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On a posteriori estimates of inverse operators for linear parabolic initial-boundary value problems

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Abstract We present numerically verified a posteriori estimates of the norms of inverse operators for linear parabolic differential equations. In case that the corresponding elliptic operator is not coercive, existing methods for a priori estimates of the inverse operators are not accurate and, usually, exponentially increase in time variable. We propose a new technique for obtaining the estimates of the inverse operator by using the finite dimensional approximation and error estimates. It enables us to obtain very sharp bounds compared with a priori estimates. We will give some numerical examples which confirm the actual effectiveness of our method.

Keywords A posteriori estimates · Galerkin finite element methods · Linear parabolic equations

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1 Introduction

We consider the following linear parabolic initial-boundary value problems,

$$\begin{cases} L_t w \equiv \dfrac{\partial w}{\partial t} - \nu \Delta w + (b \cdot \nabla) w + c w = g, & \text{in } \Omega \times J, & \text{(1a)} \\ w(x,t) = 0, & \text{on } \partial\Omega \times J, & \text{(1b)} \\ w(x,0) = 0, & \text{in } \Omega, & \text{(1c)} \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) is a bounded convex polygonal or polyhedral domain, $J = (0, T) \subset \mathbb{R}$ a bounded interval, ν a positive parameter, $b \in L^\infty\big(J; L^\infty(\Omega)\big)^d$, $c \in L^\infty\big(J; L^\infty(\Omega)\big)$ and $g \in L^2\big(J; L^2(\Omega)\big)$.

As well known, for all $g \in L^2\big(J; L^2(\Omega)\big)$, there exists a unique solution $w \in L^2\big(J; H_0^1(\Omega) \cap H^2(\Omega)\big)$ to the problem (1). Denoting the solution operator of (1) by $\mathcal{L}_t^{-1}$, it is bounded from $L^2\big(J; L^2(\Omega)\big)$ into $L^2\big(J; H_0^1(\Omega)\big)$. In this paper, we present a numerical method to compute a positive constant $C_{\mathcal{L}_t^{-1}}$ s.t.

$$\left\| \mathcal{L}_t^{-1} \right\|_{\mathcal{L}\left(L^2(J;L^2(\Omega)),L^2(J;H_0^1(\Omega))\right)} \le C_{\mathcal{L}_t^{-1}}. \tag{2}$$

It is not so difficult to determine such a constant, by some theoretical consideration (e.g. [10]), which we call '*a priori estimates*'. However, in general, $C_{\mathcal{L}_t^{-1}}$ obtained by existing a priori methods is exponentially dependent on the time interval J unless that the corresponding elliptic operator to the right-hand side of (1a) is coercive [3]. For example, in case of $b = 0$, the following a priori estimate is easily derived [10],

$$\left\| \mathcal{L}_t^{-1} \right\|_{\mathcal{L}\left(L^2(J;L^2(\Omega)),L^2(J;H_0^1(\Omega))\right)} \le \exp(\beta T) \frac{C_p}{\nu}, \tag{3}$$

where C_p is a Poincaré constant and β a nonnegative parameter defined as $\beta \equiv \max\{\sup_{\Omega \times J}(-c), 0\}$. Therefore, if the function c takes negative value, then the right-hand side of (3) becomes very large and it leads to an over-estimation of the inverse operator $\mathcal{L}_t^{-1}$, which yields the worse results for various purposes using the norm bounds. For example, applying the estimates (2) or (3), we can develop a numerical verification method for solutions of nonlinear problem (4) in a similar way as in [7].

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u = f(t, x, u, \nabla u), & \text{in } \Omega \times J, & \text{(4a)} \\ u(x,t) = 0, & \text{on } \partial\Omega \times J, & \text{(4b)} \\ u(x,0) = u_0, & \text{in } \Omega. & \text{(4c)} \end{cases}$$

In the verification process, the estimation of the norm for $\mathcal{L}_t^{-1}$ plays an essential role. In order to obtain a successful and efficient verification, we usually need to estimate it as small as possible. As described later, our *a posteriori estimates* will present more accurate bounds than a priori bounds (3) or similar result in [4]. Particularly, it will be illustrated by some numerical examples that our method could remove the exponential dependency on T even though c is negative. Therefore, our a posteriori approach is more efficient than the existing a priori method.

