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On spatial and material settings of thermo-hyperelastodynamics for open systems

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Summary. The present treatise aims at deriving a general framework for the thermodynamics of open systems typically encountered in chemo- or biomechanical applications. Due to the fact that an open system is allowed to constantly gain or lose mass, the classical conservation law of mass has to be recast into a balance equation balancing the rate of change of the current mass with a possible in- or outflux of matter and an additional volume source term. The influence of the generalized mass balance on the balance of momentum, kinetic energy, total energy and entropy is highlighted. Thereby, special focus is dedicated to the strict distinction between a volume specific and a mass specific format of the balance equations which is of no particular relevance in classical thermodynamics of closed systems. The change in density furnishes a typical example of a local rearrangement of material inhomogeneities which can be characterized most elegantly in the material setting. The set of balance equations for open systems will thus be derived for both, the spatial and the material motion problem. Thereby, we focus on the one hand on highlighting the remarkable duality between both approaches. On the other hand, special emphasis is placed on deriving appropriate relations between the spatial and the material motion quantities.

The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful.

Aristotle, Metaphysica

1 Introduction

It is a well-established fact, that in classical non-relativistic continuum mechanics, each part of a body can be assigned a specific mass which never changes no matter how the body is moved, accelerated or deformed. Although valid for most practical applications, the statement of the "conservation of mass" is nothing but a mere definition. Yet, there exist particular problem classes for which the conservation of mass is no longer appropriate. Typical examples can be found in chemomechanical or biomechanical applications. In both cases, the apparent changes in mass result from confining attention to only a part of the overall matter present. Thus one might argue, that these problems can be overcome naturally by using the "theory of mixtures", as proposed by Truesdell and Toupin [50] §155 or Bowen [3]. Therein, the loss or gain of mass of one constituent is compensated by the others while the mass of the overall mixture itself remains constant.

Nevertheless, it is possible to think of classes of problems for which it might seem more reasonable to restrict focus to one single constituent which is allowed to exchange mass, momentum, energy and entropy with its environment, i.e. the "outside world". This approach typically falls within the category of "thermodynamics of open systems". Following the line of thought introduced by Maugin [36] §2.1, such systems can be understood as being enclosed by a permeable, deformable and diathermal membrane. A classical example of an open system that can be found in nearly every textbook of mechanics is furnished by the motion of a burning body typically encountered in rocket propulsion, see, e.g. Truesdell and Toupin [50] §155, Müller [39] §1.4.6 or Haupt [20] §3.5. The behavior of open-pored hard tissues under quasi-static loading is another example found in biomechanics, see, e.g. Cowin and Hegedus [6], Carter and Beaupré [4] or Krstin, Nackenhorst and Lammering [26]. In soft tissue mechanics, proliferation, hyperplasia, hypertrophy and atrophy can be considered as typical examples of mass sources on the microlevel, while migration or cell movement might cause an additional mass flux, as illustrated in the classical overview monographs by Taber [49], Humphrey [22], and Humphrey and Rajagopal [23].

The first continuum model for open systems in the context of biomechanics has been presented by Cowin and Hegedus [6], [21] under the name of "theory of adaptive elasticity". Nowadays, most of the biomechanical models and the related numerical simulations are based on this theory for which the set of common balance equations has been enhanced basically by additional volume sources, see, e.g. Beaupré, Orr and Carter [1], Weinans, Huiskes and Grootenboer [51], Harrigan and Hamilton [18], [19]. Only recently, Epstein and Maugin [10] have proposed the "theory of volumetric growth" for which the exchange with the environment is not a priori restricted to source terms by allowing for additional fluxes of mass, momentum, energy and entropy through the domain boundary, see also Kuhl and Steinmann [28]. Its numerical realization within the context of the finite element method has been illustrated recently by Kuhl and Steinmann [29].

Typically, the process or growth encountered in open systems will be accompanied by the development of inhomogeneities responsible for residual stresses in the body. The interpretation of growth as "local rearrangement of material inhomogeneities" suggests the formulation of the governing equations in the material setting as proposed by Epstein and Maugin [10]. The appealing advantage of the material motion point of view is that local inhomogeneities such as abrupt changes in density are reflected elegantly by the governing equations which result from a complete projection of the standard balance equations onto the so-called material manifold. The material motion point of view originally dates back to the early works of Eshelby [11] on defect mechanics. It was elaborated in detail by Chadwick [5], Eshelby [12], and Rogula [40] and has attracted an increasing attention only recently as documented by the trendsetting textbooks by Maugin [34], Gurtin [17], Kienzler and Herrmann [25] and also by Silhavý [43] or by the recent publications by Epstein and Maugin [9], Maugin and Trimarco [37], Maugin [35], Gurtin [16]. Thereby, the remarkable duality between the spatial or "direct motion problem" and the material or "inverse motion problem" as pointed out originally by Shield [42] is of particular importance. Our own attempts along these lines are documented in [30], [44], [45], [46], [47] and [48] to which we refer for further motivation of the material motion point of view in the context of fracture and defect mechanics.

This presentation aims at presenting a general framework for the thermodynamics of open systems highlighting the striking duality between the spatial and the material motion approach. Thereby, we shall consider the most general formulation by allowing for mass exchanges not only through the supply of mass within the domain itself but also through the in- or outflux of mass through the domain boundary. The influence of a non-constant mass on all the other balance equations will be discussed for both, the spatial and the material motion framework. Thereby, particular emphasis is dedicated to the strict distinction between the "volume specific" and the "mass specific" format. Unlike in classical mechanics of closed systems, the balance equations for the mechanics of open systems will be shown to differ considerably in the volume specific and the mass specific context. In contrast to former models and in the autors' opinion as a benefit, the mass specific format introduced herein is free from explicit open system contributions, thus taking the standard format typically encountered in classical thermodynamics.

To illustrate the nature of the mechanics of open systems, we begin by reviewing the classical example of rocket propulsion in Chap. 2. After briefly summarizing the relevant kinematics of continuum mechanics in Chap. 3, we introduce the balance of mass for open systems in Chap. 4. The notions of "volume specific" and "mass specific" format will be defined in Chap. 5 for a generic prototype balance law. Having introduced the balance of momentum in the spatial and the material motion context in Chap. 6, we can derive the balance of kinetic energy as a useful byproduct in Chap. 7. The balance of energy and entropy will be highlighted in Chaps. 8 and 9 whereby the latter naturally lends itself to the formulation of the dissipation inequality which is shown to place further restrictions on the constitutive response functions. The derivation of appropriate constitutive equations is briefly sketched for the classical model problem of thermo-hyperelasticity in Chap. 10.

Throughout the entire derivation, we apply a two-step strategy. First, the well-known balance equations of the classical spatial motion problem are discussed. Next, guided by arguments of duality, and beauty in the sense of our leitmotif, we shall formally introduce the material motion balance equations in complete analogy to the corresponding spatial motion versions. In a second step, appropriate transformations between both settings are set up helping to identify the introduced material motion quantities in terms of their spatial motion counterparts.

2 Motivation

To illustrate the nature of open systems and the corresponding mechanics, we consider the classical example of the loss of mass through combustion and ejection during rocket propulsion. Thereby, the rocket head, the subsystem of the rocket hull plus the amount of fuel present, can be understood as an open system constantly losing mass due to the process of combustion and ejection. Consequently, the balance of mass of the rocket head balances the time rate of change of the rocket head mass $\mathfrak{m}$ with the rate of mass ejection $\mathfrak{R}$.

$$\mathrm{D}_t \mathfrak{m} = \mathfrak{R}. \tag{2.1}$$

The case of combustion and ejection is characterized through negative growth $\mathfrak{R} \leq 0$ since the mass of the rocket head decreases with time. The related balance of momentum states that the time rate of change of the rocket head momentum $\mathfrak{p}$ is equal to the total force $\mathfrak{f}$ acting on it

$$\mathrm{D}_t \mathfrak{p} = \mathfrak{f} \quad \text{with} \quad \mathfrak{f} = \mathfrak{f}^{\text{closed}} + \mathfrak{f}^{\text{open}} \quad \text{and} \quad \mathfrak{f}^{\text{open}} = \bar{\mathfrak{f}}^{\text{open}} + \mathfrak{v}\mathfrak{R}. \tag{2.2}$$

Thereby, the total force can be interpreted as the sum of the closed system contribution $\mathfrak{f}^{\text{closed}}$ and the open system contribution $\mathfrak{f}^{\text{open}}$. The latter can be understood as the sum of a reduced open system term $\bar{\mathfrak{f}}^{\text{open}}$ and explicit effects due to the added or in this case removed amount of mass $\mathfrak{v}\mathfrak{R}$. Note, that this version of the balance of momentum will be referred to as "volume specific" version in the sequel. The momentum $\mathfrak{p}$ of the rocket head is defined as the rocket head velocity $\mathfrak{v}$ weighted by its actual mass $\mathfrak{m}$. Consequently, the material time derivative of the momentum $\mathfrak{p}$ can be evaluated with the help of the chain rule with

$$\mathfrak{p} = \mathfrak{m}\mathfrak{v} \quad \text{thus} \quad \mathrm{D}_t\mathfrak{p} = \mathfrak{m}\mathrm{D}_t\mathfrak{v} + \mathfrak{v}\mathrm{D}_t\mathfrak{m} = \mathfrak{m}\mathrm{D}_t\mathfrak{v} + \mathfrak{v}\mathfrak{R}. \tag{2.3}$$

A reduced form of the balance of momentum follows from subtracting the balance of mass (2.1) weighted by the rocket head velocity $\mathfrak{v}$ from the volume specific balance of momentum (2.2).

$$\mathfrak{m}\mathrm{D}_t\mathfrak{v} = \bar{\mathfrak{f}} \quad \text{with} \quad \bar{\mathfrak{f}} = \mathfrak{f}^{\text{closed}} + \bar{\mathfrak{f}}^{\text{open}} = \mathfrak{f} - \mathfrak{v}\mathfrak{R}. \tag{2.4}$$

The above equation, which we will refer to as "mass specific" version of the balance of momentum, defines the reduced force $\bar{\mathfrak{f}}$, the overall force responsible for changes in the rocket velocity, as the sum of the closed system contributions, i.e. the mechanical forces $\mathfrak{f}^{\text{closed}}$, and a reduced open system forces term $\bar{\mathfrak{f}}^{\text{open}}$, the so-called propulsive force. However, of course, the standard balance equations hold for the overall closed system composed of the rocket head and the exhausted mass. The balance of momentum of this overall system requires that the sum of the rate of change of rocket head momentum $\mathrm{D}_t\mathfrak{p}$ minus the rate of change of the momentum of the ejected mass $\bar{\mathfrak{v}}\mathfrak{R}$ be in equilibrium with closed system force term $\mathfrak{f}^{\text{closed}}$,

$$\mathrm{D}_t\mathfrak{p} - \bar{\mathfrak{v}}\mathfrak{R} = \mathfrak{f}^{\text{closed}}, \tag{2.5}$$

whereby $\bar{\mathfrak{v}}$ denotes the total velocity of the ejected mass, see Goldstein [13] §1.6. From the above equations, the reduced open system force term $\bar{\mathfrak{f}}^{\text{open}}$, which is responsible for the rocket thrust can be identified as the force caused by the difference of the velocity of the ejection $\bar{\mathfrak{v}}$ with respect to the rocket head velocity $\mathfrak{v}$.

$$\mathfrak{f}^{\text{open}} = \bar{\mathfrak{v}}\mathfrak{R} \quad \text{thus} \quad \bar{\mathfrak{f}}^{\text{open}} = [\bar{\mathfrak{v}} - \mathfrak{v}]\mathfrak{R}. \tag{2.6}$$

In Ref. [10], the propulsive term $[\bar{\mathfrak{v}} - \mathfrak{v}]\mathfrak{R}$ is referred to as "irreversible" contribution while the extra force $\mathfrak{v}\mathfrak{R}$ generated by the ejection leaving the system at the same velocity as the remaining rocket head is then denoted as "reversible" contribution. In what follows, we will generalize the above considerations to the continuum mechanics of open systems.

3 Kinematics

To clarify the following discussions, we shall strictly distinguish between the terminology of parametrization, reference, description and motion, as suggested by Steinmann [45], [47]. Thereby, any quantity expressed in terms of the spatial coordinate $\mathbf{x}$ as $\{\bullet\}(\mathbf{x},t)$ will be referred to as spatial parametrization of $\{\bullet\}$, while the material parametrization $\{\bullet\}(\mathbf{X},t)$ is formulated in terms of the material coordinate $\mathbf{X}$. Irrespective of the parametrization, we will distinguish between the spatial and material reference of a scalar- or tensor-valued quantity denoted as $\{\bullet\}_t$ or $\{\bullet\}_0$, respectively. Thereby, the former relates to the spatial domain $\mathscr{B}_t$ while the latter represents a quantity in the material domain $\mathscr{B}_0$. Moreover, for tensor-valued quantities, we shall distinguish between the spatial, the material and the two-point description. While tensorial quantities in the spatial description are elements of the tangent or cotangent space to $\mathscr{B}_t$, tensorial quantities in the material description are elements of the tangent or cotangent space to $\mathscr{B}_0$. Tensorial quantities in the two-point description are elements of the tangent or cotangent spaces to $\mathscr{B}_t$ and $\mathscr{B}_0$. Finally, we will discuss all balance equations (except for the balance of mass) in the spatial and the material motion context. Thereby, the classical spatial motion problem, which is sometimes introduced as "direct motion problem" is based on the idea of following "physical particles" from a fixed material position $\mathbf{X}$ through the ambient space. In contrast to this, within the material motion or "inverse motion problem", "physical particles" are followed through the ambient material at fixed spatial position $\mathbf{x}$.

3.1 Spatial motion problem

The spatial motion problem is characterized through the spatial motion map

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t) : \quad \mathscr{B}_0 \to \mathscr{B}_t \tag{3.1}$$

mapping the material placement $\mathbf{X}$ of a "physical particle" in the material configuration $\mathscr{B}_0$, to the spatial placement $\mathbf{x}$ of the same "physical particle" in the spatial configuration $\mathscr{B}_t$, see Fig. 1. The related spatial deformation gradient $\mathbf{F}$ and its Jacobian J

$$\mathbf{F} = \nabla_X \boldsymbol{\varphi}(\mathbf{X}, t) : \quad T\mathscr{B}_0 \to T\mathscr{B}_t \quad J = \det \mathbf{F} > 0 \tag{3.2}$$

define the linear tangent map from the fixed material tangent space $T\mathscr{B}_0$ to the time-dependent tangent space $T\mathscr{B}_t$. The right spatial motion Cauchy–Green strain tensor $\mathbf{C}$,

$$\mathbf{C} = \mathbf{F}^t \cdot \mathbf{g} \cdot \mathbf{F}, \tag{3.3}$$

i.e. the spatial motion pull back of the covariant spatial metric $\mathbf{g}$, can be introduced as a typical strain measure of the material motion problem. Moreover, with the material time derivative D_t of an arbitrary quantity $\{\bullet\}$ at fixed material placement $\mathbf{X}$

$$\mathrm{D}_t\{\bullet\} = \partial_t\{\bullet\}|_X, \tag{3.4}$$

the spatial velocity $\mathbf{v}$ is introduced as the material time derivative of the spatial motion map $\boldsymbol{\varphi}$,

$$\mathbf{v} = \mathrm{D}_t \boldsymbol{\varphi}(\mathbf{X}, t). \tag{3.5}$$

Its material gradient is equal to the material time derivative of the spatial deformation gradient $\mathbf{F}$ while its spatial gradient will be denoted as $\mathbf{l}$ in the sequel

$$\mathrm{D}_t \mathbf{F} = \nabla_X \mathbf{v} \quad \mathbf{l} = \nabla_x \mathbf{v}. \tag{3.6}$$

With these definitions at hand, e.g., the material time derivative of the spatial motion Jacobian J can be expressed through the well-known Euler identity $\mathrm{D}_t J = J \operatorname{div} \mathbf{v}$ with $\operatorname{div} \mathbf{v} = \mathbf{F}^{-t} : \mathrm{D}_t \mathbf{F}$ denoting the spatial divergence of the spatial velocity $\mathbf{v}$.

3.2 Material motion problem

Accordingly, the material motion map $\boldsymbol{\Phi}$ with

$$\mathbf{X} = \boldsymbol{\Phi}(\mathbf{x}, t) : \quad \mathscr{B}_t \to \mathscr{B}_0 \tag{3.7}$$

defines the mapping of the spatial placement of a "physical particle" $\mathbf{x}$ in the spatial configuration $\mathscr{B}_t$ to the material placement of the same "physical particle" in the material configuration $\mathscr{B}_0$, see Fig. 2. The related linear tangent map from the fixed spatial tangent space $T\mathscr{B}_t$ to

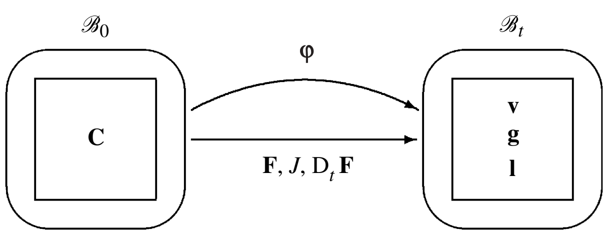

Fig. 1. Spatial motion problem: Kinematics

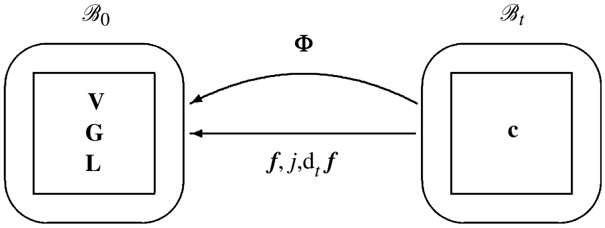

Fig. 2. Material motion problem: Kinematics

the time-dependent tangent space $T\mathscr{B}_0$ is defined through the material deformation gradient $\mathbf{f}$ and its Jacobian j.

$$\mathbf{f} = \nabla_x \boldsymbol{\Phi}(\mathbf{x}, t) \; : \quad T\mathscr{B}_t \to T\mathscr{B}_0 \quad j = \det \mathbf{f} > 0. \tag{3.8}$$

The material motion pull back of the covariant material metric $\mathbf{G}$ introduces the right material motion Cauchy–Green strain tensor $\mathbf{c}$.

$$\mathbf{c} = \mathbf{f}^t \cdot \mathbf{G} \cdot \mathbf{f}. \tag{3.9}$$

With the definition of the spatial time derivative d_t of a quantity $\{\bullet\}$ at fixed spatial placements $\mathbf{x}$

$$\mathrm{d}_t\{\bullet\} = \partial_t\{\bullet\}|_x \tag{3.10}$$

the material velocity $\mathbf{V}$ can be defined as spatial time derivative of the material motion map $\boldsymbol{\Phi}$,

$$\mathbf{V} = \mathrm{d}_t \boldsymbol{\Phi}(\mathbf{x}, t). \tag{3.11}$$

Its spatial gradient is equal to the spatial time derivative of the material motion deformation gradient $\mathbf{f}$ while the material gradient of the material velocity will be denoted as $\mathbf{L}$

$$\mathrm{d}_t \mathbf{f} = \nabla_x \mathbf{V} \quad \mathbf{L} = \nabla_X \mathbf{V}. \tag{3.12.1,2}$$

Consequently, the spatial time derivative of the material motion Jacobian j can be expressed as $\mathrm{d}_t j = j \,\mathrm{Div}\, \mathbf{V}$, whereby $\mathrm{div}\, \mathbf{V} = \mathbf{f}^{-t} : \mathrm{d}_t \mathbf{f}$ denotes the material divergence of the material velocity $\mathbf{V}$.

