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Uncertain linguistic harmonic mean operators and their applications to multiple attribute group decision making

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Abstract In this paper, some uncertain linguistic aggregation operators called uncertain linguistic weighted harmonic mean (ULWHM) operator, uncertain linguistic ordered weighted harmonic mean operator and uncertain linguistic hybrid harmonic mean (ULHHM) operator are proposed. An approach to multiple attribute group decision making (MAGDM) with uncertain linguistic information is developed based on the ULWHM and the ULHHM operators. Finally, a practical application of the developed approach to MAGDM problem with uncertain linguistic information is given.

Keywords Multiple attribute group decision making (MAGDM) · Uncertain linguistic aggregation operator · Uncertain linguistic ordered weighted harmonic mean (ULOWHM) operator · Uncertain linguistic hybrid harmonic mean (ULHHM) operator

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1 Introduction

Decision making problems generally consist of finding the most desirable alternative(s) from a given alternative set. The increasing complexity of the socio-economic

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environment makes it less and less possible for single decision maker to consider all relevant aspects of a problem [11,14]. As a result, many decision making processes, in the real world, take place in group settings. Group decision making problems follow a common resolution scheme composed by the following two phases:

(1) Aggregation phase: it combines the individual preferences to obtain a collective preference.
(2) Exploitation phase: it orders the collective preference values to obtain the best alternative(s).

Recently, there are a number of studies on the group decision making with linguistic preference relations [2–10,12,15,19].

Herrera et al. [8,9] combined the linguistic ordered weighted averaging (LOWA) operator with linguistic preference relations and the concept of dominance and non-dominance to show its use in the field of group decision making, and presented three models of group decision making based on LOWA operator, and presented a consensus model in complete linguistic framework for group decision making. Herrera and Herrera-Viedma [5] analyzed the steps to follow linguistic decision analysis of group decision making problem with linguistic preference relations. Herrera and Martinez [10] developed a linguistic representation model for representing the linguistic information with the 2-tuples without loss of information. Motivated by this idea, Xu [15] proposed some linguistic aggregation operators such as linguistic geometric (LG) operator, linguistic weighted geometric (LWG) operator, linguistic ordered weighted geometric (LOWG) operator and linguistic hybrid geometric (LHG) operator, and developed an approach to group decision making with linguistic relations, which is straightforward and has no loss of information. Xu [19] defined two generalized induced linguistic aggregation operators, including generalized induced linguistic ordered weighted averaging (GILOWA) operator and generalized induced linguistic ordered weighted geometric (GILOWG) operator, and proved that the induced linguistic ordered weighted averaging (ILOWA) operator and LOWA operator are the special cases of the GILOWA operator, and induced linguistic ordered weighted geometric (ILOWG) operator and LOWG operator are the special cases of the GILOWG operator. Harmonic mean is widely used to aggregate central tendency data. In the existing literature, the harmonic mean is generally considered as a fusion technique of numerical data, in the real-life situations, the input data sometimes cannot be obtained exactly, but linguistic data can be given. Therefore, "how to aggregate linguistic data by using the harmonic mean?" is an interesting research topic and is worth paying attention too. To do so, Park et al. [12] developed some linguistic aggregation operators, such as linguistic weighted harmonic mean (LWHM) operator, linguistic ordered weighted harmonic mean (LOWHM) operator and linguistic hybrid harmonic mean (LHHM) operator, and presented an approach to group decision making based on the developed operators. However, in many situations, the decision makers either are willing to provide only uncertain linguistic information, or take the input arguments as the form of uncertain linguistic variables because of time pressure, lack of knowledge, or data, and their limited expertise related to the problem domain. Therefore, some authors devoted their attentions to this issue. Xu [16] proposed uncertain linguistic aggregation operators such as uncertain linguistic weighted averaging (ULWA) operator, uncertain linguistic ordered weighted averaging (ULOWA) operator and uncertain linguistic hybrid averaging (ULHA) operator, and developed an approach to multiple group decision making with uncertain linguistic information. Xu [17] proposed some uncertain linguistic aggregation operators including the uncertain linguistic geometric mean (ULGM) operator, uncertain linguistic weighted geometric mean (ULWGM) operator, and induced uncertain linguistic ordered weighted geometric (IULOWG) operator, and developed an approach to group decision making with uncertain multiplicative linguistic relation. Based on induced ordered weighted averaging (IOWA) operator proposed by Yager and Filev [20], Xu [18] introduced induced uncertain linguistic ordered weighted averaging (IULOWA) operator which take as their argument pair, called ULOWA pair, in which one component is used to induce an ordering over the second components which are given in the form of uncertain linguistic variables, and applied the IULOWA operator to group decision making with uncertain linguistic information. Xu [19] proposed two generalized induced uncertain linguistic aggregation operators, including generalized induced uncertain linguistic ordered weighted averaging (GIULOWA) operator and generalized induced uncertain linguistic ordered weighted geometric (GIULOWG) operator, and showed that the IULOWA operator and ULOWA operator are the special cases of the GIULOWA operator, and IULOWG operator and ULOWGM operator are the special cases of the GIULOWG operator.

The aim of this paper is to propose new uncertain linguistic aggregation operators, and to develop an approach to multiple attribute group decision making (MAGDM) with uncertain linguistic information. To do so, the remainder of this paper is arranged in following sections. Section 2 reviews some operational laws of uncertain linguistic variables and the formula for comparison between two uncertain linguistic variables. Section 3 develops some uncertain linguistic aggregation operators, such as uncertain linguistic weighted harmonic mean (ULWHM) operator, uncertain linguistic ordered weighted harmonic mean (ULOWHM) operator and uncertain linguistic hybrid harmonic mean (ULHHM) operator, and then studies some desirable properties of the operators. Section 4 presents an approach to group decision making based on the developed operators. Section 5 illustrates the presented approach with a practical example. Section 6 ends the paper with some concluding remarks.

2 Some operational laws of uncertain linguistic variables

Let $S = \{s_i : i = 0, 2, \ldots, t\}$ be a finite and totally ordered discrete term set. Any label, s_i, represents a possible value for a linguistic variable, and it must have the following characteristics [7]:

(1) The set is ordered: $s_i \geq s_j$ if $i \geq j$;
(2) There is the negation operator: $\text{neg}(s_i) = s_j$ such that $j = t - i$.
(3) Max operator: $\max(s_i, s_j) = s_i$ if $s_i \geq s_j$;
(4) Min operator: $\min(s_i, s_j) = s_i$ if $s_i \leq s_j$.