2 Function spaces and projections

In this section, we introduce some function spaces and finite dimensional projections. Let $S_h(\Omega) \subset H_0^1(\Omega)$ be a finite dimensional subspace dependent on the parameter h. For example, h stands for the mesh size when we use the finite element method. Let $\{\phi_i\}_{1\le i\le n}$ be a basis for $S_h(\Omega)$. Then, letting $P_h^0 : L^2(\Omega) \to S_h(\Omega)$ be the orthogonal L^2-projection, we extend it to the projection $P_h^0 : L^2\big(J; L^2(\Omega)\big) \to L^2\big(J; S_h(\Omega)\big)$ by

$$\Big(u(t) - P_h^0 u(t), v_h\Big)_{L^2(\Omega)} = 0, \quad \forall v_h \in S_h(\Omega), \quad \text{a.e. } t \in J. \tag{5}$$

Here, $(\cdot,\cdot)_{L^2(\Omega)}$ means L^2 inner product on Ω. It is easy to show that

$$\left\| P_h^0 \right\|_{\mathcal{L}\left(L^2(J;L^2(\Omega)),\ L^2(J;L^2(\Omega))\right)} \le 1. \tag{6}$$

Also using the orthogonal H_0^1-projection $P_h^1 : H_0^1(\Omega) \to S_h(\Omega)$, we define $P_h^1 : L^2\big(J; H_0^1(\Omega)\big) \to L^2\big(J; S_h(\Omega)\big)$ by

$$\Big(u(t) - P_h^1 u(t), v_h\Big)_{H_0^1(\Omega)} = 0, \quad \forall v_h \in S_h(\Omega), \quad \text{a.e. } t \in J. \tag{7}$$

Here, H_0^1 inner product on Ω is defined as $(u,v)_{H_0^1(\Omega)} := (\nabla u, \nabla v)_{L^2(\Omega)^d}$. Further, we define the function space $V^1(J) \subset H^1(J)$ by

$$V^1(J) := \Big\{u \in H^1(J)\ ;\ u(0) = 0\Big\}$$

and the inner product $(u,v)_{V^1(J)} := \big(u', v'\big)_{L^2(J)}$.

And define $V^1\big(J; L^2(\Omega)\big) \subset H^1\big(J; L^2(\Omega)\big)$ by

$$V^1\big(J; L^2(\Omega)\big) := \Big\{u \in H^1\big(J; L^2(\Omega)\big)\ ;\ u(0) = 0, \quad \text{in } \ L^2(\Omega)\Big\}$$

and the inner product $(u,v)_{V^1\left(J;L^2(\Omega)\right)} := (u_t, v_t)_{L^2\left(J;L^2(\Omega)\right)}$, where $u_t := \frac{\partial u}{\partial t}$ and $(u,v)_{L^2\left(J;L^2(\Omega)\right)}$ means L^2 inner product on $\Omega \times J$. Further, $L^2\big(J; H_0^1(\Omega)\big)$ is a Hilbert space with the inner product $(u,v)_{L^2\left(J;H_0^1(\Omega)\right)} := (\nabla u, \nabla v)_{L^2\left(J;L^2(\Omega)\right)^d}$.

Let $V := V^1\big(J; L^2(\Omega)\big) \cap L^2\big(J; H_0^1(\Omega)\big)$ and $X(\Omega) := \{u \in L^2(\Omega);\ \triangle u \in L^2(\Omega)\}$.

We assume that the following estimates hold for P_h^1.

Assumption 1 There exists a constant $C(h) > 0$ satisfying

$$\left\| u - P_h^1 u \right\|_{H_0^1(\Omega)} \le C(h) \left\| \triangle u \right\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega) \cap X(\Omega), \tag{8}$$

$$\left\| u - P_h^1 u \right\|_{L^2(\Omega)} \le C(h) \left\| u - P_h^1 u \right\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{9}$$

In many cases, the explicit values for $C(h)$ satisfying Assumption 1 are decidable. For examples, see [6].

3 Constructive a priori error estimates for a simple problem

In this section, we discuss the a priori estimate for linear simple equations, which is important to estimate a more general problem (1).
For a given constant $\nu > 0$ and $g \in L^2\big(J; L^2(\Omega)\big)$, we consider a linear heat equation of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \triangle u = g, & \text{in } \Omega \times J, \quad (10a) \\ u(x,t) = 0, & \text{on } \partial\Omega \times J, \quad (10b) \\ u(x,0) = 0, & \text{in } \Omega. \quad (10c) \end{cases}$$

We define the weak solution of (10) by

$$\left(\tfrac{\partial u}{\partial t}(t), v\right)_{L^2(\Omega)} + \nu \left(\nabla u(t), \nabla v\right)_{L^2(\Omega)^d} = \left(g(t), v\right)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega), \quad \text{a.e. } t \in J, \tag{11}$$

It is known that (11) has a unique solution $u \in V$.

