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The effects of side walls on axial flow in rectangular ducts with suction and injection

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Summary. The effects of the side walls on the flows in ducts with suction and injection are examined. Three illustrative examples are considered. The first example considers the effect of the side walls on the flow over a porous plate. It is shown that the presence of the side walls provides a solution for both injection and suction, although, in the absence of the side walls, a solution exists only in the case of suction. The second example considers the flow between two porous plates and the third example the flow in a rectangular duct with two porous walls. Analytical solutions are obtained for the velocity, the volume flux across a plane normal to the flow and the vorticity. In order to show the effects of the side walls for the flow on a rectangular duct, a comparison of these quantities with those in the flow between two parallel porous plates is established. These three examples show that there are pronounced effects of the side walls on the flows in ducts with suction and injection.

1 Introduction

The effects of the side walls on the porous plates are examined. For this purpose, three illustrative examples are given. These are: the asymptotic flow along a porous plate with uniform suction bounded by two side walls, the flow between two parallel porous plates with suction at one plate and injection at the other, and the flow in a rectangular duct with impermeable lateral boundaries and upper and lower porous boundaries at which uniform suction and injection are imposed. In order to show the effects of the side walls on the velocity, the flux across a plane normal to the flow and the vorticity are calculated. These three examples show that there are pronounced effects of the side walls on the flows in ducts with suction and injection.

The flow of fluids over boundaries of porous materials has many applications in practice such as boundary-layer control. A surprisingly simple solution of the Navier-Stokes equations can be obtained for the flow over a porous plane boundary at which there is a uniform suction velocity. The solution was found by Griffith and Meredith [1], and it is an exact solution of the Navier-Stokes equations. The velocity profile given by this solution has been called the asymptotic suction profile [2]. The asymptotic suction profile is also the limiting form of the velocity distribution for an arbitrary boundary layer flow when the rate of suction tends to infinity, even when the velocity at infinity and the suction velocity are functions of the distance along the surface [2]. It is clear from the discussion that this solution is very important in the boundary layer theory not only for Newtonian fluids but also for non-Newtonian fluids [3] and electrically conducting fluids [4]. There is no solution of the Navier-Stokes equations for flow over a porous plane boundary at which there is a uniform injection velocity. The reason is that the boundary layer in the case of injection at the plane boundary grows gradually. However, if the porous plate is bounded by two side walls, then, a solution of the Navier-Stokes equations can be found for the uniform injection case. This is due to the effects of the side walls. In this case the pressure gradient is constant, on the other hand, in the absence of the side walls the pressure is constant. A pressure gradient in the presence of the side walls is necessary to overcome the effects of the side walls. If the porous plate is moving with a constant velocity, an exact solution of the Navier-Stokes equations is obtained with a constant pressure and zero velocity at infinity.

The second illustrative example is the flow between two parallel porous plates with uniform injection at the upper plate and uniform suction at the lower plate. When the distance between two parallel plates goes to infinity, the velocity distribution reduces to that for the asymptotic suction flow, and when the suction velocity goes to zero, it tends to that for the Poiseuille flow. The velocity distribution depends on a single parameter called cross-Reynolds number which is based on the distance between two plates and the suction velocity. For large values of this parameter, the flow near the lower plate has a boundary-layer character, and the velocity varies sharply [5]. It is important to note that the vorticity for large values of this parameter is concentrated near the lower plate and it has a constant value across the channel [6]. The flow between two parallel porous plates provides a comparison with the flow in a rectangular duct in the presence of suction and injection.

The flow in a rectangular duct with uniform suction and injection has been examined by Mehta and Jain [7], and Sai and Rao [8]. The form of the velocity given in [8] is not convenient for use in this paper and therefore cannot be used to show the effects of the side walls. For this reason, a new form for the velocity is given. It is found that the velocity has two terms; the first term shows the velocity of the flow between two parallel porous plates and the second term denotes the effect of the side walls. The velocity field depends on two parameters, one is the cross-Reynolds number based on the suction velocity and the distance between two porous walls, and the other is the aspect ratio b/c (Fig. 4). When the cross-Reynolds number goes to zero, the velocity reduces to that for flow in a rectangular duct without porous walls. When the aspect ratio b/c goes to zero, the velocity tends to that for the flow between two parallel porous walls.

