\mathbf{B} \ominus \mathbf{B}$. This may be written using the singular value decomposition in the form:

$$\begin{aligned} &\mathbf{B} \ominus \mathbf{B} = \mathfrak{U}\mathbf{\Sigma}\mathbf{V}^T, \mathfrak{U}, \mathbf{V}\ 9 \times 9 \text{ orthogonal matrices}, \\ &\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_1 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{\Sigma}_1 = \mathbf{diag}(\mathfrak{s}_i), \end{aligned} \tag{53}$$

$\mathfrak{s}_i$ the i-th non-vanishing singular value of $\mathbf{B} \ominus \mathbf{B}$.

The Morse-Penrose inverse is

$$(\mathbf{B} \ominus \mathbf{B})^{\mathrm{I}} = \mathbf{V}\mathbf{\Sigma}^{\mathrm{I}}\mathfrak{U}^T, \quad \mathbf{\Sigma}^{\mathrm{I}} = \begin{bmatrix} \mathbf{\Sigma}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{54}$$

Now

$$\begin{aligned} \mathbf{B}_2 \ominus \mathbf{B}_2 &= \mathbb{Q}(t) \otimes \mathbb{Q}(t)\mathbf{B}_1 \ominus \mathbf{B}_1\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t) \\ &= \mathbb{Q}(t) \otimes \mathbb{Q}(t)\mathfrak{U}\mathbf{\Sigma}\mathbf{V}^T\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t). \end{aligned} \tag{55}$$

But

$$\begin{aligned} [\mathbb{Q}(t) \otimes \mathbb{Q}(t)\mathfrak{U}][\mathbb{Q}(t) \otimes \mathbb{Q}(t)\mathfrak{U}]^T &= [\mathbb{Q}(t) \otimes \mathbb{Q}(t)\mathfrak{U}][\mathfrak{U}^T\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t)] \\ &= \mathbf{I}_9 \\ [\mathbf{V}^T\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t)][[\mathbf{V}^T\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t)]^T]^T &= [[\mathbf{V}^T\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t)][\mathbb{Q}(t) \otimes \mathbb{Q}(t)]^T\mathbf{V}] \\ &= \mathbf{I}_9. \end{aligned} \tag{56}$$

It follows that $\mathbb{Q}(t) \otimes \mathbb{Q}(t)\mathfrak{U}$ and $\mathbf{V}^T\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t)$ are orthogonal matrices. Consequently,

$$\begin{aligned} [\mathbf{B}_2 \ominus \mathbf{B}_2]^{\mathrm{I}} &= [\mathbb{Q}(t) \otimes \mathbb{Q}(t)\mathfrak{U}\mathbf{\Sigma}\mathbf{V}^T\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t)]^{\mathrm{I}} \\ &= \mathbb{Q}(t) \otimes (t)[\mathbf{V}\mathbf{\Sigma}^{\mathrm{I}}\mathfrak{U}^T]\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t) \\ &= \mathbb{Q}(t) \otimes \mathbb{Q}(t)[[\mathbf{B}_1 \ominus \mathbf{B}_1]^{\mathrm{I}}]\mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t). \end{aligned} \tag{57}$$

It follows that

$$\left(\frac{d\ell}{d\mathbf{b}}\right)_2 = \mathbb{Q}(t) \otimes \mathbb{Q}(t)\left(\frac{d\ell}{d\mathbf{b}}\right)_1 \mathbb{Q}^T(t) \otimes \mathbb{Q}^T(t). \tag{58}$$

We conclude that $\overset{\square}{\log}(\mathbf{B})$ is objective.

We now seek the stress И which is conjugate to $\log(\mathbf{B})$ (and $\log(\mathbf{V})$). Now

$$\mathrm{trace}(И\overset{\square}{\log}\mathbf{B}) = \mathrm{trace}\left(И\mathrm{ITEN22}\left(\frac{d\ell}{d\mathbf{b}}\right)\overset{o}{\mathbf{B}}\right). \tag{59}$$

The desired stress is obtained as

$$\begin{aligned}
\mathrm{VEC}^T(\boldsymbol{\Upsilon})\mathrm{VEC}(\overset{o}{\mathbf{B}}) &= \mathrm{VEC}^T(\text{И})\frac{d\ell}{d\mathbf{b}}\mathrm{VEC}(\overset{o}{\mathbf{B}}),\\
\mathrm{VEC}(\boldsymbol{\Upsilon}) &= \left(\frac{d\ell}{d\mathbf{b}}\right)^T \mathrm{VEC}(\text{И}),\\
\boldsymbol{\Upsilon} &= \mathrm{ITEN22}\left(\left(\frac{d\ell}{d\mathbf{b}}\right)^T\right)\text{И},\\
\text{И} &= \mathrm{ITEN22}\left(\left[\frac{d\ell}{d\mathbf{b}}\right]^{-T}\right)\boldsymbol{\Upsilon}.
\end{aligned} \tag{60}$$