We now define the linear differential operator $\triangle_t : V \cap L^2\big(J; X(\Omega)\big) \to L^2\big(J; L^2(\Omega)\big)$ by $\triangle_t := \frac{\partial}{\partial t} - \nu \triangle$. Then we have the following estimate [5, Lemma 2].

Lemma 2 *It holds that*

$$\left\| \tfrac{\partial}{\partial t} u \right\|_{L^2\left(J; L^2(\Omega)\right)} \le \left\| \triangle_t u \right\|_{L^2\left(J; L^2(\Omega)\right)}, \quad \forall u \in V \cap L^2\big(J; X(\Omega)\big). \tag{12}$$

Defining the P_h^V-projection $P_h^V : V \to V^1\big(J; S_h(\Omega)\big)$ by

$$\left(\tfrac{\partial}{\partial t}\big(u(t) - P_h^V u(t)\big), v_h\right)_{L^2(\Omega)} + \nu \left(\nabla\big(u(t) - P_h^V u(t)\big), \nabla v_h\right)_{L^2(\Omega)^d} = 0, \quad \forall v_h \in S_h(\Omega), \quad \text{a.e. } t \in J. \tag{13}$$

We have the following estimates [5].

Lemma 3 *We have*

$$\left\| \frac{\partial}{\partial t} P_h^V u \right\|_{L^2\left(J;L^2(\Omega)\right)} \le \|\triangle_t u\|_{L^2\left(J;L^2(\Omega)\right)}, \quad \forall u \in V \cap L^2\big(J; X(\Omega)\big). \tag{14}$$

Also, we have the following error estimates.

Theorem 4 *Under Assumption 1, the following inequality holds.*

$$\left\| u - P_h^V u \right\|_{L^2\left(J;H_0^1(\Omega)\right)} \le \frac{2C(h)}{\nu} \|\triangle_t u\|_{L^2\left(J;L^2(\Omega)\right)}, \quad \forall u \in V \cap L^2\big(J; X(\Omega)\big), \tag{15}$$

Proof See [5, Lemma 2]. □

Note that the statements in Lemma 2 and 3 and Theorem 4 are valid not only for the case of $d = 1$ but also $d = 2$ and 3, because the concerning arguments in [5] are exactly same as to the multi-dimensional case.

The Aubin-Nitsche trick is a well-known technique to obtain a higher order a priori L^2 error estimate than H_0^1 estimates by considering the dual problem. We now present a priori L^2 error estimates using the Aubin-Nitsche trick by [3] for applied numerical verification methods. Consider the conjugate problem of (10),

$$\begin{cases} \dfrac{\partial w}{\partial t} + \nu \triangle w = u_\perp, & \text{in } \Omega \times J, \quad (16a)\\ w(x,t) = 0, & \text{on } \partial\Omega \times J, \quad (16b)\\ w(x,T) = 0, & \text{in } \Omega, \quad (16c) \end{cases}$$

where $u_\perp := u - P_h^V u$. Let w be a weak solution of (16) satisfying

$$\left(\tfrac{\partial}{\partial t} w, v\right)_{L^2(\Omega)} - \nu \left(\nabla w, \nabla v\right)_{L^2(\Omega)^d} = (u_\perp, v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega), \quad \text{a.e. } t \in J, \tag{17}$$

By the variable transformation $s := T - t$ and setting $V_*^1\big(J; L^2(\Omega)\big) := \big\{w \in H^1\big(J; L^2(\Omega)\big)\,;\ w(T) = 0, \text{ in } L^2(\Omega)\big\}$, it is seen that (17) has a unique solution $w \in V_*^1\big(J; L^2(\Omega)\big) \cap L^2\big(J; H_0^1(\Omega)\big)$.

Next, let $w_h \in V_*^1\big(J; S_h(\Omega)\big)$ be the semi-discrete approximate solution of the (17) satisfying

$$\begin{aligned} &\left(\tfrac{\partial}{\partial t} w_h, v_h\right)_{L^2(\Omega)} - \nu \left(\nabla w_h, \nabla v_h\right)_{L^2(\Omega)^d} = (u_\perp, v_h)_{L^2(\Omega)}, \\ &\forall v_h \in S_h(\Omega), \quad \text{a.e. } t \in J. \end{aligned} \tag{18}$$

Theorem 5 *Under Assumption 1, we have the following inequality,*

$$\left\| u - P_h^V u \right\|_{L^2\left(J;L^2(\Omega)\right)} \le 4C(h) \left\| u - P_h^V u \right\|_{L^2\left(J;H_0^1(\Omega)\right)}, \quad \forall u \in V. \tag{19}$$