In order to show the effects of the side walls on the flow in a rectangular duct, it is compared with the flow between two parallel porous plates. The effects of the side walls for the flow in a rectangular duct with suction and injection are determined by the cross-Reynolds number and the aspect ratio b/c. For a given value of the cross-Reynolds number, the effect of the side walls is greatest for a duct with a square cross-section for which the aspect ratio is equal to 1. When this ratio goes to zero the effect of the side walls disappears.

The volume flux across a plane normal to the flow in a rectangular duct with suction and injection for a given value of the cross-Reynolds number decreases approximately linearly with the aspect ratio. The volume flux for a given value of the aspect ratio decreases with the cross-Reynolds number.

The vorticity in a rectangular duct with suction and injection has three components. In order to compare the flow in a rectangular duct with the flow between two parallel porous plates, the component of the vorticity perpendicular to the velocity is considered. This component of the vorticity depends on the cross-Reynolds number and the aspect ratio. When the cross-Reynolds number goes to zero, the vorticity reduces to that for the flow in a rectangular duct without porous walls. When the aspect ratio goes to zero, the vorticity reduces to that for the flow between two parallel porous plates. It is found that for large values of the cross-Reynolds number the variation of the vorticity near the region of suction is sharp and has a boundary-layer character. In the other regions of the duct, the vorticity has a constant value. It is found that for a given value of the cross-Reynolds number the effect of the side walls is greatest for a duct with a square cross-section for which the aspect ratio is equal to 1. When the aspect ratio goes to zero the effect of the side walls vanishes. It is clearly understood from the discussion that the effect of the side walls is very important in practice [9], [10]. The effects of the side walls in the Couette flow were investigated by Shermann [6]. For practical purposes one wishes to know, for example, how the stress exerted on the bottom wall varies with the distance between the side walls, for a given value of the aspect ratio of the channel. If this ratio is large enough, for example, in the order of 5, the stress is nearly uniform and the measured value would be unaffected by the presence of the side walls.

2 The effects of the side walls on the asymptotic suction flow

The physical model of the asymptotic suction flow is illustrated in Fig. 1. Rectangular Cartesian coordinates are used. The x-axis is taken along the plate and the y-axis is normal to it. It is assumed that the plate is infinitely long so that the components of the velocity depend on y only. By the continuity equation the y-component of the velocity remains constant and equals to its value $-V$, say $V > 0$, at the wall. The pressure remains constant throughout the flow. The conditions on the plate and infinity are

$$u = 0 \quad \text{at} \quad y = 0; \quad u \to U \quad \text{as} \quad y \to \infty.$$

Then the Navier-Stokes equations reduce to

$$-Vu' = vu'',$$

where primes denote differentiation with respect to y. The appropriate solution is

$$\frac{u}{U} = 1 - e^{\frac{-Vy}{v}}. \tag{2.1}$$

As $v \to 0$ the disturbance to the main stream becomes more and more concentrated in a boundary layer at the plate. It is known that for the uniform injection case ($V < 0$), the governing equation does not give a solution for the velocity, since the vorticity layer over the plate grows gradually. However, it will be shown that if the porous plate is bounded by two side walls, one can then find a solution for the uniform injection due to the effects of the side walls.

The physical model of the flow in the presence of the side walls is illustrated in Fig. 2. The side walls are at $z = \pm c$. The governing equation is

