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The interaction of an edge dislocation with an inclusion of arbitrary shape analyzed by the Eshelby inclusion method

J.-Y. Shi and **Z.-H. Li,** Shanghai, China

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Summary. The interaction of an edge dislocation with an inclusion of arbitrary shape is analyzed based on the Eshelby equivalent inclusion method. A general solution to determine the force on the dislocation is obtained, from which a set of simple approximate formulae is also suggested.

1 Introduction

The study of interaction between dislocation and inclusion has received considerable interest over years because the interaction is evidently important for understanding the mechanical behavior of many materials. The study is traditionally based on the solution of appropriate boundary value problems in the linear theory of elasticity, and only for a few and highly idealized cases there were obtained analytical solutions. Among these are the circular inclusion [1]–[4], the elliptical inclusion [5]–[7], the lamellar inclusion [8], and a surface layer [9]. Furthermore, no generalizations that could be used to construct quantitative physical theories have emerged because of the intricacy of results in the individual situations. Thus, the knowledge of the interaction between dislocation and inclusion comes by slow accumulation of results for special cases, rather than by the establishment of general propositions.

A general solution of the interaction between dislocation and inclusion of arbitrary configuration is very difficult to obtain based on the linear theory of elasticity. The present study investigates the interaction of an edge dislocation with an inclusion of arbitrary shape. The analysis is based on the Eshelby equivalent inclusion theory. A general approximate solution to determine the force on the dislocation is obtained, from which a set of simple approximate formulae is suggested. It is shown that, in comparison with corresponding classical solutions for some special cases, these simple formulas have satisfactory accuracy.

2 Model and formulation

The physical problem to be studied is shown in Fig. 1. A straight edge dislocation whose line coincides with the z-axis of a Cartesian coordinate system is located at the point (0,0,0). An inclusion of arbitrary shape is near the edge dislocation, within the dislocation stress-strain

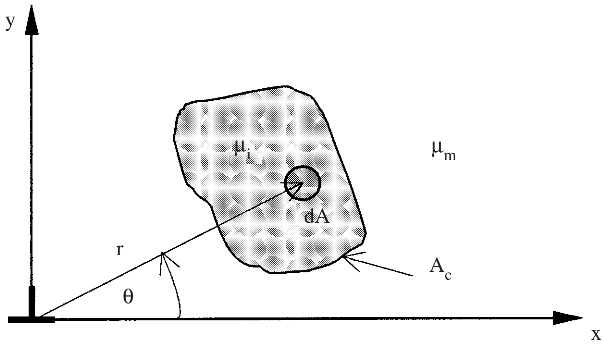

Fig. 1. An edge dislocation near an inclusion of arbitrary shape

field, which is assumed to be unperturbed by the inclusion. The inclusion will undergo an equivalent transformation strain $\mathbf{e^T}$ induced by the dislocation strain due to the inhomogeneity between matrix material and the inclusion. Now consider a differential element dA within the inclusion. According to the Eshelby equivalent inclusion approach, the transformation strain in dA can be expressed by [10], [11]

$$\mathbf{e^T} = [(\mathbf{C_i} - \mathbf{C_m})\mathbf{S} + \mathbf{C_m}]^{-1}(\mathbf{C_m} - \mathbf{C_i})\mathbf{e^A}, \tag{1}$$

where $\mathbf{S}$ is the Eshelby tensor, dependent solely upon the inclusion shape and the Poisson's ratio v of the matrix material. $\mathbf{C_i}$ and $\mathbf{C_m}$ are the elastic tensors of the inclusion and the matrix material, respectively. $\mathbf{e^A}$ is the strain field of the edge dislocation with a Burger's vector b [12]:

$$\mathbf{e^A} = \left\{-\frac{b(1-2v)\sin\theta}{4\pi r(1-v)},\ -\frac{b(1-2v)\sin\theta}{4\pi r(1-v)},\ 0,\ \frac{b\cos\theta}{4\pi r(1-v)},\ 0,\ 0\right\} \tag{2}$$

for plane strain. As shown in (1), the equivalent transformation strain $\mathbf{e^T}$ in the inclusion varies with the dislocation strain $\mathbf{e^A}$, and is not zero for an inhomogeneous inclusion ($\mathbf{C_i} \neq \mathbf{C_m}$).

For simplicity, it is assumed that the inclusion and the matrix material are isotropic and their Poisson's ratios are the same, denoted by v. Then we have

$$\mathbf{C_i} = \alpha\mathbf{C_m}, \tag{3}$$

where

$$\alpha = \frac{\mu_i}{\mu_m}. \tag{4}$$

μ_i and μ_m are the shear moduli of the inclusion and the matrix material, respectively.