For example, S can be defined so as its elements are uniformly distributed on a scale on which a total order is defined:

$$S = \{s_0 = \text{extremely poor},\ s_1 = \text{very poor},\ s_2 = \text{poor},\ s_3 = \text{slightly poor},$$
$$s_4 = \text{fair},\ s_5 = \text{slightly good},\ s_6 = \text{good},\ s_7 = \text{very good},$$
$$s_8 = \text{extremely good}\}.$$

To preserve all the given information, we extend the discrete term set S to a continuous linguistic term set $\bar{S} = \{s_\alpha : s_0 \leq s_\alpha \leq s_t, \alpha \in [0, t]\}$, where, if $s_\alpha \in S$, then we call s_α an original linguistic term, otherwise, we call s_α the virtual linguistic term [17]. The decision maker, in general, uses the original linguistic terms to evaluate alternatives, and the virtual linguistic terms can only appear in operations.

Let $\tilde{s} = [s_\alpha, s_\beta]$, where $s_\alpha, s_\beta \in \bar{S}$, s_α and s_β are the lower and upper limits, respectively. We call $\tilde{s}$ the uncertain linguistic variables. Let $\tilde{S}$ be the set of all the uncertain linguistic variables.

Consider any three uncertain linguistic variables $\tilde{s} = [s_\alpha, s_\beta]$, $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$, and let $\lambda \in [0, 1]$, then we define their operations as follows:

(1) $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \oplus [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} \oplus s_{\alpha_2}, s_{\beta_1} \oplus s_{\beta_2}] = [s_{\alpha_1+\alpha_2}, s_{\beta_1+\beta_2}]$;
(2) $\lambda\tilde{s} = \lambda[s_\alpha, s_\beta] = [\lambda s_\alpha, \lambda s_\beta] = [s_{\lambda\alpha}, s_{\lambda\beta}]$;
(3) $\frac{1}{\tilde{s}} = \frac{1}{[s_\alpha, s_\beta]} = [\frac{1}{s_\beta}, \frac{1}{s_\alpha}] = [s_{\frac{1}{\beta}}, s_{\frac{1}{\alpha}}]$.

In order to compare uncertain linguistic variables, Xu [17] provided the following definition:

Definition 1 Let $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$ be two uncertain linguistic variables, and let $\text{len}(\tilde{s}_1) = \beta_1 - \alpha_1$ and $\text{len}(\tilde{s}_2) = \beta_2 - \alpha_2$, then the degree of possibility of $\tilde{s}_1 \geq \tilde{s}_2$ is defined as

$$p(\tilde{s}_1 \geq \tilde{s}_2) = \frac{\max\{0, \text{len}(\tilde{s}_1) + \text{len}(\tilde{s}_2) - \max(\beta_2 - \alpha_1, 0)\}}{\text{len}(\tilde{s}_1) + \text{len}(\tilde{s}_2)}. \tag{1}$$

From Definition 1, we can easily get the following results:

(1) $0 \leq p(\tilde{s}_1 \geq \tilde{s}_2) \leq 1, 0 \leq p(\tilde{s}_2 \geq \tilde{s}_1) \leq 1$;
(2) $p(\tilde{s}_1 \geq \tilde{s}_2) + p(\tilde{s}_2 \geq \tilde{s}_1) = 1$. Especially, $p(\tilde{s}_1 \geq \tilde{s}_1) = p(\tilde{s}_2 \geq \tilde{s}_2) = \frac{1}{2}$.

Wei and Yi [13] introduced the concept of fuzzy triangular linguistic variable as follow:

Definition 2 Let $\hat{s} = (s_\alpha, s_\beta, s_\gamma)$, where $s_\alpha, s_\beta, s_\gamma \in \bar{S}$, s_α, s_β, and s_γ are the lower, modal and upper values of $\hat{s}$, respectively, then we called $\hat{s}$ a triangular fuzzy linguistic variable, which characterized by the following membership function:

$$\mu_{\hat{s}}(s_\theta) = \begin{cases} 0, & s_1 \leq s_\theta \leq s_\alpha, \\ \frac{d(s_\theta, s_\alpha)}{d(s_\beta, s_\alpha)}, & s_\alpha \leq s_\theta \leq s_\beta, \\ \frac{d(s_\theta, s_\gamma)}{d(s_\beta, s_\gamma)}, & s_\beta \leq s_\theta \leq s_\gamma, \\ 0, & s_\gamma \leq s_\theta \leq s_t, \end{cases} \tag{2}$$

where $d(s_\alpha, s_\beta) = |\beta - \alpha|$ is the distance between s_α and s_β.

Clearly, s_β gives the maximal grade of $\mu_{\hat{s}}(s_\theta)(\mu_{\hat{s}}(s_\beta) = 1)$, s_α and s_γ are the lower and upper bounds with limit in the field of possible evaluation. If $s_\alpha = s_\beta = s_\gamma$, then $\hat{s}$ is reduced to a linguistic variable. If $s_\alpha = s_\beta$ or $s_\beta = s_\gamma$, then $\hat{s}$ is reduced to an uncertain linguistic variable.

In the following, Wei and Yi [13] introduced a formula for comparing triangular fuzzy linguistic variables.

Definition 3 Let $\hat{s}_1 = (s_{\alpha_1}, s_{\beta_1}, s_{\gamma_1})$ and $\hat{s}_2 = (s_{\alpha_2}, s_{\beta_2}, s_{\gamma_2})$ be any two triangular fuzzy linguistic variables, then the degree of possibility of $\hat{s}_1 \geq \hat{s}_2$ is defined as

$$p(\hat{s}_1 \geq \hat{s}_2) = \lambda \max\left\{1 - \max\left[\frac{d(s_{\beta_2}, s_{\alpha_1})}{d(s_{\beta_1}, s_{\alpha_1}) + d(s_{\beta_2}, s_{\alpha_2})}, 0\right], 0\right\} + (1-\lambda)\max\left\{1 - \max\left[\frac{d(s_{\gamma_2}, s_{\beta_1})}{d(s_{\gamma_1}, s_{\beta_1}) + d(s_{\gamma_2}, s_{\beta_2})}, 0\right], 0\right\}. \quad (3)$$

Definition 4 The α-cut of a triangular fuzzy linguistic variable is a subset of $\bar{S}$ and is denoted by

$$[\hat{s}]_\alpha = \{s_\theta \in \bar{S} : \mu_{\hat{s}}(s_\theta) \geq \alpha\}, \quad (4)$$

where $\mu_{\hat{s}}(s_\theta)$ is the membership function of $\hat{s}$ and $\alpha \in [0, 1]$.

