Meteorol Atmos Phys 82, 287–301 (2003)
DOI 10.1007/s00703-001-0593-8

Meteorology and Atmospheric Physics

Printed in Austria

[1] Deutscher Wetterdienst, Offenbach, Germany
[2] Dipartimento di Ingegneria Civile ed Ambientale, Università degli Studi di Trento, Trento, Italy

Review of numerical methods for nonhydrostatic weather prediction models

J. Steppeler[1], **R. Hess**[1], **U. Schättler**[1], and **L. Bonaventura**[2]

Received July 6, 2001; revision October 12, 2001
Published online: November 21, 2002

Summary

Currently available computer power allows to run operational numerical weather prediction models at resolutions higher than 10 km. The aim of such high resolution modeling is the prediction of local weather, including orographically induced winds and local precipitation patterns. In this range the hydrostatic approximation is no longer valid and nonhydrostatic models have to be used instead. For several decades these models have been developed for research purposes only, but operational application is now reality. In this paper, the numerical methods used in current nonhydrostatic forecast models will be reviewed and some promising techniques in this field will be discussed. Special attention is given to aspects such as the choice of the vertical coordinate, the efficiency of algebraic solvers for semi-implicit time discretizations, and accurate and non-oscillatory advection schemes.

1. Introduction

Operational numerical weather prediction (NWP) models are currently close to the 10 km horizontal resolution threshold, beyond which the hydrostatic approximation becomes inaccurate. Therefore, many NWP centers have been doing research in nonhydrostatic (NH) modeling over the last ten years. Some of the numerical methods used for NH models are adaptations of methods well tested in the hydrostatic case. Many of these techniques are presented in reviews such as Arakawa and Mesinger (1976), and Pielke (1984). However, some specific features of NH dynamics and some more recent numerical developments require special consideration.

In most NH models, for example, the vertical coordinate is fixed in time and based on geometrical height, whereas it is mostly time dependent in hydrostatic modeling. Fully elastic models require numerical methods that handle sound waves appropriately, in order to avoid excessive time-step restrictions. Semi-implicit time integration methods can be chosen here, which lead to three dimensional Helmholtz equations as opposed to two dimensional ones, which are solved in hydrostatic semi-implicit schemes. The solution of the three dimensional equations require highly efficient algorithms, that must be applicable on parallel computers.

Furthermore, special numerical techniques are required for forecasts in the meso-γ scale which is resolved with the very high resolution that can be reached by NH models. The most important among these are small scale orographically induced winds, explicit representation of convection, and local weather, such as local precipitation and fog. Orography in high resolution models in general is steeper than for larger scales. Therefore, spurious numerical effects caused by the deformation of the terrain-following coordinates are potentially present and must be avoided. Explicit forecasts of convection require numerical methods that can handle highly divergent flows. Numerical methods have to be appropriate for this situation. Prediction of local weather features such as rain and fog requires accurate and non oscillatory moisture advection schemes. The accuracy of second- or first-order schemes, which are present in many current operational models, may not be sufficient for the applications indicated.

The issue of conservation of mass, and possibly of other quantities such as energy or momentum, may become important in this respect, even though most current NH models neglect this. Furthermore, models that resolve convective clouds can be used for nowcasting purposes over time ranges of 3 to 6 hrs. Assimilation of highly resolved observations, such as radar, is essential for such applications. Due to the strong time dependence, these indirect observations need to be assimilated via continuous assimilation schemes, for which a high efficiency is required. This important numerical issue is not subject of this review but is treated by Park and Županski (2002).

This paper reviews and discusses features of the dynamical cores of NH numerical models, with special attention to the discretization approaches for the fully elastic equations applied in operational models. A number of important aspects will not be considered here, such as, for example, top and lateral boundary conditions. These aspects are currently treated in the same way as in hydrostatic models. It will, however, be attempted to discuss numerical techniques which may become important for the future high resolution dynamical cores. These include methods for the solution of Helmholtz equations and adaptive meshes. In particular, techniques which are being applied successfully in areas like oceanography or computational aerodynamics will be considered. Some of these approaches are likely to be relevant in the development of the next generation of operational NH models. Current operational NH models are described in Sect. 2. Problems related to the coordinate choices, discretization grids, and vertical coordinates are discussed in Sect. 3. Split-explicit and semi-implicit time discretizations are discussed in Sect. 4, along with numerical methods for the solution of the resulting sparse linear systems. The problem of advection schemes for NH models is discussed in Sect. 5 and some perspectives for the future developments are presented in Sect. 6.

2. The state of the art of operational nonhydrostatic models

NH models have been developed since the early 70s for research purposes in the atmospheric sciences community. Thorough reviews of earlier work can be found in Pielke (1984). In the past, the anelastic approximation was often used, which can be justified on the basis of a scale analysis. In this approach, the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

is replaced by the diagnostic equation

$$\nabla \cdot (\rho_0 \mathbf{u}) = 0, \tag{2}$$

where ρ_0 is a reference density profile.

Various anelastic models developed by German universities and research centers in the 80s have been reviewed in Schlünzen (1994). The purpose of this section is rather to review models that are currently run in operational or quasi-operational mode. The models mentioned in this paper are summarized in the Appendix. All these models employ the fully elastic equations, where the continuity equation is used without further approximation before discretization.

The model equations are often formulated in advection form, using temperature and pressure as prognostic variables, along with the velocity components and moisture fields. As an example, the equations employed in the *Lokal Modell* (LM) (see Steppeler et al, 2002) can be written in Cartesian (x, y, z) coordinates in the dry adiabatic case as

$$\begin{aligned}
\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla v + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y}, \\
\frac{\partial w}{\partial t} + \mathbf{v} \cdot \nabla w &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \\
\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p &= -(c_{pd}/c_{vd})\, p\, D, \\
\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T &= \frac{1}{\rho\, c_{pd}}\left(\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p\right).
\end{aligned} \tag{3}$$

Here, u, v and w are the three components of the velocity vector $\mathbf{v}$, p is pressure and T the absolute temperature. ρ is the density of moist air, connected to the pressure p by the equation of state $p = \rho RT$ with the gas constant R for dry air. c_{pd} and c_{vd} are the specific heats of dry air at constant pressure and constant volume, g is the gravity acceleration and f is the Coriolis parameter. $D \equiv \nabla \cdot \mathbf{v}$ denotes the three-dimensional wind divergence.

Some models, such as MM5 and LM, violate mass conservation due to numerical approximations to (3) and approximations concerning the heating rates, which are considered as zero in (3).

2.1 Split-explicit, Eulerian models

A number of NH models employing Eulerian discretization of advection have been used in an operational or quasi-operational mode to produce numerical forecasts in real time. Models of this kind are ARPS (Xue et al, 2000), the JMA-MRI model (Saito, 1997), LM (Steppeler et al, 2002), MM5 (Dudhia, 1993), and RAMS (Cotton et al, 2002). The robustness and the computer time required by these models are not too different from current hydrostatic Eulerian operational models, if proper care in the implementation is taken.

The numerical methods used in these models are different from those employed in most operational hydrostatic models. While for hydrostatic models the use of the semi-implicit method is quite standard, sometimes in combination with a semi-Lagrangian treatment of advection, NH models are mostly based on the *split-explicit* scheme introduced by Klemp and Wilhelmson (1978) and on three time levels, leapfrog time discretization (see Sect. 5). In order to make the small time step size of the split-explicit method independent from the vertical grid spacing, which is much smaller than the horizontal one in operational NWP applications, vertical discretization is usually treated implicitly. Although this method is quite well established and currently the best option to reach operational efficiency at the meso-γ scale, the more complicated approaches, such as three dimensional semi-implicit methods could well be faster, if sufficient research is invested. Two time level schemes have a potential for greater efficiency and accuracy of moisture treatment, but encounter some difficulties, as pointed out by Skamarock and Klemp (1994).