Combining (1) and (3), it gives

$$\mathbf{e^T} = \mathbf{L}\mathbf{e^A}, \tag{5}$$

where

$$\mathbf{L} = [(\alpha - 1)\mathbf{S} + \mathbf{I}]^{-1}(1 - \alpha), \tag{6}$$

where $\mathbf{I}$ is the unit tensor. Thus, the tensor $\mathbf{L}$ relates the equivalent transformation strain $\mathbf{e^T}$ in the inclusion to the dislocation strain $\mathbf{e^A}$ without going into the details of the form of $\mathbf{C_i}$ and $\mathbf{C_m}$ tensors.

For a differential element with circular section inside the inclusion, the components of the Eshelby tensor are given by [12]

$$S_{1111} = S_{2222} = \frac{5-4v}{8(1-v)}, \quad S_{1122} = S_{2211} = \frac{4v-1}{8(1-v)}, \tag{7.1}$$

$$S_{1133} = S_{2233} = \frac{\nu}{2(1-\nu)}, \quad S_{1212} = \frac{3-4\nu}{4(1-\nu)}, \tag{7.2}$$

$$S_{1313} = S_{2323} = \frac{1}{2}. \tag{7.3}$$

Other components of the Eshelby tensor are zero. Substituting (7) into (6) yields:

$$L_{1111} = L_{2222} = \frac{(1-\alpha)(1-\nu)(3-4\nu+5\alpha-4\nu\alpha)}{(1+\alpha-2\nu)(1+3\alpha-4\nu\alpha)}, \tag{8.1}$$

$$L_{1122} = L_{2211} = \frac{(1-\alpha)^2(1-\nu)(1-4\nu)}{(1+\alpha-2\nu)(1+3\alpha-4\nu\alpha)}, \tag{8.2}$$

$$L_{1133} = L_{2233} = \frac{(1-\alpha)^2\nu}{(1+\alpha-2\nu)}, \quad L_{3333} = (1-\alpha), \tag{8.3}$$

$$L_{1212} = \frac{4(1-\alpha)(1-\nu)}{(1+3\alpha-4\nu\alpha)}, \quad L_{1313} = L_{2323} = \frac{2(1-\alpha)}{1+\alpha}. \tag{8.4}$$

And other components of the $\mathbf{L}$ tensor are zero. Combining (2), (5) and (8), the transformation strain $\mathbf{e^T}$ in dA is given by

$$\mathbf{e^T} = \frac{b(1-\alpha)}{\pi r}\left\{-\frac{(1-2\nu)\sin\theta}{2(1+\alpha-2\nu)},\ -\frac{(1-2\nu)\sin\theta}{2(1+\alpha-2\nu)},\ 0,\ \frac{\cos\theta}{(1+3\alpha-4\alpha\nu)},\ 0,\ 0\right\} \tag{9}$$

for plane strain.

The elastic interaction energy per unit length in z-direction between the differential element and the dislocation is

$$dU_{int} = \boldsymbol{\sigma}^{\mathbf{A}}\mathbf{e^T}dA,$$

where $\boldsymbol{\sigma}^{\mathbf{A}}$ is the stress field of the edge dislocation

$$\boldsymbol{\sigma}^{\mathbf{A}} = \frac{-\mu_m b}{2\pi r(1-\nu)}\{\sin\theta,\ \sin\theta,\ 2\nu\sin\theta,\ -\cos\theta,\ 0,\ 0\}. \tag{10}$$

Then we have:

$$dU_{int} = \frac{1}{r^2}\left(\frac{1}{2}C_1 + C_2\sin^2\theta\right)dA, \tag{11}$$

where

$$C_1 = \frac{\mu_m b^2(1-\alpha)}{\pi^2(1-\nu)(1+3\alpha-4\alpha\nu)}, \tag{12.1}$$

$$C_2 = \frac{\mu_m b^2\alpha(1-\alpha)(1-4\nu)}{\pi^2(1+\alpha-2\nu)(1+3\alpha-4\alpha\nu)}. \tag{12.2}$$

From (11) we gain the force acting on the dislocation unit length,

$$dF_r = \frac{\partial(dU_{int})}{\partial r} = -\frac{1}{r^3}(C_1 + 2C_2\sin^2\theta)\,dA. \tag{13}$$

The total forces on the dislocation along the x- and y-directions are, respectively, the glide and climb forces:

$$F_{glide} = \int\limits_{A_c} dF_x = -C_1 \int\limits_{A_c} \frac{\cos\theta}{r^3}\, dA - C_2 \int\limits_{A_c} \frac{\sin 2\theta \sin\theta}{r^3}\, dA, \tag{14}$$

$$F_{climb} = \int\limits_{A_c} dF_y = -C_1 \int\limits_{A_c} \frac{\sin\theta}{r^3}\, dA - 2C_2 \int\limits_{A_c} \frac{\sin^3\theta}{r^3}\, dA. \tag{15}$$

The integration is carried out over the whole domain A_c occupied by the inclusion. A positive value of F is corresponding to repulsion.