The lower and upper points of any α-cut, $[\hat{s}]_\alpha$, are represented by $[\hat{s}]_\alpha^L$ and $[\hat{s}]_\alpha^U$, respectively, and suppose that both are finite.

Remark 1 If $\hat{s} = [[\hat{s}]_\alpha^L, [\hat{s}]_\alpha^U]$, then by choosing $\alpha = 1$ we can identify the modal value of $\hat{s}$, and by $\alpha = 0$ we can identify the lower and upper values of $\hat{s}$.

3 Some new uncertain linguistic aggregation operators

Recently, Park et al. [12] extended the OWHM operator [1] to accommodate the situation where the input arguments are linguistic variables.

Definition 5 A LOWHM operator of dimension n is a mapping LOWHM : $\bar{S}^n \to \bar{S}$, which has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, such that

$$\text{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} = \frac{1}{s_{\frac{w_1}{\beta_1}} \oplus s_{\frac{w_2}{\beta_2}} \oplus \cdots \oplus s_{\frac{w_n}{\beta_n}}} = \frac{1}{s_{\sum_{j=1}^n \frac{w_j}{\beta_j}}}, \quad (5)$$

where s_{β_j} is the jth largest of the s_{α_i}.

Sometimes, however, the decision makers are willing or able to provide only uncertain linguistic information because of time pressure, lack of knowledge or data, and their limited expertise related to the problem. In the following, we shall develop some operators for aggregating uncertain linguistic information.

Definition 6 Let ULHM:$\tilde{S}^n \to \tilde{S}$, if

$$\mathrm{ULHM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{n}{\frac{1}{\tilde{s}_1} \oplus \frac{1}{\tilde{s}_2} \oplus \cdots \oplus \frac{1}{\tilde{s}_n}} \tag{6}$$

where $\tilde{s} \in \tilde{S}$, $i = 1, 2, \ldots, n$, then ULHM is called the uncertain linguistic harmonic mean (ULHM) operator.

Example 1 Given the collection of uncertain linguistic variables: $\tilde{s}_1 = [s_2, s_3]$, $\tilde{s}_2 = [s_1, s_2]$, $\tilde{s}_3 = [s_3, s_4]$, $\tilde{s}_4 = [s_4, s_5]$, then by (6) and the operational laws of uncertain linguistic variables, we have

$$\begin{aligned}\mathrm{ULHM}(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4) &= \frac{4}{\frac{1}{\tilde{s}_1} \oplus \frac{1}{\tilde{s}_2} \oplus \frac{1}{\tilde{s}_3} \oplus \frac{1}{\tilde{s}_4}} \\ &= \frac{4}{\frac{1}{[s_2,s_3]} \oplus \frac{1}{[s_1,s_2]} \oplus \frac{1}{[s_3,s_4]} \oplus \frac{1}{[s_4,s_5]}} \\ &= [s_{1.92}, s_{3.13}].\end{aligned}$$

Definition 7 Let ULWHM : $\tilde{S}^n \to \tilde{S}$, if

$$\mathrm{ULWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{w_1}{\tilde{s}_1} \oplus \frac{w_2}{\tilde{s}_2} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_n}}, \tag{7}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector of $\tilde{s}_j (j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, then ULWHM is called the ULWHM operator.

Especially, if $w_i = 1, w_j = 0, j \neq i$, then $\mathrm{ULWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{s}_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the ULWHM operator is reduced to the ULHM operator. Furthermore, the ULWHM operator has the following property similar to that of the LWHM operator:

$$\min_j(\tilde{s}_j) \leq \mathrm{ULWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \max_j(\tilde{s}_j).$$

Example 2 Given the collection of uncertain linguistic variables: $\tilde{s}_1 = [s_2, s_3]$, $\tilde{s}_2 = [s_1, s_2]$, $\tilde{s}_3 = [s_3, s_4]$, $\tilde{s}_4 = [s_4, s_5]$, and let $w = (0.3, 0.2, 0.3, 0.2)^T$ be the weight vector of $\tilde{s}_j$ $(j = 1, 2, 3, 4)$, then by (7), we have

$$\text{ULWHM}_w(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4) = \frac{1}{\frac{0.3}{\tilde{s}_1} \oplus \frac{0.2}{\tilde{s}_2} \oplus \frac{0.3}{\tilde{s}_3} \oplus \frac{0.2}{\tilde{s}_4}} = \frac{1}{\frac{0.3}{[s_2, s_3]} \oplus \frac{0.2}{[s_1, s_2]} \oplus \frac{0.3}{[s_3, s_4]} \oplus \frac{0.2}{[s_4, s_5]}} = [s_{2.00}, s_{3.17}].$$

Definition 8 An ULOWHM operator of dimension n is a mapping ULOWHM : $\tilde{S}^n \to \tilde{S}$, which has associated weighting vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$. Furthermore:

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{w_1}{\tilde{r}_1} \oplus \frac{w_2}{\tilde{r}_2} \oplus \cdots \oplus \frac{w_n}{\tilde{r}_n}}, \tag{8}$$

where $\tilde{r}_j$ is the jth largest of the $\tilde{s}_i$, $\tilde{s}_i \in \tilde{S}$.

Especially, if $w_i = 1, w_j = 0, j \neq i$, then $\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{s}_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the ULOWHM operator is reduced to the ULHM operator. The weighting vector $w = (w_1, w_2, \ldots, w_n)^T$ can be determined by using some weight determining methods like the normal distribution based method (see, Refs. [18–20] for more details).

To rank these arguments $\tilde{s}_i (i = 1, 2, \ldots, n)$, we first compared each argument $\tilde{s}_i$ with all arguments $\tilde{s}_j (j = 1, 2, \ldots, n)$ by using (1), and let $p_{ij} = p(\tilde{s}_i \geq \tilde{s}_j)$. Then we construct a complementary matrix $\mathbf{P} = (p_{ij})_{n \times n}$ where:

$$p_{ij} \geq 0, \quad p_{ij} + p_{ji} = 1, \quad p_{ii} = \frac{1}{2}, \quad i, j = 1, 2, \ldots, n.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have $p_i = \sum_{j=1}^{n} p_{ij}, i = 1, 2, \ldots, n$. Then, in accordance with the values of $p_i (i = 1, 2, \ldots, n)$, we rank the arguments $\tilde{s}_i (i = 1, 2, \ldots, n)$ in descending order.