Proof For arbitrary $u \in V$, we put $u_\perp := u - P_h^V u$. Let $w \in V_*^1\big(J; L^2(\Omega)\big) \cap L^2\big(J; H_0^1(\Omega)\big)$ be the solution of (17). For almost every $t \in J$, by taking the test function $v = u_\perp(t) \in H_0^1(\Omega)$ in (17), we have

$$\begin{aligned}\|u_\perp(t)\|^2_{L^2(\Omega)} &= \left(\tfrac{\partial}{\partial t} w, u_\perp\right)_{L^2(\Omega)} - \nu \left(\nabla w, \nabla u_\perp\right)_{L^2(\Omega)^d} \\ &= \tfrac{d}{dt}\left(w, u_\perp\right)_{L^2(\Omega)} - \left(w, \tfrac{\partial}{\partial t} u_\perp\right)_{L^2(\Omega)} - \nu \left(\nabla w, \nabla u_\perp\right)_{L^2(\Omega)^d}.\end{aligned}$$

By some simple calculations, taking into account the definition of P_h^1- and P_h^V-projection, we obtain the following equality

$$\begin{aligned}\|u_\perp(t)\|^2_{L^2(\Omega)} = \tfrac{d}{dt}\left(w_h, u_\perp\right)_{L^2(\Omega)} + \left(\tfrac{\partial}{\partial t}(w - w_h), u - P_h^1 u\right)_{L^2(\Omega)} \\ -\nu\left(\nabla(w - w_h), \nabla(u - P_h^1 u)\right)_{L^2(\Omega)^d}.\end{aligned}$$

Integrating this on J, we get

$$\begin{aligned}&\|u_\perp\|^2_{L^2\left(J; L^2(\Omega)\right)} \\ &= \left(\tfrac{\partial}{\partial t}(w - w_h), u - P_h^1 u\right)_{L^2\left(J; L^2(\Omega)\right)} - \nu\left(\nabla(w - w_h), \nabla(u - P_h^1 u)\right)_{L^2\left(J; L^2(\Omega)\right)^d} \\ &\le \|w - w_h\|_{V^1(J; L^2)} \left\|u - P_h^1 u\right\|_{L^2(J; L^2)} + \nu \|w - w_h\|_{L^2(J; H_0^1)} \left\|u - P_h^1 u\right\|_{L^2(J; H_0^1)}\end{aligned} \tag{20}$$

where we used the properties $u_\perp(0) = 0$ and $w_h(T) = 0$. By the fact that the error of P_h^1-projection in the H_0^1 norm is minimized, we have from (9)

$$\left\|u - P_h^1 u\right\|_{L^2\left(J; L^2(\Omega)\right)} \le C(h) \left\|u - P_h^V u\right\|_{L^2\left(J; H_0^1(\Omega)\right)}.$$

Next, since we can obtain the same error estimates in Theorem 4 for the dual problem, we have

$$\|w - w_h\|_{L^2\left(J; H_0^1(\Omega)\right)} \le \frac{2C(h)}{\nu} \left\|\tfrac{\partial}{\partial t} w + \nu \triangle w\right\|_{L^2\left(J; L^2(\Omega)\right)} = \frac{2C(h)}{\nu} \|u_\perp\|_{L^2\left(J; L^2(\Omega)\right)}.$$

Finally, from Lemmas 2 and 3, we have

$$\|w - w_h\|_{V^1\left(J; L^2(\Omega)\right)} \le 2 \|\triangle_t w\|_{L^2\left(J; L^2(\Omega)\right)} = 2 \|u_\perp\|_{L^2\left(J; L^2(\Omega)\right)}.$$

Therefore, (20) implies

$$\|u_\perp\|^2_{L^2\left(J; L^2(\Omega)\right)} \le 4C(h) \|u_\perp\|_{L^2\left(J; L^2(\Omega)\right)} \|u_\perp\|_{L^2\left(J; H_0^1(\Omega)\right)}.$$

□

4 A numerically verified a posteriori estimates for solutions of parabolic problems

In this section, we discuss the a posteriori estimates for solutions of (1), especially, an efficient computation of $C_{\mathcal{L}_t^{-1}}$ in (2).