3 Some special cases

Some simply approximate formulas can be obtained for several special inclusion shapes:

(a) For a small circular inclusion of radius R centered at $(r, 0)$ on the x-axis, the maximum glide force on the inclusion is approximated from (14) by

$$F_{glide} = -\frac{\mu_m b^2 (1-\alpha)}{\pi R (1-v)(1+3\alpha-4\alpha v)} \frac{1}{\beta^3}, \tag{16}$$

where $\beta = r/R$. Equation (16) is just the approximation of the classical solution [1]:

$$F = -\frac{\mu_m b^2}{\pi R(\kappa+1)} \frac{1}{\beta^3} \left(\frac{B+A}{1-1/\beta^2} + 3A - B \right) \tag{17}$$

for $\beta \gg 1$. Here,

$$A = \frac{1-\alpha}{1+\alpha\kappa}, \quad B = \frac{(1-\alpha)\kappa}{\kappa+\alpha}, \quad \kappa = 3-4v.$$

When the inclusion is on the y-axis, we can obtain the maximum climb force from (15),

$$F_{climb} = -(C_1 + 2C_2)\frac{\pi}{R\beta^3}. \tag{18}$$

(b) A lamellar inclusion of length $2l$ and width $w(1 \gg w)$ perpendicular to the x axis located with its center at $(r_0, 0)$ as shown in Fig. 2a. By use of $dA = wrd\,\theta/\cos\theta$ in (15) and (14), we have $F_{climb} = 0$, and

$$\begin{aligned} F_{glide} &= \int\limits_{A_c} dF_x = -2\frac{w}{r_0^2}\left[C_1 \int\limits_0^{\theta_0} \cos^2\theta\, d\theta + 2C_2 \int\limits_0^{\theta_0} \sin^2\theta \cos^2\theta\, d\theta \right] \\ &= -\frac{w}{r_0^2}\left[C_1\left(\theta_0 + \frac{1}{2}\sin 2\theta_0\right) + \frac{1}{2}C_2\left(\theta_0 - \frac{1}{4}\sin 4\theta_0\right)\right], \end{aligned} \tag{19}$$

where $\theta_0 = \arctan l/r_0$. When $l/r_0 \gg 1$, one obtains

$$F_{glide} = -\frac{w\pi}{2r_0^2}\left(C_1 + \frac{1}{2}C_2\right). \tag{20}$$

(c) A lamellar inclusion of length l and width $w(l \gg w)$ lies on the x-axis, Fig. 2b. The distance of the near dislocation end of the inclusion to the dislocation is r_0. By use of $dA = w\,dr$ and $\sin\theta = 0$, $\cos\theta = 1$ in (15) and (14), we have $F_{climb} = 0$, and

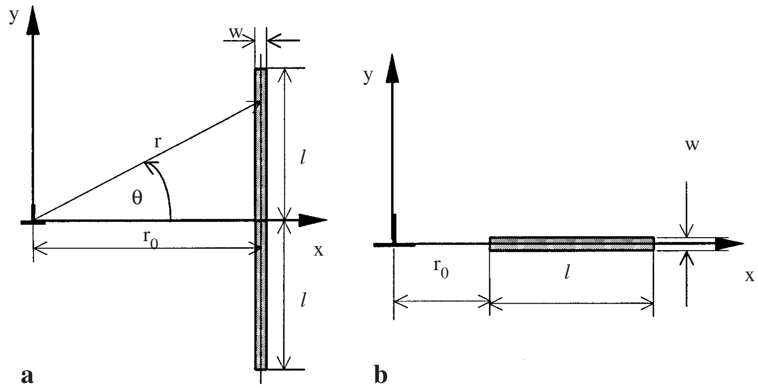

Fig. 2. The interaction of a lamellar inclusion with an edge dislocation; **a** the distance of the dislocation to the center of the inclusion is r_0, **b** the distance of the dislocation to the end of the inclusion is r_0