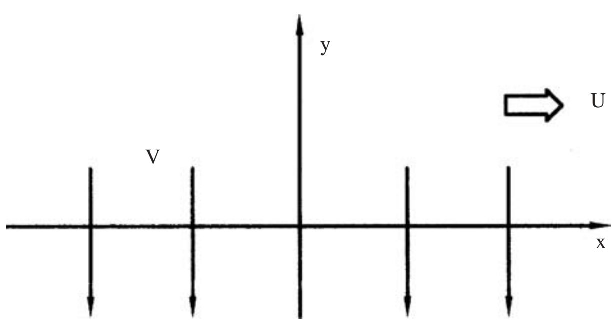


Fig. 1. Physical model

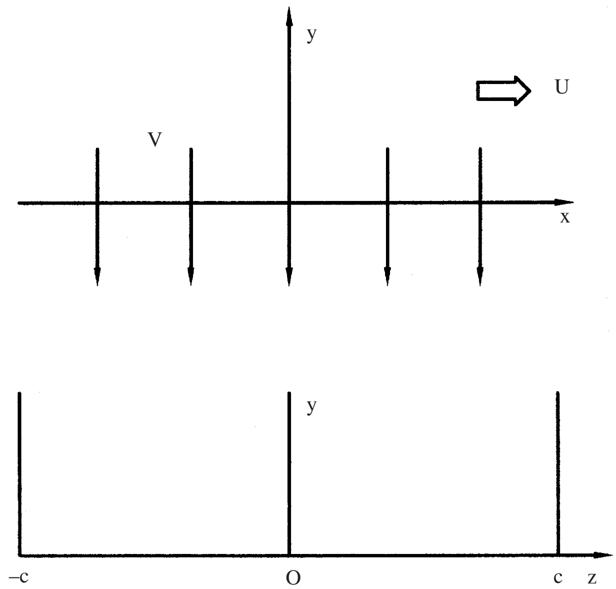


Fig. 2. Flow geometry. The side walls are at $z = \pm c$

$$-V\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{2.2}$$

where dp/dx is the pressure gradient, and it can be easily shown that dp/dx is constant. The boundary conditions are

$$u(0, z) = 0, \quad u(y, \pm c) = 0.$$

The boundary conditions suggest a solution for the velocity in the following form:

$$u(y, z) = \sum_0 f_n(y)\cos\mu_n z, \tag{2.3}$$

where $\mu_n = (2n+1)\pi/2c$ and $2c$ is the distance between the side walls. $f_n(y)$ satisfies the differential equation

$$f_n'' + \frac{V}{\nu}f_n' - \mu_n^2 f_n = \frac{1}{\mu}\frac{dp}{dx}\frac{4(-1)^n}{(2n+1)\pi},$$

where primes denote differentiation with respect to y. Using the form of $f_n(y)$obtained by the differential equation, Eq. (2.3) can be written as

$$u(y, z) = \left(-\frac{1}{2\mu}\frac{dp}{dx}c^2\right)\frac{32}{\pi^3}\sum_0 \frac{(-1)^n}{(2n+1)^3}(1 - e^{\alpha_n y})\cos\mu_n z, \tag{2.4}$$

where $\alpha_n = -(V/2) - [(V^2/4\nu^2) + \mu_n^2]^{1/2}$. When c goes to infinity one can find the asymptotic suction case which is

$$u = U\left(1 - e^{-\frac{Vy}{\nu}}\right),$$

where for $c \to \infty$, $\alpha_n = -V/\nu$ and $-(dp/dx)c^2/2\mu = U$. When V goes to zero, Eq. (2.4) reduces to

$$u = \left(-\frac{1}{2\mu}\frac{dp}{dx}c^2\right)\frac{32}{\pi^3}\sum_0 \frac{(-1)^n}{(2n+1)^3}(1 - e^{-\mu_n y})\cos\mu_n z.$$

If the boundary conditions for the flow illustrated in Fig. 2 are in the following form:

$$u(0,z) = U, \quad u(y,\pm c) = 0,$$

the velocity distribution, in the absence of the pressure gradient, becomes

$$u = \frac{4U}{\pi}\sum_{0}\frac{(-1)^n}{(2n+1)}e^{-\alpha_n y}\cos\mu_n z.$$

As it is expected the velocity vanishes at infinity.

It is clearly seen from Eq. (2.4) that there are pronounced effects of the side walls. Indeed, at $y/c = 1$, for $Vc/\nu = 0.5$, $u/-(dp/dx)c^2/2\mu$ becomes 0.9740, accurate to four decimal places and for $Vc/\nu = 5$, it is about 0.6615, which can be comparable with 0.6321 that corresponds to the case in the absence of the side walls. It is very important that the velocity distribution given by Eq. (2.4) can also be used for the injection case ($V < 0$). This is due to the effect of the side walls.