Example 3 Given the collection of uncertain linguistic variables: $\tilde{s}_1 = [s_2, s_3], \tilde{s}_2 = [s_1, s_3], \tilde{s}_3 = [s_2, s_4], \tilde{s}_4 = [s_3, s_4]$. To rank these arguments, we first compare each argument $\tilde{s}_i$ with all arguments $\tilde{s}_j (j = 1, 2, \ldots, n)$ by using (1), let $p_{ij} = p(\tilde{s}_i \geq \tilde{s}_j)(j = 1, 2, 3, 4)$, then we utilize these possibility degrees to construct the following matrix $\mathbf{P} = (p_{ij})_{4 \times 4}$:

$$\mathbf{P} = \begin{pmatrix} 0.500 & 0.667 & 0.333 & 0.000 \\ 0.333 & 0.500 & 0.250 & 0.000 \\ 0.667 & 0.750 & 0.500 & 0.333 \\ 1.000 & 1.000 & 0.667 & 0.500 \end{pmatrix}.$$

Summing all elements in each line of matrix **P**, we have

$$p_1 = 1.500, \quad p_2 = 1.083, \quad p_3 = 2.250, \quad p_4 = 3.167.$$

Then we rank the arguments $\tilde{s}_i (i = 1, 2, 3, 4)$ in descending order in accordance with the values of $p_i (i = 1, 2, 3, 4)$:

$$\tilde{r}_1 = \tilde{s}_4 = [s_3, s_4], \quad \tilde{r}_2 = \tilde{s}_3 = [s_2, s_4], \quad \tilde{r}_3 = \tilde{s}_1 = [s_2, s_3], \quad \tilde{r}_4 = \tilde{s}_2 = [s_1, s_3].$$

Suppose that the weighting vector $w = (w_1, w_2, w_3, w_4)^T$ of the ULOWHM operator is $w = (0.3, 0.2, 0.3, 0.2)^T$, then by (8), we get

$$\begin{aligned} \text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4) &= \frac{1}{\frac{0.3}{\tilde{s}_4} \oplus \frac{0.2}{\tilde{s}_3} \oplus \frac{0.3}{\tilde{s}_1} \oplus \frac{0.2}{\tilde{s}_2}} \\ &= \frac{1}{\frac{0.3}{[s_3,s_4]} \oplus \frac{0.2}{[s_2,s_4]} \oplus \frac{0.3}{[s_2,s_3]} \oplus \frac{0.2}{[s_1,s_3]}} \\ &= [s_{1.82}, s_{3.42}]. \end{aligned}$$

Based on Definition 5, we have the following properties of the ULOWHM operator:

Theorem 1 *Let $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n$ be a collection of uncertain linguistic variables and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the ULOWHM operator with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$; then we have the following.*

(1) (*Idempotency*): *If all $\tilde{s}_j$ $(j = 1, 2, \ldots, n)$ are equal, i.e., $\tilde{s}_j = \tilde{s}$ for all j, then*

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{s}.$$

(2) (*Boundedness*):

$$\min_j(\tilde{s}_j) \leq \text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \max_j(\tilde{s}_j).$$

(3) (*Monotonicity*): *If $\tilde{s}_j \leq \tilde{s}_j^*$, for all j, then*

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \text{ULOWHM}_w(\tilde{s}_1^*, \tilde{s}_2^*, \ldots, \tilde{s}_n^*).$$

(4) (*Commutativity*): *If $(\tilde{s}_1', \tilde{s}_2', \ldots, \tilde{s}_n')$ is a permutation of $(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n)$, then*

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \text{ULOWHM}_w(\tilde{s}_1', \tilde{s}_2', \ldots, \tilde{s}_n').$$

Proof (1) Since $\tilde{s}_j = \tilde{s}$, for all i, we have

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{w_1}{\tilde{r}_1} \oplus \frac{w_2}{\tilde{r}_2} \oplus \cdots \oplus \frac{w_n}{\tilde{r}_n}} = \frac{1}{\frac{w_1}{\tilde{s}} \oplus \frac{w_2}{\tilde{s}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}}} = \tilde{s}.$$

(2) Let $\max_j(\tilde{s}_j) = \tilde{s}_k$ and $\min_j(\tilde{s}_j) = \tilde{s}_l$, then

$$\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{w_1}{\tilde{r}_1} \oplus \frac{w_2}{\tilde{r}_2} \oplus \cdots \oplus \frac{w_n}{\tilde{r}_n}} \leq \frac{1}{\frac{w_1}{\tilde{s}_k} \oplus \frac{w_2}{\tilde{s}_k} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_k}} = \tilde{s}_k,$$

$$\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{w_1}{\tilde{r}_1} \oplus \frac{w_2}{\tilde{r}_2} \oplus \cdots \oplus \frac{w_n}{\tilde{r}_n}} \geq \frac{1}{\frac{w_1}{\tilde{s}_l} \oplus \frac{w_2}{\tilde{s}_l} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_l}} = \tilde{s}_l.$$

Hence

$$\min_j(\tilde{s}_j) \leq \mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \max_j(\tilde{s}_j).$$

(3) Since $\tilde{s}_j \leq \tilde{s}_j^*$, for all j, it follows that $\tilde{r}_j \leq \tilde{r}_j^*$, then

$$\begin{aligned}\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) &= \frac{1}{\frac{w_1}{\tilde{r}_1} \oplus \frac{w_2}{\tilde{r}_2} \oplus \cdots \oplus \frac{w_n}{\tilde{r}_n}} \\ &\leq \frac{1}{\frac{w_1}{\tilde{r}_1^*} \oplus \frac{w_2}{\tilde{r}_2^*} \oplus \cdots \oplus \frac{w_n}{\tilde{r}_n^*}} \\ &= \mathrm{ULOWHM}_w(\tilde{s}_1^*, \tilde{s}_2^*, \ldots, \tilde{s}_n^*).\end{aligned}$$

(4) Since $(\tilde{s}_1', \tilde{s}_2', \ldots, \tilde{s}_n')$ is a permutation of $(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n)$, we have $\tilde{r}_j = \tilde{r}_j'$, for all j, then

$$\begin{aligned}\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) &= \frac{1}{\frac{w_1}{\tilde{r}_1} \oplus \frac{w_2}{\tilde{r}_2} \oplus \cdots \oplus \frac{w_n}{\tilde{r}_n}} \\ &= \frac{1}{\frac{w_1}{\tilde{r}_1'} \oplus \frac{w_2}{\tilde{r}_2'} \oplus \cdots \oplus \frac{w_n}{\tilde{r}_n'}} \\ &= \mathrm{ULOWHM}_w(\tilde{s}_1', \tilde{s}_2', \ldots, \tilde{s}_n').\end{aligned}$$

Besides the above properties, the ULOWHM operator has the following desirable results.