3.3 Spatial vs. material motion problem

The spatial and the material motion problem are related through the identity maps in $\mathscr{B}_0$ and $\mathscr{B}_t$

$$\mathbf{id}_{\mathscr{B}_0} = \boldsymbol{\Phi} \circ \boldsymbol{\varphi}(\mathbf{X}, t) = \boldsymbol{\Phi}(\boldsymbol{\varphi}(\mathbf{X}, t), t) \quad \mathbf{id}_{\mathscr{B}_t} = \boldsymbol{\varphi} \circ \boldsymbol{\Phi}(\mathbf{x}, t) = \boldsymbol{\varphi}(\boldsymbol{\Phi}(\mathbf{x}, t), t) \tag{3.13.1,2}$$

with $\circ$ denoting the composition of mappings. Consequently, the spatial and the material deformation gradient are simply related by their inverses

$$\mathbf{F}^{-1} = \mathbf{f} \circ \boldsymbol{\varphi}(\mathbf{X}, t) = \mathbf{f}(\boldsymbol{\varphi}(\mathbf{X}, t), t) \quad \mathbf{f}^{-1} = \mathbf{F} \circ \boldsymbol{\Phi}(\mathbf{x}, t) = \mathbf{F}(\boldsymbol{\Phi}(\mathbf{x}, t), t). \tag{3.14.1,2}$$

Note, that the total differentials of the spatial and material identity map[1] yield the following fundamental relations between spatial and material velocities, see Maugin [34].

$$\mathbf{V} = -\mathbf{f} \cdot \mathbf{v} \quad \mathbf{v} = -\mathbf{F} \cdot \mathbf{V}. \tag{3.15.1,2}$$

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[1]
$$\begin{aligned} \mathrm{d}\mathbf{X} &= \mathrm{d}_t\boldsymbol{\Phi}\,\mathrm{d}t + \nabla_x\boldsymbol{\Phi}\cdot\mathrm{d}\mathbf{x} & &= \mathrm{D}_t\boldsymbol{\varphi}\,\mathrm{d}t + \nabla_X\boldsymbol{\varphi}\cdot\mathrm{d}\mathbf{X} \\ &= \mathrm{d}_t\boldsymbol{\Phi}\,\mathrm{d}t + \nabla_x\boldsymbol{\Phi}\cdot[\mathrm{D}_t\boldsymbol{\varphi}\,\mathrm{d}t + \nabla_X\boldsymbol{\varphi}\cdot\mathrm{d}\mathbf{X}] & &= \mathrm{D}_t\boldsymbol{\varphi}\,\mathrm{d}t + \nabla_X\boldsymbol{\varphi}\cdot[\mathrm{d}_t\boldsymbol{\Phi}\,\mathrm{d}t + \nabla_x\boldsymbol{\Phi}\cdot\mathrm{d}\mathbf{x}] \\ &= \mathbf{V}\,\mathrm{d}t + \mathbf{f}\cdot[\mathbf{v}\,\mathrm{d}t + \mathbf{F}\cdot\mathrm{d}\mathbf{X}] & &= \mathbf{v}\,\mathrm{d}t + \mathbf{F}\cdot[\mathbf{V}\,\mathrm{d}t + \mathbf{f}\cdot\mathrm{d}\mathbf{x}] \end{aligned}$$

In the following, we will distinguish between scalar-valued or tensorial quantities with material reference $\{\bullet\}_0$ and spatial reference $\{\bullet\}_t$ whereby the integration of a quantity in material reference over the material domain $\mathscr{B}_0$ yields the identical result as the integration of a quantity in spatial reference over the spatial domain $\mathscr{B}_t$

$$\int_{\mathscr{B}_0} \{\bullet\}_0 \,\mathrm{d}V = \int_{\mathscr{B}_t} \{\bullet\}_t \,\mathrm{d}v. \tag{3.16}$$

The above equation introduces the well-known transformation formulae

$$\{\bullet\}_0 = J\{\bullet\}_t \quad \{\bullet\}_t = j\{\bullet\}_0. \tag{3.17.1,2}$$

Moreover, the equivalence of a vector- or tensor-valued surface contribution $\{\square\}$ on the boundary of the material domain $\partial\mathscr{B}_0$ and the corresponding contribution $\{\diamondsuit\}$ on the spatial boundary $\partial\mathscr{B}_t$ as

$$\int_{\partial\mathscr{B}_0} \{\square\} \cdot \mathrm{d}\mathbf{A} = \int_{\partial\mathscr{B}_t} \{\diamondsuit\} \cdot \mathrm{d}\mathbf{a} \tag{3.18}$$

can be transformed into the following relation

$$\int_{\mathscr{B}_0} \mathrm{Div}\{\square\} \,\mathrm{d}V = \int_{\mathscr{B}_t} \mathrm{div}\{\diamondsuit\} \,\mathrm{d}v \tag{3.19}$$

through the application of Gauss' theorem. Clearly, the material and spatial flux terms $\{\square\}$ and $\{\diamondsuit\}$ are related through the well-known Nanson's formula.

$$\{\square\} = J\{\diamondsuit\} \cdot \mathbf{F}^{-t} \quad \{\diamondsuit\} = j\{\square\} \cdot \mathbf{f}^{-t}. \tag{3.20.1,2}$$

In the dynamic context, the subscripts $\delta = D, d$ will be assigned to the material and spatial flux terms as $\{\square\}_\delta$ and $\{\diamondsuit\}_\delta$ indicating that the corresponding flux refers either to the material or to the spatial time derivative D_t or d_t of the related balanced quantity. The notion of the material time derivative D_t of a scalar- or vector-valued function $\{\bullet\}$ as introduced by Euler relates the material and the spatial time derivative D_t and d_t through the individual convective terms $\nabla_x\{\bullet\} \cdot \mathbf{v}$ and $\nabla_X\{\bullet\} \cdot \mathbf{V}$,

$$\mathrm{D}_t\{\bullet\} = \mathrm{d}_t\{\bullet\} + \nabla_x\{\bullet\} \cdot \mathbf{v} \quad \mathrm{d}_t\{\bullet\} = \mathrm{D}_t\{\bullet\} + \nabla_X\{\bullet\} \cdot \mathbf{V}. \tag{3.21.1,2}$$

From the above equations, we obtain the differential form of the spatial and material motion version of Reynold's transport theorem.

$$j\,\mathrm{D}_t\{\bullet\}_0 = \mathrm{d}_t\{\bullet\}_t + \mathrm{div}\big(\{\bullet\}_t \otimes \mathbf{v}\big) \quad J\,\mathrm{d}_t\{\bullet\}_t = \mathrm{D}_t\{\bullet\}_0 + \mathrm{Div}\big(\{\bullet\}_0 \otimes \mathbf{V}\big). \tag{3.22.1,2}$$

The global form of the spatial motion version of Reynold's transport theorem (3.22.1), which originally goes back to Kelvin in 1869, states that the rate of change of the quantity $\{\bullet\}_0$ over a material volume $\mathscr{B}_0$ equals the rate of change of the quantity over a spatial volume $\mathscr{B}_t$ being the instantaneous configuration of $\mathscr{B}_0$ plus the flux through the boundary surface $\partial\mathscr{B}_t$.

4 Balance of mass

While in classical mechanics of closed systems, the amount of matter contained in a body $\mathscr{B}_0$ generally does not change, the mass of a body can no longer be considered a conservation property within the thermodynamics of open systems. Accordingly, the balance of mass plays a key role within the present theory. It can be used to transform the volume specific version of any other balance law into its mass specific counterpart. In abstract terms, the local balance of mass states that the appropriate rate of change of the density ρ_τ with $\tau = 0, t$ is equal to the sum of the divergence of the related mass flux $\mathbf{M}_\delta$ or $\mathbf{m}_\delta$ with $\delta = D, d$ and the mass source $\mathscr{M}_\tau$, see Fig. 3. It should be emphasized that most theories for open systems except for the one developed by Epstein and Maugin [10] a priori exclude the influx of mass as $\mathbf{M}_D = \mathbf{0}$ and $\mathbf{m}_D = \mathbf{0}$. Although according to the "equivalence of surface and volume sources" as stated by Truesdell and Toupin [50] §157, it is in principle possible to express any influx $\mathbf{M}_\delta$ or $\mathbf{m}_\delta$ through an equivalent source term of the form $\mathrm{Div}\,\mathbf{M}_\delta$ or $\mathrm{div}\,\mathbf{m}_\delta$, we shall allow for independent flux terms to keep the underlying theory as general as possible for the time being. Thus, the balance of mass with material reference and material parametrization takes the following form

$$\mathrm{D}_t\,\rho_0 = \mathrm{Div}\,\mathbf{M}_D + \mathscr{M}_0. \tag{4.1}$$

With the standard Piola transforms

$$\rho_0 = J\rho_t \quad \mathscr{M}_0 = J\,\mathscr{M}_t \quad \mathbf{M}_D = J\,\mathbf{m}_D \cdot \mathbf{F}^{-t} \tag{4.2.1,2,3}$$

the push forward of the different terms in Eq. (4.1) yields the balance of mass with spatial reference and material parametrization,

$$j\,\mathrm{D}_t\,\rho_0 = \mathrm{div}\,\mathbf{m}_D + \mathscr{M}_t. \tag{4.3}$$

Note, that in classical continuum mechanics, Eq. (4.2.1) is usually referred to as "equation of material continuity". The application of Reynold's transport theorem (3.22.1)

$$j\,\mathrm{D}_t\,\rho_0 = \mathrm{d}_t\,\rho_t + \mathrm{div}(\rho_t\mathbf{v}) \quad \text{and} \quad \mathbf{m}_d = \mathbf{m}_D - \rho_t\mathbf{v} \tag{4.4.1,2}$$

introduce the spatial parametrization of the balance of mass.

$$\mathrm{d}_t\,\rho_t = \mathrm{div}\,\mathbf{m}_d + \mathscr{M}_t \tag{4.5}$$

Equation (4.4.1) which has been introduced as "spatial continuity equation" by Euler as early as 1757, represents one of the basic equations in classical fluid mechanics. The above statement with spatial reference and spatial parametrization can easily be transformed into the local balance of mass with material reference and spatial parametrization by applying the related Piola transforms

$$\rho_t = j\,\rho_0 \quad \mathscr{M}_t = j\,\mathscr{M}_0 \quad \mathbf{m}_d = j\,\mathbf{M}_d \cdot \mathbf{f}^{-t} \tag{4.6.1,2,3}$$

together with the classical pull back formalism

$$J\,\mathrm{d}_t\,\rho_t = \mathrm{Div}\,\mathbf{M}_d + \mathscr{M}_0. \tag{4.7}$$

The application of Reynold's transport theorem (3.22.2) with

$$J\,\mathrm{d}_t\,\rho_t = \mathrm{D}_t\,\rho_0 + \mathrm{Div}(\rho_t\mathbf{V}) \quad \text{and} \quad \mathbf{M}_D = \mathbf{M}_d - \rho_0\mathbf{V} \tag{4.8.1,2}$$

can be used to finally retransform equation (4.7) into the original version (4.1). In summary, four different versions of the balance of mass can be distinguished.

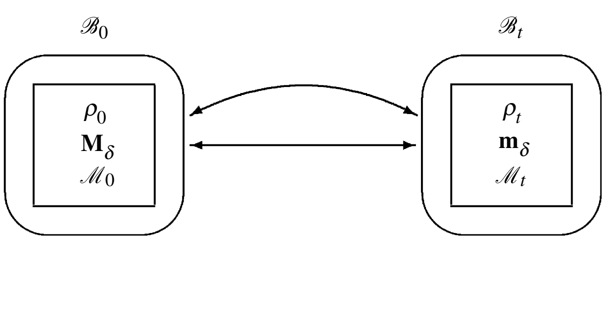

Fig. 3. Balance of mass: densities, mass fluxes and mass sources

$$\begin{array}{llrl} \text{mr} & \text{mp} & \mathrm{D}_t\,\rho_0 & = \mathrm{Div}\,\mathbf{M}_D + \mathscr{M}_0 \\ \text{sr} & \text{mp} & j\,\mathrm{D}_t\,\rho_0 & = \mathrm{div}\,\mathbf{m}_D + \mathscr{M}_t \\ \text{mr} & \text{sp} & J\,\mathrm{d}_t\,\rho_t & = \mathrm{Div}\,\mathbf{M}_d + \mathscr{M}_0 \\ \text{sr} & \text{sp} & \mathrm{d}_t\,\rho_t & = \mathrm{div}\,\mathbf{m}_d + \mathscr{M}_t \end{array}. \tag{4.9}$$

Thereby, the mass fluxes $\mathbf{M}_\delta$ and $\mathbf{m}_\delta$ can be understood as the sum of a "convective contribution" $\bar{\mathbf{M}}_\delta$ and $\bar{\mathbf{m}}_\delta$ and the open system contribution through the influx of mass $\mathbf{R}$ or $\mathbf{r} = j\mathbf{R}\cdot\mathbf{f}^{-t}$,

$$\begin{array}{ll} \mathbf{M}_D = \bar{\mathbf{M}}_D + \mathbf{R} & \bar{\mathbf{M}}_D = \mathbf{0} \\ \mathbf{m}_D = \bar{\mathbf{m}}_D + \mathbf{r} & \bar{\mathbf{m}}_D = \mathbf{0} \\ \mathbf{M}_d = \bar{\mathbf{M}}_d + \mathbf{R} & \bar{\mathbf{M}}_d = +\rho_0\mathbf{V} \\ \mathbf{m}_d = \bar{\mathbf{m}}_d + \mathbf{r} & \bar{\mathbf{m}}_d = -\rho_t\mathbf{v} \end{array}, \tag{4.10}$$

while the corresponding extra mass sources $\mathscr{M}_\tau$ are formally given as follows

$$\begin{array}{ll} \mathscr{M}_0 = \bar{\mathscr{M}}_0 + \mathscr{R}_0 & \bar{\mathscr{M}}_0 = 0 \\ \mathscr{M}_t = \bar{\mathscr{M}}_t + \mathscr{R}_t & \bar{\mathscr{M}}_t = 0 \end{array}. \tag{4.11}$$

For further elaborations, it proves convenient to independently introduce the abbreviations m_τ and M_τ solely taking into account the effects of convection of mass as present in classical continuum mechanics.

$$\begin{array}{ll} \mathsf{m}_0 = \mathrm{Div}\,\bar{\mathbf{M}}_D + \bar{\mathscr{M}}_0 & = 0 \\ \mathsf{m}_t = \mathrm{div}\,\bar{\mathbf{m}}_D + \bar{\mathscr{M}}_t & = 0 \\ \mathsf{M}_0 = \mathrm{Div}\,\bar{\mathbf{M}}_d + \bar{\mathscr{M}}_0 & = +\mathrm{Div}(\rho_0\mathbf{V}) \\ \mathsf{M}_t = \mathrm{div}\,\bar{\mathbf{m}}_d + \bar{\mathscr{M}}_t & = -\mathrm{div}(\rho_t\mathbf{v}) \end{array}. \tag{4.12}$$

In the following, the convective terms m_τ and M_τ, which vanish for the spatial motion problem but are nonzero in the material motion case, will prove instrumental to highlight the dualities between the spatial and the material motion problem. In particular, we will make use of the definition of M_0 as $\mathsf{M}_0 = \nabla_X\rho_0\cdot\mathbf{V} + \rho_0\mathbf{f}^{-t} : \mathrm{d}_t\mathbf{f}$. With their help, the four fundamental versions of the balance of mass (4.9) can be reformulated in the following form, which is particularly tailored to our needs since closed and open system contributions are clearly separated.

$$\begin{array}{llrl} \text{mr} & \text{mp} & \mathrm{D}_t\,\rho_0 & = \mathrm{Div}\,\mathbf{R} + \mathscr{R}_0 + \mathsf{m}_0 \\ \text{sr} & \text{mp} & j\,\mathrm{D}_t\,\rho_0 & = \mathrm{div}\,\mathbf{r} + \mathscr{R}_t + \mathsf{m}_t \\ \text{mr} & \text{sp} & J\,\mathrm{d}_t\,\rho_t & = \mathrm{Div}\,\mathbf{R} + \mathscr{R}_0 + \mathsf{M}_0 \\ \text{sr} & \text{sp} & \mathrm{d}_t\,\rho_t & = \mathrm{div}\,\mathbf{r} + \mathscr{R}_t + \mathsf{M}_t \end{array}. \tag{4.13}$$

Note, that by making use of the balance of mass, the volume specific forms of Reynold's transport theorem (3.22) can be transformed into corresponding mass specific formulations.

$$\rho_t\,\mathrm{D}_t\{\bullet\} = \rho_t\,\mathrm{d}_t\{\bullet\} + \mathrm{div}(\rho_t\{\bullet\}\otimes\mathbf{v}) + [\mathsf{M}_t - \mathsf{m}_t]\{\bullet\} \tag{4.14.1}$$

$$\rho_0\,\mathrm{d}_t\{\bullet\} = \rho_0\,\mathrm{D}_t\{\bullet\} + \mathrm{Div}(\rho_0\{\bullet\}\otimes\mathbf{V}) + [\mathsf{m}_0 - \mathsf{M}_0]\{\bullet\}. \tag{4.14.2}$$

Herein, $\{\bullet\}$ denotes the mass specific density of a scalar- or vector-valued quantity which is related to its volume specific density as $\{\bullet\}_\tau = \rho_\tau\{\bullet\}$. While the volume specific version of Reynold's transport theorem (3.22) will lateron be applied to relate the spatial and material motion quantities in the volume specific format, the mass specific version of the transport theorem (4.14) will serve to relate the corresponding mass specific quantities.

5 Generic balance law

In what follows, we will illustrate how the different versions of a balance law can be derived from one another. For the sake of transparency, we will restrict ourselves to the discussion of the local or differential forms of the master balance law

$$\mathrm{D}_t\{\bullet\}_0 = \mathrm{Div}\{\square\}_D + \{\circ\}_0 \tag{5.1.1}$$

$$\mathrm{d}_t\{\bullet\}_t = \mathrm{div}\{\diamondsuit\}_d + \{\circ\}_t \tag{5.1.2}$$

which can of course be derived from the related global or integral form

$$\mathrm{D}_t \int_{\mathscr{B}_0} \{\bullet\}_0 \, \mathrm{d}V = \int_{\partial\mathscr{B}_0} \{\square\}_D \cdot \mathrm{d}\mathbf{A} + \int_{\mathscr{B}_0} \{\circ\}_0 \, \mathrm{d}V \tag{5.2.1}$$

$$\mathrm{d}_t \int_{\mathscr{B}_t} \{\bullet\}_t \, \mathrm{d}v = \int_{\partial\mathscr{B}_t} \{\diamondsuit\}_d \cdot \mathrm{d}\mathbf{a} + \int_{\mathscr{B}_t} \{\circ\}_t \, \mathrm{d}v \tag{5.2.2}$$

if sufficient smoothness criteria are fulfilled by the related fields of the balance quantity $\{\bullet\}_\tau$ itself, the related fluxes $\{\square\}_\delta$ and $\{\diamondsuit\}_\delta$ and the related source terms $\{\circ\}_\tau$. In classical continuum mechanics, it is not necessary to distinguish between volume and mass specific representations of the balance equations. In this context, in a material parametrization with material reference, which is commonly referred to as Lagrangian formulation, not only the mass flux and source but also the convective terms vanish. Consequently, the rate of change of the density of any quantity $\mathrm{D}_t\{\bullet\}_0 = \rho_0 \mathrm{D}_t\{\bullet\}$, e.g., the momentum density or the energy density, is equivalent to the rate of change of $\{\bullet\}$ weighted by the material density ρ_0 since $\mathrm{D}_t\,\rho_0 = 0$. Within the thermodynamics of open systems, however, the volume and the mass specific version of the balance laws differ considerably since $\mathrm{D}_t\,\rho_0 \neq 0$. In the following, we will derive a prototype set of balance laws in the volume and in the mass specific format. Particular interest will be dedicated to the fact, that the mass specific version of a balance law takes the standard format known from classical continuum mechanics, merely enhanced by the effects of convection of mass.

5.1 Volume specific version

In the volume specific version of a balance law, the quantity to be balanced $\{\bullet\}_\tau$ can either be given in a material or spatial reference as $\{\bullet\}_0 = \rho_0\{\bullet\}$ or $\{\bullet\}_t = \rho_t\{\bullet\}$. It is balanced with the sum of the divergence of the corresponding fluxes $\{\square\}_\delta$, $\{\diamondsuit\}_\delta$ and the volume sources $\{\circ\}_\tau$, see Fig. 4. In analogy to the balance of mass, we start with the formulation with material reference and material parametrization

$$\mathrm{D}_t\{\bullet\}_0 = \mathrm{Div}\{\square\}_D + \{\circ\}_0. \tag{5.3}$$

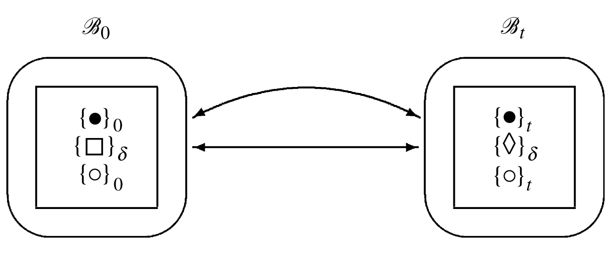

Fig. 4. Generic balance law: quantities to be balanced, fluxes and sources

Its individual terms can be pushed forward by making use of the related Piola transforms

$$\{\bullet\}_0 = J\{\bullet\}_t \quad \{\circ\}_0 = J\{\circ\}_t \quad \{\Box\}_D = J\{\Diamond\}_D \cdot \mathbf{F}^{-t} \tag{5.4.1–3}$$

to render the formulation with spatial reference and material parametrization,

$$j\,\mathrm{D}_t\{\bullet\}_0 = \mathrm{div}\{\Diamond\}_D + \{\circ\}_t. \tag{5.5}$$

The application of Reynold's transport theorem (3.22.1) with

$$j\,\mathrm{D}_t\{\bullet\}_0 = \mathrm{d}_t\{\bullet\}_t + \mathrm{div}\big(\{\bullet\}_t \otimes \mathbf{v}\big) \quad \text{and} \quad \{\Diamond\}_d = \{\Diamond\}_D - \{\bullet\}_t \otimes \mathbf{v} \tag{5.6.1, 2}$$

yield the corresponding version with spatial reference and spatial parametrization.

$$\mathrm{d}_t\{\bullet\}_t = \mathrm{div}\{\Diamond\}_d + \{\circ\}_t. \tag{5.7}$$

The related Piola transforms

$$\{\bullet\}_t = j\{\bullet\}_0 \quad \{\circ\}_t = j\{\circ\}_0 \quad \{\Diamond\}_d = j\{\Box\}_d \cdot \mathbf{f}^{-t} \tag{5.8.1–3}$$

together with the classical pull back formalism yield the general form of a balance law with material reference and spatial parametrization.

$$J\,\mathrm{d}_t\{\bullet\}_t = \mathrm{Div}\{\Box\}_d + \{\circ\}_0. \tag{5.9}$$

The original equation (5.3) can eventually be rederived by applying the appropriate version of Reynold's transport theorem (3.22.2)

$$J\,\mathrm{d}_t\{\bullet\}_t = \mathrm{D}_t\{\bullet\}_0 + \mathrm{Div}\big(\{\bullet\}_0 \otimes \mathbf{V}\big) \quad \text{and} \quad \{\Box\}_D = \{\Box\}_d - \{\bullet\}_0 \otimes \mathbf{V}. \tag{5.10.1, 2}$$

The closed loop of transformations inherit to any balance equation is illustrated in Fig. 5. In summary, each balance equation can be expressed in four different ways

$$\begin{array}{lll} \text{mr} & \text{mp} & \mathrm{D}_t\{\bullet\}_0 = \mathrm{Div}\{\Box\}_D + \{\circ\}_0 \\ \text{sr} & \text{mp} & j\,\mathrm{D}_t\{\bullet\}_0 = \mathrm{div}\{\Diamond\}_D + \{\circ\}_t \\ \text{mr} & \text{sp} & J\,\mathrm{d}_t\{\bullet\}_t = \mathrm{Div}\{\Box\}_d + \{\circ\}_0 \\ \text{sr} & \text{sp} & \mathrm{d}_t\{\bullet\}_t = \mathrm{div}\{\Diamond\}_d + \{\circ\}_t \end{array} . \tag{5.11}$$

Therein, the flux terms $\{\Box\}_\delta$ and $\{\Diamond\}_\delta$ with either $\delta = D$ for the spatial motion problem or $\delta = d$ for the material motion problem are related to the corresponding Neumann boundary conditions in terms of the standard closed system surface contributions $\blacksquare^{closed}$ and $\blacklozenge^{closed}$ and the open system supplements $\blacksquare^{open}$ and $\blacklozenge^{open}$.

$$\begin{array}{ll} \{\Box\}_\delta \cdot \mathbf{N} = \blacksquare^{closed} + \blacksquare^{open} & \blacksquare^{open} = \bar{\blacksquare}^{open} + [\{\bullet\} \otimes \mathbf{R}] \cdot \mathbf{N} \\ \{\Diamond\}_\delta \cdot \mathbf{n} = \blacklozenge^{closed} + \blacklozenge^{open} & \blacklozenge^{open} = \bar{\blacklozenge}^{open} + [\{\bullet\} \otimes \mathbf{r}] \cdot \mathbf{n} \end{array} \tag{5.12}$$