All models mentioned above use terrain following coordinates which are fixed in time. Mostly this is a σ-coordinate based on reference pressure or Gal-Chen type coordinates (see Sect. 3). Some of the models use second order centered differences. As an option, higher-order advection schemes are also implemented sometimes. For all models the grids are uniformly structured, and mostly of Arakawa-B or C type (Arakawa, 1966).

The models described here have a computational speed similar to that of Eulerian hydrostatic models and are therefore quite attractive for operational or quasi-operational use. With the exception of JMA-MRI, the models have been implemented on parallel computers with distributed memory (see Sathye et al, 1997, for ARPS, Schättler et al, 2000, for LM, Michalakes, 1997, for MM5, and Tremback and Walko, 1997, for RAMS). The parallelization strategy used is the decomposition of the horizontal domain, where each processor solves the model equations on its own subdomain. Because the split-explicit method only involves grid points of the finite difference stencil, all communications are local, i.e. an exchange is necessary only between adjacent domains. This is a substantial advantage on massively parallel distributed memory computers.

2.2 Semi-implicit, semi-Lagrangian methods

Semi-implicit and semi-Lagrangian methods have been developed for hydrostatic models to overcome limitations posed on the time step due to stability reasons. The first fully elastic semi-implicit semi-Lagrangian NH model has been introduced in Tanguay et al (1990). In this paper it was demonstrated that this approach is also applicable to NH models using the fully elastic equations (3). This model employed height as the vertical coordinate and used the semi-Lagrangian approach only for horizontal advection, while a standard finite difference discretization was applied in the vertical direction. However, tests have been carried out without topography only.

This attempt paved the way for the development of other fully elastic, semi-Lagrangian models in the Canadian research community. Benoit et al (1997) developed the MC2 as a research community mesoscale model by extending Robert's and Tanguay's work with features necessary for realistic three-dimensional simulations. A first test version of this model was developed by Pinty et al (1995). One of the main differences between the MC2 and the former model is the introduction of the terrain-following vertical coordinate of the Gal-Chen type.

The GEM model (see, e.g., Staniforth and Côté, 1995; Côté et al, 1998) has been developed by the Atmospheric Environment Service of Canada in order to replace both their previous global and regional models. GEM is a unified model which can be used simultaneously for global and local forecasting. Semi-implicit and semi-Lagrangian discretization is employed. A stretched Cartesian grid allows for higher resolution over an area of interest and the hydrostatic pressure vertical coordinate introduced by Laprise (1992) is employed. Although devised so as to be developed into a fully nonhydrostatic model, it has been running operationally only in hydrostatic mode so far.

In Europe, a semi-implicit semi-Lagrangian model is developed at the UK Met Office, see Cullen et al (1997). This model is based on previous work on mesoscale NH models, see, e.g. Cullen (1990) and Golding (1992). It is a unified forecast and climate model, which is meant to replace the existing unified model described in Cullen (1993). In the new model, some of the conventional meteorological approximations (such as the *shallow atmosphere* approximation) are not applied.

Care has to be taken when implementing semi-implicit semi-Lagrangian models on parallel computers. A parallel version of MC2 is described in Thomas et al (1997). In semi-Lagrangian methods the derivatives of scalar fields are approximated along a parcel trajectory that arrives at a grid point at the end of a time step. Values of the departure point of this trajectory, which usually is not a grid point, are computed by interpolation of the surrounding grid points. When decomposing the horizontal domain, the departure point (and the necessary surrounding grid points) could reside in any of the surrounding subdomains and a trajectory could even traverse several subdomains. In the MC2 a fixed overlap strategy is implemented, where all grid points are exchanged between subdomains that might be necessary to calculate the values for the departure points. For this purpose the maximum possible wind speed is estimated for the whole integration time.

While the semi-Lagrangian advection still only needs a nearest neighbor exchange, many numerically efficient solvers for the Helmholtz equation in semi-implicit methods require a global communication. In the MC2 a flexible variant of the generalized minimal residual algorithm (FGMRES) is used. On the one hand there are local communications necessary between neighboring subdomains because of matrix-vector products and also global communications for the calculation of scalar products. The efficiency and performance therefore depends much on good parallel implementations of these methods.

2.3 Other operational models

The NH version of the spectral model ALADIN has been developed on Laprise's idea that hydrostatic-pressure coordinates could also be used in nonhydrostatic atmospheric models (Laprise, 1992). Bubnova et al (1995) implemented a scheme that uses hydrostatic pressure as an independent variable and that remains very close to a primitive equation system. The gravity and sound waves are filtered through a semi-implicit algorithm, where the discrete linear operators have the same form as in the hydrostatic dynamics. To handle the orography, a terrain following hybrid coordinate is introduced.

Based on the NCEP meso-model, Janjic et al (2001) proposed and tested an alternative approach for nonhydrostatic modeling. It is based on relaxing the hydrostatic approximation in a hydrostatic model using a vertical coordinate based on hydrostatic pressure. Thus the model can also be applied to nonhydrostatic motions. The nonhydrostatic dynamics is introduced through an add-on module that can be turned on and off depending on the resolution. The system of the nonhydrostatic equations is splitted into the hydrostatic system and a set of equation that allows the computation of corrections appearing due to the vertical acceleration.

3. Grids and vertical coordinates

The high resolution which is necessary in order to resolve NH effects results in very large computational grids and in a steeper orography. The size of the required grids will be much larger than in current hydrostatic models. (see also the analysis in Lindzen and Fox-Rabinowitz, 1989).

Regarding the choice of the discretization grid, a question that arises naturally is whether the memory requirements necessary to achieve a given accuracy can be reduced either by local grid refinement in the areas of interest or by less traditional choices of the discretization grid. Some steps towards the development of adaptive meshes have already been taken, which are described in Sect. 3.1.

In contrast to hydrostatic models, where use of a hybrid pressure based vertical coordinate is the general choice, height based terrain following coordinates are nowadays the most established choice, but various difficulties have been encountered over very steep mountains. It is still unclear to which extent the more complex orographies of high resolution models will require different approaches, some of which have already been successful applied to other environmental flows. The prognostic variables often combine a horizontal C-grid arrangement with vertical staggering after Lorenz (1960) (all scalar variables at the cell center and vertical velocity at the cell side). This arrangement has been questioned in Arakawa and Moorthi (1987) as a source of spurious computational modes and now some models (see, e.g. Cullen et al, 1997) employ instead the *Charney-Phillips* staggering introduced in Charney and Phillips (1953). In this grid arrangement, potential temperature (or temperature) is given at the cell side, at the same location as the vertical velocity.

With unstructured grids discretization is often based on triangles and finite volumes or finite elements. Unstructured grids can be refined in a natural way during the construction phase. Complications for dynamical grids are the dynamical adaptation of the grids, the grid partitioning and the load balance process for simulations on parallel computers. A number of simulations exist, mostly in oceanography (Behrens, 1998) and technical engineering. A finite difference discretization has been proposed for NH models of estuaries and lagoons in Casulli and Walters (2000), where orthogonal grids are introduced that extend Arakawa-C staggering to unstructured grids.

The simple data arrays of structured grids facilitate implementation and provide efficient memory access and thus numerical operations and are therefore used for atmospheric simulations. The atmosphere is a homogeneous medium and also the lateral boundaries do not require the flexibility of unstructured grids.

3.1 Adaptive and unstructured grids

Because of their high potential to save computation time and the possibility to run even higher resolutions, a variety of different grid adaptation approaches have been considered (see, e.g. Arney and Flaherty, 1990; Bai and Brandt, 1987; Behrens, 1998; Berger and Oliger, 1984; McCormick, 1989; Skamarock et al, 1989; Fulton, 1997). These range from statically and one-way nesting where an area of specific interest is more highly resolved to dynamically adaptive grids that are automatically adjusted to the dynamical weather conditions being simulated and require two-way feedbacks. Whereas adaptive grids are almost standard for computational intensive industrial applications as CFD or structural mechanics, operational meteorological models have at best very restricted adaptive features (e.g., MM5, RAMS and LM).