Theorem 2 *Let $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n$ be a collection of uncertain linguistic variables and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the ULOWHM operator with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$; then we have the following.*

(1) *If $w = (1, 0, \ldots, 0)^T$, then*

$$\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \max_j(\tilde{s}_j).$$

(2) *If $w = (0, 0, \ldots, 1)^T$, then*

$$\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \min_j(\tilde{s}_j).$$

(3) *If* $w_j = 1$ *and* $w_i = 0$ $(i \neq j)$, *then*

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{r}_j,$$

where $\tilde{r}_j$ *is the* j*th largest of* $\tilde{s}_j$ $(j = 1, 2, \ldots, n)$.

Proof (1) Since $w = (1, 0, \ldots, 0)^T$, we have

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{1}{\tilde{r}_1} \oplus \frac{0}{\tilde{r}_2} \oplus \cdots \oplus \frac{0}{\tilde{r}_n}} = \tilde{r}_1 = \max_j(\tilde{s}_j).$$

(2) Since $w = (0, 0, \ldots, 1)^T$, we have

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{0}{\tilde{r}_1} \oplus \frac{0}{\tilde{r}_2} \oplus \cdots \oplus \frac{1}{\tilde{r}_n}} = \tilde{r}_n = \min_j(\tilde{s}_j).$$

(3) Since $w_j = 1$ and $w_i = 0$ $(i \neq j)$, we have

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{0}{\tilde{r}_1} \oplus \frac{0}{\tilde{r}_2} \oplus \cdots \oplus \frac{1}{\tilde{r}_j} \oplus \cdots \oplus \frac{0}{\tilde{r}_n}} = \tilde{r}_j.$$

Clearly, the fundamental characteristic of the ULWHM operator is that it considers the importance of each given uncertain linguistic variable, whereas the fundamental characteristic of the ULOWHM operator is the reordering step, and it weights all the ordered positions of uncertain linguistic variables instead of weighting the given uncertain linguistic variables themselves. In the following, by combining the advantages of the ULWHM and ULOWHM operators, we develop a ULHHM operator that weights both the given uncertain linguistic variables and their ordered positions.

Definition 9 An ULHHM operator of dimension n is a mapping ULHHM : $\tilde{S}^n \to \tilde{S}$, which has associated weighting vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^n w_j = 1$, such that

$$\text{ULHHM}_{\omega,w}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{w_1}{\tilde{r}_1} \oplus \frac{w_2}{\tilde{r}_2} \oplus \cdots \oplus \frac{w_n}{\tilde{r}_n}}, \tag{9}$$

where $\tilde{r}_j$ is the jth largest of the weighted uncertain linguistic variables $\dot{\tilde{s}}_i (\dot{\tilde{s}}_i = n\omega_i \tilde{s}_i, i = 1, 2, \ldots, n)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\tilde{s}_i (i = 1, 2, \ldots, n)$ with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, and n is the balancing coefficient.

Especially, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\tilde{s}}_i = \tilde{s}_i$, $i = 1, 2, \ldots, n$, in this case, the ULHHM operator is reduced to ULOWHM operator; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the ULHHM operator is reduced to ULWHM operator.

Example 4 Given the collection of uncertain linguistic variables: $\tilde{s}_1 = [s_2, s_3], \tilde{s}_2 = [s_1, s_3], \tilde{s}_3 = [s_2, s_4], \tilde{s}_4 = [s_3, s_4]$, and let $w = (0.3, 0.2, 0.3, 0.2)^T$ be the weight vector of $\tilde{s}_j (j = 1, 2, 3, 4)$. Then we get the weighted uncertain linguistic variables:

$$\dot{\tilde{s}}_1 = 4 \times 0.3 \times [s_2, s_3] = [s_{2.4}, s_{3.6}],$$
$$\dot{\tilde{s}}_2 = 4 \times 0.2 \times [s_1, s_3] = [s_{0.8}, s_{2.4}],$$
$$\dot{\tilde{s}}_3 = 4 \times 0.3 \times [s_2, s_4] = [s_{2.4}, s_{4.8}],$$
$$\dot{\tilde{s}}_4 = 4 \times 0.2 \times [s_3, s_4] = [s_{2.4}, s_{3.2}].$$

By using (1), we construct the following matrix:

$$\mathbf{P} = \begin{pmatrix} 0.500 & 1.000 & 0.333 & 0.600 \\ 0.000 & 0.500 & 0.000 & 0.000 \\ 0.667 & 1.000 & 0.500 & 0.750 \\ 0.400 & 1.000 & 0.250 & 0.500 \end{pmatrix}.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have

$$p_1 = 2.433, \quad p_2 = 0.500, \quad p_3 = 2.917, \quad p_4 = 2.150.$$

Then we rank the arguments $\tilde{s}_i (i = 1, 2, 3, 4)$ in descending order in accordance with the values of $p_i (i = 1, 2, 3, 4)$:

$$\tilde{r}_1 = \tilde{s}_3 = [s_2, s_4], \quad \tilde{r}_2 = \tilde{s}_1 = [s_2, s_3], \quad \tilde{r}_3 = \tilde{s}_4 = [s_3, s_4], \quad \tilde{r}_4 = \tilde{s}_2 = [s_1, s_3].$$