First, we define the $n \times n$ matrices L_ϕ, D_ϕ and Q_ϕ associated with the equation (1) by

$$L_{\phi,i,j} := \left(\phi_j, \phi_i\right)_{L^2(\Omega)}, \qquad D_{\phi,i,j} := \left(\nabla\phi_j, \nabla\phi_i\right)_{L^2(\Omega)^d}, \tag{21}$$

$$Q_{\phi,i,j} := \nu\left(\nabla\phi_j, \nabla\phi_i\right)_{L^2(\Omega)^d} + \left((b\cdot\nabla)\phi_j, \phi_i\right)_{L^2(\Omega)} + \left(c\phi_j, \phi_i\right)_{L^2(\Omega)}. \tag{22}$$

Note that D_ϕ and L_ϕ are symmetric positive definite matrices. Let Y_ϕ and Z_ϕ be the Cholesky factors of D_ϕ and L_ϕ, respectively, i.e.,

$$D_\phi = Y_\phi Y_\phi^T \qquad \text{and} \qquad L_\phi = Z_\phi Z_\phi^T.$$

Next, we consider the semi-discrete finite element solution $w_h \in V^1\big(J; S_h(\Omega)\big)$ of (1) defined by

$$\begin{aligned}\left(\tfrac{\partial w_h}{\partial t}(t), v_h\right)_{L^2(\Omega)} &+ \nu\left(\nabla w_h(t), \nabla v_h\right)_{L^2(\Omega)^d}\\ &+ \Big(\big(b(t)\cdot\nabla\big)w_h(t) + c(t)w_h(t), v_h\Big)_{L^2(\Omega)}\\ &= \left(g(t), v_h\right)_{L^2(\Omega)}, \quad \forall v_h \in S_h(\Omega), \quad \text{a.e. } t \in J.\end{aligned} \tag{23}$$

Defining two vector functions α and β by $w_h = \sum_{i=1}^n \alpha_i(t)\phi_i(x)$ and $\beta_i := (g, \phi_i)_{L^2(\Omega)}$ $(i = 1, \ldots, n)$, (23) can be rewritten as

$$\left(L_\phi \tfrac{d}{dt} + Q_\phi(t)\right)\alpha(t) = \beta(t), \quad \text{a.e. } t \in J.$$

Therefore, if we can compute $\left(L_\phi \frac{d}{dt} + Q_\phi\right)^{-1}$ then we can estimate w_h.

We now define the positive constant

$$M_\phi^{10}(h) = \left\| Y_\phi^T\left(L_\phi \tfrac{d}{dt} + Q_\phi\right)^{-1} Z_\phi \right\|_{\mathcal{L}\left(L^2(J)^n,\ L^2(J)^n\right)}. \tag{24}$$

We can compute an upper bound of $M_\phi^{10}(h)$ by applying the numerical method introduced in [2]. Also we define positive constants $C_b := \left\|\sqrt{b_1^2 + \cdots + b_d^2}\right\|_{L^\infty\left(J;L^\infty(\Omega)\right)}$, $C_1 := C_b + C_p\|c\|_{L^\infty\left(J;L^\infty(\Omega)\right)}$ and $C_2 := C_b + 4C(h)\|c\|_{L^\infty\left(J;L^\infty(\Omega)\right)}$.

Theorem 6 *Under Assumption 1, let $\kappa_\phi > 0$ be satisfy*

$$\kappa_\phi := 2C(h)C_2\big(1 + C_1 M_\phi^{10}(h)\big) < \nu. \tag{25}$$

Then, we have following estimates,

$$\left\|\mathcal{L}_t^{-1}\right\|_{\mathcal{L}\left(L^2(J;L^2(\Omega)),\ L^2(J;H_0^1(\Omega))\right)} \le \frac{\nu M_\phi^{10}(h) + 2C(h) + 2C(h)C_1 M_\phi^{10}(h)}{\nu - \kappa_\phi}. \quad (26)$$

Proof For an arbitrary $g \in L^2\big(J; L^2(\Omega)\big)$, we put $u := \mathcal{L}_t^{-1} g \in V \cap L^2\big(J; X(\Omega)\big)$ and decompose it into two parts, namely, the finite and infinite dimensional parts, as follows

$$\frac{\partial u}{\partial t} - \nu \triangle u + (b\cdot\nabla)u + cu = g \quad \Longleftrightarrow \quad u = \triangle_t^{-1}\big(-(b\cdot\nabla)u - cu + g\big)$$

$$\Longleftrightarrow \begin{cases} P_h^V u = P_h^V \triangle_t^{-1}\big(-(b\cdot\nabla)u - cu + g\big), & (27a)\\ (I - P_h^V)u = (I - P_h^V)\triangle_t^{-1}\big(-(b\cdot\nabla)u - cu + g\big). & (27b)\end{cases}$$

Denoting $u_\perp := u - P_h^V u$ and $\partial_t := \frac{\partial}{\partial t}$, by some simple calculations using (27a), it is readily seen, for almost every $t \in J$ and arbitrary $v_h \in S_h(\Omega)$, that

$$\left(\partial_t P_h^V u, v_h\right)_{L^2(\Omega)} + \nu\left(\nabla P_h^V u, \nabla v_h\right)_{L^2(\Omega)^d} + \left((b\cdot\nabla)P_h^V u + cP_h^V u, v_h\right)_{L^2(\Omega)}$$
$$= \left(P_h^0\big(-(b\cdot\nabla)u_\perp - cu_\perp + g\big), v_h\right)_{L^2(\Omega)}. \quad (28)$$