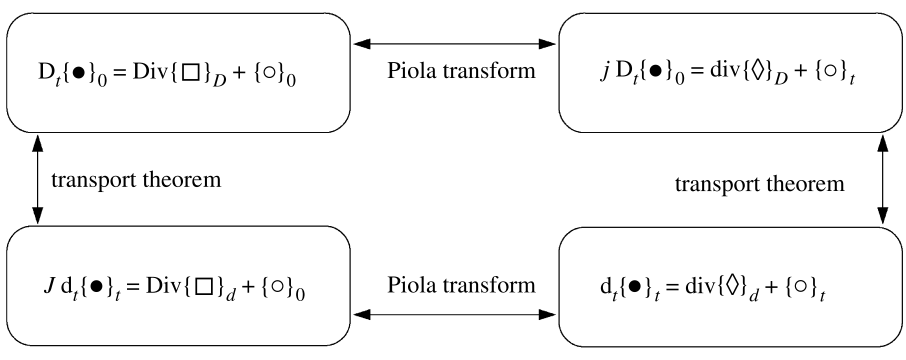

Fig. 5. Transformation of balance laws

Moreover, the source terms are composed of the standard closed and the additional open system contributions

$$\begin{array}{ll} \{\circ\}_0 = \{\circ\}_0^{closed} + \{\circ\}_0^{open} & \{\circ\}_0^{open} = \{\bar{\circ}\}_0^{open} + \{\bullet\}\mathscr{R}_0 - \nabla_X\{\bullet\}\cdot\mathbf{R} \\ \{\circ\}_t = \{\circ\}_t^{closed} + \{\circ\}_t^{open} & \{\circ\}_t^{open} = \{\bar{\circ}\}_t^{open} + \{\bullet\}\mathscr{R}_t - \nabla_x\{\bullet\}\cdot\mathbf{r} \end{array}. \tag{5.13}$$

5.2 *Mass specific version*

Each balance law can be transformed into a mass specific version balancing the rate of change of the mass specific quantity $\{\bullet\} = \{\bullet\}_\tau/\rho_\tau$ with the corresponding reduced flux $\{\bar{\square}\}_\delta$ and $\{\bar{\diamond}\}_\delta$ and the reduced source terms $\{\bar{\circ}\}_\tau$, whereby $\tau = 0, t$ and $\delta = d, D$. The mass specific version can be derived by subtracting $\{\bullet\}$ times the balance of mass (4.13) from the corresponding volume specific version of the balance law (5.11). Consequently, we obtain the following remarkably simple generic forms of the mass specific balance laws,

$$\begin{array}{llrcl} \text{mr} & \text{mp} & \rho_0\,\mathrm{D}_t\{\bullet\} & = & \mathrm{Div}\{\bar{\square}\}_D + \{\bar{\circ}\}_0 - \mathsf{m}_0\{\bullet\} \\ \text{sr} & \text{mp} & \rho_t\,\mathrm{D}_t\{\bullet\} & = & \mathrm{div}\{\bar{\diamond}\}_D + \{\bar{\circ}\}_t - \mathsf{m}_t\{\bullet\} \\ \text{mr} & \text{sp} & \rho_0\,\mathrm{d}_t\{\bullet\} & = & \mathrm{Div}\{\bar{\square}\}_d + \{\bar{\circ}\}_0 - \mathsf{M}_0\{\bullet\} \\ \text{sr} & \text{sp} & \rho_t\,\mathrm{d}_t\{\bullet\} & = & \mathrm{div}\{\bar{\diamond}\}_d + \{\bar{\circ}\}_t - \mathsf{M}_t\{\bullet\} \end{array}, \tag{5.14}$$

whereby the reduced flux terms $\{\bar{\square}\}_\delta$ and $\{\bar{\diamond}\}_\delta$ are related to their overall counterparts as $\{\bar{\square}\}_\delta = \{\square\}_\delta - \{\bullet\}\otimes\boldsymbol{R}$ and $\{\bar{\diamond}\}_\delta = \{\diamond\}_\delta - \{\bullet\}\otimes\mathbf{r}$. Again, the reduced fluxes are related to the corresponding Neumann boundary conditions in terms of the standard closed system contributions $\blacksquare^{closed}$ and $\blacklozenge^{closed}$ and the reduced open system supplements $\bar{\blacksquare}^{open}$ and $\bar{\blacklozenge}^{open}$.

$$\{\bar{\square}\}_\delta\cdot\mathbf{N} = \blacksquare^{closed} + \bar{\blacksquare}^{open} \tag{5.15.1}$$

$$\{\bar{\diamond}\}_\delta\cdot\mathbf{n} = \blacklozenge^{closed} + \bar{\blacklozenge}^{open}. \tag{5.15.2}$$

For the spatial motion problem, these Neumann boundary conditions are given for the fluxes denoted by $\delta = D$ while for the material motion problem, we can formally introduce Neumann type of boundary conditions for the fluxes $\delta = d$. In a similar way, the reduced source terms of the mass specific balance equations are composed of closed and reduced open system contributions

$$\{\bar{\circ}\}_0 = \{\circ\}_0^{closed} + \{\bar{\circ}\}_0^{open} \tag{5.16.1}$$

$$\{\bar{\circ}\}_t = \{\circ\}_t^{closed} + \{\bar{\circ}\}_t^{open}. \tag{5.16.2}$$

Note, that the mass specific format is free from all the explicit extra terms caused by the changes in mass. The influence of the open system manifests itself only implicitly through the prescribed boundary terms $\bar{\blacksquare}^{open}$ and $\bar{\blacklozenge}^{open}$ and the prescribed volume sources $\{\bar{\circ}\}_\tau^{open}$. The convective influence introduced through the m_τ and M_τ terms, however, is also present in the closed system case.

Remark 5.1: It is worth noting, that in the "theory of volumetric growth" derived earlier by Epstein and Maugin [10], the source terms in Eq. (5.11) are introduced in the following form

$$\begin{array}{ll} \{\circ\}_0 = \{\circ\}_0^{closed} + \{\circ\}_0^{open} & \{\circ\}_0^{open} = \{\hat{\circ}\}_0^{open} + \{\bullet\}\mathscr{R}_0 \\ \{\circ\}_t = \{\circ\}_t^{closed} + \{\circ\}_t^{open} & \{\circ\}_t^{open} = \{\hat{\circ}\}_t^{open} + \{\bullet\}\mathscr{R}_t \end{array}.$$

Consequently, the gradient terms $-\nabla_X\{\bullet\}\cdot\mathbf{R}$ and $-\nabla_x\{\bullet\}\cdot\mathbf{r}$ that are part of the volume specific definition (5.13) in our formulation appear with a positive sign in the definition of the reduced source terms of Epstein and Maugin [10] pertaining to Eq. (5.14).

$$\begin{aligned}\{\bar{\circ}\}_0 &= \{\circ\}_0^{closed} + \{\hat{\circ}\}_0^{open} + \nabla_X\{\bullet\}\cdot\mathbf{R}\\ \{\bar{\circ}\}_t &= \{\circ\}_t^{closed} + \{\hat{\circ}\}_t^{open} + \nabla_x\{\bullet\}\cdot\mathbf{r}\end{aligned}.$$

The different introduction of these source terms is visible in every balance equation and finally results in a significantly different dissipation inequality. The line of thought followed within this paper is believed to be more convenient especially in pointing out the duality between the spatial and the material motion problem. However, both formulations can be understood as a natural extension of the classical formulation of growth, the "theory of adaptive elasticity" by Cowin and Hegedus [6] which does not include any flux of mass since $\mathbf{R}=\mathbf{0}$ and $\mathbf{r}=\mathbf{0}$.

6 Balance of momentum

Keeping in mind the derivation of the generic balance laws of the preceding chapter, we now elaborate their specification to yield the balance of linear momentum. Unlike the balance of mass, the balance of momentum takes different forms in the spatial and the material motion context due to the vector-valued nature of the balanced quantity. Consequently, we will discuss the spatial and the material motion problem in separate subchapters.

6.1 Volume specific version

6.1.1 Spatial motion problem

The balance of momentum, which can be understood as the continuum version of Newton's axiom for a system of discrete particles, balances the rate of change of the spatial momentum density $\mathbf{p}_\tau$ with the spatial or rather physical forces generated by a change in actual spatial placement of "physical particles". These can essentially be divided into two types, namely the contact or surface forces represented by the momentum fluxes $\mathbf{\Pi}_\delta^t$ and $\boldsymbol{\sigma}_\delta^t$ and the at-a-distance forces, i.e. the momentum sources $\mathbf{b}_\tau$. The volume specific momentum density $\mathbf{p}_\tau$ of the spatial motion problem is canonically defined as spatial covector given through the partial derivative of the volume specific kinetic energy density K_τ

$$K_\tau = \frac{1}{2}\rho_\tau\mathbf{v}\cdot\mathbf{g}\cdot\mathbf{v} \tag{6.1}$$

with respect to the spatial velocity $\mathbf{v}$.

$$\mathbf{p}_\tau = \partial_\mathbf{v}K_\tau = \rho_\tau\mathbf{g}\cdot\mathbf{v}. \tag{6.2}$$

The volume specific balance of momentum with material reference and material parametrization can thus be expressed as

$$\mathrm{D}_t\mathbf{p}_0 = \mathrm{Div}\,\mathbf{\Pi}_D^t + \mathbf{b}_0, \tag{6.3}$$

whereby $\mathbf{\Pi}_D^t$ is referred to as the classical two-field first Piola–Kirchhoff stress tensor in standard continuum mechanics. With the help of the well-known Piola transforms

$$\mathbf{p}_0 = J\mathbf{p}_t \quad \mathbf{b}_0 = J\mathbf{b}_t \quad \boldsymbol{\Pi}^t_D = J\boldsymbol{\sigma}^t_D \cdot \mathbf{F}^{-t} \tag{6.4.1–3}$$

the individual terms of Eq. (6.3) can be pushed forward to the spatial configuration

$$j\mathrm{D}_t\mathbf{p}_0 = \mathrm{div}\,\boldsymbol{\sigma}^t_D + \mathbf{b}_t. \tag{6.5}$$

Note, that the corresponding momentum flux $\boldsymbol{\sigma}^t_D$ is commonly denoted as Cauchy stress tensor in standard continuum mechanics. The application of Reynold's transport theorem (3.22.1)

$$j\mathrm{D}_t\mathbf{p}_0 = \mathrm{d}_t\mathbf{p}_t + \mathrm{div}(\mathbf{p}_t \otimes \mathbf{v}) \quad \text{and} \quad \boldsymbol{\sigma}^t_d = \boldsymbol{\sigma}^t_D - \mathbf{p}_t \otimes \mathbf{v} \tag{6.6.1, 2}$$

yield the balance of momentum with spatial reference and spatial parametrization.

$$\mathrm{d}_t\mathbf{p}_t = \mathrm{div}\,\boldsymbol{\sigma}^t_d + \mathbf{b}_t. \tag{6.7}$$

The definition of the stress tensor $\boldsymbol{\sigma}^t_d$ reflects the convective nature of the above equation in terms of the "transport of linear momentum" $\mathbf{p}_t \otimes \mathbf{v}$. The individual terms of equation (6.7) which is typically applied in classical fluid mechanics can be pulled back to the material configuration with the help of the related Piola transforms

$$\mathbf{p}_t = j\mathbf{p}_0 \quad \mathbf{b}_t = j\mathbf{b}_0 \quad \boldsymbol{\sigma}^t_d = j\boldsymbol{\Pi}^t_d \cdot \mathbf{f}^{-t} \tag{6.8.1–3}$$

thus leading to the following expression

$$J\mathrm{d}_t\mathbf{p}_t = \mathrm{Div}\,\boldsymbol{\Pi}^t_d + \mathbf{b}_0. \tag{6.9}$$

Finally, the starting point version of the balance of momentum (6.3) can be recovered though the application of Reynold's transport theorem (3.22.2)

$$J\mathrm{d}_t\mathbf{p}_t = \mathrm{D}_t\mathbf{p}_0 + \mathrm{Div}(\mathbf{p}_0 \otimes \mathbf{V}) \quad \text{and} \quad \boldsymbol{\Pi}^t_D = \boldsymbol{\Pi}^t_d - \mathbf{p}_0 \otimes \mathbf{V}. \tag{6.10.1, 2}$$

In summary, the four different versions of the volume specific balance of momentum can be distinguished for the spatial motion problem

$$\begin{array}{lll} \text{mr} & \text{mp} & \mathrm{D}_t\,\mathbf{p}_0 = \mathrm{Div}\,\boldsymbol{\Pi}^t_D + \mathbf{b}_0 \\ \text{sr} & \text{mp} & j\mathrm{D}_t\,\mathbf{p}_0 = \mathrm{div}\,\boldsymbol{\sigma}^t_D + \mathbf{b}_t \\ \text{mr} & \text{sp} & J\mathrm{d}_t\,\mathbf{p}_t = \mathrm{Div}\,\boldsymbol{\Pi}^t_d + \mathbf{b}_0 \\ \text{sr} & \text{sp} & \mathrm{d}_t\,\mathbf{p}_t = \mathrm{div}\,\boldsymbol{\sigma}^t_d + \mathbf{b}_t \end{array}. \tag{6.11}$$

On the Neumann boundary, the normal projection of the momentum fluxes $\boldsymbol{\Pi}^t_D$ and $\boldsymbol{\sigma}^t_D$ is required to be in equilibrium with the corresponding closed and the open system spatial stress vector contributions $\mathbf{t}^{closed}_\tau$ and $\mathbf{t}^{open}_\tau$.

$$\begin{array}{ll} \boldsymbol{\Pi}^t_D \cdot \mathbf{N} = \mathbf{t}^{closed}_0 + \mathbf{t}^{open}_0 & \mathbf{t}^{open}_0 = \bar{\mathbf{t}}^{open}_0 + [\mathbf{p} \otimes \mathbf{R}] \cdot \mathbf{N} \\ \boldsymbol{\sigma}^t_D \cdot \mathbf{n} = \mathbf{t}^{closed}_t + \mathbf{t}^{open}_t & \mathbf{t}^{open}_t = \bar{\mathbf{t}}^{open}_t + [\mathbf{p} \otimes \mathbf{r}] \cdot \mathbf{n} \end{array}. \tag{6.12}$$

Correspondingly, the momentum sources $\mathbf{b}_\tau$ can be understood as the sum of the closed and the open system volume force contributions $\mathbf{b}^{closed}_\tau$ and $\mathbf{b}^{open}_\tau$

$$\begin{array}{ll} \mathbf{b}_0 = \mathbf{b}^{closed}_0 + \mathbf{b}^{open}_0 & \mathbf{b}^{open}_0 = \bar{\mathbf{b}}^{open}_0 + \mathbf{p}\mathscr{R}_0 - \nabla_X\mathbf{p} \cdot \mathbf{R} \\ \mathbf{b}_t = \mathbf{b}^{closed}_t + \mathbf{b}^{open}_t & \mathbf{b}^{open}_t = \bar{\mathbf{b}}^{open}_t + \mathbf{p}\mathscr{R}_t - \nabla_x\mathbf{p} \cdot \mathbf{r} \end{array}. \tag{6.13}$$

6.1.2 Material motion problem

Generally speaking, the balance of momentum of the material motion problem follows from a complete projection of the classical version of the standard momentum balance (6.11) onto the material manifold. For the particular case of a thermo-hyperelastic material, this projection is illustrated in detail in chap. 10.3. For the time being, we will introduce the balance of momentum of the material motion problem in a more abstract way. For that purpose, we make use of the definition of the volume specific material motion momentum density based on the related volume specific kinetic energy density K_τ.

$$K_\tau = \frac{1}{2}\rho_\tau \mathbf{V}\cdot\mathbf{C}\cdot\mathbf{V}. \tag{6.14}$$

Consequently, the rate of change of the volume specific material momentum densitiy $\mathbf{P}_\tau$,

$$\mathbf{P}_\tau = \partial_{\mathbf{V}} K_\tau = \rho_\tau \mathbf{C}\cdot\mathbf{V}, \tag{6.15}$$

which is typically referred to as "pseudomomentum" by Maugin [34] is balanced with the momentum fluxes $\boldsymbol{\pi}^t_\delta$ and $\boldsymbol{\Sigma}^t_\delta$ and the momentum sources $\mathbf{B}_\tau$. The balance of momentum of the material motion problem with spatial reference and spatial parametrization can thus be postulated as

$$\mathrm{d}_t\mathbf{P}_t = \operatorname{div}\boldsymbol{\pi}^t_d + \mathbf{B}_t, \tag{6.16}$$

whereby the related Piola transforms

$$\mathbf{P}_t = j\mathbf{P}_0 \quad \mathbf{B}_t = j\mathbf{B}_0 \quad \boldsymbol{\pi}^t_d = j\boldsymbol{\Sigma}^t_d\cdot\mathbf{f}^{-t} \tag{6.17.1–3}$$

and a pull back to the material configuration yield the following expression

$$J\mathrm{d}_t\mathbf{P}_t = \operatorname{Div}\boldsymbol{\Sigma}^t_d + \mathbf{B}_0. \tag{6.18}$$

In the honor of Eshelby who originally introduced the material momentum flux $\boldsymbol{\Sigma}^t$ as energy momentum tensor, $\boldsymbol{\Sigma}^t_\delta$ is nowadays often referred to as the dynamic generalization of the classical Eshelby stress tensor in the related literature. The application of Reynold's transport theorem (3.22.2)

$$J\mathrm{d}_t\mathbf{P}_t = \mathrm{D}_t\mathbf{P}_0 + \operatorname{Div}(\mathbf{P}_0\otimes\mathbf{V}) \quad \text{and} \quad \boldsymbol{\Sigma}^t_D = \boldsymbol{\Sigma}^t_d - \mathbf{P}_0\otimes\mathbf{V} \tag{6.19.1,2}$$

lead to the formulation with material reference and material parametrization

$$\mathrm{D}_t\mathbf{P}_0 = \operatorname{Div}\boldsymbol{\Sigma}^t_D + \mathbf{B}_0. \tag{6.20}$$

The individual terms of the latter can again be transformed by the related Piola transforms

$$\mathbf{P}_0 = J\mathbf{P}_t \quad \mathbf{B}_0 = J\mathbf{B}_t \quad \boldsymbol{\Sigma}^t_D = J\boldsymbol{\pi}^t_D\cdot\mathbf{F}^{-t} \tag{6.21.1–3}$$

and pushed forward to the spatial configuration

$$j\mathrm{D}_t\mathbf{P}_0 = \operatorname{div}\boldsymbol{\pi}^t_D + \mathbf{B}_t. \tag{6.22}$$

Again, the application of Reynold's transport theorem (3.22.1)

$$j\mathrm{D}_t\mathbf{P}_0 = \mathrm{d}_t\mathbf{P}_t + \operatorname{div}(\mathbf{P}_t\otimes\mathbf{v}) \quad \text{and} \quad \boldsymbol{\pi}^t_d = \boldsymbol{\pi}^t_D - \mathbf{P}_t\otimes\mathbf{v} \tag{6.23.1,2}$$

can be used to gain back the original formulation (6.16). The four different versions of the volume specific balance of momentum of the material motion problem are summarized in the following

$$\begin{array}{llrcl} \text{mr} & \text{mp} & \mathrm{D}_t\,\mathbf{P}_0 & = & \operatorname{Div}\boldsymbol{\Sigma}^t_D + \mathbf{B}_0 \\ \text{sr} & \text{mp} & j\,\mathrm{D}_t\,\mathbf{P}_0 & = & \operatorname{div}\boldsymbol{\pi}^t_D + \mathbf{B}_t \\ \text{mr} & \text{sp} & J\,\mathrm{d}_t\,\mathbf{P}_t & = & \operatorname{Div}\boldsymbol{\Sigma}^t_d + \mathbf{B}_0 \\ \text{sr} & \text{sp} & \mathrm{d}_t\,\mathbf{P}_t & = & \operatorname{div}\boldsymbol{\pi}^t_d + \mathbf{B}_t \end{array}. \tag{6.24}$$

To illustrate the duality with the spatial motion problem, we can formally introduce the following Neumann type boundary conditions relating the momentum fluxes $\boldsymbol{\pi}^t_d$ and $\boldsymbol{\Sigma}^t_d$ to the sum of the closed and the open system material stress vector contributions $\mathbf{T}^{closed}_\tau$ and $\mathbf{T}^{open}_\tau$

$$\begin{array}{ll} \boldsymbol{\pi}_d^t \cdot \mathbf{n} = \mathbf{T}_t^{closed} + \mathbf{T}_t^{open} & \mathbf{T}_t^{open} = \bar{\mathbf{T}}_t^{open} + [\mathbf{P} \otimes \mathbf{r}] \cdot \mathbf{n} \\ \boldsymbol{\Sigma}_d^t \cdot \mathbf{N} = \mathbf{T}_0^{closed} + \mathbf{T}_0^{open} & \mathbf{T}_0^{open} = \bar{\mathbf{T}}_0^{open} + [\mathbf{P} \otimes \mathbf{R}] \cdot \mathbf{N} \end{array} . \tag{6.25}$$

Moreover, the volume specific momentum sources $\mathbf{B}_\tau$ can be expressed as the sum of the closed and the open system material force contributions $\mathbf{B}_\tau^{closed}$ and $\mathbf{B}_\tau^{open}$

$$\begin{array}{ll} \mathbf{B}_t = \mathbf{B}_t^{closed} + \mathbf{B}_t^{open} & \mathbf{B}_t^{open} = \bar{\mathbf{B}}_t^{open} + \mathbf{P}\mathscr{R}_t - \nabla_x \mathbf{P} \cdot \mathbf{r} \\ \mathbf{B}_0 = \mathbf{B}_0^{closed} + \mathbf{B}_0^{open} & \mathbf{B}_0^{open} = \bar{\mathbf{B}}_0^{open} + \mathbf{P}\mathscr{R}_0 - \nabla_X \mathbf{P} \cdot \mathbf{R} \end{array} . \tag{6.26}$$

6.1.3 Spatial vs. material motion problem

In order to distinguish the balance of momentum of the spatial and the material motion problem, the former has been introduced as the "balance of physical momentum" while the latter is referred to as the "balance of pseudomomentum" by Maugin [34]. The balance of momentum of the material motion problem (6.24) can be interpreted as a projection of the corresponding spatial motion balance Eqs. (6.11) onto the material manifold $\mathscr{B}_0$. In this respect, the spatial and the material momentum densities are clearly related via the spatial and the material deformation gradient $\mathbf{F}$ and $\mathbf{f}$

$$\begin{array}{ll} \mathbf{p}_0 = -\mathbf{f}^t \cdot \mathbf{P}_0 & \mathbf{P}_0 = -\mathbf{F}^t \cdot \mathbf{p}_0 \\ \mathbf{p}_t = -\mathbf{f}^t \cdot \mathbf{P}_t & \mathbf{P}_t = -\mathbf{F}^t \cdot \mathbf{p}_t \end{array} . \tag{6.27}$$