Suppose that the weighting vector $w = (w_1, w_2, w_3, w_4)^T$ of the ULHHM operator is $w = (0.3, 0.2, 0.3, 0.2)^T$, then by (9), we get

$$\begin{aligned} \text{ULHHM}_w(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4) &= \frac{1}{\frac{0.3}{\tilde{s}_3} \oplus \frac{0.2}{\tilde{s}_1} \oplus \frac{0.3}{\tilde{s}_4} \oplus \frac{0.2}{\tilde{s}_2}} \\ &= \frac{1}{\frac{0.3}{[s_2, s_4]} \oplus \frac{0.2}{[s_2, s_3]} \oplus \frac{0.3}{[s_3, s_4]} \oplus \frac{0.2}{[s_1, s_3]}} \\ &= [s_{1.82}, s_{3.53}]. \end{aligned}$$

4 An approach to multiple attribute group decision making

In this section, we consider a MAGDM problem, let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of n feasible alternatives and $G = \{G_1, G_2, \ldots, G_m\}$ be a set of m attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$, where $w_i \geq 0$ and $\sum_{i=1}^m w_i = 1$, and let $D = \{d_1, d_2, \ldots, d_l\}$ be the set of decision makers, whose weight vector is $v = (v_1, v_2, \ldots, v_l)^T$, where $v_k \geq 0$ and $\sum_{k=1}^l v_k = 1$. The decision maker $d_k \in D$ may provide the uncertain linguistic decision matrix $R_k = (\tilde{r}_{ij}^{(k)})_{m \times n}$, where $\tilde{r}_{ij}^{(k)}$ is an attribute value, which takes the form of uncertain linguistic variable, of the alternative $x_j \in X$ with respect to the attribute $G_i \in G$ for all $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, l$.

In the following, we shall develop an approach based on the ULWHM and ULHHM operators to MAGDM with uncertain linguistic preference relations.

Step 1. Utilize the ULWHM operator:

$$\tilde{r}_j^{(k)} = \text{ULWHM}_w\left(\tilde{r}_{1j}^{(k)}, \tilde{r}_{2j}^{(k)}, \ldots, \tilde{r}_{mj}^{(k)}\right) = \frac{1}{\frac{w_1}{\tilde{r}_{1j}^{(k)}} \oplus \frac{w_2}{\tilde{r}_{2j}^{(k)}} \oplus \cdots \oplus \frac{w_n}{\tilde{r}_{mj}^{(k)}}} \quad (10)$$

to aggregate all the elements in the jth column of R_k and get the overall attribute value $\tilde{r}_j^{(k)}$ of the alternative x_j corresponding to the decision maker d_k.

Step 2. Utilize the ULHHM operator:

$$\tilde{r}_j = \text{ULHHM}_\omega\left(\tilde{r}_j^{(1)}, \tilde{r}_j^{(2)}, \ldots, \tilde{r}_j^{(l)}\right) = \frac{1}{\frac{\omega_1}{\dot{\tilde{r}}_j^{\sigma(1)}} \oplus \frac{\omega_2}{\dot{\tilde{r}}_j^{\sigma(2)}} \oplus \cdots \oplus \frac{\omega_l}{\dot{\tilde{r}}_j^{\sigma(l)}}} \quad (11)$$

to aggregate the overall attribute values $\tilde{r}_j^{(k)}(k = 1, 2, \ldots l)$ corresponding to the decision maker $d_k(k = 1, 2, \ldots, l)$ and get the collective overall attribute value $\tilde{r}_j$, where $\dot{\tilde{r}}_j^{\sigma(k)}$ is the kth largest of the weighted data $\dot{\tilde{r}}_j^{(k)}(\dot{\tilde{r}}_j^{(k)} = lv_k\tilde{r}_j^{(k)}, k = 1, 2, \ldots, l)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_l)^T$ is the weighting vector of the ULHHM operator with $\omega_k \geq 0$ and $\sum_{k=1}^{l} \omega_k = 1$.

Step 3. Compare each $\tilde{r}_j$ with all $\tilde{r}_i(i = 1, 2, \ldots, n)$ by using (1), and let $p_{ij} = p(\tilde{r}_i \geq \tilde{r}_j)$, and then construct a possibility matrix $\mathbf{P} = (p_{ij})_{n\times n}$, where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5, i, j = 1, 2, \ldots, n$. Summing all elements in each line of matrix $\mathbf{P}$, we have $p_i = \sum_{j=1}^{n} p_{ij}, i = 1, 2, \ldots, n$, and then reorder $\tilde{r}_j(j = 1, 2, \ldots, n)$ in descending order in accordance with the values of $p_j(j = 1, 2, \ldots, n)$.

Step 4. Rank all the alternatives $x_j(j = 1, 2, \ldots n)$ by the ranking of $\tilde{r}_j(j = 1, 2, \ldots, n)$, and then select the most desirable one.

Step 5. End.

5 Illustrative example

In this section, we use a MAGDM problem of determining what kind of air-conditioning systems should be installed in a library (adapted from [21]) to illustrate the proposed approach.

Table 1 Decision matrix R_1

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_5, s_7]$	$[s_3, s_4]$	$[s_2, s_4]$	$[s_4, s_5]$	$[s_2, s_3]$
G_2	$[s_4, s_5]$	$[s_1, s_3]$	$[s_3, s_4]$	$[s_3, s_5]$	$[s_4, s_6]$
G_3	$[s_2, s_4]$	$[s_5, s_6]$	$[s_1, s_3]$	$[s_6, s_7]$	$[s_4, s_5]$
G_4	$[s_3, s_4]$	$[s_2, s_3]$	$[s_3, s_5]$	$[s_2, s_3]$	$[s_3, s_4]$

Table 2 Decision matrix R_2

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_3, s_5]$	$[s_4, s_5]$	$[s_1, s_2]$	$[s_3, s_5]$	$[s_1, s_3]$
G_2	$[s_2, s_4]$	$[s_2, s_3]$	$[s_3, s_5]$	$[s_2, s_4]$	$[s_4, s_5]$
G_3	$[s_1, s_2]$	$[s_2, s_3]$	$[s_1, s_2]$	$[s_2, s_4]$	$[s_5, s_6]$
G_4	$[s_3, s_5]$	$[s_4, s_6]$	$[s_2, s_3]$	$[s_1, s_3]$	$[s_4, s_6]$

Table 3 Decision matrix R_3

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_2, s_3]$	$[s_3, s_5]$	$[s_1, s_3]$	$[s_2, s_3]$	$[s_4, s_5]$
G_2	$[s_3, s_4]$	$[s_1, s_3]$	$[s_4, s_5]$	$[s_3, s_4]$	$[s_3, s_4]$
G_3	$[s_1, s_3]$	$[s_3, s_5]$	$[s_2, s_3]$	$[s_4, s_5]$	$[s_3, s_4]$
G_4	$[s_2, s_3]$	$[s_2, s_4]$	$[s_4, s_5]$	$[s_1, s_2]$	$[s_2, s_4]$

A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor offers five feasible alternatives, which might be adapted to the physical structure of the library. The offered air-conditioning system must take a decision according to the following four attributes:

(1) G_1 is performance.
(2) G_2 is maintainability.
(3) G_3 is flexibility.
(4) G_4 is safety.