Since $P_h^V u$ and $P_h^0\big(-(b\cdot\nabla)u_\perp - cu_\perp + g\big)$ are elements in $V^1\big(J; S_h(\Omega)\big)$ and $L^2\big(J; S_h(\Omega)\big)$, respectively, they are represented by the linear combinations of the basis of $S_h(\Omega)$. Namely, there exists $\alpha := (\alpha_1, \ldots, \alpha_n)^T \in V^1(J)^n$ and $\beta := (\beta_1, \ldots, \beta_n)^T \in L^2(J)^n$ such that

$$P_h^V u(x,t) = \sum_{i=1}^n \alpha_i(t)\phi_i(x), \quad P_h^0\big(-(b\cdot\nabla)u_\perp - cu_\perp + g\big)(x,t) = \sum_{i=1}^n \beta_i(t)\phi_i(x).$$

Therefore, (28) is rewritten by using α and β of the form

$$L_\phi \alpha' + Q_\phi \alpha = L_\phi \beta, \quad (29)$$

where the matrices L_ϕ and Q_ϕ are defined by (21) and (22), respectively. By (29), we have

$$\left\|P_h^V u(t)\right\|_{H_0^1(\Omega)}^2 = \alpha(t)^T D_\phi \alpha(t)$$
$$= \left(Y_\phi^T \alpha(t)\right)^T Y_\phi^T \left(L_\phi \tfrac{d}{dt} + Q_\phi(t)\right)^{-1} Z_\phi \left(Z_\phi^T \beta(t)\right).$$

Integrating both sides of the above in t and by using (24), we get

$$\begin{aligned}
&\left\| P_h^V u \right\|^2_{L^2\left(J;H_0^1(\Omega)\right)} \\
&= \int_J \left(Y_\phi^T \alpha(t)\right)^T Y_\phi^T \left(L_\phi \tfrac{d}{dt} + Q_\phi(t)\right)^{-1} Z_\phi \left(Z_\phi^T \beta(t)\right) dt \\
&\le \left\| Y_\phi^T \alpha \right\|_{L^2(J)^n} \left\| Y_\phi^T \left(L_\phi \tfrac{d}{dt} + Q_\phi\right)^{-1} Z_\phi \right\|_{\mathcal{L}\left(L^2(J)^n,\ L^2(J)^n\right)} \left\| Z_\phi^T \beta \right\|_{L^2(J)^n} \\
&\le \left\| P_h^V u \right\|_{L^2\left(J;H_0^1(\Omega)\right)} M_\phi^{10}(h) \left\| P_h^0\left(-(b\cdot\nabla)u_\perp - cu_\perp + g\right)\right\|_{L^2\left(J;L^2(\Omega)\right)}.
\end{aligned}$$

Therefore, by (6), we have

$$\left\| P_h^V u \right\|_{L^2\left(J;H_0^1(\Omega)\right)} \le M_\phi^{10}(h) \left\| -(b\cdot\nabla)u_\perp - cu_\perp + g \right\|_{L^2\left(J;L^2(\Omega)\right)}.$$

Moreover, from (19), we obtain

$$\left\| P_h^V u \right\|_{L^2\left(J;H_0^1(\Omega)\right)} \le C_2 M_\phi^{10}(h) \left\| u_\perp \right\|_{L^2\left(J;H_0^1(\Omega)\right)} + M_\phi^{10}(h) \left\| g \right\|_{L^2\left(J;L^2(\Omega)\right)}. \tag{30}$$

Next, taking the $L^2\left(J;\, H_0^1(\Omega)\right)$ norm of (27b) with (15), we have

$$\begin{aligned}
&\left\| u_\perp \right\|_{L^2\left(J;H_0^1(\Omega)\right)} = \left\| (I - P_h^V)\triangle_t^{-1}\left(-(b\cdot\nabla)u - cu + g\right)\right\|_{L^2\left(J;H_0^1(\Omega)\right)} \\
&\le \frac{2C(h)}{\nu} \left\| -(b\cdot\nabla)u - cu + g \right\|_{L^2\left(J;L^2(\Omega)\right)} \\
&\le \frac{2C(h)}{\nu} C_b \left(\left\| P_h^V u \right\|_{L^2\left(J;H_0^1(\Omega)\right)} + \left\| u_\perp \right\|_{L^2\left(J;H_0^1(\Omega)\right)} \right) + \frac{2C(h)}{\nu} \left\| g \right\|_{L^2\left(J;L^2(\Omega)\right)} \\
&\quad + \frac{2C(h)}{\nu} \left\| c \right\|_{L^\infty\left(J;L^\infty(\Omega)\right)} \left(C_p \left\| P_h^V u \right\|_{L^2\left(J;H_0^1(\Omega)\right)} + \left\| u_\perp \right\|_{L^2\left(J;L^2(\Omega)\right)} \right).
\end{aligned}$$