At this point, it proves convenient to additively decompose the dynamical stress measures $\boldsymbol{\Pi}_D^t$ $\boldsymbol{\sigma}_D^t$, $\boldsymbol{\pi}_d^t$ and $\boldsymbol{\Sigma}_d^t$ into the static stress measures $\boldsymbol{\Pi}^t$, $\boldsymbol{\sigma}^t$, $\boldsymbol{\pi}^t$ and $\boldsymbol{\Sigma}^t$ and additional contributions stemming from the volume specific kinetic energy density K_τ, see Steinmann [47].

$$\begin{array}{ll} \boldsymbol{\Pi}_D^t = \boldsymbol{\Pi}^t - \mathrm{D}_{\mathbf{F}} K_0 & \boldsymbol{\Sigma}_d^t = \boldsymbol{\Sigma}^t - K_0 \mathbf{I} + \mathbf{F}^t \cdot \mathrm{d}_{\mathbf{F}} K_0 \\ \boldsymbol{\sigma}_D^t = \boldsymbol{\sigma}^t - K_t \mathbf{I} + \mathbf{f}^t \cdot \mathrm{D}_{\mathbf{f}} K_t & \boldsymbol{\pi}_d^t = \boldsymbol{\pi}^t - \mathrm{d}_{\mathbf{f}} K_t \end{array} . \tag{6.28}$$

With the help of the partial derivative of the kinetic energy with respect to the deformation gradients[2], we conclude that the dynamic stress measures of the spatial motion problem $\boldsymbol{\Pi}_D^t = \boldsymbol{\Pi}^t$ and $\boldsymbol{\sigma}_D^t = \boldsymbol{\sigma}^t$ remain unaffected by these additional contributions. In a similar manner, the associated volume forces $\mathbf{b}_\tau$ and $\mathbf{B}_\tau$ can be introduced as the sum of an external and an internal static contribution and an additional dynamic term whereby the latter can be expressed in terms of the explicit derivative $\partial_{\boldsymbol{\varphi}}$ and $\partial_{\boldsymbol{\Phi}}$ of the volume specific kinetic energy density K_τ

$$\begin{array}{ll} \mathbf{b}_0 = \mathbf{b}_0^{ext} + \mathbf{b}_0^{int} + \partial_{\boldsymbol{\varphi}} K_0 & \mathbf{B}_0 = \mathbf{B}_0^{ext} + \mathbf{B}_0^{int} + \partial_{\boldsymbol{\Phi}} K_0 \\ \mathbf{b}_t = \mathbf{b}_t^{ext} + \mathbf{b}_t^{int} + \partial_{\boldsymbol{\varphi}} K_t & \mathbf{B}_t = \mathbf{B}_t^{ext} + \mathbf{B}_t^{int} + \partial_{\boldsymbol{\Phi}} K_t \end{array} . \tag{6.29}$$

While the standard forces $\mathbf{b}_\tau$ perform work over positional changes relative to the ambient space, the configurational forces $\mathbf{B}_\tau$ perform work over positional changes relative to the ambient material. The latter have originally been introduced by Eshelby [11] as forces acting on defects. As a matter of fact, the internal forces, which are sometimes also interpreted as a measure of inhomogeneity in the material motion context, vanish identically for the spatial motion problem as $\mathbf{b}_\tau^{int} = \mathbf{0}$. Likewise, the dynamic contributions can only be found in the material motion context since $\partial_{\boldsymbol{\varphi}} K_\tau = \mathbf{0}$ whereas $\partial_{\boldsymbol{\Phi}} K_\tau = \partial_{\boldsymbol{\Phi}} \rho_\tau K$. Note, that in the above decomposition, the external forces $\mathbf{b}_\tau^{ext}$ and $\mathbf{B}_\tau^{ext}$ are composed of a closed and an open system contribution while the internal forces $\mathbf{b}_\tau^{int}$ and $\mathbf{B}_\tau^{int}$ can be understood as a natural outcome of the particular underlying constitutive assumption

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2 $$\begin{array}{ll} \mathrm{D}_{\mathbf{F}} K_0 = \mathbf{0} & \mathbf{F}^t \cdot \mathrm{d}_{\mathbf{F}} K_0 = \mathbf{P}_0 \otimes \mathbf{V} \\ \mathbf{f}^t \cdot \mathrm{D}_{\mathbf{f}} K_t = K_t \mathbf{I} & \mathrm{d}_{\mathbf{f}} K_t = K_t \mathbf{F}^t + \mathbf{P}_t \otimes \mathbf{v} \end{array}$$

Remark 6.1: It shall be emphasized, that the balance of momentum of the spatial motion problem which is often introduced as the "physical force balance" can be interpreted as a natural consequence of the invariance of the working under changes in spatial observer, see, e.g. Gurtin [17] §4b. The balance of momentum of the material motion problem can be understood as the "balance of configurational forces". In complete analogy to the spatial motion problem, it follows from invariance requirements posed on the working under changes in material observer, see Gurtin [17] §5c.

6.2. Mass specific version

6.2.1 Spatial motion problem

The mass specific version of the balance of momentum is based on the mass specific kinetic energy density

$$K = \frac{1}{2}\mathbf{v}\cdot\mathbf{g}\cdot\mathbf{v} \tag{6.30}$$

defining the quantity to be balanced as its partial derivative with respect to the spatial velocity $\mathbf{v}$.

$$\mathbf{p} = \partial_{\mathbf{v}}K = \mathbf{g}\cdot\mathbf{v}. \tag{6.31}$$

The rate of change of the mass specific spatial motion momentum density $\mathbf{p}$, i.e. the covariant spatial velocity, is balanced with the reduced momentum fluxes $\bar{\mathbf{\Pi}}^t_\delta$ and $\bar{\boldsymbol{\sigma}}^t_\delta$, the reduced momentum sources $\bar{\mathbf{b}}_\tau$ and a convective contribution in terms of m_τ which vanishes for the spatial motion problem. By subtracting the balance of mass (4.13) weighted by the momentum density $\mathbf{p}$ from the volume specific momentum balance (6.11), we can derive the four different versions of the mass specific momentum balance.

$$\begin{array}{llrcl} \text{mr} & \text{mp} & \rho_0\mathrm{D}_t\mathbf{p} & = & \mathrm{Div}\,\bar{\mathbf{\Pi}}^t_D + \bar{\mathbf{b}}_0 - \mathsf{m}_0\mathbf{p} \\ \text{sr} & \text{mp} & \rho_t\mathrm{D}_t\mathbf{p} & = & \mathrm{div}\,\bar{\boldsymbol{\sigma}}^t_D + \bar{\mathbf{b}}_t - \mathsf{m}_t\mathbf{p} \\ \text{mr} & \text{sp} & \rho_0\mathrm{d}_t\mathbf{p} & = & \mathrm{Div}\,\bar{\mathbf{\Pi}}^t_d + \bar{\mathbf{b}}_0 - \mathsf{m}_0\mathbf{p} \\ \text{sr} & \text{sp} & \rho_t\mathrm{d}_t\mathbf{p} & = & \mathrm{div}\,\bar{\boldsymbol{\sigma}}^t_d + \bar{\mathbf{b}}_t - \mathsf{m}_t\mathbf{p} \end{array}. \tag{6.32}$$

Note, that the reduced momentum fluxes $\bar{\mathbf{\Pi}}^t_\delta$ and $\bar{\boldsymbol{\sigma}}^t_\delta$ which are related to the overall momentum fluxes $\mathbf{\Pi}^t_\delta$ and $\boldsymbol{\sigma}^t_\delta$ through $\bar{\mathbf{\Pi}}^t_\delta = \mathbf{\Pi}^t_\delta - \mathbf{p}\otimes\mathbf{R}$ and $\bar{\boldsymbol{\sigma}}^t_\delta = \boldsymbol{\sigma}^t_\delta - \mathbf{p}\otimes\mathbf{r}$ are determined by the closed and open system spatial stress vector contributions $\mathbf{t}^{closed}_\tau$ and $\bar{\mathbf{t}}^{open}_\tau$ on the Neumann boundary,

$$\begin{aligned} \bar{\mathbf{\Pi}}^t_D\cdot\mathbf{N} &= \mathbf{t}^{closed}_0 + \bar{\mathbf{t}}^{open}_0, \\ \bar{\boldsymbol{\sigma}}^t_D\cdot\mathbf{n} &= \mathbf{t}^{closed}_t + \bar{\mathbf{t}}^{open}_t, \end{aligned} \tag{6.33}$$

while the reduced spatial momentum sources $\bar{\mathbf{b}}_\tau$ are given as the sum of the classical closed system volume force contributions $\mathbf{b}^{closed}_\tau$ and the reduced open system contributions $\bar{\mathbf{b}}^{open}_\tau$,

$$\begin{aligned} \bar{\mathbf{b}}_0 &= \mathbf{b}^{closed}_0 + \bar{\mathbf{b}}^{open}_0 \\ \bar{\mathbf{b}}_t &= \mathbf{b}^{closed}_t + \bar{\mathbf{b}}^{open}_t \end{aligned} \tag{6.34}$$

whereby $\mathbf{b}^{closed}_\tau$ contributes to the external, the internal and the kinetic contributions $\mathbf{b}^{ext}_\tau$, $\mathbf{b}^{int}_\tau$ and $\partial_{\boldsymbol{\varphi}}K_\tau$ while $\bar{\mathbf{b}}^{open}_\tau$ contributes exclusively to the external sources $\mathbf{b}^{ext}_\tau$.

6.2.2 Material motion problem

In a similar way, the mass specific balance of momentum of the material motion problem is based on the material version of the mass specific kinetic energy density

$$K = \frac{1}{2}\mathbf{V}\cdot\mathbf{C}\cdot\mathbf{V} \tag{6.35}$$

defining the mass specific material momentum density.

$$\mathbf{P} = \partial_{\mathbf{V}}K = \mathbf{C}\cdot\mathbf{V}. \tag{6.36}$$

Its rate of change is balanced with the reduced momentum fluxes $\bar{\boldsymbol{\pi}}^t_\delta$ and $\bar{\boldsymbol{\Sigma}}^t_\delta$, the reduced momentum sources $\bar{\mathbf{B}}_\tau$ and the additional M_τ-term taking into account the convective contributions. Again, four different versions can be derived as the difference of the volume specific balance of momentum (6.24) and the balance of mass (4.13) weighted by the mass specific material momentum density $\mathbf{P}$, i.e., the covariant material velocity with the appropriate metric $\mathbf{C}$.

$$\begin{array}{lll} \text{mr} & \text{mp} & \rho_0 \mathrm{D}_t\mathbf{P} = \mathrm{Div}\,\bar{\boldsymbol{\Sigma}}^t_D + \bar{\mathbf{B}}_0 - \mathsf{M}_0\mathbf{P} \\ \text{sr} & \text{mp} & \rho_t \mathrm{D}_t\mathbf{P} = \mathrm{div}\,\bar{\boldsymbol{\pi}}^t_D + \bar{\mathbf{B}}_t - \mathsf{M}_t\mathbf{P} \\ \text{mr} & \text{sp} & \rho_0 \mathrm{d}_t\mathbf{P} = \mathrm{Div}\,\bar{\boldsymbol{\Sigma}}^t_d + \bar{\mathbf{B}}_0 - \mathsf{M}_0\mathbf{P} \\ \text{sr} & \text{sp} & \rho_t \mathrm{d}_t\mathbf{P} = \mathrm{div}\,\bar{\boldsymbol{\pi}}^t_d + \bar{\mathbf{B}}_t - \mathsf{M}_t\mathbf{P} \end{array}. \tag{6.37}$$

Thereby, the corresponding reduced momentum fluxes $\bar{\boldsymbol{\pi}}^t_\delta$ and $\bar{\boldsymbol{\Sigma}}^t_\delta$ which are related to the overall momentum fluxes $\boldsymbol{\pi}^t_\delta$ and $\boldsymbol{\Sigma}^t_\delta$ as $\bar{\boldsymbol{\pi}}^t_\delta = \boldsymbol{\pi}^t_\delta - \mathbf{P}\otimes\mathbf{r}$ and $\bar{\boldsymbol{\Sigma}}^t_\delta = \boldsymbol{\Sigma}^t_\delta - \mathbf{P}\otimes\mathbf{R}$ are formally determined by the corresponding material stress vectors $\mathbf{T}^{closed}_t$ and $\bar{\mathbf{T}}^{open}_t$ through the Neumann boundary conditions

$$\begin{aligned} \bar{\boldsymbol{\pi}}^t_d\cdot\mathbf{n} &= \mathbf{T}^{closed}_t + \bar{\mathbf{T}}^{open}_t \\ \bar{\boldsymbol{\Sigma}}^t_d\cdot\mathbf{N} &= \mathbf{T}^{closed}_0 + \bar{\mathbf{T}}^{open}_0 \end{aligned}. \tag{6.38}$$

Moreover, the reduced material volume forces $\bar{\mathbf{B}}_\tau$ are given as the sum of the standard closed system material volume forces $\mathbf{B}^{closed}_\tau$ and the reduced open system contribution $\bar{\mathbf{B}}^{open}_\tau$

$$\begin{aligned} \bar{\mathbf{B}}_t &= \mathbf{B}^{closed}_t + \bar{\mathbf{B}}^{open}_t \\ \bar{\mathbf{B}}_0 &= \mathbf{B}^{closed}_0 + \bar{\mathbf{B}}^{open}_0 \end{aligned}. \tag{6.39}$$

Again, the closed system part $\mathbf{B}^{closed}_\tau$ contributes to the external, the internal and the kinetic contributions $\mathbf{B}^{ext}_\tau$, $\mathbf{B}^{int}_\tau$ and $\partial_{\boldsymbol{\Phi}}K_\tau$ while the open system term $\bar{\mathbf{B}}^{open}_\tau$ only contributes to the external sources $\mathbf{B}^{ext}_\tau$.

6.2.3 Spatial vs. material motion problem

Similar to the kinematic relation (3.15) between the contravariant velocities $\mathbf{v}$ and $\mathbf{V}$, the covariant mass specific spatial and material momentum densities $\mathbf{p}$ and $\mathbf{P}$ are related via the corresponding deformation gradients

$$\mathbf{p} = -\mathbf{f}^t\cdot\mathbf{P} \qquad \mathbf{P} = -\mathbf{F}^t\cdot\mathbf{p}. \tag{6.40}$$

Moreover, the additive decomposition of the dynamic momentum fluxes introduced for the volume specific case can be transferred to the mass specific context, thus

$$\begin{array}{ll} \bar{\boldsymbol{\Pi}}^t_D = \bar{\boldsymbol{\Pi}}^t - \mathrm{D}_{\mathbf{F}}K_0 & \bar{\boldsymbol{\Sigma}}^t_d = \bar{\boldsymbol{\Sigma}}^t - K_0\mathbf{I} + \mathbf{F}^t\cdot\mathrm{d}_{\mathbf{F}}K_0 \\ \bar{\boldsymbol{\sigma}}^t_D = \bar{\boldsymbol{\sigma}}^t - K_t\mathbf{I} + \mathbf{f}^t\cdot\mathrm{D}_{\mathbf{f}}K_t & \bar{\boldsymbol{\pi}}^t_d = \bar{\boldsymbol{\pi}}^t - \mathrm{d}_{\mathbf{f}}K_t. \end{array} \tag{6.41}$$

Correspondingly, the additive decomposition of the volume forces $\mathbf{b}_\tau$ and $\mathbf{B}_\tau$ is likewise valid for the reduced volume forces $\bar{\mathbf{b}}_\tau$ and $\bar{\mathbf{B}}_\tau$.

$$\begin{array}{ll} \bar{\mathbf{b}}_0 = \bar{\mathbf{b}}_0^{ext} + \bar{\mathbf{b}}_0^{int} + \partial_{\boldsymbol{\varphi}} K_0 & \bar{\mathbf{B}}_0 = \bar{\mathbf{B}}_0^{ext} + \bar{\mathbf{B}}_0^{int} + \partial_{\boldsymbol{\Phi}} K_0 \\ \bar{\mathbf{b}}_t = \bar{\mathbf{b}}_t^{ext} + \bar{\mathbf{b}}_t^{int} + \partial_{\boldsymbol{\varphi}} K_t & \bar{\mathbf{B}}_t = \bar{\mathbf{B}}_t^{ext} + \bar{\mathbf{B}}_t^{int} + \partial_{\boldsymbol{\Phi}} K_t \end{array}. \tag{6.42}$$

7 Balance of kinetic energy

The balance of kinetic energy can be interpreted as a particular weighted form of the balance of momentum, i.e., weighted by the appropriate velocity field $\mathbf{v}$ or $\mathbf{V}$ modified by a weighted version of the balance of mass in case of open systems. Thus, the balance of kinetic energy does not constitute an independent balance law. Yet, it proves significant to discuss it in detail since it will help to introduce work conjugate stress and strain pairs. Moreover, the balance of kinetic energy will be used to identify the external and internal mechanical power which are essential for our further thermodynamical considerations.

7.1 Volume specific version

7.1.1 Spatial motion problem

The material time derivative of the volume specific kinetic energy density $K_0 = 1/2\rho_0 \mathbf{v} \cdot \mathbf{g} \cdot \mathbf{v}$ can be expressed as follows

$$\mathrm{D}_t K_0 = \mathbf{v} \cdot [\mathrm{D}_t \mathbf{p}_0 - \partial_{\boldsymbol{\varphi}} K_0] - \mathrm{D}_{\mathbf{F}} K_0 : \mathrm{D}_t \mathbf{F} - K[\mathrm{D}_t \rho_0 - \mathsf{m}_0], \tag{7.1}$$

whereby the second, the third and the fifth term vanish identically for the spatial motion problem as $\partial_{\boldsymbol{\varphi}} K_0 = \mathbf{0}$, $\mathrm{D}_{\mathbf{F}} K_0 = \mathbf{0}$ and $\mathsf{m}_0 = 0$. With the projection of the volume specific balance of momentum (6.3) with the spatial velocity $\mathbf{v}$

$$\mathbf{v} \cdot \mathrm{D}_t \mathbf{p}_0 = \mathrm{Div}(\mathbf{v} \cdot \boldsymbol{\Pi}_D^t) + \mathbf{v} \cdot [\mathbf{b}_0^{ext} + \mathbf{b}_0^{int} + \partial_{\boldsymbol{\varphi}} K_0] - [\boldsymbol{\Pi}^t - \mathrm{D}_{\mathbf{F}} K_0] : \mathrm{D}_t \mathbf{F} \tag{7.2}$$

and the balance of mass (4.13.1) weighted by the mass specific kinetic energy density K

$$K \mathrm{D}_t \rho_0 = \mathrm{Div}(K\mathbf{R}) + K\mathscr{R}_0 - \nabla_X K \cdot \mathbf{R} + \mathsf{m}_0 K, \tag{7.3}$$

Eq. (7.1) can be rewritten in explicit form

$$\mathrm{D}_t K_0 = \mathrm{Div}(\mathbf{v} \cdot \boldsymbol{\Pi}_D^t - K\,\mathbf{R}) + \mathbf{v} \cdot \mathbf{b}_0^{ext} - K\mathscr{R}_0 + \nabla_X K \cdot \mathbf{R} - \boldsymbol{\Pi}^t : \mathrm{D}_t \mathbf{F} + \mathbf{v} \cdot \mathbf{b}_0^{int}. \tag{7.4}$$

In what follows, it will prove convenient to reformulate the above equation in terms of the reduced momentum flux $\bar{\boldsymbol{\Pi}}_D^t$ and the reduced momentum source $\bar{\mathbf{b}}_0^{ext}$ which can be related to their overall counterparts $\boldsymbol{\Pi}_D^t$ and $\mathbf{b}_0^{ext}$ through the following identities

$$\mathrm{Div}(\mathbf{v} \cdot \boldsymbol{\Pi}_D^t) = \mathrm{Div}(\mathbf{v} \cdot \bar{\boldsymbol{\Pi}}_D^t) + \mathrm{Div}(2K\mathbf{R}) \tag{7.5.1}$$

$$\mathbf{v} \cdot \mathbf{b}_0^{ext} = \mathbf{v} \cdot \bar{\mathbf{b}}_0^{ext} + 2K\mathscr{R}_0 - \mathbf{v} \cdot \nabla_X \mathbf{p} \cdot \mathbf{R} \tag{7.5.2}$$

$$\boldsymbol{\Pi}^t : \mathrm{D}_t \mathbf{F} = \bar{\boldsymbol{\Pi}}^t : \mathrm{D}_t \mathbf{F} + \mathbf{p} \cdot \nabla_X \mathbf{v} \cdot \mathbf{R}. \tag{7.5.3}$$

With the help of the above equations and the identity $\mathbf{v} \cdot \nabla_X \mathbf{p} + \mathbf{p} \cdot \nabla_X \mathbf{v} = 2\nabla_X K$ following from $\mathbf{v} \cdot \mathbf{p} = 2K$, Eq. (7.4) can be reformulated in the following way

$$\mathrm{D}_t K_0 = \mathrm{Div}(\mathbf{v} \cdot \bar{\boldsymbol{\Pi}}_D^t + K\,\mathbf{R}) + \mathbf{v} \cdot \bar{\mathbf{b}}_0^{ext} + K\mathscr{R}_0 - \nabla_X K \cdot \mathbf{R} - \bar{\boldsymbol{\Pi}}^t : \mathrm{D}_t \mathbf{F} + \mathbf{v} \cdot \bar{\mathbf{b}}_0^{int}. \tag{7.6}$$

As stated already be Stokes as early as 1857, the rate of increase of the kinetic energy is equal to the input of external mechanical power minus the internal mechanical power, i.e., in the spatial motion context the stress power. The righthand side of the above equation thus motivates the identification of the volume specific external and internal mechanical power p_0^{ext} and p_0^{int},

$$\begin{aligned}\mathsf{p}_0^{ext} &:= \mathrm{Div}(\mathbf{v}\cdot\bar{\mathbf{\Pi}}_D^t + K\mathbf{R}) + \mathbf{v}\cdot\bar{\mathbf{b}}_0^{ext} + K\mathscr{R}_0 - \nabla_X K\cdot\mathbf{R}\\ \mathsf{p}_0^{int} &:= \bar{\mathbf{\Pi}}^t : \mathrm{D}_t\mathbf{F} - \mathbf{v}\cdot\bar{\mathbf{b}}_0^{int},\end{aligned} \tag{7.7}$$

whereby p_0^{ext} characterizes the total rate of working of mechanical actions on the body. This rate of working consists of the flux contribution $\mathbf{v}\cdot\bar{\mathbf{\Pi}}_D^t + K\mathbf{R}$ and the source term $\mathbf{v}\cdot\bar{\mathbf{b}}_0^{ext} + K\mathscr{R}_0 - \nabla_X K\cdot\mathbf{R}$. The internal mechanical power p_0^{int} includes the production term for the kinetic energy as $\bar{\mathbf{\Pi}}^t : \mathrm{D}_t\mathbf{F} - \mathbf{v}\cdot\bar{\mathbf{b}}_0^{int}$. The definition of the latter suggests the interpretation of the reduced momentum flux $\bar{\mathbf{\Pi}}^t$ and the material time derivative of the spatial motion deformation gradient $\mathrm{D}_t\mathbf{F}$ as work conjugate pairs. To highlight the duality with the material motion problem, the internal force contribution has been included in the definition of the internal mechanical power p_0^{int} although this terms vanishes identically in the spatial motion case as $\bar{\mathbf{b}}_0^{int} = \mathbf{0}$. With the above abbreviations at hand, the balance of kinetic energy can be rewritten in the following form

$$\mathrm{D}_t K_0 = \mathsf{p}_0^{ext} - \mathsf{p}_0^{int} \tag{7.8}$$

which has been denoted as the local form the "theorem of energy", by Maugin [34]. With the related Piola transforms $K_0 = JK_t$, $\mathsf{p}_0^{ext} = J\mathsf{p}_t^{ext}$ and $\mathsf{p}_0^{int} = J\mathsf{p}_t^{int}$ and the volume specific version of Reynold's transport theorem (3.22), we easily obtain the alternative versions of the volume specific kinetic energy balance of the spatial motion problem with spatial reference and spatial parametrization.