6 Application of the Truesdell rate

In the previous sections, Kronecker-product relations were used in formulating transformations furnishing the stresses conjugate to the classical strains $\mathbf{C}$, $\boldsymbol{\varepsilon}$, $\mathbf{U}$, $\mathbf{B}$, $\exists$, $\mathbf{V}$, and $\log(\mathbf{B})$ ($\log(\mathbf{V})$), and to the pseudostrains $\mathbf{F}$ and з. Here, motivated by attractive properties of the Truesdell rate, we introduce a nonclassical Eulerian strain with a Truesdell rate, as well as its conjugate stress and an objective stress rate. Further the objective stress rate is subordinate to the Truesdell rate in a manner to be explained.

The Truesdell stress flux $\overset{\vee}{\boldsymbol{\tau}}$ of the Cauchy stress has the interesting *corotational* property that

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{\sigma} &= \frac{d}{dt}(J\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T})\\
&= J\mathbf{F}^{-1}\overset{\vee}{\boldsymbol{\tau}}\mathbf{F}^{-T},\ J = \det(\mathbf{F}) = \det{}^{1/2}(\mathbf{B}),
\end{aligned} \tag{61}$$

$$\overset{\vee}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} + \mathrm{trace}(\mathbf{D})\boldsymbol{\tau} - \mathbf{L}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{L}^T.$$

It may be seen that, in nonlinear formulations, the Truesdell stress increment is proportional to the increment of the 2nd Piola-Kirchhoff stress, without any additional terms due to corotation. This is in contrast to the Jaumann rate and all other objective rates which are not subordinate to the Truesdell rate.

Now

$$\begin{aligned}
\mathbf{F}_2 &= \mathbf{Q}\mathbf{F}_1,\\
\mathbf{F}_2^T &= \mathbf{F}_1^T\mathbf{Q}^T,\\
\mathbf{F}_2^{-1} &= \mathbf{F}_1^{-1}\mathbf{Q}^T,\\
\mathbf{F}_2^{-T} &= \mathbf{Q}\mathbf{F}_1^{-T}.
\end{aligned} \tag{62}$$

Consider a nonclassical Eulerian strain defined implicitly by

$$\boldsymbol{\varepsilon} = J\mathbf{F}^{-1}\text{з}\mathbf{F}^{-T}, \tag{63.1}$$

from which we conclude that

$$\dot{\boldsymbol{\varepsilon}} = J\mathbf{F}^{-1}\overset{\vee}{\text{з}}\mathbf{F}^{-T}, \quad \overset{\vee}{\text{з}}\ \ \text{Truesdellrateof з}. \tag{63.2}$$

The work per unit *undeformed* volume satisfies

$$\begin{aligned}
\dot{w}_o &= \mathrm{trace}(\boldsymbol{\sigma}\dot{\boldsymbol{\varepsilon}}) \\
&= \mathrm{trace}(\boldsymbol{\sigma} J\mathbf{F}^{-1}\overset{\vee}{3}\mathbf{F}^{-T}) \\
&= \mathrm{trace}(J\mathbf{F}^{-T}\boldsymbol{\sigma}\mathbf{F}^{-1}\overset{\vee}{3}) \\
&= \mathrm{trace}\left(J^2\mathbf{F}^{-T}\mathbf{F}^{-1}\frac{\mathbf{F}\boldsymbol{\sigma}\mathbf{F}^T}{J}\mathbf{F}^{-T}\mathbf{F}^{-1}\overset{\vee}{3}\right) \\
&= \mathrm{trace}(J^2\mathbf{B}^{-1}\boldsymbol{\tau}\mathbf{B}^{-1}\overset{\vee}{3}) \\
&= \mathrm{trace}(\boldsymbol{\tau} J^2\mathbf{B}^{-1}\overset{\vee}{3}\mathbf{B}^{-1}). \qquad (63.3)
\end{aligned}$$