In atmospheric applications, adaptive grids are typically formed with rectangular patches of higher resolution that are nested in the original grid. Simple application of grid adaptation is the nesting of higher resolution regional models into lower resolved global models using one-way interaction from the global to the regional model only. Most of the models mentioned in Sect. 2 support self nesting procedures in order to refine the resolution in predefined areas. A number of them already provide two-way interactive coupling with feedback from the fine to the coarse grid (MM5, RAMS, LM).

Approaches to dynamically adapt the resolution to the simulated meteorological conditions exist as well. Based on a refinement criterion the areas to be highly resolved are determined during simulation only. Dynamical grid adaptation requires two-way interaction in order to improve the accuracy of the global grid and to remove the fine grid once it is not required any more. Grid adaptation with more or less dynamical rectangular patches have been implemented by Skamarock et al (1997), Fulton (1997), and also for parallel computers by Michalakes (1997). Grid partitioning and load distribution for parallel computers with distributed memory were investigated for a shallow water model in Hess (1999).

Other approaches of structured adaptive grids are continuous refinements, where the resolution is smoothly increased to the certain region of interest or techniques where gridpoints are slightly moved (e.g., Dietachmayer and Droegemeier, 1992; Staniforth and Côté, 1995; Côté et al, 1998).

3.2 Vertical coordinates

Most NH models use vertical coordinates derived from the natural height coordinate by an appropriate normalization, which allows to map the computational domain onto a unit height box. Such coordinates, also known as *terrain following* vertical coordinates, have been used extensively since the work of Gal-Chen and Somerville (1975), and Clark (1977). More specifically, given an orographic profile $h(x, y)$ and some constant a general normalized coordinate can be defined as

$$\sigma = \begin{cases} \frac{z-h}{z_F-h} z_F & h \le z \le z_F \\ z & z_F \le z \le z_T. \end{cases}$$

Here, z_F denotes some reference level above the maximum orographic height and z_T denotes the height of the domain top. If $z_T = z_F$, the original adapted coordinate of Gal-Chen and Somerville (1975) is obtained, otherwise a hybrid coordinate results, which reverts to the natural height coordinate above the reference level z_F. Another approach is the use of sigma levels based on the reference pressure. This method is used in MM5. LM allows a rather general coordinate where time independent layers can be prescribed as a table of z-values.

The choice of these terrain-following coordinates has the advantage that the boundary layer is uniformly resolved and orography is resolved according to the model resolution. Furthermore, the computational domain is rectangular, which allows for simple data structures and efficient model implementations. On the other hand, the resulting coordinate system is non orthogonal and is strongly deformed over steep orography, thus resulting in a series of potential problems, in part analogous to those highlighted in Gary (1973), and Sundqvist (1976) for the case of the pressure based coordinates. For example, numerical experiments carried out with LM at DWD showed that spurious flows develop over a steep mountain because of purely numerical reasons, if a stable atmosphere at rest is assumed as an initial state. Furthermore, the Helmholtz equation derived in semi-implicit models using such coordinates in general is asymmetric, thus requiring the application of computationally more costly linear solvers (see the discussion in Sect. 4).

One attempt to overcome these difficulties is the so-called η-coordinate proposed by Mesinger (see, e.g. Mesinger et al, 1988). This method describes the orography as a construction of building blocks. Because of the discontinuity of the orographic function it does not have sufficient convergence properties (see Kröner, 1997). Models based on the η-coordinate were shown to have major difficulties at high resolution. An analysis of inaccuracies arising in a NH η-coordinate model has been carried out in Gallus and Klemp (2000).

Another possible solution to the problems connected to steep orography is the use of height as vertical coordinate, without any terrain following normalization, coupled to finite volume or finite elements discretization of the Euler equations, along the lines of what is done in many typical CFD application areas, such as for example computational aerodynamics. The natural coordinate z suffers from obvious difficulties in obtaining a uniform resolution of the boundary layer. Following standard schemes from CFD, such problems are, however, solvable, using variable grids together with the finite volume approach. Various types of finite volume discretization approaches for oceanic circulation models have been reviewed and analyzed in Adcroft et al (1997) in order to study their behavior over orography. In fact, it was shown for example in Montavon (1997) how a commercial CFD code could be adapted to reproduce quite reasonable lee-wave patterns. However, the straightforward application of finite volume techniques to atmospheric flows has resulted so far in methods whose computational efficiency is not compatible with the requirements of operational models. In the context of estuarine modeling, the hybrid finite-difference finite volume approach described in

Casulli (1990), and Casulli and Cattani (1994) has proven to yield rather efficient, accurate and robust semi-implicit models. In this approach, only the divergence term in the pressure equation is discretized by a simplified finite volume method (see, e.g. Eqs. (5) in Sect. 4), thus resulting in a symmetric Helmholtz equation to be solved. This technique has also been successfully extended to NH flows in estuaries, see, e.g. Casulli and Stelling (1998). Furthermore, an accurate and efficient computation of idealized stratified atmospheric flow over two-dimensional orography was achieved in Bonaventura (2000) with a semi-implicit semi-Lagrangian model based on the same principle. Extensions and improvements of this technique are currently investigated at DWD (Steppeler et al, 2002).

The pressure based σ-coordinates, widely employed in large scale models based on the primitive equations, have also been applied to NH models, see, e.g. Durran and Klemp (1983), Dudhia (1993), and Xue and Thorpe (1991). A vertical coordinate representing hydrostatic pressure has been proposed in Laprise (1992). This coordinate can be defined as

$$\eta = \frac{\pi - \pi_T}{\pi_S - \pi_T}, \tag{4}$$

where π_T, π_S represent pressure values, respectively, at the top and at the bottom of the considered domain, and π is computed at each timestep from the hydrostatic relation $\frac{\partial \pi}{\partial z} = -\rho g$. This type of vertical coordinate is practically advantageous, because it allows to reuse physical parameterization libraries available from hydrostatic models, as well as allowing a simple way to switch from a hydrostatic to NH model (see, e.g. Côté et al, 1998). The hydrostatic pressure coordinate has been incorporated in the GEM model and in the ALADIN model of Météo France.

4. Split-explicit and semi-implicit methods

With explicit time discretizations the size of stable time steps is limited by the CFL-condition that basically states that the fastest waves must not propagate more than the mesh size within one time step. In NH models using the fully elastic equations disturbances propagate at the speed of sound and because of the high local resolution time steps are restricted to some seconds only. A way to reduce these severe time step restrictions is the operator splitting scheme known as the *split-explicit* method (also referred to as mode splitting in other modeling communities). In this approach, the elastic equations are solved by splitted computation of the slow advective tendencies and of the equation terms giving rise to fast sound wave solutions. In the case of the two-dimensional Euler equations (see, e.g. Holton, 1992)

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x},\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y},\\
&\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \gamma p_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.
\end{aligned} \tag{5}$$