7.1.2 Material motion problem

In complete analogy to the spatial motion problem, the spatial time derivative of the volume specific kinetic energy density $K_t = 1/2\rho_t\mathbf{V}\cdot\mathbf{C}\cdot\mathbf{V}$ is given in the following form

$$\mathrm{d}_t K_t = \mathbf{V}\cdot[\mathrm{d}_t\mathbf{P}_t - \partial_{\mathbf{\Phi}}K_t] - \mathrm{d}_{\mathbf{f}}K_t : \mathrm{d}_t\mathbf{f} - K[\mathrm{d}_t\rho_t - \mathsf{M}_t]. \tag{7.9}$$

Note, however, that in contrast to the spatial motion problem, the terms $\partial_{\mathbf{\Phi}}K_t$, $\mathrm{d}_{\mathbf{f}}K_t$ and M_t do not vanish for the material motion problem. With the projection of the volume specific balance of momentum (6.16) with the material velocity $\mathbf{V}$

$$\mathbf{V}\cdot\mathrm{d}_t\mathbf{P}_t = \mathrm{div}\big(\mathbf{V}\cdot\boldsymbol{\pi}_d^t\big) + \mathbf{V}\cdot\big[\mathbf{B}_t^{ext} + \mathbf{B}_t^{int} + \partial_{\mathbf{\Phi}}K_t\big] - \big[\boldsymbol{\pi}^t - \mathrm{d}_{\mathbf{f}}K_t\big] : \mathrm{d}_t\mathbf{f} \tag{7.10}$$

and the balance of mass (4.13.4) weighted by the mass specific kinetic energy density K

$$K\mathrm{d}_t\rho_t = \mathrm{div}(K\mathbf{r}) + K\mathscr{R}_t - \nabla_x K\cdot\mathbf{r} + \mathsf{M}_t K \tag{7.11}$$

the above stated balance of kinetic energy takes the following explicit form

$$\mathrm{d}_t K_t = \mathrm{div}\big(\mathbf{V}\cdot\boldsymbol{\pi}_d^t - K\mathbf{r}\big) + \mathbf{V}\cdot\mathbf{B}_t^{ext} - K\mathscr{R}_t + \nabla_x K\cdot\mathbf{r} - \boldsymbol{\pi}^t : \mathrm{d}_t\mathbf{f} + \mathbf{V}\cdot\mathbf{B}_t^{int}. \tag{7.12}$$

Similar to the spatial motion problem, we will reformulate the above equation by making use of the fundamental relations between the reduced and non-reduced flux and source terms

$$\mathrm{div}(\mathbf{V}\cdot\boldsymbol{\pi}_d^t) = \mathrm{div}(\mathbf{V}\cdot\bar{\boldsymbol{\pi}}_d^t) + \mathrm{Div}(2K\mathbf{r}) \tag{7.13.1}$$

$$\mathbf{V}\cdot\mathbf{B}_t^{ext} = \mathbf{V}\cdot\bar{\mathbf{B}}_t^{ext} + 2K\mathscr{R}_t - \mathbf{V}\cdot\nabla_x\mathbf{P}\cdot\mathbf{r} \tag{7.13.2}$$

$$\boldsymbol{\pi}^t : \mathrm{d}_t\mathbf{f} = \bar{\boldsymbol{\pi}}^t : \mathrm{d}_t\mathbf{f} + \mathbf{P}\cdot\nabla_x\mathbf{V}\cdot\mathbf{r} \tag{7.13.3}$$

which eventually render the following expression

$$\mathrm{d}_t K_t = \mathrm{div}\big(\mathbf{V}\cdot\bar{\boldsymbol{\pi}}_d^t + K\mathbf{r}\big) + \mathbf{V}\cdot\bar{\mathbf{B}}_t^{ext} + K\mathscr{R}_t - \nabla_x K\cdot\mathbf{r} - \bar{\boldsymbol{\pi}}^t : \mathrm{d}_t\mathbf{f} + \mathbf{V}\cdot\bar{\mathbf{B}}_t^{int}. \tag{7.14}$$

Notice the remarkable duality of the above expression with its spatial motion counterpart (7.6). This beautiful analogy, see again our leitmotif, is only possible due to our specific choice of volume sources. The identification of the material motion external and internal mechanical power P_t^{ext} and P_t^{int}

$$\mathsf{P}_t^{ext} := \mathrm{div}\big(\mathbf{V}\cdot\bar{\boldsymbol{\pi}}_d^t + K\mathbf{r}\big) + \mathbf{V}\cdot\bar{\mathbf{B}}_t^{ext} + K\mathscr{R}_t - \nabla_x K\cdot\mathbf{r} \tag{7.15.1}$$

$$\mathsf{P}_t^{int} := \bar{\boldsymbol{\pi}}^t : \mathrm{d}_t\mathbf{f} - \mathbf{V}\cdot\bar{\mathbf{B}}_t^{int} \tag{7.15.2}$$

allows for the following shorthanded notation of Eq. (7.14).

$$\mathrm{d}_t K_t = \mathsf{P}_t^{ext} - \mathsf{P}_t^{int}. \tag{7.16}$$

Similar to the spatial motion problem, the definition of the material motion internal power P_t^{int} motivates the definition of the reduced momentum flux $\bar{\boldsymbol{\pi}}^t$ and the spatial time derivative of the material motion deformation gradient $\mathrm{d}_t\mathbf{f}$ as work conjugate pairs. The appropriate Piola transforms $K_t = jK_0$, $\mathsf{P}_t^{ext} = j\mathsf{P}_0^{ext}$ and $\mathsf{P}_t^{int} = j\mathsf{P}_0^{int}$ and the application of the volume specific version of Reynold's transport theorem (3.22) could be used to derive the alternative formulations with material reference and material parametrization.

7.1.3 Spatial vs. material motion problem

A comparison of the spatial and the material motion problem (7.8) and (7.16) based on the volume specific version of Reynold's transport theorem (3.22) defines the following identities

$$\mathsf{p}_0^{ext} - \mathsf{p}_0^{int} = \mathsf{P}_0^{ext} - \mathsf{P}_0^{int} - \mathrm{Div}(K_0\mathbf{V}) \tag{7.17.1}$$

$$\mathsf{P}_t^{ext} - \mathsf{P}_t^{int} = \mathsf{p}_t^{ext} - \mathsf{p}_t^{int} - \mathrm{div}(K_t\mathbf{v}). \tag{7.17.2}$$

7.2 Mass specific version

7.2.1 Spatial motion problem

To investigate the mass specific version of the balance of kinetic energy, we need to evaluate the material time derivative of the mass specific kinetic energy density $K = 1/2\mathbf{v}\cdot\mathbf{g}\cdot\mathbf{v}$ which can easily be derived by subtracting the weighted balance of mass (7.3) from the material time derivative of the volume specific kinetic energy density K_0 given in Eq. (7.6) as

$$\rho_0\mathrm{D}_t K = \mathrm{D}_t K_0 - K\mathrm{D}_t\rho_0 \tag{7.18}$$

and thus

$$\rho_0\mathrm{D}_t K = \mathrm{Div}\big(\mathbf{v}\cdot\bar{\boldsymbol{\Pi}}_D^t\big) + \mathbf{v}\cdot\bar{\mathbf{b}}_0^{ext} - \mathsf{m}_0 K - \bar{\boldsymbol{\Pi}}^t : \mathrm{D}_t\mathbf{F} + \mathbf{v}\cdot\bar{\mathbf{b}}_0^{int}. \tag{7.19}$$

Consequently, we can identify the mass specific external and internal mechanical power $\bar{\mathsf{p}}_0^{ext}$ and $\bar{\mathsf{p}}_0^{int}$,

$$\bar{\mathsf{p}}_0^{ext} := \mathsf{p}_0^{ext} - K\mathrm{D}_t\rho_0 + \mathsf{m}_0 K = \mathrm{Div}(\mathbf{v}\cdot\bar{\boldsymbol{\Pi}}_D^t) + \mathbf{v}\cdot\bar{\mathbf{b}}_0^{ext} \tag{7.20.1}$$

$$\bar{\mathsf{p}}_0^{int} := \mathsf{p}_0^{int} = \bar{\boldsymbol{\Pi}}^t : \mathrm{D}_t\mathbf{F} - \mathbf{v}\cdot\bar{\mathbf{b}}_0^{int} \tag{7.20.2}$$

keeping in mind that the internal force term $\bar{\mathbf{b}}_0^{int} = \mathbf{0}$ vanishes identically for the spatial motion case and has only been included to stress the duality with the definition based on the material motion problem. With the above definitions at hand, the mass specific balance of kinetic energy can be expressed in the following form

$$\rho_0\mathrm{D}_t K = \bar{\mathsf{p}}_0^{ext} - \bar{\mathsf{p}}_0^{int} - \mathsf{m}_0 K. \tag{7.21}$$

It is worth noting, that the difference of the mass and volume specific formulation manifests itself only in the definition of the external mechanical power while the mass specific internal power is identical to its volume specific counterpart as $\bar{\mathsf{p}}_0^{int} = \mathsf{p}_0^{int}$. To derive the alternative formulations of Eq. (7.21) we have to make use of the related Piola transforms with $\bar{\mathsf{p}}_0^{ext} = J\bar{\mathsf{p}}_t^{ext}$ and $\bar{\mathsf{p}}_0^{int} = J\bar{\mathsf{p}}_t^{int}$ and the Euler theorem (3.21).

7.2.2 Material motion problem

In analogy to the spatial motion problem, the spatial time derivative of the mass specific kinetic energy density $K = 1/2\mathbf{V}\cdot\mathbf{C}\cdot\mathbf{V}$ is given as the difference of the volume specific version (7.14) and the corresponding weighted balance of mass (7.11)

$$\rho_t \mathrm{d}_t K = \mathrm{d}_t K_t - K \mathrm{d}_t \rho_t. \tag{7.22}$$

Consequently, the explicit form

$$\rho_t \mathrm{d}_t K = \mathrm{div}(\mathbf{V}\cdot\bar{\boldsymbol{\pi}}_d^t) + \mathbf{V}\cdot\bar{\mathbf{B}}_t^{ext} - \mathsf{M}_t K - \bar{\boldsymbol{\pi}}^t : \mathrm{d}_t\mathbf{f} + \mathbf{V}\cdot\bar{\mathbf{B}}_t^{int} \tag{7.23}$$

defines the mass specific external and internal power $\bar{\mathsf{P}}_t^{ext}$ and $\bar{\mathsf{P}}_t^{int}$ as

$$\bar{\mathsf{P}}_t^{ext} := \mathsf{P}_t^{ext} - K\mathrm{d}_t\rho_t + \mathsf{M}_t K = \mathrm{div}(\mathbf{V}\cdot\bar{\boldsymbol{\pi}}_d^t) + \mathbf{V}\cdot\bar{\mathbf{B}}_t^{ext} \tag{7.24.1}$$

$$\bar{\mathsf{P}}_t^{int} := \mathsf{P}_t^{int} = \bar{\pi}^t : \mathrm{d}_t\mathbf{f} - \mathbf{V}\cdot\bar{\mathbf{B}}_t^{int}, \tag{7.24.2}$$

whereby again, the mass specific internal power is defined to be identical to its volume specific counterpart as $\bar{\mathsf{P}}_t^{int} = \mathsf{P}_t^{int}$. The mass specific balance of kinetic energy of the material motion problem

$$\rho_t \mathrm{d}_t K = \bar{\mathsf{P}}_t^{ext} - \bar{\mathsf{P}}_t^{int} - \mathsf{M}_t K \tag{7.25}$$

could alternatively be reformulated with the related Piola transforms $\bar{\mathrm{P}}_t^{ext} = j\bar{\mathsf{P}}_0^{ext}$ and $\bar{\mathsf{P}}_t^{int} = j\bar{\mathsf{P}}_0^{int}$ and the corresponding versions of the Euler theorem (3.21).

7.2.3 Spatial vs. material motion problem

A comparison of the spatial and the material motion formulations (7.21) and (7.25) based on the mass specific version of Reynold's transport theorem (4.14) reveals the following identities

$$\bar{\mathsf{p}}_0^{ext} - \bar{\mathsf{p}}_0^{int} = \bar{\mathsf{P}}_0^{ext} - \bar{\mathsf{P}}_0^{int} - \mathrm{Div}(K_0\mathbf{V}) \tag{7.26.1}$$

$$\bar{\mathsf{P}}_t^{ext} - \bar{\mathsf{P}}_t^{int} = \bar{\mathsf{p}}_t^{ext} - \bar{\mathsf{p}}_t^{int} - \mathrm{div}(K_t\mathbf{v}). \tag{7.26.2}$$

Remarkably, the difference of the spatial and the material motion quantities, $\mathrm{Div}(K_0\mathbf{V})$ or $\mathrm{div}(K_t\mathbf{v})$, respectively, is identical for the volume specific and the mass specific formulation, compare (7.17).

8 Balance of energy

The balance of total energy as a representation of the first law of thermodynamics balances the rate of change of the volume specific total energy density $E_\tau = \rho_\tau E$ as the sum of the kinetic and internal energy density $E_\tau = K_\tau + I_\tau$ with the external power. In classical continuum mechanics of closed systems, this external power is composed of a purely mechanical contribution p_τ^{ext} or P_τ^{ext} and a non-mechanical thermal contribution q_τ^{ext} or Q_τ^{ext}. Therefore, the balance of energy is sometimes referred to as "principle of interconvertibility of heat and mechanical work'', a notion which goes back to Carnot 1832. However, when dealing with open systems, we have to generalize the definition of the non-mechanical external power by including an additional external open system contribution in the definition of q_τ^{ext} and Q_τ^{ext}.

8.1. Volume specific version

8.1.1 Spatial motion problem

For the spatial motion problem, the rate of change of the volume specific total energy density $E_0 = \rho_0 E$ can be expressed in the following form

$$\mathrm{D}_t E_0 = \mathrm{Div}(\mathbf{v} \cdot \bar{\mathbf{\Pi}}_D^t - \bar{\mathbf{Q}}_D + E\mathbf{R}) + \mathbf{v} \cdot \bar{\mathbf{b}}_0^{ext} + \bar{\mathcal{Q}}_0 + E\mathcal{R}_0 - \nabla_X E \cdot \mathbf{R}. \tag{8.1}$$

Thereby, in addition to the purely mechanical external power p_0^{ext} already defined in Eq. (7.7), we have included the non-mechanical external power q_0^{ext} accounting for the classical thermal effects of the closed system and the additional open system effects as an additional non-mechanical supply of energy.

$$\mathsf{q}_0^{ext} := \mathrm{Div}(-\mathbf{Q}_D) + \mathcal{Q}_0. \tag{8.2}$$

Similar to the mechanical power, the non-mechanical power consists of a flux and a source contribution, denoted by $\mathbf{Q}_D$ and $\mathcal{Q}_0$, respectively. The former is composed of the reduced outward non-mechanical energy flux $\bar{\mathbf{Q}}_D$ modified by the explicit extra flux due to the open system $I\mathbf{R}$, while the latter is the sum of the reduced non-mechanical energy source $\bar{\mathcal{Q}}_0$ enhanced by the explicit effects of the open system $I\mathcal{R}_0$ and $\nabla_X I \cdot \mathbf{R}$

$$\mathsf{p}_0^{ext} := \mathrm{Div}(\mathbf{v} \cdot \bar{\mathbf{\Pi}}_D^t + K\mathbf{R}) + \mathbf{v} \cdot \bar{\mathbf{b}}_0^{ext} + K\mathcal{R}_0 - \nabla_X K \cdot \mathbf{R}, \tag{8.3.1}$$

$$\mathsf{q}_0^{ext} := \mathrm{Div}(-\bar{\mathbf{Q}}_D + I\mathbf{R}) + \bar{\mathcal{Q}}_0 + I\mathcal{R}_0 - \nabla_X I \cdot \mathbf{R}. \tag{8.3.2}$$

Equation (8.1) can thus be reformulated in the following concise form

$$\mathrm{D}_t E_0 = \mathsf{p}_0^{ext} + \mathsf{q}_0^{ext} \tag{8.4}$$

which for the classical closed system case dates back to the early works of Duhem in 1892. The appropriate Piola transforms $E_0 = JE_t$, $\mathsf{p}_0^{ext} = J\mathsf{p}_t^{ext}$, $\mathsf{q}_0^{ext} = J\mathsf{q}_t^{ext}$, $\mathbf{Q}_D = J\mathbf{q}_D \cdot \mathbf{F}^{-t}$ and $\bar{\mathbf{Q}}_D = J\bar{\mathbf{q}}_D \cdot \mathbf{F}^{-t}$ can be used together with the application of the volume specific transport theorem to derive the alternative formulations of Eq. (8.4). On the Neumann boundary, the non-mechanical energy fluxes of Piola–Kirchhoff and Cauchy type $\mathbf{Q}_D$ and $\mathbf{q}_d$ are given in terms of the normal projection of the classical heat flux related to the closed system q_τ^{closed} and the additional open system contribution q_τ^{open}

$$\begin{aligned} \mathbf{Q}_D \cdot \mathbf{N} &= q_0^{closed} + q_0^{open} \quad q_0^{open} = \bar{q}_0^{open} - I\mathbf{R} \cdot \mathbf{N}, \\ \mathbf{q}_D \cdot \mathbf{n} &= q_t^{closed} + q_t^{open} \quad q_t^{open} = \bar{q}_t^{open} - I\mathbf{r} \cdot \mathbf{n} \quad . \end{aligned} \tag{8.5}$$

Moreover, the non-mechanical energy sources $\mathcal{Q}_\tau$ can be understood as the sum of the classical heat source of the closed system $\mathcal{Q}_\tau^{closed}$ and an additional non-mechanical energy source taking into account the nature of the open system $\mathcal{Q}_\tau^{open}$

$$\begin{aligned} \mathcal{Q}_0 &= \mathcal{Q}_0^{closed} + \mathcal{Q}_0^{open} \quad \mathcal{Q}_0^{open} = \bar{\mathcal{Q}}_0{}^{open} + I\mathcal{R}_0 - \nabla_X I \cdot \mathbf{R}, \\ \mathcal{Q}_t &= \mathcal{Q}_t^{closed} + \mathcal{Q}_t^{open} \quad \mathcal{Q}_t^{open} = \bar{\mathcal{Q}}_t^{open} + I\mathcal{R}_t - \nabla_x I \cdot \mathbf{r} \quad . \end{aligned} \tag{8.6}$$

By subtracting the balance of kinetic energy (7.8) from the total energy balance (8.4), we obtain in addition the balance equation of the internal energy density $I_0 = E_0 - K_0$,

$$\mathrm{D}_t I_0 = \mathsf{p}_0^{int} + \mathsf{q}_0^{ext} \tag{8.7}$$

which will be useful for our further thermodynamical considerations.