3 Flow between two parallel porous plates

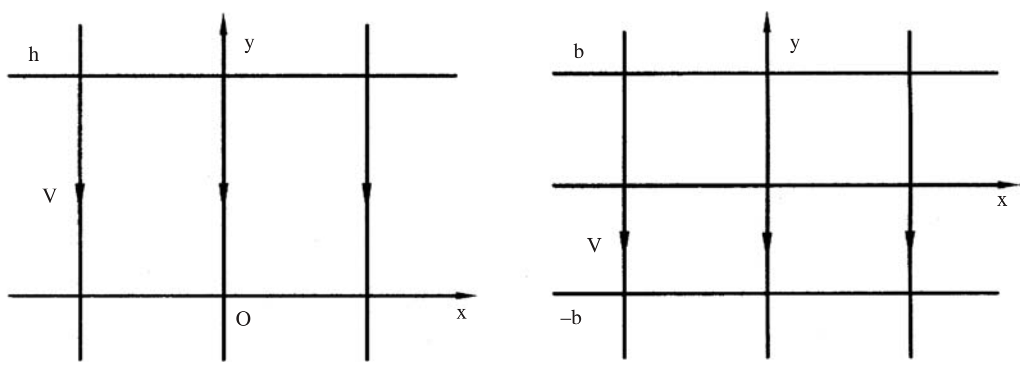
Two physical models, one of them in the limiting case reduces to the asymptotic suction flow, and the other, in the limiting case, gives the Poiseuille flow, are illustrated in Fig. 3. The governing equation is

$$-V\frac{du}{dy} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{d^2u}{dy^2},$$

where dp/dx is the pressure gradient, and it can be easily shown that dp/dx is constant. The boundary conditions are $u(\pm b) = 0$. The solution can be written in four different forms,

$$\frac{u}{-\frac{1}{2\mu}\frac{dp}{dx}b^2} = \frac{2}{\lambda}\left(1 - \eta + \frac{e^{-\lambda} - e^{-\lambda\eta}}{\sinh\lambda}\right), \tag{3.1}$$

$$\frac{u}{-\frac{1}{2\mu}\frac{dp}{dx}b^2} = \frac{2}{\lambda}\left(\frac{\cosh\lambda}{\sinh\lambda} - \eta - \frac{e^{-\lambda\eta}}{\sinh\lambda}\right), \tag{3.2}$$



a **b**

Fig. 3. **a** Flow geometry; when h goes to infinity it gives the asymptotic suction flow, **b** flow geometry; when V goes to zero it gives the Poiseuille flow

$$\frac{u}{-\frac{1}{2\mu}\frac{dp}{dx}b^2} = \frac{2}{\lambda}\left(\frac{e^{\lambda} - e^{-\lambda\eta}}{\sinh\lambda} - 1 - \eta\right), \tag{3.3}$$

$$\frac{u}{-\frac{1}{2\mu}\frac{dp}{dx}b^2} = \frac{2\cosh\lambda}{\lambda\sinh\lambda}\left(1 - \frac{\cosh\lambda\eta}{\cosh\lambda}\right) - \frac{2}{\lambda}\left(\eta - \frac{\sinh\lambda\eta}{\sinh\lambda}\right), \tag{3.4}$$

where $\lambda = Vb/v$ and $\eta = y/b$. When λ goes to zero, the velocity field reduces to the case of the Poiseuille flow. Throughout the paper the simplest form of the velocity given by Eq. (3.1) is used. For large values of λ, Eq. (3.1) reduces to

$$\frac{u}{-\frac{1}{2\mu}\frac{dp}{dx}b^2} = \frac{2}{\lambda}(1 - \eta).$$

This form of the velocity satisfies the boundary condition at $\eta = 1$, but it does not satisfy the one at $\eta = -1$. This shows that there is a boundary layer at $\eta = -1$ where the velocity profile varies sharply [5]. It is important to note that the vorticity for large values of λ is concentrated near $\eta = -1$ and has a constant value across the channel [6]. The vorticity is given in the following form:

$$\frac{\omega}{-(dp/dx)b/\mu} = \frac{1}{\lambda}\left(1 - \frac{\lambda e^{-\lambda\eta}}{\sinh\lambda}\right). \tag{3.5}$$

It is interesting that the difference between the values of the vorticities at the upper and lower boundaries equals 2. This shows that this difference does not depend on the properties of the fluid and the type of boundaries.

The volume flux across a plane normal to the flow is given by

$$\frac{Q(\lambda)}{Q(0)} = \frac{3}{\lambda}\left(\frac{1}{\tanh\lambda} - \frac{1}{\lambda}\right), \tag{3.6}$$

where $Q(0) = -2(dp/dx)b^3/3\mu$ which corresponds to the Poiseuille flow with impermeable walls. The results obtained by the examination of the flow between two parallel porous plates will be used, in the next paragraph, for comparison with the flow in a rectangular duct with suction and injection.

