Computing 66, 179–197 (2001)

Computing

Uniform Pointwise Convergence of Difference Schemes for Convection-Diffusion Problems on Layer-Adapted Meshes*

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Received December 14, 1999; revised September 13, 2000

Abstract

We consider two convection-diffusion boundary value problems in conservative form: for an ordinary differential equation and for a parabolic equation. Both the problems are discretized using a four-point second-order upwind space difference operator on arbitrary and layer-adapted space meshes. We give ε-uniform maximum norm error estimates $O\big(N^{-2}\ln^2 N(+\tau)\big)$ and $O\big(N^{-2}(+\tau)\big)$, respectively, for the Shishkin and Bakhvalov space meshes, where N is the space meshnodes number, τ is the time mesh-interval. The smoothness condition for the Bakhvalov mesh is replaced by a weaker condition.

AMS Subject Classifications: 65L10, 65L12, 65L70, 65M06, 65M12, 65M15.

Key Words: Convection-diffusion problems, four-point upwind difference scheme, singular perturbation, Shishkin mesh, Bakhvalov mesh.

1. Introduction

This paper is concerned with ε-uniform numerical methods for the two model boundary value problems: for an ordinary differential equation

$$Lu := -\varepsilon\frac{\partial^2}{\partial x^2}u - \frac{\partial}{\partial x}(p(x)u) = f(x) \quad \text{for } 0 < x < 1, \quad u(0) = g_0, \quad u(1) = g_1, \tag{1.1}$$

and for a parabolic equation

$$\begin{aligned} &\frac{\partial}{\partial t}u + Lu = f(x,t) \quad \text{for } 0 < x < 1, \quad 0 < t \le 1, \\ &u(x,0) = \varphi(x) \quad \text{for } 0 \le x \le 1, \\ &u(0,t) = g_0(t), \quad u(1,t) = g_1(t) \quad \text{for } 0 < t \le 1, \end{aligned} \tag{1.2}$$

where

$$p(x) \ge \beta = \text{const} > 0 \tag{1.3}$$

*This work was supported by the Russian Foundation for Basic Research under grant No. 99-01-01056.

and $\varepsilon \in (0, 1]$ is a small parameter. Note that the results given in this paper hold for $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is a positive constant depending on the data of the problems. We assume that the data of (1.1) and (1.2) are smooth enough, particularly

$$|p'(x)| \leq P. \tag{1.4}$$

For (1.2) we also assume that $\varphi(0) = g_0(0)$, $\varphi(1) = g_1(0)$ and the compatibility conditions [11] are satisfied so that the solution has no internal layers.

It is well known [13, 15] that as $\varepsilon \to 0$, the solutions of (1.1) and (1.2) have an exponential boundary layer at $x = 0$ and, as a result, the accuracy of classical numerical methods depends on ε as well as on the space meshnodes number N. One of the approaches to constructing ε-uniform numerical methods is combining classical discretizations of differential equations with layer-adapted highly non-uniform meshes. Bakhvalov [3] was the first to use the approach. The space mesh [3] for problems (1.1) and (1.2) is as follows:

$$x_i = x(i/N), \quad i = 0, 1, \ldots, N, \tag{1.5}$$

where $x(\xi)$ is the continuous function defined by

$$x(\xi) = \begin{cases} \begin{cases} \varepsilon\lambda \ln[b/(b-\xi)] & \text{for } \xi \in [0, \theta] \\ 1 - d(1-\xi) & \text{for } \xi \in [\theta, 1] \end{cases} & \text{if} \quad \varepsilon \leq \bar{\varepsilon}_0 \\ \xi & \text{otherwise,} \end{cases}$$
$$d = d(\theta) = (1 - \varepsilon\lambda \ln[b/(b-\theta)])/(1-\theta), \tag{1.6}$$

with constants λ, $0 < \theta < b < 1$, $\bar{\varepsilon}_0 \leq b/\lambda$. Note that the mesh [3] for problems like (1.1) was considered in [12] and [1, 2], ε-uniform accuracy being obtained $O(N^{-1})$ and $O(N^{-2})$ respectively. In the mentioned papers mesh (1.5), (1.6) is assumed to be smooth, i.e. the function $x(\xi)$ is continuously differentiable and $\theta = \bar{\theta}$, defined implicitly by the nonlinear equation

$$\bar{\theta} = b - \varepsilon\lambda/d(\bar{\theta}), \tag{1.7}$$

can be computed using the following iterations [3]

$$\theta^{(0)} = 0, \quad \theta^{(k)} = b - \varepsilon\lambda/d(\theta^{(k-1)}), \quad \lim_{k\to\infty} \theta^{(k)} = \bar{\theta}, \quad 0 = \theta^{(0)} < \theta^{(1)} < \cdots < \bar{\theta}. \tag{1.8}$$

Note that the impossibility of solving the nonlinear equation exactly, when constructing the mesh, can be considered a certain drawback [19, 15]. As in [9], we replace the mesh smoothness condition implying (1.7) by the following weaker condition

$$b - \varepsilon\bar{C} < \theta < b - \varepsilon C_0 \tag{1.9}$$

with arbitrary positive constants C_0 and $\bar{C}$ satisfying $C_0 < \bar{C} < b$. Here the right-hand inequality implies $\max_i h_i = O(N^{-1})$ for mesh (1.5), (1.6), while the left-hand inequality provides ε-uniform second-order consistency in the negative W_∞^{-1} discrete norm. We point out that the choice $\theta = \bar{\theta}$ is a particular case of (1.9) as well as

$$\theta = \theta^{(1)} = b - \varepsilon\lambda, \tag{1.10}$$

which is the result of the first iteration (1.8), and both the choices generate the meshes satisfying the reasonable condition $h_i \le h_{i+1}$ (which is provided by $\theta \le \bar{\theta}$).

Shishkin [17] suggested piecewise uniform layer-adapted meshes, in particular, for problems (1.1) and (1.2) the space mesh [17] is as follows:

$$\left.\begin{aligned} &\Omega = \Big\{x_i \,|\, x_i = \begin{cases} ih & \text{for} \quad i = 0, \dots, n, \\ x_n + (i-n)H & \text{for} \quad i = n+1, \dots, N, \end{cases} \\ &\quad h = \delta/n, \quad H = (1-\delta)/(N-n), \quad n/N = b, \quad \delta = \min(\varepsilon\lambda \ln N, a) \end{aligned}\right\} \tag{1.11}$$

with constants a, $b \in (0, 1)$ and λ, and the results from [13, 17] lead to ε-uniform error estimate $O(N^{-1} \ln N)$. Recently (see, e.g., the survey [14]) on mesh (1.11) other schemes for problems like (1.1) are studied, ε-uniform accuracy being obtained of order $O(N^{-2} \ln^2 N)$.

It should be remarked that still other layer-adapted meshes were suggested to provide ε-uniform convergence [15].

We shall study difference schemes, using a four-point upwind space difference operator [6] (see also [15, I.2.1.2]), that are second-order consistent and, though do not yield M-matrices, but enjoy certain stability on arbitrary meshes unlike the second-order central-difference scheme. These schemes can be easily extended into two dimensions (unlike, e.g., three-point second-order schemes like [2, 18]). Note also that a similar many-point regularization idea leads, e.g., to the Gontcharov–Frjasinov five-point scheme [5], which works well for the Navier–Stokes equations at high Reynolds numbers.