8.1.2 Material motion problem

The balance of the volume specific total energy density $E_t = \rho_t E$ of the material motion problem can formally be stated as follows

$$\mathrm{d}_t E_t = \mathrm{div}(\mathbf{V} \cdot \bar{\boldsymbol{\pi}}_d^t - \bar{\mathbf{q}}_d + E\mathbf{r}) + \mathbf{V} \cdot \bar{\mathbf{B}}_t^{ext} + \bar{\mathscr{Q}}_t + E\mathscr{R}_t - \nabla_x E \cdot \mathbf{r}. \tag{8.8}$$

It balances the spatial rate of change of the total energy density $E_t = K_t + I_t$ with the material external mechanical power $\mathsf{P}_\mathsf{t}^{ext}$ and the external non-mechanical power $\mathsf{Q}_\mathsf{t}^{ext}$, whereby the latter consists of the material motion flux of non-mechanical energy $\mathbf{q}_d$ and the related material motion source $\mathscr{Q}_t$

$$\mathsf{Q}_\mathsf{t}^{ext} := \mathrm{div}(-\mathbf{q}_d) + \mathscr{Q}_t. \tag{8.9}$$

As for the spatial motion problem, the contributions $\mathbf{q}_d$ and $\mathscr{Q}_t$ can be expressed explicitly in terms of their reduced counterparts $\bar{\mathbf{q}}_d$ and $\bar{\mathscr{Q}}_t$ and the additional open system extra terms $I\mathbf{r}$, $I\mathscr{R}_t$ and $\nabla_x I \cdot \mathbf{r}$.

$$\mathsf{P}_\mathsf{t}^{ext} := \mathrm{div}(\mathbf{V} \cdot \bar{\boldsymbol{\pi}}_d^t + K\mathbf{r}) + \mathbf{V} \cdot \bar{\mathbf{B}}_t^{ext} + K\mathscr{R}_t - \nabla_x K \cdot \mathbf{r}, \tag{8.10.1}$$

$$\mathsf{Q}_\mathsf{t}^{ext} := \mathrm{Div}(-\bar{\mathbf{q}}_d + I\mathbf{r}) + \bar{\mathscr{Q}}_t + I\mathscr{R}_t - \nabla_x I \cdot \mathbf{r}. \tag{8.10.2}$$

Equation (8.8) can thus be rewritten as follows

$$\mathrm{d}_t E_t = \mathsf{P}_\mathsf{t}^{ext} + \mathsf{Q}_\mathsf{t}^{ext}. \tag{8.11}$$

Its alternative formulations could be derived through the application of the related Piola transforms $E_t = jE_0$, $\mathsf{P}_\mathsf{t}^{ext} = j\mathsf{P}_0^{ext}$, $\mathsf{Q}_\mathsf{t}^{ext} = j\mathsf{Q}_0^{ext}$, $\mathbf{q}_d = j\mathbf{Q}_d \cdot \mathbf{f}^{-t}$ and $\bar{\mathbf{q}}_d = j\bar{\mathbf{Q}}_d \cdot \mathbf{f}^{-t}$ in combination with the volume specific transport theorem (3.22). In complete analogy to the spatial motion problem, we can formally introduce boundary conditions for the non-mechanical energy flux $\mathbf{q}_d$ and $\mathbf{Q}_d$

$$\begin{aligned} \mathbf{q}_d \cdot \mathbf{n} &= Q_t^{closed} + Q_t^{open} & Q_t^{open} &= \bar{Q}_t^{open} - I\mathbf{r} \cdot \mathbf{n}, \\ \mathbf{Q}_d \cdot \mathbf{N} &= Q_0^{closed} + Q_0^{open} & Q_0^{open} &= \bar{Q}_0^{open} - I\mathbf{R} \cdot \mathbf{N} \end{aligned} \tag{8.12}$$

and define the non-mechanical heat sources $\mathscr{Q}_\tau$ in formal analogy to the spatial motion case

$$\begin{aligned} \mathscr{Q}_t &= \mathscr{Q}_t^{closed} + \mathscr{Q}_t^{open} & \mathscr{Q}_t^{open} &= \bar{\mathscr{Q}}_t^{open} + I\mathscr{R}_t - \nabla_x I \cdot \mathbf{r}, \\ \mathscr{Q}_0 &= \mathscr{Q}_0^{closed} + \mathscr{Q}_0^{open} & \mathscr{Q}_0^{open} &= \bar{\mathscr{Q}}_0^{open} + I\mathscr{R}_0 - \nabla_X I \cdot \mathbf{R}. \end{aligned} \tag{8.13}$$

Again, a reduction to the useful balance of internal energy density $I_t = E_t - K_t$ can be derived by subtracting the balance of kinetic energy (7.16) from the balance of total energy (8.11)

$$\mathrm{d}_t I_t = \mathsf{P}_\mathsf{t}^{int} + \mathsf{Q}_\mathsf{t}^{ext}. \tag{8.14}$$

8.1.3 Spatial vs. material motion problem

A comparison of the balance of total energy of the spatial and the material motion problem (8.4) and (8.11) together with the volume specific version of Reynold's transport theorem (3.22) reveals the following relations

$$\mathsf{p}_0^{ext} + \mathsf{q}_0^{ext} = \mathsf{P}_0^{ext} + \mathsf{Q}_0^{ext} - \mathrm{Div}(E_0 \mathbf{V}), \tag{8.15.1}$$

$$\mathsf{P}_\mathsf{t}^{ext} + \mathsf{Q}_\mathsf{t}^{ext} = \mathsf{p}_\mathsf{t}^{ext} + \mathsf{q}_\mathsf{t}^{ext} - \mathrm{div}(E_t \mathbf{v}). \tag{8.15.2}$$

Furthermore, we can state the following identities

$$\mathsf{p}_0^{int} + \mathsf{q}_0^{ext} = \mathsf{P}_0^{int} + \mathsf{Q}_0^{ext} - \mathrm{Div}(I_0\mathbf{V}), \tag{8.16.1}$$

$$\mathsf{P}_\mathsf{t}^{int} + \mathsf{Q}_\mathsf{t}^{ext} = \mathsf{p}_\mathsf{t}^{ext} + \mathsf{q}_\mathsf{t}^{ext} - \mathrm{div}(I_t\mathbf{v}) \tag{8.16.2}$$

which follow from a comparison of the different version of the volume specific balance of internal energy. Note in anticipation of chap. 8.2.3, that their closer evaluation yields the same results as the evaluation of the corresponding mass specific equations which will be elaborated later on.

8.2 Mass specific version

8.2.1 Spatial motion problem

The mass specific counterpart of the equations derived above balances the mass specific energy density $E = E_0/\rho_0$ with the mass specific external mechanical power $\bar{\mathsf{p}}_0^{ext}$ introduced in (7.20) and the mass specific non-mechanical power $\bar{\mathsf{q}}_0^{ext}$ as

$$\bar{\mathsf{q}}_0^{ext} := \mathsf{q}_0^{ext} - I\mathrm{D}_t\rho_0 + \mathsf{m}_0 I = -\mathrm{Div}\,\bar{\mathbf{Q}}_D + \bar{\mathscr{Q}}_0. \tag{8.17}$$

The corresponding balance equation

$$\rho_0\mathrm{D}_t E = \bar{\mathsf{p}}_0^{ext} + \bar{\mathsf{q}}_0^{ext} - \mathsf{m}_0 E \tag{8.18}$$

follows from subtracting the corresponding balance of mass (4.13.1) weighted by the total energy E from the volume specific energy balance (8.4.1). Alternative formulations can be derived by applying the corresponding Piola transforms with $\bar{\mathsf{p}}_0^{ext} = J\bar{\mathsf{p}}_\mathsf{t}^{ext}$ and $\bar{\mathsf{q}}_0^{ext} = J\bar{\mathsf{q}}_\mathsf{t}^{ext}$ and the Euler theorem (3.21). Again, we can relate the reduced energy fluxes $\bar{\mathbf{Q}}_D$ and $\bar{\mathbf{q}}_D$ defined through $\bar{\mathbf{Q}}_D = \mathbf{Q}_D - I\mathbf{R}$ and $\bar{\mathbf{q}}_D = \mathbf{q}_D - I\mathbf{r}$ to the classical heat flux q_τ^{closed} and the energy flux caused by additional effects of the open system $\bar{q}_\tau^{open}$

$$\bar{\mathbf{Q}}_D \cdot \mathbf{N} = q_0^{closed} + \bar{q}_0^{open}, \tag{8.19.1}$$

$$\bar{\mathbf{q}}_D \cdot \mathbf{n} = q_t^{closed} + \bar{q}_t^{open}. \tag{8.19.2}$$

Moreover, the reduced non-mechanical energy sources $\bar{\mathscr{Q}}_\tau$ are given as the sum of the classical heat source of a closed system $\mathscr{Q}_\tau^{closed}$ and the additional open system contribution to the energy $\bar{\mathscr{Q}}_\tau^{open}$

$$\bar{\mathscr{Q}}_0 = \mathscr{Q}_0^{closed} + \bar{\mathscr{Q}}_0^{open}, \tag{8.20.1}$$

$$\bar{\mathscr{Q}}_t = \mathscr{Q}_t^{closed} + \bar{\mathscr{Q}}_t^{open}. \tag{8.20.2}$$

A reduction to the balance of internal energy $I = E - K$ follows from by subtracting the balance of kinetic energy (7.21) from the balance of total energy (8.18)

$$\rho_0\mathrm{D}_t I = \bar{\mathsf{p}}_0^{int} + \bar{\mathsf{q}}_0^{ext} - \mathsf{m}_0 I. \tag{8.21}$$

Recall, that the convective mass contribution $\mathsf{m}_0 = 0$ vanishes identically for the spatial motion case. Consequently, the mass specific balance equations of total and internal energy are free from all the explicit extra terms caused by the changes in mass and, remarkably, take a similar structure as the standard balance equations for classical closed systems.

8.2.2 Material motion problem

The mass specific balance of energy of the material motion problem balances the rate of change of the mass specific energy $E = E_t/\rho_t$ with the corresponding mechanical external power $\bar{\mathsf{P}}_\mathsf{t}^{ext}$ and the non-mechanical external power $\bar{\mathsf{Q}}_\mathsf{t}^{ext}$ with

$$\bar{\mathsf{Q}}_{\mathsf{t}}^{ext} := \mathsf{Q}_{\mathsf{t}}^{ext} - I\mathrm{d}_t\rho_t + \mathsf{M}_{\mathsf{t}}I = -\mathrm{div}\,\bar{\mathbf{q}}_d + \bar{\mathscr{Q}}_t. \tag{8.22}$$

In short form, the mass specific balance of energy with spatial reference and spatial parametrization can be expressed as

$$\rho_t\mathrm{d}_tE = \bar{\mathsf{P}}_{\mathsf{t}}^{ext} + \bar{\mathsf{Q}}_{\mathsf{t}}^{ext} - \mathsf{M}_{\mathsf{t}}E \tag{8.23}$$

while alternative formulations can be derived through the corresponding Piola transforms with $\bar{\mathsf{P}}_{\mathsf{t}}^{ext} = j\bar{\mathsf{P}}_0^{ext}$ and $\bar{\mathsf{Q}}_{\mathsf{t}}^{ext} = j\bar{\mathsf{Q}}_0^{ext}$ in combination with the Euler theorem (3.21). Again, to illustrate the duality with the classical spatial motion problem, Neumann boundary conditions can formally be introduced for the reduced non-mechanical energy fluxes $\bar{\mathbf{q}}_d$ and $\bar{\mathbf{Q}}_d$ defined through $\bar{\mathbf{q}}_d = \mathbf{q}_d - I\mathbf{r}$ and $\bar{\mathbf{Q}}_d = \mathbf{Q}_d - I\mathbf{R}$ in the following way

$$\bar{\mathbf{q}}_d \cdot \mathbf{n} = Q_t^{closed} + \bar{Q}_t^{open}, \tag{8.24.1}$$

$$\bar{\mathbf{Q}}_d \cdot \mathbf{N} = Q_0^{closed} + \bar{Q}_0^{open} \tag{8.24.2}$$

while the reduced non-mechanical energy sources are given as follows:

$$\bar{\mathscr{Q}}_t = \mathscr{Q}_t^{closed} + \bar{\mathscr{Q}}_t^{open}, \tag{8.25.1}$$

$$\bar{\mathscr{Q}}_0 = \mathscr{Q}_0^{closed} + \bar{\mathscr{Q}}_0^{open}. \tag{8.25.2}$$

Finally, by subtracting the balance of kinetic energy (7.25) from the balance of total energy (8.23) the mass specific balance of internal energy $I = E - K$ can be derived.

$$\rho_t\mathrm{d}_tI = \bar{\mathsf{P}}_{\mathsf{t}}^{int} + \bar{\mathsf{Q}}_{\mathsf{t}}^{ext} - \mathsf{M}_{\mathsf{t}}I. \tag{8.26}$$

8.2.3 Spatial vs. material motion problem

By comparing the spatial motion balance Eqs. (8.18) and (8.21) with their material motion counterparts (8.23) and (8.26), we easily obtain the identities

$$\bar{\mathsf{p}}_0^{ext} + \bar{\mathsf{q}}_0^{ext} = \bar{\mathsf{P}}_0^{ext} + \bar{\mathsf{Q}}_0^{ext} - \mathrm{Div}(E_0\mathbf{V}), \tag{8.27.1}$$

$$\bar{\mathsf{P}}_{\mathsf{t}}^{ext} + \bar{\mathsf{Q}}_{\mathsf{t}}^{\ ext} = \bar{\mathsf{p}}_{\mathsf{t}}^{ext} + \bar{\mathsf{q}}_{\mathsf{t}}^{ext} - \mathrm{div}(E_t\mathbf{v}) \tag{8.27.2}$$

and

$$\bar{\mathsf{p}}_0^{int} + \bar{\mathsf{q}}_0^{ext} = \bar{\mathsf{P}}_0^{int} + \bar{\mathsf{Q}}_0^{ext} - \mathrm{Div}(I_0\mathbf{V}), \tag{8.28.1}$$

$$\bar{\mathsf{P}}_{\mathsf{t}}^{int} + \bar{\mathsf{Q}}_{\mathsf{t}}^{ext} = \bar{\mathsf{p}}_{\mathsf{t}}^{int} + \bar{\mathsf{q}}_{\mathsf{t}}^{ext} - \mathrm{div}(I_t\mathbf{v}) \tag{8.28.2}$$

by making use of the mass specific version of Reynold's transport theorem (4.14). Remarkably, the differences of the spatial and the material motion quantities $\mathrm{Div}(E_0\mathbf{V})$ or $\mathrm{div}(E_t\mathbf{v})$ as well as $\mathrm{Div}(I_0\mathbf{V})$ or $\mathrm{div}(I_t\mathbf{v})$ are identical to the volume specific case. According to Gurtin [17], we now introduce the scalar fields C_0 and C_t which are related through the corresponding Jacobians.

$$C_t = jC_0 \quad C_0 = JC_t. \tag{8.29.1,2}$$

For the time being, the fields C_τ which can be interpreted as configurational energy change are introduced by mere definition while in the following chapter they will be determined explicitly by exploiting the balance of entropy. With the help of the fields C_τ, we can set up the following relations between the spatial and material reduced non-mechanical energy fluxes

$$\bar{\mathbf{q}}_D = \bar{\mathbf{q}}_d - C_t\mathbf{v} \quad \bar{\mathbf{Q}}_d = \bar{\mathbf{Q}}_D - C_0\mathbf{V}. \tag{8.30.1,2}$$

Remarkably, in the above equations, the energy outfluxes $\bar{\mathbf{q}}_\delta$ and $\bar{\mathbf{Q}}_\delta$ are related via the configurational energy change $-C_\tau$ while the mass fluxes $\bar{\mathbf{m}}_D = \bar{\mathbf{m}}_d + \rho_t \mathbf{v}$ and $\bar{\mathbf{M}}_d = \bar{\mathbf{M}}_D + \rho_0 \mathbf{V}$ introduced in Eq. (4.10) are related in an identical format via the density ρ_τ. With the help of the configurational energy change and the definitions of the external power, the comparisons (8.27) can be restated as follows

$$\mathrm{Div}(\mathbf{v}\cdot\bar{\mathbf{\Pi}}_D) + \mathbf{v}\cdot\bar{\mathbf{b}}_0^{ext} = \mathrm{Div}(\mathbf{V}\cdot\bar{\mathbf{\Sigma}}_d + [C_0 - E_0]\mathbf{V}) + \mathbf{V}\cdot\bar{\mathbf{B}}_0^{ext}, \tag{8.31.1}$$

$$\mathrm{div}(\mathbf{V}\cdot\bar{\boldsymbol{\pi}}_d) + \mathbf{V}\cdot\bar{\mathbf{B}}_t^{ext} = \mathrm{div}(\mathbf{v}\cdot\bar{\boldsymbol{\sigma}}_D + [C_t - E_t]\mathbf{v}) + \mathbf{v}\cdot\bar{\mathbf{b}}_t^{ext}. \tag{8.31.2}$$

A comparison of the flux terms reveals the following identities in terms of the spatial and the material motion reduced dynamic momentum fluxes $\bar{\mathbf{\Pi}}_D$, $\bar{\boldsymbol{\sigma}}_D$, $\bar{\boldsymbol{\pi}}_d$ and $\bar{\mathbf{\Sigma}}_d$.

$$\mathbf{v}\cdot\bar{\boldsymbol{\sigma}}_D^t = \mathbf{v}\cdot[[E_t - C_t]\mathbf{I} - \mathbf{f}^t\cdot\bar{\boldsymbol{\pi}}_d^t], \tag{8.32.1}$$

$$\mathbf{V}\cdot\bar{\mathbf{\Sigma}}_d^t = \mathbf{V}\cdot[[E_0 - C_0]\mathbf{I} - \mathbf{F}^t\cdot\bar{\mathbf{\Pi}}_D^t]. \tag{8.32.2}$$

These can be further simplified by making use of the definition of the dynamic momentum fluxes (6.28) as $\bar{\boldsymbol{\pi}}_d = \bar{\boldsymbol{\pi}} - K_t\mathbf{F}^t - \mathbf{P}_t\otimes\mathbf{v}$ and $\bar{\mathbf{\Sigma}}_d = \bar{\mathbf{\Sigma}} - K_0\mathbf{I} + \mathbf{P}_0\otimes\mathbf{V}$ and the identities $\bar{\boldsymbol{\sigma}}_D = \bar{\boldsymbol{\sigma}}$ and $\bar{\mathbf{\Pi}}_D = \bar{\mathbf{\Pi}}$. Consequently, Eqs. (8.32) can be rewritten as follows

$$\mathbf{v}\cdot\bar{\boldsymbol{\sigma}}^t = \mathbf{v}\cdot[[E_t - C_t + K_t]\mathbf{I} - \mathbf{f}^t\cdot\bar{\boldsymbol{\pi}}^t - \mathbf{p}_t\otimes\mathbf{v}], \tag{8.33.1}$$

$$\mathbf{V}\cdot\bar{\mathbf{\Sigma}}^t = \mathbf{V}\cdot[[E_0 - C_0 + K_0]\mathbf{I} - \mathbf{F}^t\cdot\bar{\mathbf{\Pi}}^t - \mathbf{P}_0\otimes\mathbf{V}]. \tag{8.33.2}$$

Taking into account the fact that $E_\tau = K_\tau + I_\tau$ along with the orthogonality conditions

$$\mathbf{v}\cdot[2K_t\mathbf{I} - \mathbf{p}_t\otimes\mathbf{v}] = 0, \tag{8.34.1}$$

$$\mathbf{V}\cdot[2K_0\mathbf{I} - \mathbf{P}_0\otimes\mathbf{V}] = 0 \tag{8.34.2}$$

emphasized in Steinmann [45], we end up with the following tentative relations between the spatial and the material motion reduced static momentum fluxes $\bar{\mathbf{\Pi}}$, $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{\pi}}$ and $\bar{\mathbf{\Sigma}}$.

$$\bar{\boldsymbol{\sigma}}^t = [I_t - C_t]\mathbf{I} - \mathbf{f}^t\cdot\bar{\boldsymbol{\pi}}^t, \tag{8.35.1}$$

$$\bar{\mathbf{\Sigma}}^t = [I_0 - C_0]\mathbf{I} - \mathbf{F}^t\cdot\bar{\mathbf{\Pi}}^t$$

A comparison of the related source terms yields the transformation formulae between the spatial and the material motion reduced external forces,

$$\bar{\mathbf{b}}_t^{ext} = -\mathbf{f}^t\cdot\bar{\mathbf{B}}_t^{ext}, \tag{8.36.1}$$

$$\bar{\mathbf{B}}_0^{ext} = -\mathbf{F}^t\cdot\bar{\mathbf{b}}_0^{ext}, \tag{8.36.2}$$

see also Steinmann [47].

9 Balance of entropy and dissipation inequality

The first law of thermodynamics in the form of the balance of energy expresses the interconvertibility of heat and work. However, the balance of energy itself does not place any restrictions on the direction of the thermodynamical process. The second law of thermodynamics, the balance of entropy, postulates the existence of an absolute temperature and of a specific entropy as a state function. Through the internal production of the latter, which is required to either vanish for reversible processes or to be strictly positive for irreversible processes, a direction is imposed on the thermodynamical process.

9.1 Volume specific version

9.1.1 Spatial motion problem

The balance of entropy balances the volume specific entropy density $S_0 = \rho_0 S$ with the external entropy input h_0^{ext} and the internal entropy production h_0^{int}. Thereby, the former consists of the entropy flux $\mathbf{H}_D$ across the material surface $\partial\mathscr{B}_0$ and the entropy source $\mathscr{H}_0$ in the material domain $\mathscr{B}_0$

$$\mathsf{h}_0^{ext} := \mathrm{Div}(-\mathbf{H}_D) + \mathscr{H}_0. \tag{9.1}$$

Recall, that we are dealing with open systems for which a fixed material volume $\mathscr{B}_0$ is allowed to constantly gain or lose mass. Open systems naturally exhibit an additional entropy flux and entropy source caused by the added mass as pointed out earlier in the famous monograph by Schrödinger [41] §6 as well as by Malvern [33] §5.6 or only recently by Epstein and Maugin [10]. As one consequence, the external entropy flux $\mathbf{H}_D$ is introduced as the sum of the reduced external entropy flux $\bar{\mathbf{H}}_D$ enhanced by the explicit open system contribution $S\mathbf{R}$. Accordingly, the external entropy source $\mathscr{H}_0$ consists of the reduced entropy source $\bar{\mathscr{H}}_0$ modified by additional terms $S\mathscr{R}_0$ and $-\nabla_X S \cdot \mathbf{R}$ accounting for the explicit open system contribution to the entropy supply

$$\begin{aligned} &\mathsf{h}_0^{ext} := \mathrm{Div}(-\bar{\mathbf{H}}_D + S\mathbf{R}) + \bar{\mathscr{H}}_0 + S\mathscr{R}_0 - \nabla_X S \cdot \mathbf{R}, \\ &\mathsf{h}_0^{int} \geq 0. \end{aligned} \tag{9.2}$$

Just like in classical thermodynamics, the internal entropy production h_0^{int} is required to be point-wise non-negative. This condition naturally induces a direction to the thermodynamic process. Consequently, the local version of the balance of entropy of the material motion problem with material reference and material parametrization can be stated in the following form

$$\mathrm{D}_t S_0 = \mathsf{h}_0^{ext} + \mathsf{h}_0^{int}. \tag{9.3}$$

By making use of the requirement that the internal entropy production be non-negative as $\mathsf{h}_0^{int} \geq 0$, the above equation can be recast into the inequality $\mathrm{D}_t S_0 - \mathsf{h}_0^{ext} \geq 0$ which is referred to as "postulate of irreversibility" in classical thermodynamics, see Truesdell and Toupin [50] §258. Again, we can derive alternative formats of the above statement by applying the related Piola transforms $S_0 = JS_t$, $\mathsf{h}_0^{ext} = J\mathsf{h}_t^{ext}$, $\mathsf{h}_0^{int} = J\mathsf{h}_t^{int}$, $\mathbf{H}_D = J\mathbf{h}_D \cdot \mathbf{F}^{-t}$ and $\bar{\mathbf{H}}_D = J\bar{\mathbf{h}}_D \cdot \mathbf{F}^{-t}$ in combination with the volume specific version of Reynold's transport theorem. Next, we will introduce Neumann boundary conditions for the spatial motion Kirchhoff and Cauchy type entropy flux $\mathbf{H}_D$ and $\mathbf{h}_D$ in terms of the classical closed system entropy flux contribution h_τ^{closed} and the additional open system contribution h_τ^{open}

$$\begin{aligned} \mathbf{H}_D \cdot \mathbf{N} &= h_0^{closed} + h_0^{open} \qquad & h_0^{open} &= \bar{h}_0^{open} - S\mathbf{R} \cdot \mathbf{N}, \\ \mathbf{h}_D \cdot \mathbf{n} &= h_t^{closed} + h_t^{open} \qquad & h_t^{open} &= \bar{h}_t^{open} - S\mathbf{r} \cdot \mathbf{n}. \end{aligned} \tag{9.4}$$