4 Flow in a rectangular duct with suction and injection

The sides of the rectangular section are at $y = \pm b$ and $z = \pm c$. It is supposed throughout that $c \geq b$. The boundaries at $y = \pm b$ are porous, and the boundaries at $z = \pm c$ are non-porous. The flow geometry is denoted in Fig. 4. The governing equation is

$$-V\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + v\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{4.1}$$

and the boundary conditions are

$$u(\pm b, z) = 0, \quad u(y, \pm c) = 0, \tag{4.2}$$

where dp/dx is the pressure gradient along the flow direction, and it can be easily shown that dp/dx is constant. The solution of Eq. (4.1) subject to the boundary conditions (4.2) has been given by Sai and Rao [8]. However, the form of their solution is not convenient for use in this paper. Their solution cannot be used to show the effects of side walls. For this reason, a new solution of Eq. (4.1) subject to the boundary conditions (4.2) is given. The required solution is

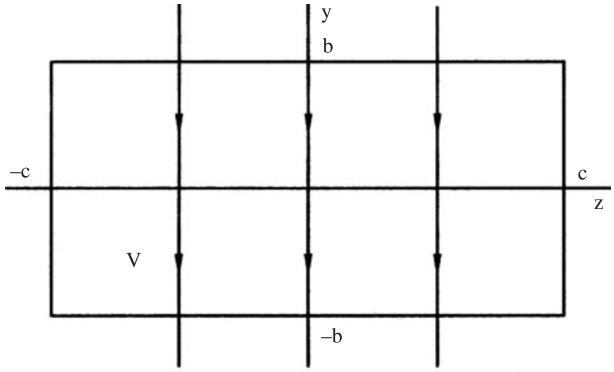


Fig. 4. Flow geometry for the flow in a rectangular duct

$$\frac{u}{-\frac{1}{2\mu}\frac{dp}{dx}b^2} = \frac{2}{\lambda}\left(1 - \eta + \frac{e^{-\lambda} - e^{-\lambda\eta}}{\sinh\lambda}\right) + e^{-\frac{\lambda\eta}{2}}\sum_0 A_n \cosh(k_n z)\cos(\lambda_n b)\eta, \tag{4.3}$$

where

$$A_n = -\frac{64(-1)^n(\lambda_n b)\cosh(\lambda/2)}{\left(4\lambda_n^2 b^2 + \lambda^2\right)^2 \cosh(k_n c)},$$

$$\lambda_n = (2n+1)\pi/2b, \quad k_n = \left(4\lambda_n^2 b^2 + \lambda^2\right)^{1/2} \Big/ 2b, \quad \lambda = Vb/v.$$

The first term in Eq. (4.3) shows the velocity field of the flow between two parallel porous walls, and the second term denotes the effects of the side walls. Figure 5 shows the variation of the velocity with y/b for various values of λ for $b/c = 0$, namely, for flow between two parallel porous plates. The flow for non-porous walls corresponds to $\lambda = 0$, then the velocity profile is symmetric. For large values of λ the symmetry of the velocity profile breaks down and the velocity near the lower wall has a boundary-layer character. Figure 6 denotes the variation of the velocity with y/b for various values of λ for $b/c = 1$, namely, for flow in a duct of square cross-section. It is clearly seen from Fig. 6 that there is a pronounced effect due to the side walls.

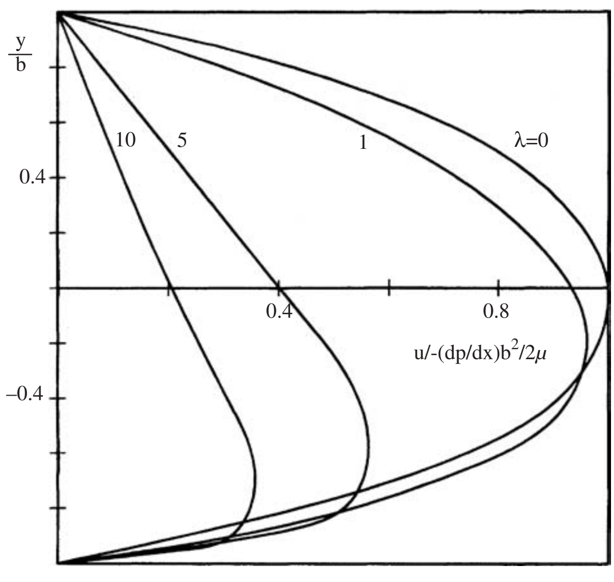


Fig. 5. The variation of the velocity with y/b for $b/c = 0$

The volume flux across a plane normal to the flow is given by

$$\frac{Q(\lambda, b/c)}{-4(dp/dx)b^3c/3\mu} = \frac{1}{\lambda}\left(\frac{1}{\tanh\lambda} - \frac{1}{\lambda}\right) - 768\frac{b}{c}\left(\cosh\frac{\lambda}{2}\right)^2 \sum_0 \frac{(\lambda_n b)^2 \tanh(k_n c)}{\left(4\lambda_n^2 b^2 + \lambda^2\right)^{7/2}},$$