Thus problem (1.1) is discretized as follows:

$$\begin{aligned} &L^N u_i^N := -\frac{A^N u_{i+1}^N - A^N u_i^N}{\hbar_i} = f_i \quad \text{for } i = 1, \dots, N-1, \\ &u_0^N = g_0, \quad u_N^N = g_1, \end{aligned} \tag{1.12}$$

where A^N is defined by

$$A^N v_i := \begin{cases} \varepsilon D^- v_i + p_{i-1/2}(v_i - 0.5 h_i D^+ v_i) & \text{for } i = 1, \dots, N-1, \\ \varepsilon D^- v_N + p_{N-1/2}(v_N - 0.5 h_N D^+ v_{N-1}) & \text{for } i = N. \end{cases} \tag{1.13}$$

Note that this scheme preserves the conservative form of the differential equation. Here and throughout the paper we use the *notation*

$$D^- v_i = \frac{v_i - v_{i-1}}{h_i}, \quad D^+ v_i = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad D v_i = \frac{v_{i+1} - v_i}{\hbar_i},$$
$$h_i = x_i - x_{i-1}, \quad \hbar_i = (h_i + h_{i+1})/2,$$

and $w_i = w(x_i)$, $w_{i-1/2} = w(x_i - h_i/2)$, $w_i^j = w(x_i, t_j)$, $w_i(t) = w(x_i, t)$ for any continuous function $w(x)$ or $w(x,t)$. Thus u_i (or u_i^j) denotes the exact solution at the meshnodes, while u_i^N (or $u_i^{N,j}$) is the computed solution.

Clearly, (1.13) implies

$$A^N v_i = \begin{cases} \varepsilon D^- v_i + p_{i-1/2}[(v_{i-1} + v_i)/2 - (h_i \hbar_i/2) D D^- v_i] & \text{for } i = 1, \ldots, N-1, \\ \varepsilon D^- v_N + p_{N-1/2}(v_{N-1} + v_N)/2 & \text{for } i = N, \end{cases} \tag{1.14}$$

i.e. A^N is a second-order approximation of the differential operator A defined by

$$A v(x) = \varepsilon \frac{\partial}{\partial x} v + p(x) v(x). \tag{1.15}$$

If $p(x) \equiv 1$ and the mesh is uniform, (1.12) turns into the well-known discretization

$$-\varepsilon D D^- u_i^N + (3u_i^N - 4u_{i+1}^N + u_{i+2}^N)/(2h) = f_i \quad \text{for } i = 1, \ldots, N-2, \tag{1.6}$$

with the first-order upwind discretization $-\varepsilon D D^- u_{N-1}^N - D^+ u_{N-1}^N = f_{N-1}$ for $i = N-1$. Solving (1.16) exactly, it can be easily checked that $u_i^N = c_0 + c_1 r_1^i + c_2 r_2^i$ with some constants c_0, c_1, c_2, where the roots $r_0 = 1$, r_1, r_2 are positive, i.e. the solution u_i^N of (1.16) never oscillates (regarding inverse-monotonicity, see Remark 2).

Note also that in [8] this scheme is studied on the Shishkin mesh (1.11) and proved to converge ε-uniformly in the discrete maximum norm, the accuracy being $O(N^{-2} \ln^2 N)$. In this paper we extend the analysis to more general meshes and our parabolic equation.

Problem (1.2) is discretized using the same four-point space operator L^N, as in (1.12):

$$\begin{aligned} &\frac{u_i^{N,j} - u_i^{N,j-1}}{\tau} + L^N u_i^{N,j} = f_i^j && \text{for } i = 1, \ldots, N-1, \quad j = 1, \ldots, K, \\ &u_i^{N,0} = \varphi_i^N && \text{for } i = 1, \ldots, N-1, \\ &u_0^{N,j} = g_0(t_j), \quad u_N^{N,j} = g_2(t_j) && \text{for } j = 0, \ldots, K. \end{aligned} \tag{1.17}$$

To our knowledge the first result of ε-uniform convergence for problems like (1.2) is by Shishkin [17] for the difference scheme with the first-order upwind space operator on the Shishkin space mesh, ε-uniform accuracy being proved $O(N^{-1} \ln^2 N + \tau)$. We also refer to [7], where a time defect-correction approach for

(1.2) is considered on the Shishkin mesh, with ε-uniform error bound $O(N^{-1}\ln^2 N + \tau^k)$, $k \geq 2$; and [10], where (1.2) is discretized using the central-difference space operator, with ε-uniform accuracy $O(N^{-2}\ln^2 N + \tau)$.

The main results of this paper (Theorems 1, 2) are ε-uniform maximum norm error estimates $O(N^{-2}\ln^2 N(+\tau))$ and $O(N^{-2}(+\tau))$ for schemes (1.12) and (1.17) on the Shishkin and Bakhvalov space meshes respectively.

Notation: Throughout the paper, C, sometimes subscripted, will denote a generic positive constant that is independent of ε and of the mesh.

Remark 1. All the results given in this paper hold for difference schemes (1.12) and (1.17) with $A^N := \bar{A}^N$ defined by

$$\bar{A}^N v_i = \begin{cases} \varepsilon D^- v_i + p_i v_i - 0.5 h_i D^+(pv)_i & \text{for } i = 1, \ldots, N-1, \\ \varepsilon D^- v_N + p_N v_N - 0.5 h_N D^+(pv)_{N-1} & \text{for } i = N. \end{cases}$$

(compare with (1.13)).

2. Two Point Boundary Value Problem

2.1. Hybrid Stability Inequality

Let $\omega = \{x_i \mid 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1\}$ be an arbitrary nonuniform mesh on [0, 1]. Throughout the paper we assume that

$$h := \max_i h_i \leq CN^{-1}, \quad H := h_{N-1} = h_N. \tag{2.1}$$

For any mesh functions v_i and w_i, we assume that $v_0 = v_N = w_0 = w_N = 0$, when these values are not defined explicitly, and use the scalar product

$$(v, w) = \sum_{i=1}^{N-1} \hbar_i v_i w_i \tag{2.2}$$

and the discrete L_∞, L_2 and W_∞^{-1} norms defined, respectively, by

$$\|v\|_\infty = \max_i |v_i|, \quad \|v\|_2 = \|v\| = \sqrt{(v, v)}, \quad \|v\|_* = \max_i \Big|\sum_{j=i}^{N-1} \hbar_j v_j\Big|.$$

Note that for any discrete function v_i on an arbitrary nonuniform mesh, we have

$$\|v\|_* \leq \|v\|_2 \leq \|v\|_\infty, \quad \|Dv\|_* \leq 2\|v\|_\infty. \tag{2.3}$$

The key to our analysis of schemes (1.12) and (1.17) is the hybrid stability inequality given by