Accordingly, the entropy sources $\mathscr{H}_\tau$ are introduced as the sum of the classical entropy source of the closed system $\mathscr{H}_\tau^{closed}$ and the additional entropy source accounting for the nature of the open system $\mathscr{H}_\tau^{open}$

$$\begin{aligned} \mathscr{H}_0 &= \mathscr{H}_0^{closed} + \mathscr{H}_0^{open} \qquad & \mathscr{H}_0^{open} &= \bar{\mathscr{H}}_0^{open} + S\mathscr{R}_0 - \nabla_X S \cdot \mathbf{R}, \\ \mathscr{H}_t &= \mathscr{H}_t^{closed} + \mathscr{H}_t^{open} \qquad & \mathscr{H}_t^{open} &= \bar{\mathscr{H}}_t^{open} + S\mathscr{R}_t - \nabla_x S \cdot \mathbf{r}\ . \end{aligned} \tag{9.5}$$

9.1.2 Material motion problem

In complete analogy to the spatial motion problem, we can formally introduce the balance of entropy for the material motion problem balancing the rate of change of the volume specific entropy density $S_t = \rho_t S$ with the external entropy input $\mathsf{H}_\mathsf{t}^{ext}$ and the internal entropy production $\mathsf{H}_\mathsf{t}^{int}$. The former can be introduced as the sum of the material motion entropy flux $\mathbf{h}_d$ and the material motion entropy source $\mathscr{H}_t$

$$\mathsf{H}_\mathsf{t}^{ext} := \mathrm{div}(-\mathbf{h}_d) + \mathscr{H}_t. \tag{9.6}$$

Again, the contributions $\mathbf{h}_d$ and $\mathscr{H}_t$ will be expressed in terms of their reduced counterparts $\bar{\mathbf{h}}_d$ and $\bar{\mathscr{H}}_t$ and the explicit open system extra terms $S\mathbf{r}$, $S\mathscr{R}_t$ and $\nabla_x S \cdot \mathbf{r}$,

$$\mathsf{H}_\mathsf{t}^{ext} := \mathrm{div}(-\bar{\mathbf{h}}_d + S\mathbf{r}) + \bar{\mathscr{H}}_t + S\mathscr{R}_t - \nabla_x S \cdot \mathbf{r} \tag{9.7.1}$$

$$\mathsf{H}_\mathsf{t}^{int} \geq 0 \tag{9.7.2}$$

giving rise to the material motion entropy balance of the following form

$$\mathrm{d}_t S_t = \mathsf{H}_\mathsf{t}^{ext} + \mathsf{H}_\mathsf{t}^{int}. \tag{9.8}$$

The related transport theorem and the corresponding Piola transforms with $S_t = jS_0$, $\mathsf{H}_\mathsf{t}^{ext} = j\mathsf{H}_0^{ext}$, $\mathsf{H}_\mathsf{t}^{int} = j\mathsf{H}_0^{int}$, $\mathbf{h}_d = j\mathbf{H}_d \cdot \mathbf{f}^{-t}$ and $\bar{\mathbf{h}}_d = j\bar{\mathbf{H}}_d \cdot \mathbf{f}^{-t}$ can be used to derive alternative formats of the statement (9.8). Again, we can formally introduce Neumann boundary conditions for the material motion entropy fluxes $\mathbf{h}_d$ and $\mathbf{H}_d$

$$\begin{aligned} \mathbf{h}_d \cdot \mathbf{n} &= H_t^{closed} + H_t^{open} & H_t^{open} &= \bar{H}_t^{open} - S\mathbf{r} \cdot \mathbf{n}, \\ \mathbf{H}_d \cdot \mathbf{N} &= H_0^{closed} + H_0^{open} & H_0^{open} &= \bar{H}_0^{open} - S\mathbf{R} \cdot \mathbf{N} \end{aligned} \tag{9.9}$$

and define the material motion entropy sources $\mathscr{H}_\tau$ in complete analogy to the spatial motion case

$$\begin{aligned} \mathscr{H}_t &= \mathscr{H}_t^{closed} + \mathscr{H}_t^{open} & \mathscr{H}_t^{open} &= \bar{\mathscr{H}}_t^{open} + S\mathscr{R}_t - \nabla_x S \cdot \mathbf{r}, \\ \mathscr{H}_0 &= \mathscr{H}_0^{closed} + \mathscr{H}_0^{open} & \mathscr{H}_0^{open} &= \bar{\mathscr{H}}_0^{open} + S\mathscr{R}_0 - \nabla_X S \cdot \mathbf{R}. \end{aligned} \tag{9.10}$$

9.1.3 Spatial vs. material motion problem

A comparison of the rate of change of the entropy density based on the spatial and the material motion problem (9.3) and (9.8) with the help of the volume specific version of Reynold's transport theorem (3.22) reveals the following identities

$$\mathsf{h}_0^{ext} + \mathsf{h}_0^{int} = \mathsf{H}_0^{ext} + \mathsf{H}_0^{int} - \mathrm{Div}(S_0\mathbf{V}), \tag{9.11.1}$$

$$\mathsf{H}_\mathsf{t}^{ext} + \mathsf{H}_\mathsf{t}^{int} = \mathsf{h}_\mathsf{t}^{ext} + \mathsf{h}_\mathsf{t}^{int} - \mathrm{div}(S_t\mathbf{v}). \tag{9.11.2}$$

In addition, we will make use of the natural but crucial assumption that the internal entropy production is independent of the particular type of motion problem considered by postulating that

$$\mathsf{h}_0^{int} = \mathsf{H}_0^{int} \tag{9.12.1}$$

$$\mathsf{H}_\mathsf{t}^{int} = \mathsf{h}_\mathsf{t}^{int}, \tag{9.12.2}$$

see, e.g. Steinmann [47]. It will turn out in the sequel, that this assertion essentially determines the connection between the constitutive relations of the spatial and the material motion problem.

Remark 9.1: Alternatively, the balance of entropy can be derived from the energy balance rather than being introduced as a mere definition. This approach has been suggested by Green and Naghdi [14], [15] who interprete the balance of entropy as a natural consequence of the invariance of working under changes of a thermal motion observer. A similar approach has been followed for the material motion problem only recently by Kalpakides and Dascalu [24].

9.2 Mass specific version

9.2.1 Spatial motion problem

The mass specific counterpart of the above equations states, that the rate of change of the mass specific entropy $S = S_0/\rho_0$ be in equilibrium with the mass specific external entropy input $\bar{\mathsf{h}}_0^{ext}$ and the mass specific internal entropy production $\bar{\mathsf{h}}_0^{int}$ which are introduced in the following way

$$\bar{\mathsf{h}}_0^{ext} := \mathsf{h}_0^{ext} - S\mathrm{D}_t\rho_0 + \mathsf{m}_0 S = -\mathrm{Div}\,\bar{\mathbf{H}}_D + \bar{\mathscr{H}}_0 \qquad (9.13.1)$$

$$\bar{\mathsf{h}}_0^{int} := \mathsf{h}_0^{int} \geq 0. \qquad (9.13.2)$$

The resulting mass specific balance of entropy

$$\rho_0 \mathrm{D}_t S = \bar{\mathsf{h}}_0^{ext} + \bar{\mathsf{h}}_0^{int} - \mathsf{m}_0 S \qquad (9.14)$$

which can be derived by subtracting S times the balance of mass (4.13) from the volume specific balance of entropy (9.3) can be recast into the related alternative forms by applying the Piola transforms $\bar{\mathsf{h}}_0^{ext} = J\bar{\mathsf{h}}_t^{ext}$ and $\bar{\mathsf{h}}_0^{int} = J\bar{\mathsf{h}}_t^{int}$ and the Euler theorem (3.21). Moreover, we can relate the reduced entropy fluxes $\bar{\mathbf{H}}_D$ and $\bar{\mathbf{h}}_D$ defined through $\bar{\mathbf{H}}_D = \mathbf{H}_D - S\mathbf{R}$ and $\bar{\mathbf{h}}_D = \mathbf{h}_D - S\mathbf{r}$ to the classical entropy flux of the closed system h_τ^{closed} and the entropy flux caused by additional effects of the open system $\bar{h}_\tau^{open}$

$$\bar{\mathbf{H}}_D \cdot \mathbf{N} = h_0^{closed} + \bar{h}_0^{open}, \qquad (9.15.1)$$

$$\bar{\mathbf{h}}_D \cdot \mathbf{n} = h_t^{closed} + \bar{h}_t^{open}. \qquad (9.15.2)$$

Accordingly, the reduced entropy sources $\bar{\mathscr{H}}_0 = \mathscr{H}_0 - S\mathscr{R}_0 + \nabla_X S \cdot \mathbf{R}$ and $\bar{\mathscr{H}}_t = \mathscr{H}_t - S\mathscr{R}_t + \nabla_x S \cdot \mathbf{r}$ are given as the sum of the classical closed system entropy source $\mathscr{H}_\tau^{closed}$ and the additional open system contribution to the entropy source $\bar{\mathscr{H}}_\tau^{open}$

$$\begin{aligned} \bar{\mathscr{H}}_0 &= \mathscr{H}_0^{closed} + \bar{\mathscr{H}}_0^{open}, \\ \bar{\mathscr{H}}_t &= \mathscr{H}_t^{closed} + \bar{\mathscr{H}}_t^{open}. \end{aligned} \qquad (9.16)$$

For further elaborations, it proves convenient to set up relations between the reduced entropy flux $\bar{\mathbf{H}}_D$ and the reduced non-mechanical energy flux $\bar{\mathbf{Q}}_D$ as well as between the reduced entropy source $\bar{\mathscr{H}}_0$ and the reduced non-mechanical energy source $\bar{\mathscr{Q}}_0$ in terms of the absolute temperature θ

$$\bar{\mathbf{H}}_D = \frac{1}{\theta}\bar{\mathbf{Q}}_D + \mathbf{S} \qquad (9.17.1)$$

$$\bar{\mathscr{H}}_0 = \frac{1}{\theta}\bar{\mathscr{Q}}_0 + \mathscr{S}_0. \qquad (9.17.2)$$

Thereby, the above equations can be understood as a generalization of the ideas of Cowin and Hegedus [6] who have suggested to include the additional entropy source $\mathscr{S}_0$ accounting for changes in entropy caused by changes in mass that are not considered implicitly through the changes in energy $\bar{\mathscr{Q}}_0$. To keep the underlying theory as general as possible, we suggest to additionally include an extra entropy flux $\mathbf{S}$ accounting for the in- or outflux of entropy that is not implicitly included in the reduced energy flux term $\bar{\mathbf{Q}}_D$. This extra entropy flux resembles the exposition in Maugin [36] §3.3. and §4.7 and has to be determined by a constitutive equation. Both, the additional entropy flux and source $\mathbf{S}$ and $\mathscr{S}_0$, which we shall summarize in the term $\mathsf{s}_0 = -\mathrm{Div}\,\mathbf{S} + \mathscr{S}_0$ in the sequel, can be understood as an explicit representation of the exchange of entropy with the "outside world". Notice, however, that these two terms are not included in the "theory of volumetric growth" by Epstein and Maugin [10], who relate the reduced entropy flux and source to the reduced non-mechanical energy flux and source as $\bar{\mathbf{H}}_D = \bar{\mathbf{Q}}_D/\theta$ and $\bar{\mathscr{H}}_0 = \hat{\mathscr{Q}}_0/\theta$ with $\hat{\mathscr{Q}}_0 = \mathscr{Q}_0 - S\mathscr{R}$. We now turn to the evaluation of the above stated second law of thermodynamics by recasting it into an appropriate form of the dissipation inequality, a statement that places further restrictions on the form of the constitutive response functions. For this purpose, we shall introduce the dissipation rate $\bar{\mathsf{d}}_0$ as the internal entropy production weighted by the absolute temperature as $\bar{\mathsf{d}}_0 := \theta\,\bar{\mathsf{h}}_0^{int} \geq 0$. With the help of Eqs. (9.17) as $\bar{\mathbf{H}}_D = \bar{\mathbf{Q}}_D/\theta + \mathbf{S}$ and $\bar{\mathscr{H}}_0 = \bar{\mathscr{Q}}_0/\theta + \mathscr{S}_0$ and the appropriate transformations[3] the dissipation rate can be reformulated yielding the spatial motion version of the Clausius–Duhem inequality in an internal energy based fashion

$$\bar{\mathsf{d}}_0 = \bar{\mathsf{p}}_0^{int} - \rho_0\,\mathrm{D}_t[I - \theta S] - \mathsf{m}_0[I - \theta S] - \rho_0 S\,\mathrm{D}_t\theta - \mathsf{s}_0\theta - \bar{\mathbf{Q}}_D \cdot \nabla_X \ln\theta \geq 0. \tag{9.18}$$

By making use of the appropriate Legendre–Fenchel transform introducing the Helmholtz free energy $\Psi = I - \theta S$, we end up with classical free energy based version of the Clausius–Duhem inequality

$$\bar{\mathsf{d}}_0 = \bar{\mathbf{\Pi}}^t : \mathrm{D}_t\mathbf{F} - \mathbf{v}\cdot\bar{\mathbf{b}}_0^{int} - \rho_0\,\mathrm{D}_t\Psi - \mathsf{m}_0\Psi - \rho_0 S\,\mathrm{D}_t\theta - \mathsf{s}_0\theta - \bar{\mathbf{Q}}_D \cdot \nabla_X \ln\theta \geq 0. \tag{9.19}$$

This formulation is particularly useful when the temperature θ rather than the entropy S is used as independent variable. Recall, that for reasons of notational comparability we have included the internal force contribution $\mathbf{v}\cdot\bar{\mathbf{b}}_0^{int} = 0$ and the term reflecting the convective effects of growth $\mathsf{m}_0\Psi = 0$ keeping in mind that both vanish identically for the spatial motion problem. In classical thermodynamics, the Clausius–Duhem inequality (9.19) is typically decomposed into a local and a conductive contribution $\bar{\mathsf{d}}_0^{loc}$ and $\bar{\mathsf{d}}_0^{con}$

$$\bar{\mathsf{d}}_0^{loc} = \bar{\mathbf{\Pi}}^t : \mathrm{D}_t\mathbf{F} - \mathbf{v}\cdot\bar{\mathbf{b}}_0^{int} - \rho_0\,\mathrm{D}_t\Psi - \mathsf{m}_0\Psi - \rho_0 S\,\mathrm{D}_t\theta - \mathsf{s}_0\theta \geq 0 \tag{9.20.1}$$

$$\bar{\mathsf{d}}_0^{con} = -\bar{\mathbf{Q}}_D \cdot \nabla_X \ln\theta \geq 0. \tag{9.20.2}$$

The conductive term $\bar{\mathsf{d}}_0^{con} \geq 0$ represents the classical Fourier inequality while the remaining local term $\bar{\mathsf{d}}_0^{loc} \geq 0$ is typically referred to as Clausius–Planck inequality. Both are required to hold separately as a sufficient condition for $\bar{\mathsf{d}}_0 \geq 0$.

9.2.2 Material motion problem

For the material motion problem, the rate of change of the mass specific entropy $S = S_t/\rho_t$ is balanced with the mass specific material motion external entropy input $\bar{\mathsf{H}}_t^{ext}$ and the internal entropy production $\bar{\mathsf{H}}_t^{int}$

$$\bar{\mathsf{H}}_t^{ext} := \mathsf{H}_t^{ext} - S\,\mathrm{d}_t\rho_t + \mathsf{M}_t S = -\mathrm{div}\,\bar{\mathbf{h}}_d + \bar{\mathscr{H}}_t, \tag{9.21.1}$$

$$\bar{\mathsf{H}}_t^{int} := \mathsf{H}_t^{int} \geq 0. \tag{9.21.2}$$

The mass specific entropy balance of the material motion problem

$$\rho_t\,\mathrm{d}_t S = \bar{\mathsf{H}}_t^{ext} + \bar{\mathsf{H}}_t^{int} - \mathsf{M}_t S \tag{9.22}$$

can be derived by subtracting S times the balance of mass (4.13) from the volume specific balance of entropy (9.8). Again, we could derive the alternative versions of the above equation through the corresponding Piola transforms $\bar{\mathsf{H}}_t^{ext} = j\bar{\mathsf{H}}_0^{ext}$ and $\bar{\mathsf{H}}_t^{int} = j\bar{\mathsf{H}}_0^{int}$ and the Euler theorem (3.21). In complete analogy to the spatial motion problem, the reduced entropy fluxes $\bar{\mathbf{h}}_d = \mathbf{h}_d - S\mathbf{r}$ and $\bar{\mathbf{H}}_d = \mathbf{H}_d - S\mathbf{R}$ can be defined through the corresponding Neumann boundary conditions as

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3 $$\begin{aligned}\bar{\mathsf{d}}_0 &= \mathsf{m}_0\theta S + \theta\rho_0\,\mathrm{D}_t S - \theta\,\bar{\mathsf{h}}_0^{ext}\\ &= \mathsf{m}_0\theta S + \theta\rho_0\,\mathrm{D}_t S - \bar{\mathsf{q}}_0^{ext} - \mathsf{s}_0\theta - \bar{\mathbf{Q}}_D\cdot\nabla_X\ln\theta\\ &= \mathsf{m}_0\theta S + \rho_0\,\mathrm{D}_t[\theta S] - \rho_0 S\,\mathrm{D}_t\theta + \bar{\mathsf{p}}_0^{int} - \mathsf{m}_0 I - \rho_0\mathrm{D}_t I - \mathsf{s}_0\theta - \bar{\mathbf{Q}}_D\cdot\nabla_X\ln\theta\end{aligned}$$

$$\bar{\mathbf{h}}_d \cdot \mathbf{n} = H_t^{closed} + \bar{H}_t^{open}, \tag{9.23.1}$$
$$\bar{\mathbf{H}}_d \cdot \mathbf{N} = H_0^{closed} + \bar{H}_0^{open} \tag{9.23.2}$$

while the reduced entropy sources are given as follows:

$$\bar{\mathscr{H}}_t = \mathscr{H}_t^{closed} + \bar{\mathscr{H}}_t^{open}, \tag{9.24.1}$$
$$\bar{\mathscr{H}}_0 = \mathscr{H}_0^{closed} + \bar{\mathscr{H}}_0^{open}. \tag{9.24.2}$$

Subsequently, we assume that the reduced material entropy flux $\bar{\mathbf{h}}_d$ and the reduced entropy source $\bar{\mathscr{H}}_t$ can be related to the corresponding energy flux $\bar{\mathbf{q}}_d$ and source $\bar{\mathscr{Q}}_t$ through the absolute temperature θ. Generalizing the ideas of Cowin and Hegedus [6], we shall again include the entropy in- or outflux $\mathbf{s}$ and the corresponding entropy source $\mathscr{S}_t$ accounting for the explicit entropy exchange with the "outside world",

$$\bar{\mathbf{h}}_d = \frac{1}{\theta}\bar{\mathbf{q}}_d + \mathbf{s} \tag{9.25.1}$$
$$\bar{\mathscr{H}}_t = \frac{1}{\theta}\bar{\mathscr{Q}}_t + \mathscr{S}_t, \tag{9.25.2}$$

whereby this extra external entropy input will be summarized in the term $\mathsf{S}_t = -\mathrm{div}\,\mathbf{s} + \mathscr{S}_t$. Next, we can again reinterpret the balance of entropy by introducing the nonnegative dissipation rate $\bar{\mathsf{D}}_t$ as $\bar{\mathsf{D}}_t := \theta\,\bar{\mathsf{H}}_t^{int} \geq 0$. By making use of Eqs. (9.25), we can transform the dissipation rate[4] into the internal energy based version of the Clausius–Duhem inequality,

$$\bar{\mathsf{D}}_t = \bar{\mathsf{P}}_t^{int} - \rho_t \mathrm{d}_t[I - \theta S] - \mathsf{M}_t[I - \theta S] - \rho_t S\,\mathrm{d}_t\theta - \mathsf{S}_t\theta - \bar{\mathbf{q}}_d \cdot \nabla_x \ln\theta \geq 0. \tag{9.26}$$

Finally, the introduction of the corresponding Legendre–Fenchel transform $\Psi = I - \theta S$ renders the more familiar free energy based version of the Clausius–Duhem inequality,

$$\bar{\mathsf{D}}_t = \bar{\boldsymbol{\pi}}^t : \mathrm{d}_t\mathbf{f} - \mathbf{V} \cdot \bar{\mathbf{B}}_t^{int} - \rho_t\,\mathrm{d}_t\Psi - \mathsf{M}_t\Psi - \rho_t S\,\mathrm{d}_t\theta - \mathsf{S}_t\theta - \bar{\mathbf{q}}_d \cdot \nabla_x \ln\theta \geq 0 \tag{9.27}$$

which can again be additively decomposed into a local and a conductive contribution $\bar{\mathsf{D}}_t^{loc}$ and $\bar{\mathsf{D}}_t^{con}$

$$\bar{\mathsf{D}}_t^{loc} = \bar{\boldsymbol{\pi}}^t : \mathrm{d}_t\mathbf{f} - \mathbf{V} \cdot \bar{\mathbf{B}}_t^{int} - \rho_t\,\mathrm{d}_t\Psi - \mathsf{M}_t\Psi - \rho_t S\,\mathrm{d}_t\theta - \mathsf{S}_t\theta \tag{9.28.1}$$
$$\bar{\mathsf{D}}_t^{con} = -\bar{\mathbf{q}}_d \cdot \nabla_x \ln\theta. \tag{9.28.2}$$

However, neither the material motion counterpart of the Fourier inequality $\bar{\mathsf{D}}_t^{con}$, nor of the material motion Clausius–Planck inequality $\bar{\mathsf{D}}_t^{loc}$ can be required to become nonnegative independently, but rather $\bar{\mathsf{D}}_t = \bar{\mathsf{D}}_t^{loc} + \bar{\mathsf{D}}_t^{con} \geq 0$.