where the first term shows the flux for flow between two parallel porous plates and the second term denotes the effect of the side walls. In the limiting case for $\lambda \to 0$, it gives the flux for flow between two non-porous walls and for $b/c \to 0$, it reduces to the flux for flow between two parallel porous walls. It is clearly seen from Fig. 7 that there is a strong effect on the flux due to the side walls.

The vorticity has three components. In order to compare with the two dimensional case, the z-component of the vorticity at $z = 0$ is considered and it is given in the following form:

$$\frac{\omega}{-(dp/dx)b/\mu} = \frac{1}{\lambda}\left(1 - \frac{\lambda e^{-\lambda\eta}}{\sinh\lambda}\right) + \frac{1}{2}e^{-(\lambda/2)\eta} \sum_0 B_n \left(\frac{\lambda}{2}\cos\lambda_n b\eta + \lambda_n b \sin\lambda_n b\eta\right), \tag{4.4}$$

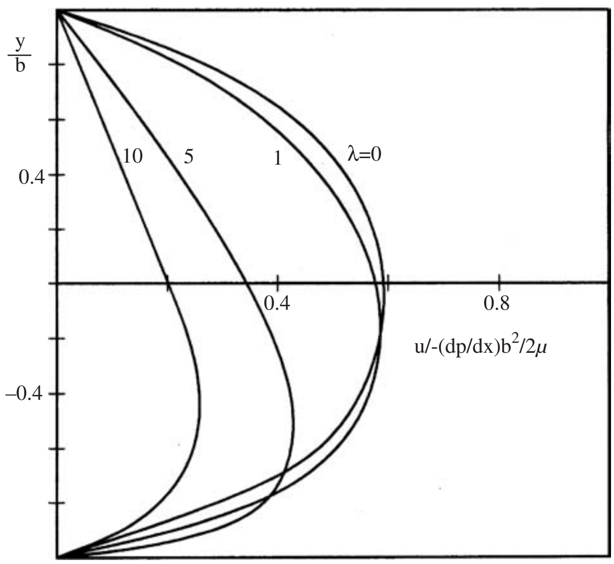


Fig. 6. The variation of the velocity with y/b for $b/c = 1$

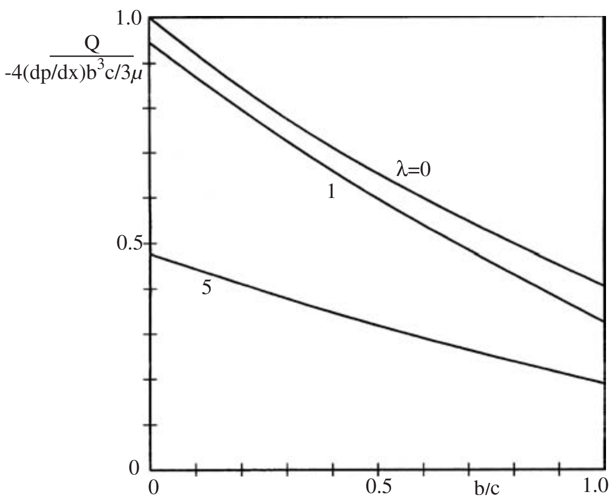


Fig. 7. The variation of the flux with b/c for various values of λ

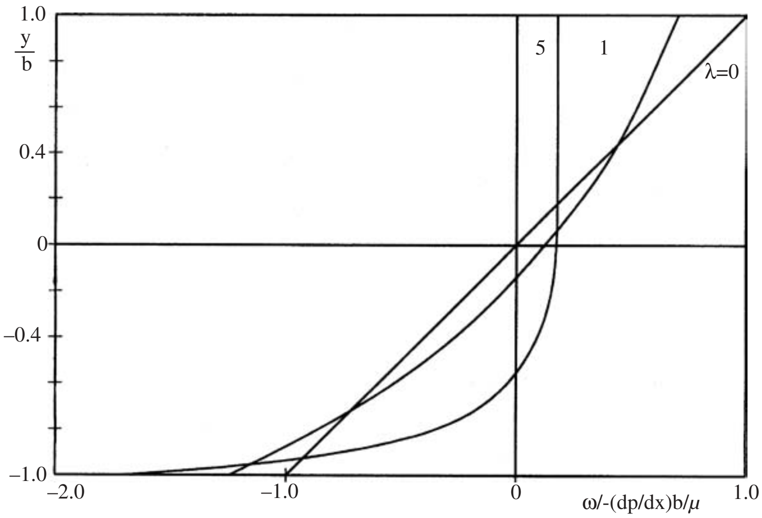


Fig. 8. The variation of the vorticity with y/b for $b/c = 0$

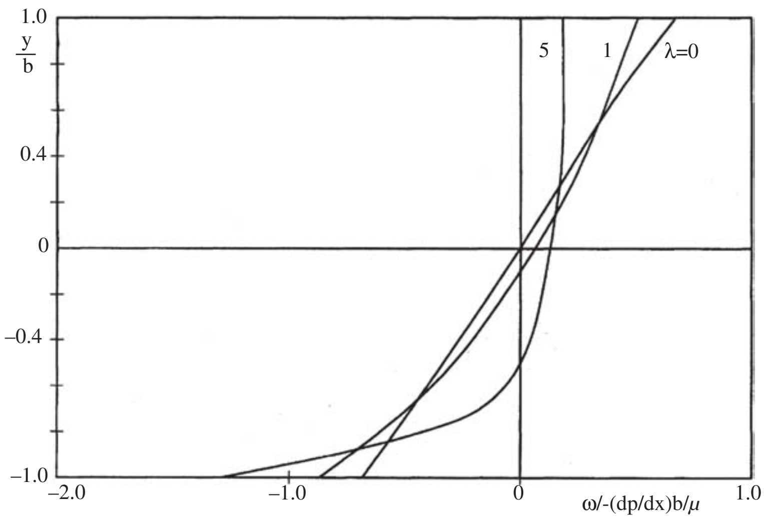


Fig. 9. The variation of the vorticity with y/b for $b/c = 1$

where

$$B_n = \frac{-64(-1)^n(\lambda_n b)\cosh(\lambda/2)}{\left(4\lambda_n^2 b^2 + \lambda^2\right)^2 \cosh(k_n c)}.$$

When λ goes to zero, Eq. (4.4) reduces to

$$\frac{\omega}{-(dp/dx)b/\mu} = \frac{y}{b} - \frac{8}{\pi^2}\sum_0 \frac{(-1)^n}{(2n+1)^2}\frac{\sin\lambda_n y}{\cosh(\lambda_n c)},$$

which is the vorticity for flow in a duct of rectangular cross-section with non-porous walls. When b/c goes to zero, Eq. (4.4) reduces to

$$\frac{\omega}{-(dp/dx)b/\mu} = \frac{1}{\lambda}\left(1 - \frac{\lambda e^{-\lambda\eta}}{\sinh\lambda}\right),$$