Here, ρ_0, p_0 denote constant reference density and pressure values, respectively. The advective tendencies $\left(\frac{\partial u}{\partial t}\right)^n_{adv} = \left(u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\right)^n$, $\left(\frac{\partial v}{\partial t}\right)^n_{adv} = \left(u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y}\right)^n$ and $\left(\frac{\partial p}{\partial t}\right)^n_{adv} = \left(u\,\frac{\partial p}{\partial x} + v\,\frac{\partial p}{\partial y}\right)^n$ at time step n are approximated first, usually by leapfrog time discretization and centered finite differences in space (see Sect. 5). The terms responsible for the propagation of sound waves are then integrated explicitly in time by forward finite differences

$$\begin{aligned}
&\frac{u^{\nu+1} - u^{\nu}}{\Delta\tau} = \left(\frac{\partial u}{\partial t}\right)^n_{adv} - \frac{1}{\rho_0}\frac{\partial p^{\nu}}{\partial x},\\
&\frac{v^{\nu+1} - v^{\nu}}{\Delta\tau} = \left(\frac{\partial v}{\partial t}\right)^n_{adv} - \frac{1}{\rho_0}\frac{\partial p^{\nu}}{\partial y},\\
&\frac{p^{\nu+1} - p^{\nu}}{\Delta\tau} + \gamma p_0\left(\frac{\partial u^{\nu+1}}{\partial x} + \frac{\partial v^{\nu+1}}{\partial y}\right) = \left(\frac{\partial p}{\partial t}\right)^n_{adv},
\end{aligned} \tag{6}$$

where $\nu = 1, \ldots, N_{\text{sub}}$ is the number of substeps to be computed, chosen in such a way that the resulting short time step $\Delta\tau = \frac{\Delta t}{N_{\text{sub}}}$ complies with a CFL-condition based on the speed of sound. This approach has been proposed in Klemp and Wilhelmson (1978) and its stability properties have been analyzed in Skamarock and Klemp (1992) (see also the discussion in Browning and Kreiss, 1994). In the three-dimensional case the vertical coordinate is normally treated implicitly (see Ikawa, 1988). Therefore these schemes can be described as one-dimensional implicit. Split-explicit discretization is attractive because it does not require the solution of large linear systems and it can be easily implemented also on parallel computers.

On the other hand, the development of unified models for global and regional forecasting is generally found to be the most efficient option for the future (see, e.g. Cullen, 1993; Côté et al, 1998). Therefore, the time discretization employed should perform well also for large scale flows in terms of efficiency and accuracy. This efficiency in the framework of the split-explicit methods is obtained by treating the buoyancy term implicitly according to Skamarock and Klemp (1992). This option is realized in LM, which therefore handles also the coarser resolutions efficiently.

Large scale flows can be simulated efficiently by using *semi-implicit* discretization, which however results into models with a somewhat more complex architecture. In its simplest first-order version, semi-implicit time discretization of Eqs. (5) is here given for the one-sided form by

$$\frac{u^{n+1}-u^n}{\Delta t} = \left(\frac{\partial u}{\partial t}\right)^n_{adv} u^n - \frac{1}{\rho_0}\frac{\partial p^{n+1}}{\partial x}$$
$$\frac{v^{n+1}-v^n}{\Delta t} = \left(\frac{\partial v}{\partial t}\right)^n_{adv} v^n - \frac{1}{\rho_0}\frac{\partial p^{n+1}}{\partial y}$$
$$\frac{p^{n+1}-p^n}{\Delta t} + \gamma p_0\left(\frac{\partial u^{n+1}}{\partial x} + \frac{\partial v^{n+1}}{\partial y}\right) = \left(\frac{\partial p}{\partial t}\right)^n_{adv} p^n, \tag{7}$$

where $\left(\frac{\partial u}{\partial t}\right)^n_{adv}$, $\left(\frac{\partial v}{\partial t}\right)^n_{adv}$, $\left(\frac{\partial p}{\partial t}\right)^n_{adv}$ represent some explicit discretization of the advective terms. The values of u^{n+1} and v^{n+1} are then substituted into the pressure equation so as to obtain

$$p^{n+1} - \frac{\Delta t^2 \gamma p_0}{\rho_0}\Delta p^{n+1} = F^n, \tag{8}$$

where all the explicit terms are collected in F^n. This Helmholtz equation has then to be solved at each timestep in order to determine p^{n+1} and update consequently the velocity values. Semi-implicit time discretization has been widely applied in hydrostatic atmospheric modeling after its introduction by Kwizak and Robert (1971), and Robert (1982), while its application to the full Euler equations for NH atmospheric modeling dates back to Tapp and White (1976). It is to be remarked that this is analogous to what is usually done in numerical modeling of incompressible or anelastic fluids in order to impose the divergence-free condition. This constraint can be taken into account in a variety of ways, resulting, among others, in the MAC method of Harlow and Welsh (1965) or in the so called *projection* methods. Comprehensive reviews of the main numerical techniques for incompressible flow can be found in many standard computational fluid dynamic (CFD) references, see, e.g. Dautray and Lions (1993), Patankar (1980), Peyret and Taylor (1983), and Quarteroni and Valli (1994). In all these methods, the numerical solution of a Poisson equation is required at each time-step, while in the case of compressible fluids analogous steps lead to the derivation of the Helmholtz equation for pressure.

In the following, the linear solvers most widely applied in NH models are reviewed and the relative advantages of the semi-implicit versus the split-explicit option are discussed.

4.1 Numerical problems and solutions for semi-implicit models

The fast solution of the Helmholtz equation (8) at each time step is the main drawback of semi-implicit methods compared to explicit schemes, especially with regard to the implementation on parallel computers. In global NWP models, the spectral method has been employed successfully, but its efficiency for high resolution modeling is increasingly being questioned. With hydrostatic models it is possible to separate the three-dimensional equation into a system of two-dimensional Helmholtz equations for each mode or level. These are then solved directly, for example by Fast Fourier decomposition with subsequent Gauss elimination for the resulting uncoupled tri-diagonal linear systems, or iteratively (see below). This separation is in general not possible for NH models, where the full three-dimensional equations must be solved.

In NH models discretization of Eq. (8) results in a large sparse linear system that is in best case symmetric, positive definite and diagonally dominant. However, it can be easily observed that semi-implicit discretization of models using terrain following coordinates yields asymmetric Helmholtz equations (see, e.g. Pinty et al, 1995; Saito, 1997, and Thomas et al, 2000). Specifically, the antisymmetric component depends in general on the steepness of the orography.

The much smaller scales in the vertical than in the horizontal direction of todays operational models lead to anisotropic dependencies and to bad condition numbers (i.e., ratios between the horizontal and vertical scales) that affect the rate of convergence for iterative solvers. Moreover, vertical discretization especially in case of steep orography may reduce the smallest vertical mesh size additionally which amplifies the problem. In order to accelerate convergence in case of anisotropic dependencies preconditioning can be extremely effective. Hereby the original equation is transformed into another one with equal solution, however better condition number. In case of strong vertical couplings line preconditioners in the vertical direction of todays operational models can be used which require the solution of tridiagonal systems (see, e.g. Thomas et al, 2000; Skamarock et al, 1997).

Although direct solvers based on block-cyclic reduction (Bunemann, 1969) have been applied to meteorological models in the past (see, e.g. Leslie and McAvaney, 1973; Haltiner and Williams, 1980), due to the size of the linear systems of high resolution models, iterative methods can almost exclusively be considered as a realistic option for operational models.

Reviews of modern iterative methods can be found, e.g. in Ortega (1988), and Golub and Van Loan (1989). Basic iterative algorithms such as the Gauss-Seidel or the successive over-relaxation (SOR) method have also been applied in NWP models (see the previous references and Fulton et al, 1986, for a review of earlier work). In SOR corrections are computed for each single equation of the system one after the other as for Gauss-Seidel relaxation, however in order to improve convergence rate, the corrections are multiplied by a constant factor. These simple methods, although robust and easy to implement and parallelize, are, however, not considered to be fast enough for high-resolution models.

More recently efficient Krylov methods have been applied. The linear system is solved by minimizing a corresponding cost functional with a recursively computed sequence of vectors that span the Krylov-subspaces. In general, without rounding errors the exact solution would be obtained after the minimization with regard to n vectors, where n is the dimension of the problem. However in practice the minimization is continued beyond n and stopped by an appropriate criterion, e.g. based on an approximate *relative* error check. (In Smolarkiewicz et al, 1997, stopping criteria for PCG methods are investigated.) Another possibility is to apply only a very limited number of vectors and repeat the minimization iteratively from the beginning. Convergence of Krylov methods can be fast but jerky: often the residual is not diminished for a number of additional vectors and then suddenly reduced by orders of magnitude. Preconditioning of Krylov methods usually can be performed within the iterative solution.