Lemma 1. *Suppose $p(x)$ satisfies* (1.3), (1.4), *and $\varepsilon \le \varepsilon_0 = 0.1\beta^2/P$. Then for any solution v_i of the discrete problem $L^N v_i = f_i$* for $i = 1, \dots, N-1$, *$v_0 = v_N = 0$ on an arbitrary nonuniform mesh satisfying* (2.1), *so that $h \le h_0 := 0.1\beta/P$, we have*

$$\|v\|_\infty \le C_0 \|f\|_*. \tag{2.4}$$

Proof: First note that, by (1.13), we have

$$A^N v_i = \begin{cases} -\frac{\varepsilon}{h_i} v_{i-1} + \left[\frac{\varepsilon}{h_i} + \left(1 + \frac{h_i}{2h_{i+1}}\right) p_{i-1/2}\right] v_i - \frac{h_i}{2h_{i+1}} p_{i-1/2} v_{i+1} & \text{for } i = 1, \dots, N-1, \\ -\left(\frac{\varepsilon}{H} - \frac{p_{N-1/2}}{2}\right) v_{N-1} + \left(\frac{\varepsilon}{H} + \frac{p_{N-1/2}}{2}\right) v_N & \text{for } i = N. \end{cases}$$

Since $L^N = -DA^N$, the discrete function v_i admits the representation

$$v_i = W_i - \frac{W_N V_i}{V_N} \quad \text{for } i = 0, \dots, N, \tag{2.5}$$

where V_i and W_i are the solutions of the following discrete problems

$$A^N V_i = 1 \quad \text{for } i = 1, 2, \dots, N, \quad V_0 = 0, \tag{2.6}$$

$$A^N W_i = \eta_i \quad \text{for } i = 1, 2, \dots, N, \quad W_0 = 0 \tag{2.7}$$

with

$$\eta_i = \sum_{j=i}^{N-1} \hbar_j f_j \quad \text{for } i = 1, 2, \dots, N-1, \quad \eta_N = 0.$$

Thus it suffices to prove that $\|v\|_\infty \le C_0 \|\eta\|_\infty$. Further, we consider the two cases.

(i) If $\varepsilon/H \ge p_{N-1/2}/2$, it can easily verified that A^N yields an M-matrix. Now, using the barrier functions $V_i^l = 0$, $V_i^u = 1/\beta$, and $W_i^{l,u} = \pm V_i \|\eta\|_\infty$, we get the bounds

$$0 < V_i \le 1/\beta, \quad |W_i| \le V_i \|\eta\|_\infty \le \|\eta\|_\infty/\beta \quad \text{for } i = 1, \dots, N,$$

which, combined with (2.5), yield (2.4) with the stability constant $C_0 = 2/\beta$.

(ii) If $\varepsilon/H < p_{N-1/2}/2$, we set $\bar{p} := p_{N-1/2}$ and, by (2.6), (2.7), have

$$V_N = \left(\frac{\bar{p}}{2} + \frac{\varepsilon}{H}\right)^{-1} \left[1 - \left(\frac{\bar{p}}{2} - \frac{\varepsilon}{H}\right) V_{N-1}\right], \quad W_N = -\left(\frac{\bar{p}}{2} + \frac{\varepsilon}{H}\right)^{-1} \left(\frac{\bar{p}}{2} - \frac{\varepsilon}{H}\right) W_{N-1}.$$

Now, eliminating V_N and W_N from (2.5), (2.6) and (2.7), we obtain

$$v_i = W_i + \frac{\left(\frac{\bar{p}}{2} - \frac{\varepsilon}{H}\right) W_{N-1} V_i}{1 - \left(\frac{\bar{p}}{2} - \frac{\varepsilon}{H}\right) V_{N-1}} \quad \text{for } i = 0, \dots, N-1, \tag{2.8}$$

where V_i and W_i, for $i = 0, \dots, N-1$, are the solutions of the slightly modified problems

$$\tilde{A}^N V_i = 1 \quad \text{for } i = 1, 2, \dots, N-2, \quad \tilde{A}^N V_{N-1} = 1 + \frac{p_{N-3/2}}{2}\left(\frac{\bar{p}}{2} + \frac{\varepsilon}{H}\right)^{-1}, \quad V_0 = 0,$$
$$\tilde{A} W_i = \eta_i \quad \text{for } i = 1, 2, \dots, N-1, \quad W_0 = 0$$

with the slightly modified operator $\tilde{A}^N$ defined by

$$\tilde{A}^N V_i := A^N V_i \quad \text{for } i = 1, \dots, N-2,$$
$$\tilde{A}^N V_{N-1} := -\frac{\varepsilon}{H} V_{N-2} + \left[\frac{\varepsilon}{H} + \frac{3p_{N-3/2}}{2} + \frac{p_{N-3/2}}{2}\left(\frac{\bar{p}}{2} + \frac{\varepsilon}{H}\right)^{-1}\left(\frac{\bar{p}}{2} - \frac{\varepsilon}{H}\right)\right] V_{N-1}.$$

Since it can be easily verified that $\tilde{A}^N$ yields an M-matrix, we shall use the barrier functions $V_i^l = 0$, $V_i^u = (5/3)/p_i$, and $W_i^{l,u} = \pm V_i \|\eta\|_\infty$ to get the bounds

$$0 \le V_i \le (5/3)/p_i, \quad |W_i| \le V_i \|\eta\|_\infty \quad \text{for } i = 1, \dots, N-1. \tag{2.9}$$

Here, in particular, we used (1.4) implying $|p(\xi_1)/p(\xi_2) - 1| \le |\xi_1 - \xi_2| P/\beta$, and also the conditions of the Lemma $\varepsilon \le \varepsilon_0$ and $h \le h_0$ implying $\varepsilon |D^-(1/p)_i| \le 0.1$, and $\tilde{A}^N V_{N-1} \le 1 + p_{N-3/2}/\bar{p} \le 2.1$, and $\tilde{A}^N V_{N-1}^u \ge (5/3)[-\varepsilon D^-(1/p)_{N-1} + 1.5 p_{N-3/2}/p_{N-1}]$. Combining bounds (2.9) with (2.8), we derive $|v_i| \le V_i \|\eta\|_\infty [1 - (p_{N-1} V_{N-1})(\bar{p}/p_{N-1})/2]^{-1}$, which yields (2.4) with $C_0 = (40/3)/\beta$. $\square$

Remark 2. Our analysis for the case (ii) implies that, if $\varepsilon \le H p_{N-1/2}/2$, the difference operator L^N is inverse-monotone.

2.2. Truncation Error and Convergence

Lemma 2. *Let $u(x)$ be the solution of* (1.1) *with sufficiently smooth $p(x)$ and $f(x)$, and u_i^N be the solution of* (1.12), (1.13) *on an arbitrary nonuniform mesh. Then, under the conditions of Lemma 1, we have*

$$\|u_i^N - u(x_i)\|_\infty \le C\left[\max_{i=1,\dots,N}\left\{h_i \hbar_i \max_{\xi \in [x_{i-1}, x_i]} |(pu)''(\xi)|\right\} + N^{-2}\right], \tag{2.10}$$

$$\|u_i^N - u(x_i)\|_\infty \le C\left[\max_{i=1,\dots,N}\left(\min\{h_i \hbar_i/\varepsilon^2, 1\} \exp\{-\gamma x_{i-1}/\varepsilon\}\right) + N^{-2}\right] \tag{2.11}$$

with an arbitrary positive constant γ, satisfying $\gamma < p(0)$, and the notation $\hbar_N := h_N$.

