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Finite central difference/finite element approximations for parabolic integro-differential equations

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Abstract We study the numerical solution of an initial-boundary value problem for parabolic integro-differential equation with a weakly singular kernel. The main purpose of this paper is to construct and analyze stable and high order scheme to efficiently solve the integro-differential equation. The equation is discretized in time by the finite central difference and in space by the finite element method. We prove that the full discretization is unconditionally stable and the numerical solution converges to the exact one with order $O(\Delta t^2 + h^l)$. A numerical example demonstrates the theoretical results.

Keywords Parabolic integro-differential equation · Finite element method · Stability · Error estimate

Mathematics Subject Classification (2000) 65M06 · 65M12 · 65M22 · 65M60 · 35M13

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1 Introduction

Consider the partial integro-differential equation with a weakly singular kernel of the form

$$\int_0^t \beta(t-s)u_t(x,s)ds - u_{xx}(x,t) = f(x,t), \quad x \in \Lambda = (0,1), \quad 0 < t \le T, \tag{1.1}$$

along with the initial conditions

$$u(x,0) = \varphi(x), \quad 0 < x < 1, \tag{1.2}$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le T. \tag{1.3}$$

Here, $u_t = \partial u/\partial t$, the kernel $\beta(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $0 < \alpha < 1$, is a singular kernel at $t = 0$ and Γ denotes the gamma function.

The integro-differential equation of the above type often occurs in applications such as heat conduction in material with memory, compression of poroviscoelastic media, population dynamics, nuclear reactor dynamics, etc. The situation studied in detail in [1] was regarded as the intermediate model between the usual heat flow model (a parabolic type) and Gurtin-Piptin heat flow model (a problem of hyperbolic type). There are lots of documents of Thomée [3,7,19,27–30], Wahlbin [7,29,30], Brunner [5], Mclean [27–29], Lubich [19], Fairweather [9–13,16,34] and Lin [4], Sanz-serna [32], Yan [4,5], Chen [7], Xu [21–26], Tang [17], Sun [18], Lin [20], Zhang [35,36]. A lot of them use FEM; finite difference methods; spectral collocation methods; spline collocation methods, etc. (Much of the work published to date has been concerned with this fractional partial differential equations (see e.g. [6–8,13,14,20,33,34] for a non-exhaustive list of reference)). Sun [18] gives a fully discrete difference scheme for the fractional diffusion-wave equation and proves that the difference scheme is uniquely solvable, unconditionally stable and convergent in L_∞ norm. The convergence order is $O(\tau^{2-\alpha} + h^2)$, where $0 < \alpha < 1$. Lin [20] constructs and analyzes stable and high order scheme to efficiently solve the time-fractional diffusion equation, which an approach is based on a finite difference scheme in time and Legendre spectral methods in space; and prove that the full discretization is unconditionally stable, and the numerical solution converges to the exact one with order $O(\Delta t^{2-\alpha} + N^{-m})$, where $0 \le \alpha \le 1$. Herein, we construct and analyze stable and high order scheme to efficiently solve the integro-differential equation which is discretized in time by the finite difference and in space by the finite element method. We prove that the full discretization is unconditionally stable and the numerical solution converges to the exact one with order $O(\Delta t^2 + h^l)$. A numerical example demonstrates the theoretical results. Throughout this paper, we assume that $f(x,t)$ in (1.1) and u_0 in (1.2) is such that the problem (1.1)–(1.3) has a unique and sufficiently smooth solution in $[0, 1] \times [0, T]$. Furthermore, we suppose that u_t is continuous in $[0, 1] \times [0, T]$. We also assume that u_{tt} and u_{ttt} are continuous in $[0, 1] \times [0, T]$, and that there exists positive constants C_0 and C_1 such that for $x \in [0, 1]$, $t \in [0, T]$,

$$|u_{tt}(t, x)| \le C_0, \quad |u_{ttt}(t, x)| \le C_1. \tag{1.4}$$

Remark 1 If $f = 0$, (1.4) is not hold at the point $t = 0$, but the following formulation

$$|u_{tt}(t, x)| \le C_0 t^{-1/2}, \quad |u_{ttt}(t, x)| \le C_1 t^{-3/2}. \tag{1.5}$$

hold (see [17] for these assumptions).