The best-known Krylov solver is the conjugate gradient (CG)-method (Hestenes and Stiefel, 1952) (with preconditioning PCG-method) for symmetric linear positive definite systems. Since standard discretization in terrain-following coordinates of the Helmholtz equation (8) yield asymmetric matrices, generalizations of CG for asymmetric problems have to be used, such as GCG, GCR, Orthomin, Orthodir, Bi-CG, Bi-CGSTAB and GCS or GMRES (see the above references, Freund et al, 1992), and, e.g., chap. 2 of Quarteroni and Valli, 1994, for a review).

Some of these linear solvers were applied successfully in the context of anelastic models, see, e.g. Kapitza and Eppel (1987); Kapitza (1988), and Kapitza and Eppel (1992). The potential of PCG based methods for atmospheric applications has been evaluated in Smolarkiewicz and Margolin (1994), Skamarock et al (1997). GMRES (Saad and Schultz, 1986) was applied in a version of the LM of DWD (Thomas et al, 2000).

Multigrid algorithms are very attractive for large scale two or three-dimensional Helmholtz-Equations, because of their potential to achieve optimal computational complexity. Very fast solutions can be achieved, however, the implementation and setup for actual applications can be difficult. In multigrid methods, sequences of coarser grids are employed in order to provide cheap initial guesses for the respective finer grids and to reduce low frequency error components effectively by simple relaxations. Fast convergence can be achieved also in case of strong vertical couplings if line-relaxations in the vertical are applied (Thole and Trottenberg, 1985). A comprehensive review of multigrid methods is given in Hackbusch (1985), and Trottenberg et al (2000).

Multigrid solvers have been applied to the algebraic problems resulting from semi-implicit discretization of hydrostatic atmospheric models in Fulton et al (1986), Barros et al (1990), and Bates et al (1990). It is also being applied in global models based on icosahedral grids, see Baumgardner and Frederickson (1985).

Assessing the effective relative efficiency of split-explicit and semi-implicit schemes is a difficult task, since it is highly dependent on the characteristics of the problem being solved, as well as on the specific solver and discretization approach that is being employed. An attempt to perform such a comparison was carried out in Saito et al (1998), where the three-dimensional semi-implicit version of the JMA-MRI model was compared to the split-explicit version of LM by means of idealized test cases of weak forcing. The conclusion of this study was that, for forecast applications, split-explicit was more efficient. It was found that for the semi-implicit option of the LM the efficiency was affected by a decreased convergence rate or stagnation of the GMRES method for realistic test cases, while it was much better for idealized tests. The efficient solution of the Helmholtz equation is a key issue for the practical use of semi-implicit models. Several aspects of the iterative solver could contribute to improved efficiency in future models, e.g. care has to be taken for the vertical discretization, since this has an effect on the convergence rate of the iterative solver. In Bonaventura (2000) a non-normalized vertical z-coordinate was applied that resulted into a semi-implicit model with a symmetric and well conditioned linear equation. It was shown that with PCG terrain independent convergence at greatly reduced computational cost was yield with respect to analogous numerical tests performed in Pinty et al (1995) over steep orography.

5. Advection schemes

The advective terms represent the main nonlinearity in the adiabatic equations of atmospheric motion, so that the accuracy and efficiency of their discretization are of essential importance. Furthermore, the issue of mass conservation has also great relevance for the transport of moisture and other chemically reacting species. As a consequence, the choice of the advection schemes has usually a strong influence on the whole model architecture.

Traditionally, quite simple Eulerian schemes such as centered finite differences with three time level leapfrog time differencing have played a major role in mesoscale and nonhydrostatic modeling, so that many of the currently operational models have inherited discretizations based on this approach. However, other techniques such as flux form schemes and semi-Lagrangian schemes have also been applied since a number of years.

Eulerian advection schemes are based on either the advective or the conservative formulation of the transport equation. If the solutions are smooth, these two formulations are mathematically equivalent. This holds for most applications to atmospheric flows, so that use of the conservative formulation is not required for correctness of the solution, in contrast to the typical CFD applications for high Mach number compressible flows.

Centered finite differences coupled to leapfrog time discretization have proven to be a quite robust modeling tool, in spite of the related numerical difficulties such as the time-step decoupling and the subsequent need for explicit time and space filtering (see, e.g. Haltiner and Williams, 1980, and Pielke, 1984). This type of advection discretization has been incorporated in several NH models, see, e.g. Klemp and Wilhelmson (1978), Durran and Klemp (1983) and is used also in the operational LM model of DWD (Steppeler et al, 2002). The centered finite differences and leapfrog discretization are combined in Dudhia (1993) with the flux form reformulation of the equations proposed in Anthes and Warner (1978). However, as observed in Gross et al (1998), the use of time filters may lead to an effective numerical diffusion of the same magnitude or larger than the turbulent diffusion arising in some environmental applications. Two time level schemes have also been proposed, coupled to centered differences space discretization, see, e.g. Wicker and Skamarock (1998). Other higher-order schemes based on the advective form of the equations, such as the Crowley advection scheme and its improvements (see, e.g. Crowley, 1986; Tremback et al, 1987) have also been applied to operational models, see, e.g. Tripoli (1992), and Cotton et al

(2001). For many of these schemes, directional splitting is used for the three-dimensional implementation. The vertical discretization is usually implicit in time, in order to avoid excessive stability restrictions.

Schemes in conservation law form allow instead to achieve exact mass and momentum conservation. Among operational models, only the MRI model (see, e.g. Saito, 1997) is based on the flux form of the equations. The advection scheme proposed in Cullen and Davies (1991) was employed in a preliminary version of the new UKMO unified model (see Cullen et al, 1997), together with a conservative reformulation of the split-explicit scheme of Gadd (1978). The MacCormack scheme was used for the discretization of momentum advection in Kapitza and Eppel (1992). High-resolution upwind-based schemes such as the PPM method have also been applied, see, e.g. Carpenter et al (1990). Other conservative schemes for atmospheric applications have been proposed and applied in Smolarkiewicz and Clark (1986), Smolarkiewicz and Grabowski (1990), Smolarkiewicz and Pudykiewicz (1992), Smolarkiewicz and Margolin (1993), Smolarkiewicz and Margolin (1997), Stevens and Bretherton (1996).

The semi-Lagrangian discretization is based on a generalization of the classical method of characteristics: the advected quantity is updated at each grid point by interpolation of the values at the previous time-step at the foot of the characteristic reaching that gridpoint at the new time-step. The semi-Lagrangian approach to the discretization of advection has first become popular in the context of large scale hydrostatic models, see, e.g. Temperton and Staniforth (1987) and Staniforth and Côté (1991). Semi-Lagrangian advection was then applied to a NH model in Tanguay et al (1990). Various NH models employing semi-Lagrangian advection have been proposed since then, see, e.g. Benoit et al (1997), Bonaventura (2000), Côté et al (1998), Côté et al (1998), Golding (1992), Pinty et al (1995), Saito (1997), Semazzi et al (1995), and Quian et al (1998). The efficiency of semi-Lagrangian techniques for application to high resolution models has been questioned in Bartello and Thomas (1996). After several attempts to use the semi-Lagrangian scheme at resolutions of about 3 km, practical experience has indicated that Courant numbers smaller than 1, or even 0.5, are necessary to achieve reasonable results at such high resolutions. This reduces very much the efficiency of such schemes. However, the semi-Lagrangian approach is also a convenient choice for a fully multidimensional advection discretization. Furthermore, it is quite advantageous for models employing the non-normalized z vertical coordinate, where cells with arbitrary thickness for the vertical discretization imply that high vertical Courant numbers can easily arise. It should also be kept in mind, that the currently observed limitation to small Courant numbers is based on practical experience and there is no argument against the development of improved semi-Lagrange schemes which are free of this drawback.