Proof: Let $z_i := u_i^N - u(x_i)$ be the error and $\psi_i := f_i - L^N u_i$ be the truncation error. Then $L^N z_i = \psi_i$ for $i = 1, \dots, N-1$, $z_0 = z_N = 0$, and Lemma 1 implies

$\|u_i^N - u(x_i)\|_\infty \le C_0\|\psi\|_*$. Further, $\|\psi\|_*$ is estimated as in [2, 9] to derive (2.10), (2.11). □

Our main result regarding problem (1.1) is given by

Theorem 1. *Let* $u(x)$ *be the solution of* (1.1), (1.3) *with sufficiently smooth* $p(x)$ *and* $f(x)$, *and* u_i^N *be the solution of* (1.12). *Let also our meshnodes be* $x_i = x(\xi_i)$ *with* $\{\xi_i\}$ *satisfying* $0 = \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = 1$, $\xi_i - \xi_{i-1} = O(N^{-1})$, *and* $\xi_N - \xi_{N-1} = \xi_{N-1} - \xi_{N-2}$, *where the function* $x(\xi)$ *is defined by a)* (1.6), (1.9) *or*

$$b)\ x(\xi) = \begin{cases} \frac{\delta}{b}\xi & \text{for } \xi \in [0,b], \\ \delta + \frac{1-\delta}{1-b}(\xi - b) & \text{for } \xi \in [b,1], \end{cases} \quad \textit{with} \quad \delta = \min(\varepsilon\lambda \ln N, a)$$

and some constants $a, b \in (0,1)$, λ. *Then, provided that* $\lambda > 2/p(0)$, *we have*

$$a)\ \|u_i^N - u(x_i)\|_\infty \le CN^{-2}; \quad b)\ \|u_i^N - u(x_i)\|_\infty \le CN^{-2}\ln^2 N.$$

Proof: These estimates are derived from bound (2.11) of Lemma 2. The right-hand terms in (2.11) for our two meshes are estimated using a slightly modified analysis [2, 9]. □

Remark 3. If $\xi_i = i/N$ for $i = 0, 1, \ldots, N$, the meshes a) and b) of Theorem 1 turn into (1.5), (1.6), (1.9) and (1.11) respectively, i.e. the meshes a) and b) of Theorem 1 are nonuniform generalizations of the Bakhvalov [3] and Shishkin [17] meshes.

3. Parabolic Problem

3.1. Truncation Error

Let K, our time discretization parameter, be a positive integer, and $\tau = 1/K$. We define the tensor-product mesh on $[0,1] \times [0,T]$

$$\omega \times \omega_\tau = \{(x_i, t_j), \quad \text{with } t_j = j\tau, \quad \text{for } i = 0,\ldots,N,\ j = 0,\ldots,K\},$$

which is uniform in time. It is assumed for the space mesh ω, in addition to (2.1), that

$$h_i \le h_{i+1} \quad \text{for } i = 1, 2, \ldots, N-1. \tag{3.1}$$

which is reasonable for problem (1.2), since its solution has a boundary layer at $x = 0$. On $\omega \times \omega_\tau$ we shall study difference scheme (1.17). For the time difference derivatives we shall use the notation

$$\delta_{\bar{t}} v_i^j = \frac{v_i^j - v_i^{j-1}}{\tau}, \quad \delta_{\bar{t}}^2 v_i^j = \frac{\delta_{\bar{t}} v_i^j - \delta_{\bar{t}} v_i^{j-1}}{\tau} = \frac{v_i^j - 2v_i^{j-1} + v_i^{j-2}}{\tau^2}.$$

Let $z_i^j := u_i^{N,j} - u(x_i, t_j)$ be the error and $\psi_i^j := f_i^j - \delta_{\bar{t}} u_i^j - L^N u_i^j$ be the truncation error. Then

$$\begin{aligned} &\delta_{\bar{t}} z_i^j + L^N z_i^j = \psi_i^j \quad \text{for } i = 1, \ldots, N-1, \quad j = 1, \ldots, K, \\ &z_0^j = z_N^j = 0 \quad \text{for } j = 0, \ldots, K, \quad z_i^0 = \varphi_i^N - \varphi(x_i) \quad \text{for } i = 0, \ldots, N. \end{aligned} \tag{3.2}$$

It is easy to check that ψ_i^j can be splitted as

$$\psi_i^j = \Psi_{1,i}^j + \Psi_{2,i}^j = \Psi_{1,i}(t_j) + \Psi_{2,i}(t_j), \tag{3.3}$$

where

$$\Psi_{1,i}(t) := -L^N u_i(t) + f_i(t) - \frac{\partial}{\partial t} u(x_i, t) \quad \text{for } 0 \le t \le 1,$$

$$\Psi_{2,i}(t) := -\left[\delta_{\bar{t}} u_i(t) - \frac{\partial}{\partial t} u(x_i, t)\right] \quad \text{for } \tau \le t \le 1, \tag{3.4}$$

and the obvious notation $\delta_{\bar{t}} v(t) = [v(t) - v(t-\tau)]/\tau$ is used. Note that the corresponding discrete functions $\Psi_{1,i}^j$ and $\Psi_{1,i}^j$ are defined for $i = 1, \ldots, N-1$ and $j = 0, \ldots, K$ or $j = 1, \ldots, K$ respectively.

Integrating (1.2) w.r.t. x over $[x_{i-1/2}, x_{i+1/2}]$ we get

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial}{\partial t} u(x,t)\, dx = \left[(Au)(x_{i+1/2}, t) - (Au)(x_{i-1/2}, t)\right] + \int_{x_{i-1/2}}^{x_{i+1/2}} f(x,t) dx,$$

which, combined with $L^N = -DA^N$, implies

$$\begin{aligned} \Psi_{1,i}(t) = D\left[A^N u_i(t) - (Au)(x_{i-1/2}, t)\right] + \left[f_i(t) - \frac{1}{\hbar_i} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x,t)\, dx\right] \\ - \left[\frac{\partial}{\partial t} u(x_i, t) - \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial}{\partial t} u(x,t)\, dx\right]. \end{aligned}$$

Now it can be easily verified that

$$\Psi_{1,i}(t) = D\eta_i(t) + [D\bar{\eta}_i(t) + \bar{\mu}_i(t)] + \tilde{\Psi}_i(t), \tag{3.5}$$

where

$$\eta_i(t) := A^N u_i(t) - (Au)(x_{i-1/2}, t), \quad \bar{\eta}_i(t) := -h_i^2 \frac{\partial}{\partial x} f(x_{i-1/2}, t)/8, \tag{3.6a}$$

$$\begin{aligned} \bar{\mu}_i(t) := \frac{1}{\hbar_i}\Bigg[\int_{x_{i-1/2}}^{x_i} dx \int_x^{x_i} ds \int_{x_{i-1/2}}^{s} \frac{\partial^2}{\partial x^2} f(\xi, t)\, d\xi \\ + \int_{x_i}^{x_{i+1/2}} dx \int_{x_i}^{x} ds \int_{s}^{x_{i+1/2}} \frac{\partial^2}{\partial x^2} f(\xi, t)\, d\xi\Bigg], \end{aligned} \tag{3.6b}$$

$$\tilde{\Psi}_i(t) := -\left[\frac{\partial}{\partial t}u(x_i,t) - \frac{1}{\hbar_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial}{\partial t}u(x,t)\,dx\right]. \tag{3.6c}$$

Thus we proved.