Hence from (19) and (30), we have

$$\begin{aligned}
&\left\| u_\perp \right\|_{L^2\left(J;H_0^1(\Omega)\right)} \\
&\le \frac{2C(h)}{\nu} \left(C_b + C_p \left\| c \right\|_{L^\infty(J;L^\infty(\Omega))} \right) \Big(C_2 M_\phi^{10}(h) \left\| u_\perp \right\|_{L^2(J;H_0^1(\Omega))} \\
&\quad + M_\phi^{10}(h) \left\| g \right\|_{L^2(J;L^2(\Omega))} \Big) \\
&\quad + \frac{2C(h)}{\nu} \left(C_b + 4C(h) \left\| c \right\|_{L^\infty(J;L^\infty(\Omega))} \right) \left\| u_\perp \right\|_{L^2(J;H_0^1(\Omega))} + \frac{2C(h)}{\nu} \left\| g \right\|_{L^2(J;L^2(\Omega))} \\
&= \frac{2C(h)}{\nu} C_2 \left(1 + C_1 M_\phi^{10}(h)\right) \left\| u_\perp \right\|_{L^2(J;H_0^1(\Omega))} + \frac{2C(h)}{\nu} \left(1 + C_1 M_\phi^{10}(h)\right) \left\| g \right\|_{L^2(J;L^2(\Omega))}.
\end{aligned}$$

Therefore, by the assumption $\kappa_\phi := 2C(h)C_2\big(1 + C_1 M_\phi^{10}(h)\big) < \nu$, it implies that

$$\|u_\perp\|_{L^2\left(J;H_0^1(\Omega)\right)} \le 2C(h)\frac{1 + C_1 M_\phi^{10}(h)}{\nu - \kappa_\phi}\|g\|_{L^2\left(J;L^2(\Omega)\right)}. \tag{31}$$

Substituting (31) into the right-hand side of (30), we have

$$\left\|P_h^V u\right\|_{L^2\left(J;H_0^1(\Omega)\right)} \le \frac{\nu M_\phi^{10}(h)}{\nu - \kappa_\phi}\|g\|_{L^2\left(J;L^2(\Omega)\right)}. \tag{32}$$

Finally, we obtain the following estimates by using (31) and (32),

$$\begin{aligned}\|u\|_{L^2\left(J;H_0^1(\Omega)\right)} &\le \left\|P_h^V u\right\|_{L^2\left(J;H_0^1(\Omega)\right)} + \|u_\perp\|_{L^2\left(J;H_0^1(\Omega)\right)}\\ &\le \frac{\nu M_\phi^{10}(h) + 2C(h) + 2C(h)C_1 M_\phi^{10}(h)}{\nu - \kappa_\phi}\|g\|_{L^2\left(J;L^2(\Omega)\right)}.\end{aligned}$$

Therefore, this proof is completed. □

5 Numerical examples

We considered the following linear problem, that is, $b = 0$ and $c = -2u_h^k$ in (1a) :

$$\begin{cases}\mathcal{L}_t w \equiv \frac{\partial}{\partial t}w - \nu\Delta w - 2u_h^k w = g, & \text{in } \Omega \times J, \quad (33a)\\ w(x,t) = 0, & \text{on } \partial\Omega \times J, \quad (33b)\\ w(x,0) = 0, & \text{in } \Omega, \quad (33c)\end{cases}$$

where u_h^k is supposed to be an approximate solution of the following nonlinear problem,

$$\begin{cases}\frac{\partial}{\partial t}u - \nu\Delta u = u^2 + f(x,t), & \text{in } \Omega \times J, \quad (34a)\\ u(x,t) = 0, & \text{on } \partial\Omega \times J, \quad (34b)\\ u(x,0) = 0, & \text{in } \Omega. \quad (34c)\end{cases}$$

Therefore, (33) corresponds to the linearized problem of (34). We only considered one dimensional case with $\Omega = (0,1)$ and adjusted the function $f(x,t)$ so that the problem (34) has the following exact solutions.