6. Recent developments and future perspectives

Models based on simple numerics were shown to be able to do relevant simulations at the meso-γ scale, such as convective cells and flow forced by small scale orography. In respect to this, it has to be asked if the simulations are accurate enough. The convergence of solutions with grid length was investigated by Steppeler et al (2002) using the LM. The convergence was rather slow, so there arises the question if the rather universal second order approximation should be abandoned in favor of third-order or even higher-order schemes. Even though the existing approaches obtain a reasonable degree of efficiency, it has to be recognized, that the more advanced numerical schemes (semi-implicit, semi-Lagrangian) used in experimental model versions aim to increase it substantially. Adaptive grids, as discussed in Sect. 3, are another development with a potential to substantially increase the efficiency of NH models. Expectations on fine scale models are not limited to the simulations mentioned above. In longer terms the simulation of local weather, such as fog, will be requested. Numerical requests associated with such developments are a rather good simulation of all moisture related processes. In particular a sufficient accuracy for the advection of moisture, absence of numerically generated oscillations and conservation of mass will be mandatory. For such expectations second order schemes for the simulation of advection will definitely be too inaccurate. The development of accurate non-oscillating advection schemes will be necessary. Experience from current models suggests that approximations of overall third order will be adequate. Current research in advanced schemes has not yet lead to models which can be used operationally. Therefore considerable more research is necessary in order to realize the potential efficiency and accuracy of such schemes.

As artificial circulations driven by numerical errors of the representation of mountains have a potential to destroy high fog, such errors will have to be avoided, for example by using a z-coordinate representation.

The computational costs of such numerical improvements has to be a point of concern. According to conventional wisdom the costs of more accurate semi-Lagrangian schemes should be recovered by using rather large time steps. As there have been practical difficulties in realizing such savings, and for the scales of interest there are also theoretical limits to the CFL-condition to be used (see, e.g. Bartello and Thomas, 1996), the possibility to save also by reducing the grid points of spatial representation should also be considered. This is based on the observation that in three dimensions a third order finite element representation of a field uses 7 points, which are distributed inhomogeneously, whereas the classical homogeneous representation has to use 27 points. As Galerkin finite element approaches of high order have proven to be unpractical, it will be necessary to use such reduced grids in connection with finite difference approaches, as done by Steppeler (1976).

Experimental global simulations at a scale of 15 km have already been done. Already existing computers offer the capacity to perform global simulations at NH scales, using resolutions of about 5 km. The authors of this report have no knowledge that such computations have actually been performed, but the implementation of NH models on global grids should be a point of concern.

It may be foreseen, that the prediction of the local weather will require the use of more moisture related fields, and also other chemical constituents (Doms, private communication). Conservation of mass of all these constituents is of primary importance. Models used currently operational or quasi operational do not have this property. A way to achieve this property is to use conservation forms of equations together with the finite volume method (Skamarock, Klemp and Dudhia, private communication).

Appendix

List of models mentioned in this paper:

Model	Institute	Reference
ALADIN	Météo-France	Široká et al (2002)
ARPS	Center for Analysis and Prediction of Storms, Oklahoma	Xue et al (2000)
GEM	Canadian Meteorological Centre	Côté et al (1998)
JMA-MRI	Japan Meteorological Agency (JMA)	Saito (1997)
LM	German Weather Center (DWD)	Steppeler et al (2002)
MC2	Canadian Meteorological Center	Benoit et al (1997)
MM5	National Center of Atmospheric Research (NCAR)	Dudhia (1993)
NCEP-MESO	National Center of Environmental Prediction (NCEP)	Janjic et al (2002)
RAMS	Colorado State University	Cotton et al (2002)
WRF	National Center of Atmospheric Research (NCAR)	private communication

References

Adcroft A, Hill C, Marshall J (1997) Representation of topography by shaved cells in a height coordinate ocean model. Mon Wea Rev 125: 2293–2315

Anthes R, Warner T (1978) Development of hydrodynamical models suitable for air pollution and other mesometeorological studies. Mon Wea Rev 106: 1045–1078

Arakawa A (1966) Computational design for long term numerical integration of the equation of fluid motion: 1. Two-dimensional incompressible flow. J Comp Phys 1: 119–143

Arakawa A, Mesinger F (1976) Numerical methods used in atmospheric models. Technical Report, Global Atmospheric Research Programme (GARP) Publication Ser. No. 17

Arakawa A, Moorthi S (1987) Baroclinic instabilities in vertically discrete systems. J Atmos Sci 45: 174–265

Arney DC, Flaherty JE (1990) An adaptive mesh-moving and local refinement method for time-dependent partial differential equations. ACM Trans Math Software 16: 48–71

Bai D, Brandt A (1987) Local mesh refinement multilevel techniques. SIAM J Sci Stat Comp 8: 109–134

Barros S, Dee D, Dickstein F (1990) A multigrid solver for semi-implicit global shallow-water models. Atmos Ocean 28: 24–47

Bartello P, Thomas S (1996) The cost-effectiveness of semi-Lagrangian advection. Mon Wea Rev 124: 2883–2897

Bates J, Semazzi F, Higgins R, Barros S (1990) Integration of the shallow water equations on the sphere using a vector semi-Lagrangian scheme with a multigrid solver. Mon Wea Rev 118: 1615–1627

Baumgardner J, Frederickson P (1985) Icosahedral discretization of the two-sphere. SIAM J Numer Anal 22: 1107–1115

Behrens J (1998) Adaptive finite element solver for the shallow water equations. Appl Numer Math 26(1–2): 217–226

Benoit R, Desgagné M, Pellerin P, Pellerin S, Chartier Y, Desjardins S (1997) The canadian MC2: a semi-Lagrangian, semi-implicit wideband atmospheric model suited for finescale process studies and simulation. Mon Wea Rev 125: 2382–2415

Berger M, Oliger J (1984) Local adaptive mesh refinement for hyperbolic partial differential equations. J Comp Phys 53: 484–512

Bonaventura L (2000) A semi-implicit, semi-Lagrangian scheme using the height coordinate for a nonhydrostatic and fully elastic model of atmospheric flows. J Comp Phys 158: 186–213

Browning G, Kreiss H (1994) Splitting methods for problems with different time-scales. Mon Wea Rev 122: 2614–2622

Bubnova R, Hello G, Benard P, Geleyn J-F (1995) Integration of the fully elastic equations cast in the hydrostatic pressure terrain-following coordinate in the framework of the ARPEGE/Aladin NWP system. Mon Wea Rev 123: 515–535

Bunemann O (1969) A compact non-iterative Poisson solver. Technical Report, Stanford University, Institute for Plasma

Carpenter R, Droegemeier K, Woodward P, Hane C (1990) Application of the piecewise parabolic method to meteorological modeling. Mon Wea Rev 118: 586–611

Casulli V (1990) Semi-implicit finite difference methods for the two dimensional shallow water equations. J Comp Phys 86: 56–74

Casulli V, Cattani E (1994) Stability, accuracy and efficiency of a semi-implicit method for three-dimensional shallow water flow. Comp Math Appl 27: 99–112

Casulli V, Stelling G (1998) Numerical simulation of three-dimensional quasi-hydrostatic, free surface flows. ASCE J Hydr Eng 124: 678–686 (forthcoming)

Casulli V, Walters R (2000) An unstructured grid, three-dimensional model based on the shallow water equations. Int J Num Meth Fluids 32: 331–348

Charney J, Phillips N (1953) Numerical integration of the quasigeostrophic equations for barotropic and simple baroclinic flows. J Meteorol 10: 71–99