Lemma 3. *For the truncation error* ψ_i^j, *we have* (3.3)–(3.5), *where* $\eta_i(t)$, $\bar{\eta}_i(t)$, $\bar{\mu}_i(t)$ *and* $\tilde{\Psi}_i(t)$ *are defined in* (3.6). *Also* $\tilde{\Psi}_i(t)$ *can be represented as*

$$\tilde{\Psi}_i(t) = -[D\tilde{\eta}_i(t) + \tilde{\mu}_i(t)], \tag{3.7}$$

where

$$\tilde{\eta}_i(t) := -h_i^2 \frac{\partial^2}{\partial x \partial t}u(x_{i-1/2},t)/8, \tag{3.8a}$$

$$\tilde{\mu}_i(t) := \frac{1}{\hbar_i}\left[\int_{x_{i-1/2}}^{x_i} dx \int_x^{x_i} ds \int_{x_{i-1/2}}^{s} \frac{\partial^3}{\partial x^2 \partial t}u(\xi,t)\,d\xi + \int_{x_i}^{x_{i+1/2}} dx \int_{x_i}^{x} ds \int_s^{x_{i+1/2}} \frac{\partial^3}{\partial x^2 \partial t}u(\xi,t)\,d\xi\right]. \tag{3.8b}$$

3.2. Stability Inequalities

Note that our four-point space difference operator L^N does not yield an M-matrix, which makes our stability analysis more difficult (we shall follow, partly, the analysis [10]). The main result of this Subsection is the hybrid stability inequality given by Lemma 5. But to prove it, we need a weaker L_2 stability stated in

Lemma 4. *Suppose* $p(x)$ *satisfies* (1.3), (1.4), *and our mesh* $\omega \times \omega_\tau$ *satisfies* (2.1), (3.1) *and* $\tau \le \tau_0 := 0.5/(1+3P)$; *then for the discrete function* y_i^j, *satisfying*

$$\delta_{\bar{t}} y_i^j + L^N y_i^j = f_i^j \quad \text{for } i = 1, \ldots, N-1,\ j = j_0+1, \ldots, K, \tag{3.9a}$$

$$y_0^j = y_N^j = 0 \quad \text{for } j = j_0, \ldots, K, \tag{3.9b}$$

we have

$$\|y^j\| \le C\left(\|y^{j_0}\| + \sqrt{\sum_{l=j_0+1}^{j} \tau \|f^l\|^2}\right) \quad \text{for } j = j_0, \ldots, K.$$

This Lemma is proved in Appendix A.

Lemma 5. *Let* y_i^j *satisfy* (3.9) *with* $j_0 = 0$, *and let* f_i^j *be splitted arbitrarily as* $f_i^j = f_{1,i}^j + f_{2,i}^j$ *for* $i = 1, \ldots, N-1$, $j = 1, \ldots, K$ *with* $f_{1,i}^0$ *also defined* (*arbitrarily*) *for* $i = 1, \ldots, N-1$; *then, under the conditions of Lemma* 4, *we have*

$$\|y^j\|_\infty \le C\Big(\|f_1^0 - L^N y^0\| + \|f_1^0\|_* + \|\delta_{\bar t} f_1^1\|_* + \|f_2^1\|_\infty + \max_{j=2,\ldots,K}\{\|\delta_{\bar t}^2 f_1^j\|_* + \|\delta_{\bar t} f_2^j\|_\infty\}\Big). \tag{3.10}$$

Remark 4. Though $f_{1,i}^0$ is defined arbitrarily, since there is $\delta_{\bar t} f_{1,i}^1$ on the right-hand side of (3.10), we need $f_{1,i}^0$ close to $f_{1,i}^1$ to get a sharp estimate. Note that we prove this Lemma to estimate the error z_i^j satisfying (3.2), where $f_i^j := \psi_i^j$ implies, by Lemma 3, the natural definition of $f_{1,i}^0 := \Psi_{1,i}^0$.

Proof: It follows from (3.9) with $f_i^j = f_{1,i}^j + f_{2,i}^j$ that y_i^j admits the representation

$$y_i^j = v_i^j + w_i^j,$$

where v_i^j and w_i^j are the solutions of the following discrete problems:

$$L^N v_i^j = f_{1,i}^j \quad \text{for } i = 1,\ldots,N-1, \quad v_0^j = v_N^j = 0 \quad \text{for } j = 0,\ldots,K, \tag{3.11}$$

$$\begin{aligned} &L^N w_i^j = f_{2,i}^j - \delta_{\bar t} v_i^j - \delta_{\bar t} w_i^j \quad \text{for } i = 1,\ldots,N-1,\ j = 1,\ldots,K, \\ &w_i^0 = y_i^0 - v_i^0 \quad \text{for } i = 0,\ldots,N, \quad w_0^j = w_N^j = 0 \quad \text{for } j = 0,\ldots,K. \end{aligned} \tag{3.12}$$

Then, applying Lemma 1 to (3.12) and recalling (2.3), we have

$$\|y^j\|_\infty \le \|v^j\|_\infty + C\big(\|\delta_{\bar t} v^j\|_\infty + \|W^j\| + \|f_2^j\|_\infty\big) \quad \text{for } j = 1,\ldots,K, \tag{3.13}$$

where $W_i^j := \delta_{\bar t} w_i^j$, defined for $i = 0,\ldots,N$, $j = 1,\ldots,K$, is the solution of the problem

$$\delta_{\bar t} W_i^j + L^N W_i^j = \delta_{\bar t} f_{2,i}^j - \delta_{\bar t}^2 v_i^j \quad \text{for } i = 1,\ldots,N-1,\ j = 2,\ldots,K, \tag{3.14a}$$

$$W_i^1 + \tau L^N W_i^1 = (f_{1,i}^0 - L^N y_i^0) + f_{2,i}^1 - \delta_{\bar t} v_i^1 \quad \text{for } i = 1,\ldots,N-1, \tag{3.14b}$$

$$W_0^j = W_N^j = 0 \quad \text{for } j = 1,\ldots,K. \tag{3.14c}$$

Note that (3.14b), which serves as an initial condition here, is derived from (3.12) for $j = 1$.

We claim that

$$\|W^1\|^2 \le C\|(f_{1,i}^0 - L^N y_i^0) + f_{2,i}^1 - \delta_{\bar t} v_i^1\| \le C(\|f_1^0 - L^N y^0\|^2 + \|\delta_{\bar t} v^1\| + \|f_2^1\|). \tag{3.15}$$

This claim is proved in Appendix B.

Further, it follows from (3.11), by Lemma 1, that $\|v^j\|_\infty \le C\|f_1^j\|_*$ for $j \ge 0$, $\|\delta_{\bar t} v^j\| \le C\|\delta_{\bar t} f_1^j\|_*$ for $j \ge 1$, $\|\delta_{\bar t}^2 v^j\| \le C\|\delta_{\bar t}^2 f_1^j\|_*$ for $j \ge 2$.