9.2.3 Spatial vs. material motion problem

In the balance of entropy, the influence of the "outside world" is reflected through the extra entropy fluxes $\mathbf{S}$ and $\mathbf{s}$ and the entropy sources $\mathscr{S}_0$ and $\mathscr{S}_t$ for the spatial and the material motion problem, respectively. While the extra entropy fluxes are related through the appropriate Piola transforms,

$$\mathbf{s} = j\mathbf{S} \cdot \mathbf{f}^{-t} \quad \mathbf{S} = J\mathbf{s} \cdot \mathbf{F}^{-t} \tag{9.29.1, 2}$$

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4 $\bar{\mathsf{D}}_t = \mathsf{M}_t\theta S + \theta\rho_t\,\mathrm{d}_t S - \theta\,\bar{\mathsf{H}}_t^{ext}$
$= \mathsf{M}_t\theta S + \theta\rho_t\,\mathrm{d}_t S - \bar{\mathsf{Q}}_t^{ext} - \mathsf{S}_t\theta - \bar{\mathbf{q}}_d \cdot \nabla_x \ln\theta$
$= \mathsf{M}_t\theta S + \rho_t\,\mathrm{d}_t[\theta S] - \rho_t S\,\mathrm{d}_t\theta + \bar{\mathsf{P}}_t^{int} - \mathsf{M}_t I - \rho_t\,\mathrm{d}_t I - \mathsf{S}_t\theta - \bar{\mathbf{q}}_d \cdot \nabla_x \ln\theta.$

the transformations between the extra entropy sources

$$\mathscr{S}_t = j\mathscr{S}_0 \quad \mathscr{S}_0 = J\mathscr{S}_t \tag{9.30.1, 2}$$

and the spatial and material motion extra external entropy input s_0 and S_t

$$\mathsf{S}_t = \mathsf{s}_t = j\,\mathsf{s}_0 \quad \mathsf{s}_0 = \mathsf{S}_0 = J\,\mathsf{S}_t \tag{9.31.1, 2}$$

are given in terms of the corresponding Jacobians. Next, by comparing the spatial and the material motion entropy balance in its mass specific format (9.14) and (9.22) with the help of the mass specific version of Reynold's transport theorem (4.14), we find the following identities which again take a remarkably similar structure as for the volume specific case compare (9.11).

$$\bar{\mathsf{h}}_0^{ext} + \bar{\mathsf{h}}_0^{int} = \bar{\mathsf{H}}_0^{ext} + \bar{\mathsf{H}}_0^{int} - \mathrm{Div}(S_0\mathbf{V}) \tag{9.32.1}$$

$$\bar{\mathsf{H}}_t^{ext} + \bar{\mathsf{H}}_t^{int} = \bar{\mathsf{h}}_t^{ext} + \bar{\mathsf{h}}_t^{int} - \mathrm{div}(S_t\mathbf{v}). \tag{9.32.2}$$

With the help of the definitions of the external entropy input of the spatial and the material motion problem h_0^{ext} in (9.2) and H_0^{ext} in (9.7) and the essential assertion that $\mathsf{h}_0^{int} = \mathsf{H}_0^{int}$ and $\mathsf{H}_t^{int} = \mathsf{h}_t^{int}$ stated in Eq. (9.12), the above identities yield the fundamental relations between the spatial and the material entropy fluxes

$$\bar{\mathbf{h}}_D = \bar{\mathbf{h}}_d - S_t\mathbf{v} \quad \bar{\mathbf{H}}_d = \bar{\mathbf{H}}_D - S_0\mathbf{V}. \tag{9.33.1, 2}$$

Recall the relation between the spatial and material non-mechanical energy fluxes introduced in (8.30) as $\bar{\mathbf{q}}_D = \bar{\mathbf{q}}_d - C_t\mathbf{v}$ and $\bar{\mathbf{Q}}_d = \bar{\mathbf{Q}}_D - C_0\mathbf{V}$. With the help Eqs. (9.17) and (9.25) relating corresponding energy and entropy fluxes through the temperature as $\bar{\mathbf{H}}_\delta = \bar{\mathbf{Q}}_\delta/\theta + \mathbf{S}$ and $\bar{\mathbf{h}}_\delta = \bar{\mathbf{q}}_\delta/\theta + \mathbf{s}$ and the relations between the extra entropy fluxes $\mathbf{S}$ and $\mathbf{s}$ as stated in Eq. (9.29), we can easily identify the configurational energy increase C_τ as the entropy density S_τ weighted by the absolute temperature θ

$$C_t = \theta S_t \quad C_0 = \theta S_0 \tag{9.34.1, 2}$$

This interpretation enables us to formulate the following relations between the external entropy input, the external non-mechanical energy, the external mechanical energy and the internal mechanical energy of the spatial and the material motion problem

$$\begin{array}{ll} \bar{\mathsf{H}}_0^{ext} = \bar{\mathsf{h}}_0^{ext} + \mathrm{Div}(S_0\mathbf{V}) & \bar{\mathsf{h}}_t^{ext} = \bar{\mathsf{H}}_t^{ext} + \mathrm{div}(S_t\mathbf{v}) \\ \bar{\mathsf{Q}}_0^{ext} = \bar{\mathsf{q}}_0^{ext} + \mathrm{Div}(\theta S_0\mathbf{V}) & \bar{\mathsf{q}}_t^{ext} = \bar{\mathsf{Q}}_t^{ext} + \mathrm{div}(\theta S_t\mathbf{v}) \\ \bar{\mathsf{P}}_0^{ext} = \bar{\mathsf{p}}_0^{ext} + \mathrm{Div}([K_0 + \mathbf{\Psi}_0]\mathbf{V}) & \bar{\mathsf{p}}_t^{ext} = \bar{\mathsf{P}}_t^{ext} + \mathrm{div}([K_t + \mathbf{\Psi}_t]\mathbf{v}) \\ \bar{\mathsf{P}}_0^{int} = \bar{\mathsf{p}}_0^{int} + \mathrm{Div}(\mathbf{\Psi}_0\mathbf{V}) & \bar{\mathsf{p}}_t^{int} = \bar{\mathsf{P}}_t^{int} + \mathrm{div}(\mathbf{\Psi}_t\mathbf{v}) \end{array}. \tag{9.35}$$

Moreover, with the relation between the non-mechanical energy fluxes (8.30) and the interpretation of the configurational energy increase (9.34), we can easily relate the spatial and the material motion version of the conductive dissipation $\bar{\mathsf{D}}_0^{con}$ and $\bar{\mathsf{d}}_t^{con}$ and set up an equivalent relation between the local dissipation terms $\bar{\mathsf{D}}_0^{loc}$ and $\bar{\mathsf{d}}_t^{loc}$

$$\begin{array}{ll} j\,\bar{\mathsf{D}}_0^{con} = \bar{\mathsf{d}}_t^{con} + S_t\nabla_x\theta\cdot\mathbf{v} & j\,\bar{\mathsf{D}}_0^{loc} = \bar{\mathsf{d}}_t^{loc} - S_t\nabla_x\theta\cdot\mathbf{v}, \\ J\,\bar{\mathsf{d}}_t^{con} = \bar{\mathsf{D}}_0^{con} + S_0\nabla_X\theta\cdot\mathbf{V} & J\,\bar{\mathsf{d}}_t^{loc} = \bar{\mathsf{D}}_0^{loc} - S_0\nabla_X\theta\cdot\mathbf{V}. \end{array} \tag{9.36}$$

Remark 9.2: Note, that at this stage, the identification of the reduced entropy fluxes and sources in terms of the reduced non-mechanical energy fluxes and sources, the absolute temperature and the additional extra terms $\bar{\mathbf{H}}_\delta = \bar{\mathbf{Q}}_\delta/\theta + \mathbf{S}$, $\bar{\mathbf{h}}_\delta = \bar{\mathbf{q}}_\delta/\theta + \mathbf{s}$ and $\bar{\mathscr{H}}_0 = \bar{\mathscr{Q}}_0/\theta + \mathscr{S}_\tau$ as introduced in Eqs. (9.17) and (9.25) is a mere constitutive assumption. Nevertheless, for particular constitutive model problems, the postulated relations can be verified through the evaluation of the dissipation inequality according to Liu [31], and Müller [38], see also Liu [32]. It will turn out that in most cases, Eqs. (9.17) and (9.25) are justified with $\mathbf{S} = \mathbf{0}$, $\mathbf{s} = \mathbf{0}$ and $\mathscr{S}_\tau = 0$. However, assuming this result from the outset might be too restrictive for complex constitutive models when diffusive processes other than heat phenomena are included, see Epstein and Maugin [10].

Remark 9.3: At first sight, the above derivations might seem to be closely related to the "theory of mixtures", see, e.g. Truesdell and Toupin [50], Bowen [3], Ehlers [8], de Boer [2], Kühn and Hauger [27], Diebels [7]. Indeed, up to the second law of thermodynamics, the balance equations for one single constituent of a mixture are formally almost identical to the balance equations for open systems. However, in the theory of mixtures, the dissipation inequality is usually stated for the mixture as a whole rather than for each individual constituent. The latter approach, which is indeed a sufficient condition, is thus felt to be too restrictive in most practical applications, see, e.g. Bowen [3]. Nevertheless, here, we shall focus on the open system itself rather than aiming at characterizing the other constituents representing the "outside world", since in our case, the constituents are not superposed at each spatial point as in the "theory of mixtures" but are rather spatially separated. In this context, recall the example of rocket propulsion due to combustion which would typically never be modelled within the mixture theory. In the present case, the influence of the "outside world" is represented through the extra terms s_0 and S_t in the spatial and the material motion dissipation inequality. In what follows, we shall apply the dissipation inequalities (9.20) and (9.28) to derive constitutive equations for the reduced momentum fluxes $\bar{\mathbf{\Pi}}^t$ and $\bar{\boldsymbol{\pi}}^t$, the entropy S and the internal forces $\bar{\mathbf{b}}_0^{int}$ and $\bar{\mathbf{B}}_t^{int}$. In addition, the evaluation of the dissipation inequalities places further restrictions related to the extra entropy terms s_0 and S_t. The underlying procedure will be highlighted in detail for the simple model problem of thermo-hyperelasticity in the following chapter.

10 Thermo-hyperelasticity

We are now in the position to exploit the second law of thermodynamics in the form of the Clausius–Duhem inequality for the thermo-hyperelastic case. We will thus restrict ourselves to a locally reversible model problem for which all the dissipation is caused exclusively by heat conduction and possibly by an additional contribution of the "outside world".

10.1 Spatial motion problem

For the spatial motion problem, we shall assume, that the free energy density Ψ_0 is a linear function of the material density ρ_0 and can thus be multiplicatively decomposed in the following way

$$\Psi_0 = \rho_0 \Psi. \tag{10.1}$$

Thereby, the free energy density Ψ can be expressed in terms of the material motion deformation gradient $\mathbf{F}$ and the absolute temperature θ with a possible explicit dependence on the material placement $\mathbf{X}$. Within the thermodynamics of open systems, the material density ρ_0 is allowed to vary in space and time is thus introduced as function of the material placement $\mathbf{X}$ and the time t.

$$\Psi = \Psi(\mathbf{F}, \theta; \mathbf{X}) \quad \rho_0 = \rho_0(\mathbf{X}, t). \tag{10.2.1, 2}$$

Consequently, the material time derivative of the free energy density can be expressed as

$$\mathrm{D}_t\Psi = \mathrm{D}_{\mathbf{F}}\Psi : \mathrm{D}_t\mathbf{F} + \mathrm{D}_\theta\Psi\,\mathrm{D}_t\theta. \tag{10.3}$$

The evaluation of the Clausius–Planck inequality (9.20.1)

$$\bar{\mathsf{d}}_0^{loc} = \left[\bar{\mathbf{\Pi}}^t - \rho_0\mathrm{D}_{\mathbf{F}}\Psi\right] : \mathrm{D}_t\mathbf{F} - \mathbf{v}\cdot\bar{\mathbf{b}}_0^{int} - \mathsf{m}_0\Psi - [\rho_0 S + \rho_0\,\mathrm{D}_\theta\Psi]\mathrm{D}_t\theta - \mathsf{s}_0\theta \geq 0 \tag{10.4}$$

with $\mathsf{m}_0 = 0$ defines the reduced first Piola–Kirchhoff stress tensor $\bar{\mathbf{\Pi}}^t$ and the mass specific entropy S as thermodynamically conjugate variables to the spatial motion deformation gradient $\mathbf{F}$ and the absolute temperature θ

$$\bar{\mathbf{\Pi}}^t = \rho_0\mathrm{D}_{\mathbf{F}}\Psi \quad S = -\mathrm{D}_\theta\Psi \quad \bar{\mathbf{b}}_0^{int} = \mathbf{0}. \tag{10.5.1–3}$$

From the dissipation inequality (10.4) we conclude, that the reduced internal forces $\bar{\mathbf{b}}_0^{int}$ of the spatial motion problem vanish identically. Furthermore, similar to Cowin and Hegedus [6], we are left with the inequality $-\mathsf{s}_0\theta = -\mathsf{S}_0\theta \geq 0$, which places additional restrictions on the constitutive assumptions for the extra external entropy input s_0 through the extra entropy flux $\mathbf{S}$ and the extra entropy source $\mathscr{S}_0$, the trivial choice being $\mathsf{s}_0 = 0$.

10.2 Material motion problem

In a similar way, the free energy density Ψ_t of the material motion problem can be assumed to be representable by the free energy Ψ weighted by the spatial density ρ_t

$$\Psi_t = \rho_t\Psi. \tag{10.6}$$

Within the material motion context, the free energy Ψ consequently depends on the material motion deformation gradient $\mathbf{f}$, the absolute temperature θ and the material placement $\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{x})$, representing a field in spatial parametrization. The spatial density ρ_t is thus a function of the material motion deformation gradient $\mathbf{f}$, the material placement $\mathbf{\Phi}$ and the time t.

$$\Psi = \Psi(\mathbf{f}, \theta, \mathbf{\Phi}) \quad \rho_t = \rho_t(\mathbf{f}, \mathbf{\Phi}; t) \tag{10.7.1, 2}$$

The spatial time derivative of the free energy Ψ thus takes the following form

$$\mathrm{d}_t\Psi = \mathrm{d}_{\mathbf{f}}\Psi : \mathrm{d}_t\mathbf{f} + \mathrm{d}_\theta\Psi\,\mathrm{d}_t\theta + \partial_{\mathbf{\Phi}}\Psi\cdot\mathrm{d}_t\mathbf{\Phi}. \tag{10.8}$$

Recall, that $\mathrm{d}_t\mathbf{\Phi} = \mathbf{V}$ by definition, compare (3.11). The evaluation of the dissipation inequality of the material motion problem expressed by Eq. (9.36) as $\bar{\mathsf{d}}_\mathsf{t}^{loc} = \bar{\mathsf{D}}_\mathsf{t}^{loc} - S_t\nabla_X\theta\cdot\mathbf{V}$

$$\begin{aligned}\bar{\mathsf{d}}_\mathsf{t}^{loc} = {} & \left[\bar{\boldsymbol{\pi}}^t - \rho_t\mathrm{d}_{\mathbf{f}}\Psi\right] : \mathrm{d}_t\mathbf{f} - \left[\bar{\mathbf{B}}_t^{int} - \rho_t S\nabla_X\theta + \rho_t\partial_{\mathbf{\Phi}}\Psi\right]\cdot\mathbf{V} - \mathsf{M}_\mathsf{t}\Psi \\ & - [\rho_t S + \rho_t\,\mathrm{d}_\theta\Psi]\mathrm{d}_t\theta - \mathsf{S}_\mathsf{t}\theta\end{aligned} \tag{10.9}$$

with $\mathsf{M}_\mathsf{t}\Psi = \Psi\partial_{\mathbf{\Phi}}\rho_t\cdot\mathbf{V} + \Psi\mathrm{d}_{\mathbf{f}}\rho_t : \mathrm{d}_t\mathbf{f}$ renders the definition of the reduced momentum flux $\bar{\boldsymbol{\pi}}^t$, the mass specific entropy S and the reduced internal forces $\bar{\mathbf{B}}_t^{int}$ of the material motion problem.

$$\bar{\boldsymbol{\pi}}^t = \rho_t\mathrm{d}_{\mathbf{f}}\Psi + \Psi\mathrm{d}_{\mathbf{f}}\rho_t \quad S = -\mathrm{d}_\theta\Psi \quad \bar{\mathbf{B}}_t^{int} = \rho_t S\,\nabla_X\theta - \rho_t\partial_{\mathbf{\Phi}}\Psi - \Psi\partial_{\mathbf{\Phi}}\rho_t. \tag{10.10.1–3}$$

Again, the remaining inequality $-\mathsf{s}_\mathsf{t}\theta = -\mathsf{S}_\mathsf{t}\theta \geq 0$ can be used to define constitutive assumptions for the extra external entropy input S_t in terms of the extra entropy flux $\mathbf{s}$ and the extra entropy source $\mathscr{S}_t$.

10.3 Spatial vs. material motion problem

The relations between the balance of momentum of the spatial motion problem and the material motion problem have already been sketched in Sect. 6.1.3. They are characterized by the complete pull back of the balance of physical momentum onto the material manifold. For a specific choice of constitutive relations, e.g. the presented thermo-hyperelastic material, the transitions between the spatial and the material motion problem can be further specified, compare, e.g. Maugin [34] or also Steinmann [47]. Equation (6.3), the spatial motion momentum balance with material reference and material parametrization

$$\mathrm{D}_t \mathbf{p}_0 = \mathrm{Div}(\bar{\mathbf{\Pi}}^t_D + \mathbf{p} \otimes \mathbf{R}) + \bar{\mathbf{b}}_0 + \mathscr{R}_0 \mathbf{p} - \nabla_X \mathbf{p} \cdot \mathbf{R} \tag{10.11}$$

serves as starting point for this derivation. Thereby, we have made use of the definitions $\mathbf{\Pi}^t_D = \bar{\mathbf{\Pi}}^t_D + \mathbf{p} \otimes \mathbf{R}$ and $\mathbf{b}_0 = \bar{\mathbf{b}}_0 + \mathscr{R}_0 \mathbf{p} - \nabla_X \mathbf{p} \cdot \mathbf{R}$. The pull back of the momentum rate term, the momentum flux term and the momentum source term yields the following results:

$$-j\mathbf{F}^t \cdot \mathrm{D}_t\, \mathbf{p}_0 = \mathrm{d}_t\big[-j\mathbf{F}^t \cdot \mathbf{p}_0\big] + \mathrm{div}(\rho_t \mathrm{d}_\mathbf{f} K) - \rho_t \partial_\mathbf{\Phi} K, \tag{10.12.1}$$

$$-j\mathbf{F}^t \cdot \mathrm{Div}\, \mathbf{\Pi}^t_D = \mathrm{div}\big(-j\mathbf{F}^t \cdot \bar{\mathbf{\Pi}}^t_D \cdot \mathbf{F}^t + \mathbf{P} \otimes \mathbf{r}\big) + j\bar{\mathbf{\Pi}}^t : \nabla_X \mathbf{F} + j[\mathbf{p} \otimes \mathbf{R}] : \nabla_X \mathbf{F}, \tag{10.12.2}$$

$$-j\mathbf{F}^t \cdot \mathbf{b}_0 = -j\mathbf{F}^t \cdot \bar{\mathbf{b}}_0 + \mathscr{R}_t \mathbf{P} - \nabla_x \mathbf{P} \cdot \mathbf{r} - j[\mathbf{p} \otimes \mathbf{R}] : \nabla_X \mathbf{F}. \tag{10.12.3}$$

Herein, we have applied the transport theorem (3.22) and the appropriate Piola transforms. Furthermore, the definition of the dynamic momentum flux $\bar{\mathbf{\Pi}}^t_D = \bar{\mathbf{\Pi}}^t$ and the kinematic compatibility condition $\nabla_X \mathbf{F}^t : \bar{\mathbf{\Pi}}^t = \bar{\mathbf{\Pi}}^t : \nabla_X \mathbf{F}$ have been included. Next, we shall assume the existence of a potential $\Psi_0 = \rho_0 \Psi$ according to the thermo-hyperelastic model problem advocated in the present chapter. Consequently, the second term of the pull back of the divergence of the momentum flux can be further specified

$$\Psi = \Psi(\mathbf{F}, \theta; \mathbf{X}) \quad j\bar{\mathbf{\Pi}}^t : \nabla_X \mathbf{F} = \mathrm{div}(\rho_t \Psi \mathbf{F}^t) + \rho_t S \nabla_X \theta - \rho_t \partial_\mathbf{\Phi} \Psi. \tag{10.13.1,2}$$

By introducing the material motion momentum density $\mathbf{P}_t$, the material motion momentum flux $\bar{\boldsymbol{\pi}}^t_d$ and the material motion source $\bar{\mathbf{B}}_t$ as

$$\mathbf{P}_t = -j\mathbf{F}^t \cdot \mathbf{p}_0, \tag{10.14.1}$$

$$\bar{\boldsymbol{\pi}}^t_d = -j\mathbf{F}^t \cdot \bar{\mathbf{\Pi}}^t_D \cdot \mathbf{F}^t + \rho_t \Psi \mathbf{F}^t - \rho_t \mathrm{d}_\mathbf{f} K, \tag{10.14.2}$$

$$\bar{\mathbf{B}}_t = -j\mathbf{F}^t \cdot \bar{\mathbf{b}}_0 + \rho_t S \nabla_X \theta + \rho_t \partial_\mathbf{\Phi}[K - \Psi], \tag{10.14.3}$$

we end up with the balance of momentum of the material motion problem with spatial reference and spatial parametrization (6.16),

$$\mathrm{d}_t \mathbf{P}_t = \mathrm{div}(\bar{\boldsymbol{\pi}}^t_d + \mathbf{P} \otimes \mathbf{r}) + \bar{\mathbf{B}}_t + \mathscr{R}_t \mathbf{P} - \nabla_x \mathbf{P} \cdot \mathbf{r}, \tag{10.15}$$

whereby we have made use of the definitions $\boldsymbol{\pi}^t_d = \bar{\boldsymbol{\pi}}^t_d + \mathbf{P} \otimes \mathbf{r}$ and $\mathbf{B}_t = \bar{\mathbf{B}}_t + \mathscr{R}_t \mathbf{P} - \nabla_x \mathbf{P} \cdot \mathbf{r}$. Again, due to the specific choice of the source terms, we can observe the remarkable duality of Eqs. (10.11) and (10.15).

11 Conclusion

We have derived a general framework for the thermodynamics of open systems. The provided set of equations is believed to be particularly useful for problems typically encountered in the fields of chemo- and biomechanics. In contrast to most existing formulations for open systems in which an interaction with the environment takes part exclusively via the exchange of source terms, we have allowed for an additional in- or outflux of matter keeping the underlying theory as general as possible. Consequently, not only the balance of mass, but also all the other balance equations had to be reconsidered. To clarify the influence of the non-constant amount of mass, we have introduced the notions of "volume specific" and "mass specific" format. Thereby, the latter is believed to be of particular interest, since the mass specific balance equations were set up in complete analogy to the classical thermodynamical case.

Throughout the entire derivation, we have followed a two-step strategy. First, we have formally introduced the balance equations for the material motion problem in complete analogy to the well-known balance equations of the classical spatial motion problem. Thereby, the quantities introduced in the material motion context, the related fluxes and sources, have initially been introduced through mere definitions guided by duality arguments in comparison to the spatial motion setting. In a second step, we focused on bridging the gap between the spatial and the material motion problem. For this purpose, the first and second law of thermodynamics have been further elaborated to yield additional useful relations between various spatial and material motion fluxes and sources. These relations give rise to further physical interpretations of the material motion problem which is believed to be particularly well-suited to characterize the nature of open systems, especially in the presence of material inhomogeneities.

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