Clark T (1977) A small-scale dynamic model using terrain following coordinate transformation. J Comp Phys 24: 186–215

Côté J, Desmarais J, Gravel S, Méthot A, Patoine A, Roch M, Staniforth A (1998) The operational CMC-MRB Global Environmental Multiscale (GEM) model, Part II: Results. Mon Wea Rev 126: 1397–1418

Côté J, Gravel S, Méthot A, Patoine A, Roch M, Staniforth A (1998) The operational CMC-MRB Global Environmental Multiscale (GEM) model, Part I: Design considerations and formulation. Mon Wea Rev 126: 1343–1395

Cotton W, Pielke R, Walko R, Liston G, Tremback C, Jiang H, McAnelly R, Harrington J, Nicholls M, Carrio G, McFadden J (2002) RAMS 2001: Current status and future directions. Meteorol Atmos Phys 82 (this issue)

Crowley W (1986) Numerical advection experiments. Mon Wea Rev 96: 1–11

Cullen M (1990) A test of a semi-implicit integration technique for a fully compressible nonhydrostatic model. Q J Roy Met Soc 116: 1253–1258

Cullen M (1993) The unified forecast/climate model. Meteorol Mag 122: 81–94

Cullen M, Davies T (1991) A conservative split-explicit scheme with fourth order horizontal advection. Q J Roy Met Soc 117: 993–1002

Cullen M, Davies T, Mawson M, James J, Couther S, Malcolm A (1997) An overview of numerical methods for the next generation UK NWP and climate models. In: Numerical methods in atmospheric and oceanic modelling. The André Robert Memorial Volume (Lin CA, Laprise R, Ritchie H, eds). NRC Research Press

Dautray R, Lions J (1993) Mathematical analysis and numerical methods for science and technology, Vol. 6: Evolution problems

Dietachmayer G, Droegemeier H (1992) Application of continuous dynamic grid adaption techniques to meteorological modeling. Am Met Soc 120: 1675–1722

Dudhia J (1993) A nonhydrostatic version of the Penn State-NCAR mesoscale model: Validation tests. Mon Wea Rev 121: 1493–1513

Durran D, Klemp J (1983) A compressible model for the simulation of moist mountain waves. Mon Wea Rev 111: 2341–2361

Freund R, Golub G, Nachtigal N (1992) Iterative solution of linear systems. Acta Numer, pp 57–100

Fulton S (1997) A comparison of multilevel adaptive methods for hurricane track prediction. Electronic Transactions on Numerical Analysis 6: 120–132

Fulton S, Ciesielski P, Schubert W (1986) Multigrid methods for elliptic problems: A review. Mon Wea Rev 114: 943–959

Gadd A (1978) A split-explicit scheme for numerical weather prediction. Q J Roy Met Soc 104: 569–582

Gal-Chen T, Somerville R (1975) On the use of a coordinate transformation for the solution of the Navier-Stokes equations. J Comp Phys 17: 209–228

Gallus W, Klemp J (2000) Behaviour of flow over steep orography. Mon Wea Rev 128: 1153–1164

Gary J (1973) Estimate of truncation error in transformed coordinates primitive equation models. J Atmos Sci 30: 223–233

Golding B (1992) An efficient non-hydrostatic model. Meteorol Atmos Phys 50: 89–103

Golub G, Van Loan C (1989) Matrix computations, 2nd edition. Baltimore: The Johns Hopkins University Press

Gross E, Casulli V, Bonaventura L, Koseff J (1998) A semi-implicit method for vertical transport in multidimensional models. Int J Num Meth Fluids 28: 157–186

Hackbusch W (1985) Multi-grid methods and applications. Springer

Haltiner G, Williams R (1980) Numerical prediction and dynamic meteorology. Wiley

Harlow F, Welsh J (1965) Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. Phys Fluids 8: 2182–2189

Hess R (1999) Dynamically adaptive multigrid on parallel computers for a semi-implicit discretization of the shallow water equations. Technical Report, GMD, St. Augustin

Hestenes M, Stiefel E (1952) Methods of conjugate gradients for solving linear systems. Nat Bur Standards J Res 49: 409–436

Holton J (1992) An introduction to dynamic meteorology, 3rd edition. San Diego: Academic Press

Ikawa M (1988) Comparison of some schemes for nonhydrostatic models with orography. J Met Soc Japan 66: 735–776

Janjic Z, Gerrity J, Nickovic S (2001) An alternative approach to nonhydrostatic modeling. Mon Wea Rev 129 (forthcoming)

Kapitza H (1988) Truncated incomplete factorizations for conjugate-gradient methods in two and three dimensions. Appl Math Comp 28: 73–87

Kapitza H, Eppel D (1987) A 3-D Poisson solver based on conjugate gradients compared to standard iterative methods and its performance on vector computers. J Comp Phys 68: 474–484

Kapitza H, Eppel D (1992) The non-hydrostatic model GESIMA, Part I. Beitr Phys Atmosph 65: 129–146

Klemp J, Wilhelmson R (1978) The simulation of three-dimensional convective storm dynamics. J Atmos Sci 35: 1070–1096

Kröner D (1997) Numerical schemes for conservation laws. Chichester New York: Wiley and Teubner

Kwizak M, Robert A (1971) A semi-implicit scheme for gridpoint atmospheric models of the primitive equations. Mon Wea Rev 99: 32–36

Laprise R (1992) The Euler equations of motion with hydrostatic pressure as an independent variable. Mon Wea Rev 120: 197–207

Leslie L, McAvaney B (1973) Comparative tests of direct and iterative methods for solving Helmholtz-type equations. Mon Wea Rev 101: 235–239

Lindzen R, Fox-Rabinowitz M (1989) Consistent vertical and horizontal resolution. Mon Wea Rev 117: 2575–2583

Lorenz E (1960) Energy and numerical weather prediction. Tellus 12: 364–373

McCormick SF (1989) Multilevel adaptive methods for partial differential equations. SIAM

Mesinger F, Janjic Z, Nickovic S, Gavrilov D, Deaven D (1988) The step-mountain coordinate: Model description and performance for cases of alpine lee cyclogenesis and for a case of an appalachian redevelopment. Mon Wea Rev 116: 1493–1518

Michalakes J (1997) A scalable parallel implementation of the penn state/NCAR mesoscale model (MM5). Parallel Comput 23: 2173–2186

Montavon C (1997) Validation of a non-hydrostatic numerical model to simulate stratified wind fields over complex topography. In: Proc. 2nd EACWE Conference (Solari G, ed), Genua

Ortega J (1988) Matrix theory. New York: Plenum Press

Park S, Županski D (2002) Short-range four-dimensional variational data assimilation. Meteorol Atmos Phys 82 (this issue)

Patankar S (1980) Numerical heat transfer and fluid flow. New York: Hemisphere

Peyret R, Taylor T (1983) Computational methods for fluid flow. New York: Springer

Pielke R (1984) Mesoscale meteorological modeling. Academic Press

Pinty J, Benoit R, Richard E, Laprise R (1995) Simple tests of a semi-implicit semi-Lagrangian model on 2D mountain wave problems. Mon Wea Rev 123: 3042–3058

Quarteroni A, Valli A (1994) Numerical approximation of partial differential equations. New York Berlin: Springer

Quian J, Semazzi F, Scroggs J (1998) A global nonhydrostatic semi-Lagrangian atmospheric model with orography. Mon Wea Rev 126: 747–771

Robert A (1982) A semi-Lagrangian and semi-implicit scheme for primitive equations. J Met Soc Japan 60: 319–324

Saad Y, Schultz M (1986) GMRES: A generalized minimal residual algorithm for solving large non-symmetric linear systems. SIAM J Sci Stat Comp 7: 856–869

Saito K (1997) Semi-implicit fully compressible version of the MRI mesoscale nonhydrostatic model. Geophys Mag 2: 109–137