Now, applying Lemma 4 to problem (3.14a), (3.14c) for W^j with $j_0 = 1$ and recalling (3.13), (3.15), we derive

$$\|y^j\|_\infty \le C\left(\|f_1^0 - L^N y^0\| + \max_j \{\|f_1^j\| + \|\delta_{\bar t} f_1^j\| + \|\delta_{\bar t}^2 f_1^j\| + \|f_2^j\|_\infty + \|\delta_{\bar t} f_2^j\|_\infty\}\right).$$

Since for any discrete function Y^j and any norm $\|\cdot\|$ we have $\|Y^j\| \le \|Y^{j_0}\| + \max_{j>j_0} \|\delta_{\bar t} Y^j\|$ for $j \ge j_0$, we get (3.10). □

3.3. Convergence

Theorem 2. *Let $u(x,t)$ be the solution of* (1.2) *with sufficiently smooth $p(x)$, $f(x,t)$ and $\varphi(x)$, and $u_i^{N,j}$ be the solution of* (1.17) *with the initial condition φ_i^N defined by the solution of*

$$L^N \varphi_i^N = (L\varphi)(x_i) \quad \textit{for} \quad i = 1, \ldots, N-1, \quad \varphi_0^N = \varphi(0), \quad \varphi_N^N = \varphi(1), \tag{3.16}$$

on the mesh $\omega \times \omega_\tau$, where the space meshnodes $x_i = x(\xi_i)$ are defined by a) (1.5), (1.6) and (1.7) or (1.10); *b)* (1.11). *Then, provided that the mesh parameter $\lambda > 2/\beta$, we have*

$$\begin{aligned} &a)\ \max_j \|u_i^{N,j} - u(x_i, t_j)\|_\infty \le C(N^{-2} + \tau);\\ &b)\ \max_j \|u_i^{N,j} - u(x_i, t_j)\|_\infty \le C(N^{-2}\ln^2 N + \tau). \end{aligned} \tag{3.17}$$

Remark 5. Our initial condition φ_i^N defined by (3.16) is artificial and caused by our analysis. On the other hand, since the analysis of Section 1 applied to problem (3.16) implies $|\varphi_i^N - \varphi_i| \le CN^{-2}$, our initial condition is only slightly different from the natural initial condition $\tilde\varphi_i^N := \varphi_i$.

Remark 6. Theorem 2 also holds for the space meshes defined as in Theorem 1 and satisfying (3.1), i.e. for meshes that can, in general, be essentially nonuniform.

Proof: Applying Lemma 5 to problem (3.2) and recalling Lemma 3, we get

$$\begin{aligned} \|z^j\|_\infty \le C\Big(&\|\Psi_1^0 - L^N(\varphi_i^N - \varphi_i)\| + \|\Psi_1^0\|_* + \|\delta_{\bar t}\Psi_1^1\|_* + \|\Psi_2^1\|_\infty\\ &+ \max_{j=2,\ldots,K}\{\|\delta_{\bar t}^2\Psi_1^j\|_* + \|\delta_{\bar t}\Psi_2^j\|_\infty\}\Big). \end{aligned}$$

The first right-hand term, by (3.16) and (1.1) at $t = 0$, vanishes:

$$\Psi_{1,i}^0 - L^N(\varphi_i^N - \varphi_i) = \left[-L^N\varphi_i + f_i(0) - \frac{\partial u}{\partial t}(x_i, 0)\right] - \left[L^N\varphi_i^N - L^N\varphi_i\right]$$
$$= (L\varphi)_i - L^N\varphi_i^N = 0.$$

Further, using the Mean Value Theorem, we obtain

$$\|z^j\|_\infty \le C\max_t\left\{\|\Psi_1(t)\|_* + \|\frac{\partial}{\partial t}\Psi_1(t)\|_* + \|\frac{\partial^2}{\partial t^2}\Psi_1(t)\|_* + \|\Psi_2(t)\|_\infty + \|\frac{\partial}{\partial t}\Psi_2(t)\|_\infty\right\}. \tag{3.18}$$

To estimate this, we shall use the following decomposition of $u(x,t)$ [17, p. 221,]

$$u(x,t) = U(x,t) + V(x,t), \ \left|\frac{\partial^{k+l}}{\partial^k x\partial^l t}U\right| \le C, \ \left|\frac{\partial^{k+l}}{\partial^k x\partial^l t}V\right| \le C\varepsilon^{-k}\exp(-\gamma x/\varepsilon), \tag{3.19}$$

for $k, l = 0, 1, 2, 3$, with any positive constant γ satisfying $\gamma < \beta$. Then, by (3.4), Taylor series expansions yield

$$\max_{t\in[\tau,1]}\left\{\|\Psi_2(t)\|_\infty + \|\frac{\partial}{\partial t}\Psi_2(t)\|_\infty\right\} \le C\tau. \tag{3.20}$$

The terms with $\Psi_{1,i}(t)$ in (3.18) are estimated, by (3.5), (2.3), as

$$\max_{t\in[0,1]}\|\frac{\partial^l}{\partial t^l}\Psi_1(t)\|_* \le C\left[\|\frac{\partial^l}{\partial t^l}\eta_i(t)\|_\infty + \|\frac{\partial^l}{\partial t^l}\bar{\eta}_i(t)\|_\infty + \|\frac{\partial^l}{\partial t^l}\bar{\mu}_i(t)\|_\infty + \|\frac{\partial^l}{\partial t^l}\tilde{\Psi}_i(t)\|_*\right] \tag{3.21}$$

for $l = 0, 1, 2$. Now we shall split $\tilde{\Psi}_i(t)$ as $\tilde{\Psi}_i(t) = \tilde{\Psi}_i^U(t) + \tilde{\Psi}_i^V(t)$, where the right-hand terms are defined as $\tilde{\Psi}_i(t)$ in (3.6) and admit the representations as (3.7), (3.8) with $U(x,t)$, $\tilde{\eta}_i^U(t)$, $\tilde{\mu}_i^U(t)$ and $V(x,t)$, $\tilde{\eta}_i^V(t)$, $\tilde{\mu}_i^V(t)$ instead of $u(x,t)$, $\tilde{\eta}_i(t)$, $\tilde{\mu}_i(t)$ respectively. By (2.3), this yields

$$\|\frac{\partial^l}{\partial t^l}\tilde{\Psi}_i(t)\|_* \le 2\left[\|\frac{\partial^l}{\partial t^l}\tilde{\eta}_i^U(t)\|_\infty + \|\frac{\partial^l}{\partial t^l}\tilde{\mu}_i^U(t)\|_\infty\right] + 2\max_{i\le\bar{\imath}}\left\{|\frac{\partial^l}{\partial t^l}\tilde{\eta}_i^V(t)| + |\frac{\partial^l}{\partial t^l}\tilde{\mu}_i^V(t)|\right\} + \max_{i\ge\bar{\imath}}|\frac{\partial^l}{\partial t^l}\tilde{\Psi}_i^V(t)|, \tag{3.22}$$

with the number $\bar{\imath}$ defined by the condition $h_{\bar{\imath}} \le \varepsilon < h_{\bar{\imath}+1}$. Further, combining (3.21) with (3.22), recalling (3.19) and using a slightly modified analysis [2, 9], we derive, by Taylor series expansions, that

$$\max_{t\in[0,1]}\|\frac{\partial^l}{\partial t^l}\Psi_1(t)\|_* \le C\left[\max_i\left(\min\{\hbar_i^2/\varepsilon^2, 1\}\exp(-\gamma x_{i-1}/\varepsilon)\right) + N^{-2}\right] \tag{3.23}$$

for $l = 0, 1, 2$. Finally, combining (3.18), (3.20) and (3.23), we get the bound

$$\max_j \|u_i^{N,j} - u(x_i, t_j)\|_\infty \le C\Big[\max_i\big(\min\{\hbar_i^2/\varepsilon^2, 1\}\exp(-\gamma x_{i-1}/\varepsilon)\big) + N^{-2} + \tau\Big],$$

which, as in the proof of Theorem 1, yields (3.17).