Remark 2 For example: an exact analytical solution:

$$u(x, t) = t^2 \sin(2\pi x),$$

the corresponding forcing term and the initial condition are, respectively, $f(x, t) = \frac{2}{\Gamma(1+\alpha)} t^{1+\alpha} \sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x)$, $\varphi(x) = 0$, are satisfy with the problem (1.1)–(1.3). The problem (1.1) possesses a weakly kernel, but the analytical solution: $u(x, t) = t^2 \sin(2\pi x)$ is unique and sufficiently smooth. Furthermore, we can prove that the solution $u(x, t) = t^2 \sin(2\pi x)$ is satisfy with (1.4).

The remainder of the article is organized as follows: First, a finite difference scheme for temporal discretization of the problem is proposed in Sect. 2, where the stability and convergence analysis is given. A detailed error analysis is carried out for the semi-discrete problem, showing that the temporal accuracy is of 2-order. Second, Sect. 3 is concerned with the discretization in space by the FEM, error estimates are provided for the fully discrete problem. A numerical experiment is presented in Sect. 4 which support the theoretical results.

2 Discretization in time: a finite difference scheme

First, we introduce a finite difference approximation to discretize the time derivative. Let $t_k = k \cdot \Delta t$, where $\Delta t = \frac{T}{K}$ is the time step. To motivate the construction of the scheme, we set the first step of the difference $u_t = \frac{u(x,t_1)-u(x,t_0)}{\Delta t}$, then we use the following formulation: for all $0 \le k \le K - 1$,

$$\begin{aligned}
&\int_0^{t_{k+1}} \frac{(t_{k+1} - s)^{\alpha-1}}{\Gamma(\alpha)} \frac{\partial u(x, s)}{\partial s} ds \\
&= \int_{t_0}^{t_1} \frac{u(x, t_1) - u(x, t_0)}{\Delta t} \frac{(t_{k+1} - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\quad + \sum_{j=1}^{k} \int_{t_j}^{t_{j+1}} \frac{(t_{k+1} - s)^{\alpha-1}}{\Gamma(\alpha)} \frac{u(x, t_{j+1}) - u(x, t_{j-1})}{2\Delta t} ds + r_{\Delta t}^{k+1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{u(x,t_1)-u(x,t_0)}{\Gamma(\alpha)\cdot\Delta t}\int_{t_0}^{t_1}\frac{1}{(t_{k+1}-s)^{1-\alpha}}ds \\
&\quad + \sum_{j=1}^{k}\frac{u(x,t_{j+1})-u(x,t_{j-1})}{2\Delta t\cdot\Gamma(\alpha)}\int_{t_j}^{t_{j+1}}\frac{1}{(t_{k+1}-s)^{1-\alpha}}ds + r_{\Delta t}^{k+1} \\
&= \frac{u(x,t_1)-u(x,t_0)}{\alpha\Gamma(\alpha)\cdot\Delta t^{1-\alpha}}[(k+1)^\alpha - k^\alpha] \\
&\quad + \sum_{j=0}^{k-1}\frac{u(x,t_{k+1-j})-u(x,t_{k-j-1})}{2\alpha\Gamma(\alpha)\cdot\Delta t^{(1-\alpha)}}[(j+1)^\alpha - j^\alpha] + r_{\Delta t}^{k+1} \\
&= b_k\frac{u(x,t_1)-u(x,t_0)}{\Gamma(\alpha+1)\cdot\Delta t^{1-\alpha}} + \frac{1}{2\Gamma(\alpha+1)}\sum_{j=0}^{k-1} b_j\frac{u(x,t_{k+1-j})-u(x,t_{k-j-1})}{\Delta t^{(1-\alpha)}} + r_{\Delta t}^{k+1},
\end{aligned}
\tag{2.1}
$$

where the notations $b_j = (j+1)^\alpha - j^\alpha$, $j = 0, 1, \ldots, k$, $r_{\Delta t}^{k+1}$ is the truncation error. It can be verified that the truncation error takes the following form:

$$
\begin{aligned}
\left|r_{\Delta t}^{k+1}\right| &= \left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\frac{t_1+t_0-2s}{(t_{k+1}-s)^{1-\alpha}}\frac{u_{tt}(s,x)}{2}ds\right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_j}^{t_{j+1}}\frac{t_{j+1}+t_{j-1}-2s}{(t_{k+1}-s)^{1-\alpha}}\ \frac{u_{tt}(s,x)}{2}ds\right| \\
&\quad + \max_{\substack{0\le t\le T\\ 0\le x\le 1}}|u_{ttt}(s,x)|\, o(\Delta t^2) \\
&\le c_u\left\{\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\frac{t_1+t_0-2s}{(t_{k+1}-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_j}^{t_{j+1}}\frac{t_{j+1}+t_{j-1}-2s}{(t_{k+1}-s)^{1-\alpha}}ds + O(\Delta t^2)\right\},
\end{aligned}
\tag{2.2}
$$

where $c_u = c(\max\limits_{\substack{0\le t\le T\\ 0\le x\le 1}}|u_{tt}(t,x)| + \max\limits_{\substack{0\le t\le T\\ 0\le x\le 1}}|u_{ttt}(t,x)|)$ is a constant depending only on u. For

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\frac{t_1+t_0-2s}{(t_{k+1}-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_j}^{t_{j+1}}\frac{t_{j+1}+t_{j-1}-2s}{(t_{k+1}-s)^{1-\alpha}}ds \\
&\quad = \frac{\Delta t^{1+\alpha}}{\Gamma(\alpha)}\left\{\frac{-1}{\alpha}(-1-2k)[k^\alpha - (k+1)^\alpha] + \frac{-2}{1+\alpha}[k^{\alpha+1} - (k+1)^{\alpha+1}]\right\}
\end{aligned}
$$

$$+\frac{\Delta t^{1+\alpha}}{\Gamma(\alpha)}\sum_{j=1}^{k}\frac{-2}{\alpha}(j-k-1)[(k-j)^{\alpha}-(k+1-j)^{\alpha}]$$
$$+\frac{-2}{1+\alpha}\sum_{j=1}^{k}[(k-j)^{\alpha+1}-(k+1-j)^{\alpha+1}]$$
$$=\frac{\Delta t^{1+\alpha}}{\Gamma(\alpha+1)}\left\{k^{\alpha}+(k+1)^{\alpha}+\frac{2}{1+\alpha}k^{\alpha+1}-\frac{2}{1+\alpha}(k+1)^{\alpha+1}\right\}$$
$$+\frac{2\Delta t^{1+\alpha}}{\Gamma(\alpha+1)}\left[1^{\alpha}+2^{\alpha}+\cdots+(k-1)^{\alpha}-\frac{1}{1+\alpha}k^{\alpha+1}\right]$$
$$=\frac{\Delta t^{1+\alpha}}{\Gamma(\alpha+1)}\left[(k+1)^{\alpha}+k^{\alpha}\right.$$
$$\left.+2((k-1)^{\alpha}+(k-2)^{\alpha}+\cdots+1^{\alpha})-\frac{2}{1+\alpha}(k+1)^{\alpha+1}\right].$$

Let

$$M(k)=(k+1)^{\alpha}+k^{\alpha}+2((k-1)^{\alpha}+(k-2)^{\alpha}+\cdots+1^{\alpha})-\frac{2}{1+\alpha}(k+1)^{\alpha+1}.$$

We find that $M(k)$ is bounded for all $0<\alpha<1$ and all $k\geq 1$.

In order to prove $M(k)$ is bounded, at first, we set a formula $N(k)$:

$$N(k)=(k+1)^{\alpha}+2(k^{\alpha}+(k-1)^{\alpha}+(k-2)^{\alpha}+\cdots+1^{\alpha})-\frac{2}{1+\alpha}(k+1)^{\alpha+1}.$$

Then we get the following lemma:

Lemma 1 *For all* $0\leq\alpha\leq 1$ *and all* $K\geq 1$, *it holds*

$$|N(K)|\leq C,$$

where C *is a constant independent of* α, K.

Proof First, for $\alpha=1$, a direct computation shows $N(K)=0$ for all $K\geq 1$. Then we prove the lemma for $0\leq\alpha<1$. It can be certified that

$$N(K)=(K+1)^{\alpha}+2(K^{\alpha}+(K-1)^{\alpha}+(K-2)^{\alpha}+\cdots+1^{\alpha})$$
$$-\frac{2}{1+\alpha}(K+1)^{\alpha+1}=\sum_{k=0}^{K}a_k,$$

where

$$a_k=(k+1)^{\alpha}+k^{\alpha}-\frac{2}{1+\alpha}[(k+1)^{\alpha+1}-k^{\alpha+1}].$$

When $K=1$, $N(1)=2^{\alpha}+2-\frac{2}{1+\alpha}\cdot 2^{\alpha+1}$, we can get $|N(1)|\leq 1$.