Saito K, Doms G, Schättler U, Steppeler J (1998) 3-D mountain waves by the Lokal-Modell of DWD and the MRI mesoscale nonhydrostatic model. Pap Meteorol Geophys 49: 7–19

Sathye A, Xue M, Bassett G, Droegemeier K (1997) Parallel weather modeling with the advanced regional prediction system. Parallel Comput 23: 2243–2256

Schättler U, Doms G, Steppeler J (2000) Requirements and problems in parallel model development at DWD. Scientific Programming 8: 13–22

Schlünzen K (1994) Mesoscale modelling in complex terrain – An overview on the German nonhydrostatic models. Beitr Phys Atmosph 67: 243–253

Semazzi F, Quian J, Scroggs J (1995) A global nonhydrostatic semi-Lagrangian atmospheric model without orography. Mon Wea Rev 123: 2534–2550

Široká M, Fischer C, Cassé V, Brožková R, Geleyn J-F (2002) The definition of mesoscale selective forecast error covariances for a limited area variational analysis. Meteorol Atmos Phys 82 (this issue)

Skamarock W, Klemp J (1992) The stability of the time-split numerical method for the hydrostatic and nonhydrostatic elastic equations. Mon Wea Rev 120: 2109–2127
Skamarock W, Klemp J (1994) Efficiency and accuracy of the Klemp Wilhelmson time splitting scheme. Mon Wea Rev 122: 2623–2630
Skamarock W, Oliger J, Street R (1989) Adaptive grid refinement for numerical weather prediction. J Comp Phys 80: 27–60
Skamarock W, Smolarkiewicz P, Klemp J (1997) Preconditioned conjugate gradient solvers for Helmholtz equations in nonhydrostatic models. Mon Wea Rev 125: 587–599
Smolarkiewicz P, Clark T (1986) The multidimensional positive definite advection transport algorithm: Further development and applications. J Comp Phys 67: 396–438
Smolarkiewicz P, Grabowski W (1990) The multidimensional positive definite advection transport algorithm: Nonoscillatory option. J Comp Phys 86: 355–375
Smolarkiewicz P, Grubisic V, Margolin L (1997) On forward-in-time differencing for fluids: Stopping criteria for iterative solutions of anelastic pressure equations. Mon Wea Rev 125: 647–654
Smolarkiewicz P, Margolin L (1993) On forward-in-time differencing for fluids: Extension to a curvilinear framework. Mon Wea Rev 121: 1847–1859
Smolarkiewicz P, Margolin L (1994) Variational solver for elliptic problems in atmospheric flows. Appl Math Comp Sci 4: 527–551
Smolarkiewicz P, Margolin L (1997) On forward-in-time differencing for fluids: An Eulerian/semi-Lagrangian nonhydrostatic model for stratified flows. Atmos Ocean 35: 127–152
Smolarkiewicz P, Pudykiewicz J (1992) A class of semi-Lagrangian approximations for fluids. J Atmos Sci 49: 2082–2096
Staniforth A, Côté J (1991) Semi-Lagrangian integration schemes for atmospheric models – A review. Mon Wea Rev 119: 2206–2223
Staniforth A, Côté J (1995) Developing an integrated atmospheric environment modelling system. In: Numerical methods for fluid dynamics Morton VKW (Baines MJ, ed). Oxford: Clarendon Press
Steppeler J (1976) The application of the second and third degree methods. J Comp Phys 22: 295–318
Steppeler J, Bitzer H, Bonaventura L (2002) Nonhydrostatic atmospheric modelling using a z-coordinate representation. Mon Wea Rev 130: 2143–2149
Steppeler J, Doms G, Schättler U, Bitzer H, Gassmann A, Damrath U, Gregoric G (2002) Meso gamma scale forecasts using the nonhydrostatic model LM. Meteorol Atmos Phys 82 (this issue)
Stevens D, Bretherton S (1996) A forward-in-time advection scheme and adaptive multilevel flow solver for nearly incompressible atmospheric flow. J Comp Phys 129: 284–295
Sundqvist H (1976) On vertical interpolation and truncation in connection with use of sigma system models. Atmosphere 14: 37–52
Tanguay M, Robert A, Laprise R (1990) A semi-implicit, semi-Lagrangian fully compressible regional forecast model. Mon Wea Rev 118: 1970–1980
Tapp M, White P (1976) A non-hydrostatic mesoscale model. Q J Roy Met Soc 102: 277–296
Temperton C, Staniforth A (1987) An efficient two-time-level semi-Lagrangian semi-implicit integration scheme. Q J Roy Met Soc 113: 1025–1039
Thole C, Trottenberg U (1985) Basic smoothing procedures for the multigrid treatment of elliptic 3D-operators. Technical Report, GMD, St. Augustin
Thomas S, Girard C, Doms G, Schättler U (2000) Semi-implicit scheme for the DWD Lokal-Modell. Meteorol Atmos Phys 73: 105–125
Thomas S, Malevsky A, Desgagné M, Benoit R, Pellerin P, Valin M (1997) Massively parallel implementation of the mesoscale compressible community model. Parallel Comput 23: 2143–2160
Tremback C, Powell J, Cotton W, Pielke R (1987) The forward in time upstream advection scheme: Extension to higher orders. Mon Wea Rev 115: 540–555
Tremback C, Walko R (1997) The Regional Atmospheric Modeling System (RAMS): Development for parallel processing architectures. In: Proc. Symp. on Regional Weather Prediction on Parallel Computers (Kallos G, Kotroni V, Lagouvardos K, eds), University of Athens, pp 43–56
Tripoli G (1992) A nonhydrostatic mesoscale model designed to simulate scale interaction. Mon Wea Rev 120: 1343–1359
Trottenberg U, Oosterlee C, Schüller A (2000) Multigrid Methods: Basics, Parallelism, and Adaptivity (forthcoming)
Wicker L, Skamarock W (1998) A time splitting scheme for the elastic equations incorporating second order Runge Kutta time differencing. Mon Wea Rev 126: 1992–1996
Xue M, Droegemeier K, Wong V (2000) The Advanced Regional Prediction System (ARPS) – A multi-scale nonhydrostatic atmospheric simulation and prediction tool. Part I: Model dynamics and verification. Meteorol Atmos Phys 75: 161–193
Xue M, Thorpe A (1991) A mesoscale numerical model using the nonhydrostatic pressure-based sigma-coordinate equations: Model experiments with dry mountain flows. Mon Wea Rev 119: 1168–1185

Authors' addresses: J. Steppeler, R. Hess and U. Schättler, Deutscher Wetterdienst, Offenbach, Germany (E-mail: juergen.steppeler@dwd.de); L. Bonaventura, Dipartimento di Ingegneria Civile ed Ambientale, Università degli Studi di Trento, Trento, Italy

Verleger: Springer-Verlag KG, Sachsenplatz 4–6, A-1201 Wien. – Herausgeber: Prof. Dr. Reinhold Steinacker, Institut für Meteorologie und Geophysik, Universität Wien, Althanstraße 14, A-1090 Wien. – Redaktion: Innrain 52, A-6020 Innsbruck. – Satz und Umbruch: Thomson Press (India) Ltd., Chennai. – Druck und Bindung: Grasl Druck&Neue Medien, A-2540 Bad Vöslau. – Verlagsort: Wien. – Herstellungsort: Bad Vöslau. – Printed in Austria.

Offenlegung gemäß § 25 Abs. 1 bis 3 Mediengesetz:
Unternehmensgegenstand: Verlag von wissenschaftlichen Büchern und Zeitschriften. An der Springer-Verlag KG ist beteiligt: Springer-Verlag GmbH & Co KG, Heidelberger Platz 3, D-14197 Berlin, als Kommanditist zu 74,04%. Geschäftsführer: Rudolf Siegle, Sachsenplatz 4–6, A-1201 Wien.