4. Numerical Results

We consider test problems (1.1) and (1.2) with $p(x) = (x+1)^3$ and the other data such that their solutions are

$$u(x) = \frac{1}{p(x)}\exp\left(-\frac{1}{\varepsilon}\int_0^x b(s)\,ds\right) + \exp(-x/2)$$

(this example is from [4]) and

$$u(x,t) = \frac{1}{p(x)}\exp\left(-\frac{1}{\varepsilon}\int_0^x p(s)ds\right)\sin 2t + \exp(-x/2)\sin t,$$

respectively.

The problems were solved numerically on the Bakhvalov space mesh (1.5), (1.6), (1.10) with $C = 2.3$, $b = 0.5$, $\bar{\varepsilon}_0 = b/\lambda$.

In Table 1 for test problem (1.1), solved using difference scheme (1.12), (1.13), we give the error in the discrete L_∞ norm in the odd lines and the numerical rate of convergence, computed by the formula $\log_2\big(\|u_i^{2N} - u(x_i)\|/\|u_i^N - u(x_i)\|\big)$, in the even lines. The numerical tests confirm ε-uniform second-order convergence claimed by Theorem 1. Note that similar results for a steady problem on the Shishkin mesh are given in [8].

Table 2 shows the maximum nodal error $\max_j \|u_i^{N,j} - u(x_i, t_j)\|_\infty$ for test problem (1.2) solved by (1.17). The numerical results correspond with the ε-uniform error estimate given by Theorem 2.

A. Appendix: Proof of Lemma 4

Without loss of generality we shall only prove the Lemma for $j_0 = 0$. Multiplying (3.9a) by y^j as in (2.2), by simple calculations, we get

$$\begin{aligned}\|y^j\|^2 &= (y^j, y^{j-1}) + \tau\big[-\big(L^N y^j, y^j\big) + (f^j, y^j)\big]\\ &= (y^j, y^{j-1}) + \tau\Big[S^j + 1.5P\|y^j\|^2 + (f^j, y^j)\Big]\end{aligned}$$

with

Table 1. Two point boundary value problem, maximum nodal error and computational rate of convergence

N	$\varepsilon = 1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	$\max_{\varepsilon}$
16	4.96e − 3	2.82e − 2	3.11e − 2	3.12e − 2	3.12e − 2	3.12e − 2
	2.00	1.86	1.88	1.88	1.88	1.88
32	1.24e − 3	7.78e − 3	8.42e − 3	8.44e − 3	8.44e − 3	8.44e − 3
	2.00	1.94	1.94	1.94	1.94	1.94
64	3.09e − 4	2.02e − 3	2.19e − 3	2.20e − 3	2.20e − 3	2.20e − 3
	2.00	1.99	1.97	1.97	1.97	1.97
128	7.71e − 5	5.10e − 4	5.59e − 4	5.60e − 4	5.60e − 4	5.60e − 4
	2.00	2.01	1.99	1.99	1.99	1.99
256	1.92e − 5	1.26e − 4	1.41e − 4	1.41e − 4	1.41e − 4	1.41e − 4
	2.00	2.04	1.99	1.99	1.99	1.99
512	4.80e − 6	3.07e − 5	3.54e − 5	3.55e − 5	3.55e − 5	3.55e − 5

Table 2. Parabolic problem, maximum nodal error

τ^{-1}	N	$\varepsilon = 1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$
16	16	9.28e − 4	1.40e − 2	1.54e − 2	1.54e − 2	1.54e − 2
	32	4.48e − 3	1.04e − 2	1.41e − 2	1.43e − 2	1.43e − 2
	64	5.39e − 3	1.30e − 2	1.58e − 2	1.59e − 2	1.59e − 2
	128	5.61e − 3	1.42e − 2	1.62e − 2	1.63e − 2	1.63e − 2
	256	5.67e − 3	1.45e − 2	1.63e − 2	1.64e − 2	1.64e − 2
	512	5.68e − 3	1.46e − 2	1.64e − 2	1.64e − 2	1.64e − 2
1024	16	4.79e − 3	2.40e − 2	2.54e − 2	2.55e − 2	2.55e − 2
	32	1.14e − 3	6.60e − 3	6.88e − 3	6.89e − 3	6.89e − 3
	64	2.21e − 4	1.59e − 3	1.65e − 3	1.66e − 3	1.66e − 3
	128	1.55e − 5	2.77e − 4	2.95e − 4	2.96e − 4	2.96e − 4
	256	7.13e − 5	1.75e − 4	2.41e − 4	2.43e − 4	2.43e − 4
	512	8.52e − 5	2.10e − 4	2.55e − 4	2.57e − 4	2.57e − 4

$$S^j := -\left(L^N y^j, y^j\right) - 1.5P\|y^j\|^2 = -\sum_{i=1}^{N} h_i\left(A^N y_i^j\right)\left(D^- y_i^j\right) - 1.5P\|y^j\|^2.$$

Here we used $L^N = -DA^N$. Further, by the Schwarz inequality for the terms (y^j, y^{j-1}) and (f^j, y^j), we have $\|y^j\|^2 \leq (1-\bar{\tau})^{-1}[\|y^{j-1}\|^2 + \tau(2S^j + \|f^j\|^2)]$ with $\bar{\tau} := (1+3P)\tau$, and consequently

$$\|y^j\|^2 \leq (1-\bar{\tau})^{-j}\left[\|y^0\|^2 + \tau\sum_{l=1}^{j}\|f^l\|^2 + 2\tau S\right] \quad \text{for } j = 1, \ldots, K, \tag{A.1}$$

where

$$S = (1-\bar{\tau})^{j-1}S^j + (1-\bar{\tau})^{j-2}S^{j-1} + \cdots + S^1. \tag{A.2}$$

Note that $\tau \leq \tau_0 = 0.5/(1+3P)$, i.e. $\bar{\tau} \leq 0.5$, implies

$$1 \leq (1-\bar{\tau})^{-j} \leq (1-\bar{\tau})^{-1/\tau} \leq (1-\bar{\tau})^{-1/\bar{\tau}} \leq 1/\bar{C} \quad \text{with } \bar{C} = 1/4. \tag{A.3}$$

Now, by (1.14), we get

$$S^j = -\varepsilon \sum_{i=1}^{N} h_i |D^- y_i^j|^2 - 0.5 \sum_{i=1}^{N} h_i p_{i-1/2} (y_{i-1}^j + y_i^j)(D^- y_i^j) + 0.5 \sum_{i=1}^{N-1} h_i^2 p_{i-1/2} (D^- y_{i+1}^j - D^- y_i^j)(D^- y_i^j) - 1.5P \|y^j\|^2.$$

The second term on the right, by (1.4), is estimated as

$$\left| \sum_{i=1}^{N} h_i p_{i-1/2} (y_{i-1}^j + y_i^j)(D^- y_i^j) \right| = \left| \sum_{i=1}^{N-1} (p_{i-1/2} - p_{i+1/2})(y_i^j)^2 \right| \le P \|y^j\|^2.$$