Note that, since trace($\boldsymbol{\tau} J\mathbf{D}$) is the work per unit undeformed volume

$$J\mathbf{B}^{-1}\overset{\vee}{3}\mathbf{B}^{-1} = \mathbf{D} \qquad (64.1)$$

$$3 = \frac{1}{2J}(\mathbf{B}^2 - \mathbf{B}). \qquad (64.2)$$

Since $\dot{w}_o = J\dot{w}$, the work per unit deformed volume is given by

$$\begin{aligned}
\dot{w} &= \mathrm{trace}(J\boldsymbol{\tau}\mathbf{B}^{-1}\overset{\vee}{3}\mathbf{B}^{-1}) \\
&= \mathrm{trace}(J\mathbf{B}^{-1}\boldsymbol{\tau}\mathbf{B}^{-1}\overset{\vee}{3}). \qquad (65)
\end{aligned}$$

The desired conjugate stress, per unit deformed volume, is

$$\boldsymbol{\Pi} = \mathbf{F}^{-T}\boldsymbol{\sigma}\mathbf{F}^{-1} = J\mathbf{B}^{-1}\boldsymbol{\tau}\mathbf{B}^{-1}. \qquad (66)$$

We may easily define an objective rate for the conjugate stress as

$$\overset{\nabla}{\boldsymbol{\Pi}} = J\mathbf{B}^{-1}\overset{\vee}{\boldsymbol{\tau}}\mathbf{B}^{-1}. \qquad (67)$$

As proof of objectivity, we note that

$$\begin{aligned}
\overset{\nabla}{\boldsymbol{\Pi}}_2 &= J_2\mathbf{B}_2^{-1}\overset{\vee}{\boldsymbol{\tau}}_2\mathbf{B}_2^{-1} \\
&= [\mathbb{Q}(t)\mathbf{B}_1^{-1}\mathbb{Q}^T(t)][J_1\mathbb{Q}(t)\overset{\vee}{\boldsymbol{\tau}}_1\mathbb{Q}^T(t)][\mathbb{Q}(t)\mathbf{B}_2^{-1}\mathbb{Q}^T(t)] \\
&= \mathbb{Q}(t)[J_1\mathbf{B}_1^{-1}\overset{\vee}{\boldsymbol{\tau}}_1\mathbf{B}_1^{-1}]\mathbb{Q}^T(t) \\
&= \mathbb{Q}(t)\overset{\nabla}{\boldsymbol{\Pi}}_1\mathbb{Q}^T(t). \qquad (68)
\end{aligned}$$

As an application, a hypoelastic constitutive relation may be written as

$$\overset{\nabla}{\boldsymbol{\Pi}} = \mathscr{D}(3)\overset{\vee}{3}, \text{ or}$$

$$\begin{aligned}
\overset{\vee}{\boldsymbol{\tau}} &= \left(\frac{1}{J}\mathbf{B}\boldsymbol{\Pi}\mathbf{B}\right)^{\vee} \\
&= \frac{1}{J}\mathbf{B}\mathscr{D}(3)\overset{\vee}{3}\mathbf{B} \\
&= \frac{1}{J}\mathbf{B}\mathscr{D}\left(\frac{1}{2J}(\mathbf{B}^2 - \mathbf{B})\right)\overset{\vee}{3}\mathbf{B} \\
&= \frac{1}{J}\mathbf{B}\mathscr{D}\left(\frac{1}{2J}(\mathbf{B}^2 - \mathbf{B})\right)[\mathbf{BDB}]\mathbf{B}. \qquad (69)
\end{aligned}$$

in which $\mathcal{D}$ is the fourth-order tangent modulus tensor. In Eq. (69) **B** can be eliminated in favor of з using

$$\mathbf{B} = \frac{1}{2}(\mathbf{I} + \sqrt{\mathbf{I} + 8J\,з}).$$

Equation (69) expresses the Truesdell flux of the Cauchy stress τ in terms of Truesdell flux of a nonclassical Eulerian strain. Further, τ is expressed in terms of the conjugate stress through the simple relation $\mathbf{\Pi} = J\mathbf{B}^{-1}\boldsymbol{\tau}\mathbf{B}^{-1}$.

7 Conclusion

Since the time rates of Eulerian strains are not objective, it is common practice to use objective rates such as the Jaumann rate. The same objective rate, of the conjugate stress, should be used, if possible, in the rate constitutive model, for example in hypoelasticity. A stress is considered conjugate to a strain if the product of the stress and an objective rate of the strain has a trace which is equal to the rate of work per unit (deformed) volume. In the current investigation, Kronecker-product relations are used to derive the stresses conjugate to the Finger strain **B**, the Euler strain $\exists$, the Eulerian (right) stretch tensor **V**, and log(**V**). In addition, using an attractive property of the Truesdell rate, a nonclassical strain з is introduced which permits a constitutive equation expressing its Truesdell rate in terms of **B** and the Truesdell rate of the Cauchy stress.

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