The five possible alternatives $x_j (j = 1, 2, 3, 4, 5)$ are to be evaluated using the uncertain linguistic variables by three decision makers (whose weight vector is $v = (0.4, 0.3, 0.3)^T$) under the above four attributes (whose weight vector $w = (0.2, 0.1, 0.3, 0.4)^T$), and construct, respectively, the decision matrices $R_k = (\tilde{r}_{ij}^{(k)})_{5\times4} (k = 1, 2, 3)$ as listed in Tables 1, 2 and 3:

To get the best alternative(s), the following steps are involved:

Step 1. Utilize the ULWHM operator to aggregate all the elements in the jth column of R_k and get the overall attribute value $\tilde{r}_j^{(k)}$:

$$\tilde{r}_1^{(1)} = [s_{2.86}, s_{4.55}], \quad \tilde{r}_2^{(1)} = [s_{2.33}, s_{3.70}], \quad \tilde{r}_3^{(1)} = [s_{1.75}, s_{3.85}],$$
$$\tilde{r}_4^{(1)} = [s_{3.03}, s_{4.17}], \quad \tilde{r}_5^{(1)} = [s_{3.03}, s_{4.17}],$$

$$\tilde{r}_1^{(2)} = [s_{1.82}, s_{3.33}],\quad \tilde{r}_2^{(2)} = [s_{2.86}, s_{4.17}],\quad \tilde{r}_3^{(2)} = [s_{1.59}, s_{2.50}],$$
$$\tilde{r}_4^{(2)} = [s_{1.49}, s_{3.70}],\quad \tilde{r}_5^{(2)} = [s_{2.27}, s_{5.00}],$$
$$\tilde{r}_1^{(3)} = [s_{1.59}, s_{3.03}],\quad \tilde{r}_2^{(3)} = [s_{2.13}, s_{4.35}],\quad \tilde{r}_3^{(3)} = [s_{2.08}, s_{3.70}],$$
$$\tilde{r}_4^{(3)} = [s_{1.64}, s_{2.86}],\quad \tilde{r}_5^{(3)} = [s_{2.63}, s_{4.17}].$$

Step 2. Utilize the ULHHM operator (suppose that its weight vector is $\omega = (0.2, 0.5, 0.3)^T$) to aggregate the overall attribute values $\tilde{r}_j^{(k)}(k = 1, 2, 3)$ corresponding to the decision maker $d_k(k = 1, 2, 3)$, and get the collective overall attribute value $\tilde{r}_j$:

$$\tilde{r}_1 = [s_{1.75}, s_{3.23}],\quad \tilde{r}_2 = [s_{2.38}, s_{4.00}],\quad \tilde{r}_3 = [s_{1.75}, s_{3.03}],$$
$$\tilde{r}_4 = [s_{1.59}, s_{3.23}],\quad \tilde{r}_5 = [s_{2.33}, s_{4.35}].$$

Step 3. Compare each $\tilde{r}_j$ with $\tilde{r}_i(i = 1, 2, 3, 4, 5)$ by using (1), and let $p_{ij} = p(\tilde{r}_i \geq \tilde{r}_j)$, and then construct a possibility matrix:

$$\mathbf{P} = \begin{pmatrix} 0.500 & 0.274 & 0.563 & 0.526 & 0.257 \\ 0.726 & 0.500 & 0.776 & 0.739 & 0.459 \\ 0.464 & 0.224 & 0.500 & 0.493 & 0.212 \\ 0.474 & 0.261 & 0.507 & 0.500 & 0.246 \\ 0.743 & 0.541 & 0.788 & 0.754 & 0.500 \end{pmatrix}.$$

Summing all elements in each line of matrix **P**, we have

$$p_1 = 2.093,\quad p_2 = 3.200,\quad p_3 = 1.893,\quad p_4 = 1.988,\quad p_5 = 3.326$$

and then we rank $\tilde{r}_j(j = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of $p_j(j = 1, 2, 3, 4, 5)$:

$$\tilde{r}_5 > \tilde{r}_2 > \tilde{r}_1 > \tilde{r}_4 > \tilde{r}_3.$$

Step 4. Rank all alternatives $x_j(j = 1, 2, 3, 4, 5)$ by the ranking $\tilde{r}_j(j = 1, 2, 3, 4, 5)$:

$$x_5 \succ x_2 \succ x_1 \succ x_4 \succ x_3$$

and thus the most desirable alternative is x_5.

6 Comparisons the proposed method with other methods

In this section, we compare the proposed method with other methods. The methods to be compared here are the methods proposed by Xu [16,17], respectively.

Table 4 Comparison with other methods

	Xu [16]	Xu [17]	Proposed method
Problem type	MAGDM	GDM	MAGDM
Application area	Evaluating university faculty	Investment of money	Air-conditioning system selection
Decision information	Uncertain linguistic decision matrix	Uncertain multiplicative decision matrix	Uncertain linguistic decision matrix
Solution method			
Aggregation stage	ULWA operator	IULOWG operator	ULWHM operator
Exploitation stage	ULHA operator	ULOWG operator	ULHHM operator
Ranking stage	Complementary matrix	Complementary matrix	Possibility matrix
Final decision	Ranking of a number of alternatives	Ranking of a number of alternatives	Ranking of a number of alternatives

Each of methods has its advantages and disadvantages and none of them can always perform better than the others in any situations. It perfectly depends on how we look at things, and not on how they are themselves.

The method proposed by Xu [17] is suitable for solving group decision making with uncertain multiplicative linguistic preference relations because the ULOWG operator combines the uncertain multiplicative linguistic variables giving weights to the values in relation to their ordering position, diminishing the importance of extreme values by increasing the importance of central ones; whereas the proposed method in this paper and the method proposed by Xu [16] are suitable for solving MAGDM with uncertain linguistic information because the ULHA operator and ULHHM operator reflect the importance degrees of both the given uncertain linguistic variables and their ordered positions. Others of relative comparison with the methods, respectively, proposed by Xu [16,17] are shown in Table 4.