Now, noting that $(a-b)b = [(a^2 - b^2) - (a-b)^2]/2$, with $a = D^- y_{i+1}^j$ and $b = D^- y_i^j$, we get

$$S^j \le -\varepsilon h_N |D^- y_N^j|^2 + \frac{1}{4} \sum_{i=1}^{N-1} h_i^2 p_{i-1/2} \Big(|D^- y_{i+1}^j|^2 - |D^- y_i^j|^2 \Big) - \frac{1}{4} \sum_{i=1}^{N-1} h_i^2 p_{i-1/2} \big| D^- y_{i+1}^j - D^- y_i^j \big|^2 - P \|y^j\|^2. \tag{A.4}$$

Setting $v_i = D^- y_i^j$, we observe, by (3.1), that

$$\begin{aligned} \sum_{i=1}^{N-1} h_i^2 p_{i-1/2} (v_{i+1}^2 - v_i^2) &\le h_N^2 p_{N-1/2} v_N^2 + \sum_{i=2}^{N} h_{i-1}^2 (p_{i-3/2} - p_{i-1/2}) v_i^2 \\ &\le h_N^2 p_{N-1/2} v_N^2 + 4P \|y^j\|^2. \end{aligned}$$

Further, combining this with (A.4), omitting some of the nonpositive terms and recalling (2.1), we derive

$$S^j \le -\varepsilon H |D^- y_N^j|^2 + \frac{p_{N-1/2}}{4} H^2 |D^- y_N^j|^2 - \frac{p_{N-1/2}}{4} H^4 \big| DD^- y_{N-1}^j \big|^2. \tag{A.5}$$

If $\varepsilon \ge p_{N-1/2} H/4$, then $S^j \le 0$, which implies $S \le 0$. Combining this with (A.1) and (A.3), we complete the proof.

Otherwise, if $\varepsilon < p_{N-1/2} H/4$, omitting the first term on the right in (A.5) and combining (A.5) with (A.2), (A.3), we obtain that

$$S \le \frac{p_{N-1}/2}{4} \sum_{l=1}^{j} \Big(|y_{N-1}^l|^2 - \bar{C} H^4 \big| DD^- y_{N-1}^l \big|^2 \Big). \tag{A.6}$$

Here we also used that $y_N^j = 0$ implies $D^- y_N^j = -y_{N-1}^j/H$. It follows from (3.9a) for $i = N-1$ that

$$|y_{N-1}^j| \le (1+\tau\tilde{p}/H)^{-1}\Big(|y_{N-1}^{j-1}| + \varepsilon\tau|DD^- y_{N-1}^j| + \tau|f_{N-1}^j|\Big)$$

with the notation $\tilde{p} := 1.5p_{N-3/2} - 0.5p_{N-1/2}$. Set $\delta := \tau\tilde{p}/H, \quad q := (1+\delta)^{-1}$. Then, by

$$(a+b+c)^2 \le (1+\delta)a^2 + (1+1/\delta)(b+c)^2 \le (1+\delta)\big[a^2 + (2/\delta)(b^2+c^2)\big],$$

we have

$$|y_{N-1}^j|^2 \le q|y_{N-1}^{j-1}|^2 + q(2/\delta)R^j \quad \text{with} \quad R^j = \varepsilon^2\tau^2|DD^- y_{N-1}^j|^2 + \tau^2|f_{N-1}^j|^2. \quad \text{(A.7)}$$

Further, using that $q + q^2 + \cdots + q^j \le q/(1-q) = 1/\delta = H/(\tau\beta)$, we derive that

$$\sum_{l=1}^{j} |y_{N-1}^l|^2 \le \frac{q}{1-q}|y_{N-1}^0|^2 + \frac{q}{1-q}\cdot\frac{2}{\delta}\sum_{l=1}^{j} R^j = \frac{H}{\tau\tilde{p}}|y_{N-1}^0|^2 + 2\left(\frac{H}{\tau\tilde{p}}\right)^2\sum_{l=1}^{j} R^j.$$

Combining this with (A.6) and (A.7), we get

$$S \le \frac{Hp_{N-1/2}}{4\tau\tilde{p}}|y_{N-1}^0|^2 + \frac{p_{N-1/2}}{4}\left(\frac{2H^2\varepsilon^2}{\tilde{p}^2} - \bar{C}H^4\right)\sum_{l=1}^{j}\big|DD^- y_{N-1}^l\big|^2 + CH^2\sum_{l=1}^{j}|f_{N-1}^l|^2.$$

Now we recall that $\bar{C} = 1/4$ and, by (1.4), $(p_{N-1/2}/\tilde{p}) \le 4/3$, which implies that $\varepsilon < (\tilde{p}H/4)(p_{N-1/2}/\tilde{p}) \le \tilde{p}H/3$. Then the second right-hand term is negative and consequently

$$2\tau S \le \frac{2}{3}\|y^0\|^2 + C\tau H\sum_{l=1}^{j}\|f^l\|^2.$$

Combining this with (A.1) and (A.3), we complete the proof.

B. Appendix: Proof of (3.15)

Setting $F_i := (f_{1,i}^0 - L^N y_i^0) + f_{2,i}^1 - \delta_{\bar{t}} v_i^1$, we prove that, under the conditions of Lemma 5, (3.14c), (3.14b) imply $\|W^1\|^2 \le C\|F\|^2$. Multiplying (3.14b) by W^1, we have

$$\|W^1\|^2 = (F, W^1) - \tau\big(L^N W^1, W^1\big).$$

The similar argument, as used in the proof of Lemma 4 (Appendix A) to derive (A.5), gives

$$
\begin{aligned}
S &:= -\left(L^N W^1, W^1\right) \\
&\le -\varepsilon H|D^- W_N^1|^2 + \frac{p_{N-1/2}}{4} H^2 |D^- W_N^1|^2 \\
&\quad - \frac{p_{N-1/2}}{4} H^4 \left|DD^- W_{N-1}^1\right|^2 + 1.5P\|W^1\|^2.
\end{aligned}
$$

If $\varepsilon \ge p_{N-1/2}H/4$, then $S \le 0$, and (3.15) is obvious. Otherwise, if $\varepsilon < p_{N-1/2}H/4$, i.e., by (1.4), $\varepsilon < \tilde{p}H/2$ with $\tilde{p} := 1.5p_{N-3/2} - 0.5p_{N-1/2}$, omitting the first term on the right and taking into consideration that $W_N^1 = 0$ implies $D^- W_N^1 = -W_{N-1}^1/H$, and that (3.14b) for $i = N-1$ yields $W_{N-1} = [H/(H+\tau\tilde{p})][\varepsilon\tau DD^- W_{N-1}^1 + F_{N-1}]$, we get

$$
\begin{aligned}
\tau S &\le \tau \frac{p_{N-1/2}}{4}\left(|W_{N-1}^1|^2 - H^4|DD^- W_{N-1}^1|^2\right) + \tau C\|W^1\|^2 \\
&\le \tau \frac{p_{N-1/2}}{4}\left(\frac{2\varepsilon^2}{\tilde{p}^2}H^2 - H^4\right)|DD^- W_{N-1}^1|^2 + \tau \frac{p_{N-1/2}}{4} \frac{2H^2}{(H+\tau\tilde{p})^2}|F_{N-1}|^2 + \tau C\|W^1\|^2 \\
&\le C\Big(H|F_{N-1}|^2 + \tau\|W^1\|^2\Big) \le C\Big(\|F\|^2 + \tau\|W^1\|^2\Big),
\end{aligned}
$$

which again yields (3.15).

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