$$F_{glide} = -C_1 w \int_{r_0}^{r_0+l} \frac{1}{r^3} dr = -\frac{1}{2} C_1 w \left(\frac{1}{r_0^2} - \frac{1}{(r_0+l)^2} \right); \tag{21}$$

when $l/r_0 \gg 1$, it gives

$$F_{glide} = -\frac{1}{2r_0^2} C_1 w. \tag{22}$$

4 Discussion and conclusions

The Eshelby approach is mathematically rigorous for an infinite matrix containing a single ellipsoidal inclusion. When the inclusion undergoes a uniform stress-free transformation strain, the stress and strain within the inclusion are uniform. However, in order to utilize the approach in more realistic situations, there has been considerable activity in extending the Eshelby approach to a variety of practical problems, such as the interaction of two ellipsoidal inclusions [13], the behavior of a hybrid composite [14] and short fibre reinforced composites [11], and the calculation of the stress fields inside a non-ellipsoidal inclusion which are not uniform [15], to cite only a few examples. In the present study, we extend the Eshelby approach to the case of an inclusion with arbitrary shape within the dislocation stress-strain field. Either the non-ellipsoidal shape of the inclusion considered or the nonuniform dislocation stress-strain field will result in a nonuniform stress-strain field within the inclusion. However, we assume that the Eshelby theory can be used to each differential element within the inclusion, which undergoes uniform transformation strain determined by Eq. (1).

Although the fundamental equation (13) is derived based on an approximate application of the Eshelby equivalent inclusion theory, the resultant solution for a circular inclusion (16) is very close to the classical one.

To assess the possible error range of the approximate approach used in the above analysis, we consider an extreme case: the finite inclusion shown in Fig. 1 degenerates into a semi-infinite plane. From (14) we have the force acting on the dislocation:

$$F^{present} = -\frac{\pi}{4r_0}(2C_1 + C_2) = -\frac{\mu_m b^2 (1-\alpha)(2+3\alpha-4\nu-5\alpha\nu+4\alpha\nu^2)}{4\pi r_0 (1-\nu)(1+3\alpha-4\nu\alpha)(1+\alpha-2\nu)}, \tag{23}$$

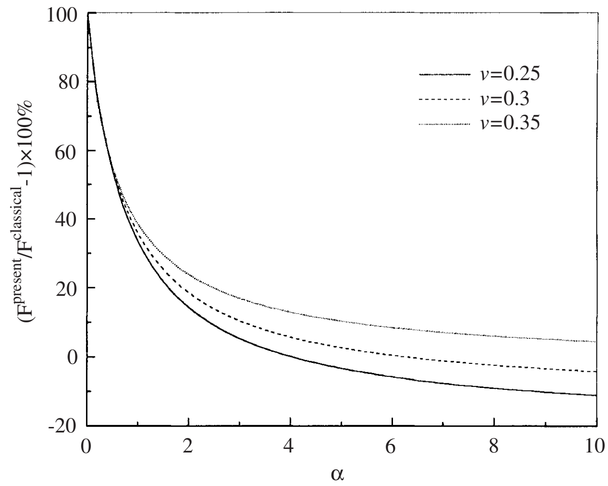

Fig. 3. The relative error of the present solution to the classical one for the case that an inclusion degenerates into a semi-infinite plane with different moduli ratios

where r_0 is the distance of the dislocation to the semi-infinite plane. The classical solution for the problem is given by [9]:

$$F^{classical} = -\frac{\mu_m b^2(1-\alpha)(3+5\alpha-4\nu-12\alpha\nu+8\alpha\nu^2)}{4\pi r_0(1-\nu)(1+3\alpha-4\nu\alpha)(3+\alpha-4\nu)}. \tag{24}$$

Figure 3 displays the relative error of the present solution to the classical one as a function of the moduli ratio α for $\nu = 0.25$, 0.3 and 0.35. The maximum error is 100% when the inclusion becomes a semi-infinite space, which may be viewed as the limit of the error for a finite inclusion. For a small finite inclusion the fundamental equation will give a much more accurate result.

The special advantage of the present solution is that the fundamental equation, in integral form, can be conventionally used to treat the interaction of an edge dislocation with an inclusion of arbitrary shape, which is difficult, even impossible to treat by the rigorous solution of appropriate boundary value problems in the linear theory of elasticity.

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Authors' address: J.-Y. Shi and Z.-H. Li, School of Civil Engineering and Mechanics, Shanghai Jiaotong University, 200240, Shanghai Minhang, P.R. China (E-mail: zhli@sjtu.edu.cn)