7 Conclusion and discussion

In this paper, we have developed some new aggregation operators including the ULWHM operator, the ULOWHM operator and the ULHHM operator. The ULOWHM operator, which is an extension of Chen et al.'s OWHM operator, can be used in situations where the input arguments are uncertain linguistic variables. The ULHHM operator generalizes both ULWHM operator and the ULOWHM operator, and reflects the importance degrees of both the given arguments and their ordered positions. Based on the ULWHM and ULHHM operators, we have proposed an approach to MAGDM with uncertain linguistic information. We have also applied the proposed approach to the problem of determining what kind of air-conditioning systems should be installed in the library. Furthermore, Wei and Yi [13] proposed some harmonic aggregation operators for aggregating triangular fuzzy linguistic information, such as the fuzzy linguistic weighted harmonic mean (FLWHM) operator, fuzzy linguistic ordered weighted harmonic mean (FLOWHM) operator and fuzzy linguistic hybrid harmonic mean (FLHHM) operator, and developed an approach to MAGDM with triangular fuzzy linguistic variables. From Definition 4, since an uncertain linguistic variable can be thought as an α-cut of triangular fuzzy linguistic variable, each triangular fuzzy linguistic variable is transform to an uncertain linguistic variable. Therefore, by Definition 4 and Remark 1, we can use the ULHM, ULWHM, ULOWHM, and ULHHM operators for aggregating triangular fuzzy linguistic information, and thus we can use the our approach for solving MAGDM problems with triangular fuzzy linguistic environment.

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Appendix A: Fuzzy linguistic harmonic mean aggregation operators [13]

A FLHHM operator is defined as follows:

$$\text{FLHHM}_{\omega,w}(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n) = \frac{1}{\frac{w_1}{\hat{r}_1} \oplus \frac{w_2}{\hat{r}_2} \oplus \cdots \oplus \frac{w_n}{\hat{r}_n}},$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the associated weighting vector such that $w_j \in [0, 1]$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$, and $\hat{r}_j$ is the jth largest element of the weighted triangular fuzzy linguistic variables $\dot{\hat{s}}_i$ ($\dot{\hat{s}}_i = \frac{\hat{s}_i}{n\omega_i}$, $i = 1, 2, \ldots, n$), $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\tilde{s}_i (i = 1, 2, \ldots, n)$ with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$, and n is the balancing coefficient, then the function FLHHM is called the FLHHM operator of dimension n.

Especially, if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then $\dot{\hat{s}}_i = \hat{s}_i$, $i = 1, 2, \ldots, n$, in this case, the FLHHM operator is reduced to the fuzzy linguistic ordered weighted harmonic mean (FLOWHM) operator; if $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the FLHHM operator is reduced to the fuzzy linguistic weighted harmonic mean (FLWHM) operator.

Appendix B: An approach to multiple attribute group decision making under triangular fuzzy linguistic environment [13]

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of n alternatives and $G = \{G_1, G_2, \ldots, G_m\}$ be a set of m attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$, where $w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$, and let $D = \{d_1, d_2, \ldots, d_l\}$ be the set of decision makers, whose weight vector is $v = (v_1, v_2, \ldots, v_l)^T$, where $v_k \geq 0$ and $\sum_{k=1}^{l} v_k = 1$. The decision maker $d_k \in D$ may provide the uncertain linguistic decision matrix $R_k = (\hat{r}_{ij}^{(k)})_{m \times n}$, where $\hat{r}_{ij}^{(k)}$ is an attribute value, which takes the form of triangular fuzzy linguistic variable, of the alternative $x_j \in X$ with respect to the attribute $G_i \in G$ for all $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, l$.

Step 1. Utilize the FLWHM operator:

$$\hat{r}_j^{(k)} = \text{FLWHM}_w \left(\hat{r}_{1j}^{(k)}, \hat{r}_{2j}^{(k)}, \ldots, \hat{r}_{mj}^{(k)}\right)$$

to aggregate all the elements in the jth column of R_k and get the overall attribute value $\hat{r}_j^{(k)}$ of the alternative x_j corresponding to the decision maker d_k.

Step 2. Utilize the FLHHM operator:

$$\hat{r}_j = \text{FLHHM}_{v,\omega}\left(\hat{r}_j^{(1)}, \hat{r}_j^{(2)}, \ldots, \hat{r}_j^{(l)}\right) = \frac{1}{\frac{\omega_1}{\dot{\hat{r}}_j^{\sigma(1)}} \oplus \frac{\omega_2}{\dot{\hat{r}}_j^{\sigma(2)}} \oplus \cdots \oplus \frac{\omega_l}{\dot{\hat{r}}_j^{\sigma(l)}}}$$

to aggregate the overall attribute values $\hat{r}_j^{(k)} (k = 1, 2, \ldots l)$ corresponding to the decision maker $d_k (k = 1, 2, \ldots, l)$ and get the collective overall attribute value $\hat{r}_j$, where $\dot{\hat{r}}_j^{\sigma(k)}$ is the kth largest of the weighted data $\dot{\hat{r}}_j^{(k)} \left(\dot{\hat{r}}_j^{(k)} = \frac{\hat{r}_j^{(k)}}{l v_k}, k = 1, 2, \ldots, l\right)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_l)^T$ is the weighting vector of the FLHHM operator with $\omega_k \geq 0$ and $\sum_{k=1}^{l} \omega_k = 1$.

Step 3. Compare each $\hat{r}_j$ with all $\hat{r}_i (i = 1, 2, \ldots, n)$ by using (3), and let $q_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct a possibility matrix $\mathbf{Q} = (q_{ij})_{n \times n}$, where $q_{ij} \geq 0, q_{ij} + q_{ji} = 1, q_{ii} = 0.5, i, j = 1, 2, \ldots, n$. Summing al elements in each line of matrix $\mathbf{Q}$, we have $q_i = \sum_{j=1}^{n} q_{ij}, i = 1, 2, \ldots, n$, and then reorder $\hat{r}_j (j = 1, 2, \ldots, n)$ in descending order in accordance with the values of $q_j (j = 1, 2, \ldots, n)$.

Step 4. Rank all the alternatives $x_j (j = 1, 2, \ldots n)$ by the ranking of $\hat{r}_j (j = 1, 2, \ldots n)$, and then select the most desirable one.

Step 5. End.

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