- $u(x,t) = 0.5t\sin(\pi x)$, $\nu = 0.1$, (Example 1);
- $u(x,t) = \sin(\pi t)\sin(\pi x)$, $\nu = 0.1$, (Example 2);

We used the finite element subspace $S_h(\Omega)$ spanned by piecewise linear functions with $n = 5$ and u_h^k is taken as the linear interpolation of u. Then, the values of constants can be taken as $C(h) = h/\pi$, $C_p = 1/\pi$ and $C_b = 0$. Moreover, we have,

Table 1 κ_ϕ and M_ϕ^{10}

	T	0.5	1.0	1.5	2.0
Example 1	κ_ϕ	0.0119	0.0263	0.0451	0.0697
	M_ϕ^{10}	2.2978	3.1871	4.2178	5.1665
Example 2	κ_ϕ	0.0552	0.0622	0.0643	0.0656
	M_ϕ^{10}	2.1326	3.5926	4.0394	4.2974

Fig. 1 Example 1

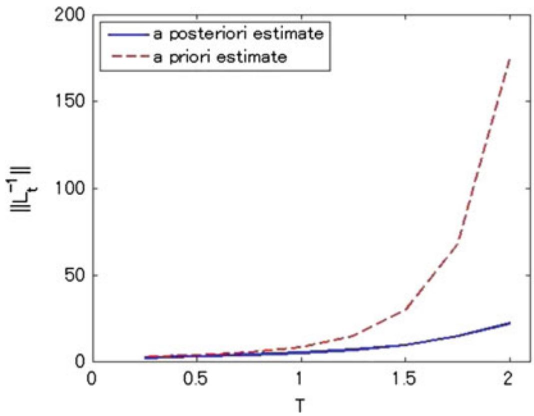
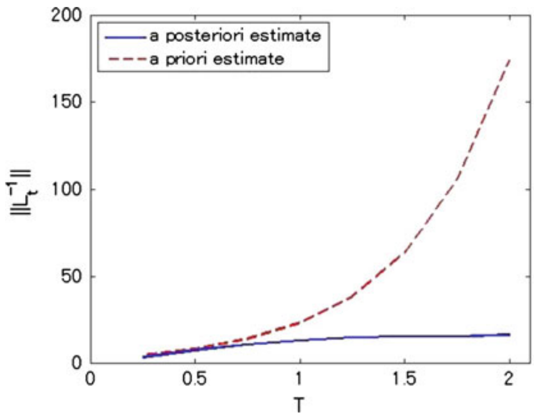


Fig. 2 Example 2



$$\|c\|_{L^\infty\left(J;L^\infty(\Omega)\right)} = 2\left\|u_h^k\right\|_{L^\infty\left(J;L^\infty(\Omega)\right)} \leq \begin{cases} T & \text{(Example 1)} \\ 2 & \text{(Example 2).} \end{cases}$$

And computational results for κ_ϕ and M_ϕ^{10} in Theorem 6 are shown in Table 1. We compared a posteriori estimates $C_{\mathcal{L}_t^{-1}}$ computed by the right-hand side of (26) with a priori estimated values $\exp(\beta T)\frac{C_p}{\nu}$ in (3). Numerical results of Example 1 and 2 are given in Figs. 1 and 2, respectively. All computations are carried out on a Dell Precision T7500 (Intel Xeon x5680, 72GB of memory) with MATLAB R2010b.

The computation errors have been taken into account by using INTLAB [8], a Toolbox of MATLAB.

Remark 7 In these figures, the values computed by our numerical method are always smaller than the a priori estimates. Particularly, both results by our method look like no exponential dependency on T for $T \leq 2$. The constant $M_{\phi}^{10}(h)$ defined by (24) is considered as an approximate value of the exact norm for $\mathcal{L}_t^{-1}$. In case of Example 1, $M_{\phi}^{10}(h)$ is rapidly, exponentially-like, increasing from $T \approx 5$. Hence the exact value for the norm of $\mathcal{L}_t^{-1}$ should be actually exponential-like increasing in time. On the other hand, in Example 2, which also has non-coercive elliptic part, we could not observe such an exponential dependency at all for a long time, large T, in the computation of $M_{\phi}^{10}(h)$. Therefore, we can say our verified computational result should be more truly represent the actual behavior of the inverse norm.

6 Conclusions

We presented a posteriori estimates of inverse operators for linear parabolic differential equations (1). Our method uses an approximate operator norm and error estimates of semi-discrete approximation as well as verified estimation of the inverse operator for linear ordinary differential equations. The numerical results show that our method gives more accurate value than existing a priori estimate. Particularly, there is a possibility to remove the exponential dependency on time, even if the corresponding elliptic problem is not coercive, which confirms us the actual usefulness of our approach for the numerical verification of solutions for complicated nonlinear evolution problems.

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