which is the vorticity for flow between two parallel porous walls. Figure 8 shows the variation of the z-component of the vorticity at $z = 0$ with y/b for various values of λ for $b/c = 0$, namely, for flow between two parallel porous walls. The flow for non-porous walls corresponds to $\lambda = 0$, where the vorticity profile is a straight line. For large values of λ the variation of the vorticity near the lower wall is sharp and it has a boundary-layer character. In the other regions of the duct the vorticity has a constant value. Figure 9 denotes the variation of the vorticity with y/b for various values of λ for $b/c = 1$, namely, for flow in a duct of square cross-section. It is clearly seen from Figs. 8 and 9 that there is a pronounced effect due to the side walls.

4 Conclusions

In order to show the effects of the side walls on the flows in ducts with suction and injection, three illustrative examples are given. The first example considers the flow over a porous plate with side walls. It is found that the effect of the side walls provides a flow in the cases of both injection and suction. However, it is well known that the flow over a porous plate occurs only in the case of suction. The second example is devoted to the flow between two parallel porous plates with suction at one plate and injection at the other. The third example considers the flow in a rectangular duct with two porous walls. In order to show the effects of the side walls, a comparison of the velocity, the volume flux across a plane normal to the flow and the vorticity with those for flow between two parallel porous plates is made. Three examples considered show that there are pronounced effects of the side walls on the flows in ducts.

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