Table 1 The value of $N(K)$

	$K = 10$	$K = 100$	$K = 1{,}000$	$K = 5{,}000$	$K = 10{,}000$
$\alpha = 0.01$	−0.9816	−0.9818	−0.9818	−0.9818	−0.9818
$\alpha = 0.2$	−0.6941	−0.6985	−0.6992	−0.6993	−0.6993
$\alpha = 0.5$	−0.3894	−0.4074	−0.4131	−0.4146	−0.4149
$\alpha = 0.8$	−0.1598	−0.1909	−0.2105	−0.2197	−0.2228
$\alpha = 0.99$	−0.0088	−0.0124	−0.0160	−0.0185	−0.0195

When $K \geq 2$, by using Taylor series expansion, we have

$$
\begin{aligned}
|a_k| &= k^\alpha \left| \left(1+\frac{1}{k}\right)^\alpha + 1 - \frac{2k}{1+\alpha}\left[\left(1+\frac{1}{k}\right)^{\alpha+1} - 1\right]\right| \\
&= k^\alpha \left| 1+\alpha\frac{1}{k} + \frac{\alpha(\alpha-1)}{2!}\frac{1}{k^2} + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\frac{1}{k^3} + \cdots + 1 - \frac{2k}{1+\alpha}\left[1+(1+\alpha)\frac{1}{k}\right.\right. \\
&\quad \left.\left. + \frac{(1+\alpha)\alpha}{2!}\frac{1}{k^2} + \frac{(1+\alpha)\alpha(\alpha-1)}{3!}\frac{1}{k^3} + \frac{(1+\alpha)\alpha(\alpha-1)(\alpha-2)}{4!}\frac{1}{k^4} + \cdots - 1\right]\right| \\
&= k^\alpha \left| \frac{\alpha(\alpha-1)}{k^2}\left[\left(\frac{1}{2!}-\frac{2}{3!}\right) + \frac{(\alpha-2)}{k}\left(\frac{1}{3!}-\frac{2}{4!}\right) + \frac{(\alpha-2)(\alpha-3)}{k^2}\left(\frac{1}{4!}-\frac{2}{5!}\right) + \cdots\right]\right| \\
&\leq \frac{\alpha(1-\alpha)}{k^{2-\alpha}\cdot 3!}\left| 1 + \frac{2(2-\alpha)}{4}\frac{1}{k} + \frac{3(2-\alpha)(3-\alpha)}{20k^2} + \cdots \right| \\
&\leq \frac{1}{3!}\alpha(1-\alpha)\frac{1}{k^{2-\alpha}}\left(1+\frac{1}{k}+\frac{1}{k^2}+\cdots\right) \leq \frac{2}{3!}\alpha(1-\alpha)\frac{1}{k^{2-\alpha}} \leq \frac{1}{k^{2-\alpha}}.
\end{aligned}
$$

It is well known that the series $\sum_{k=1}^{\infty}\frac{1}{k^p}$ converges for all $p > 1$. Here for $0 \leq \alpha < 1$, we have $2-\alpha > 1$, so that the series $\sum_{k=2}^{\infty}\frac{1}{k^{2-\alpha}}$ converges.

Therefore, $|N(K)|$ is bounded, the proof is completed. □

More precise bound of $N(K)$ can be obtained by numerical computations. Our numerical tests show that $-1 \leq N(k) \leq 0, \forall 0 \leq \alpha \leq 1, k = 1, 2, \ldots$. As can be seen clearly from Table 1 and Fig. 1, where the limit of $|N(K)|$ as K tends to infinity as a function of α is shown in Fig. 1.

Now we turn to consider $M(k)$. Obviously, $M(k) \leq N(K)$, then we can get $M(k)$ is bounded and $|M(K)| \leq 1$.

As a result, it holds

$$\left| r_{\Delta t}^{k+1} \right| \leq \frac{c_u}{\Gamma(\alpha+1)} \cdot \Delta t^{1+\alpha}. \tag{2.3}$$

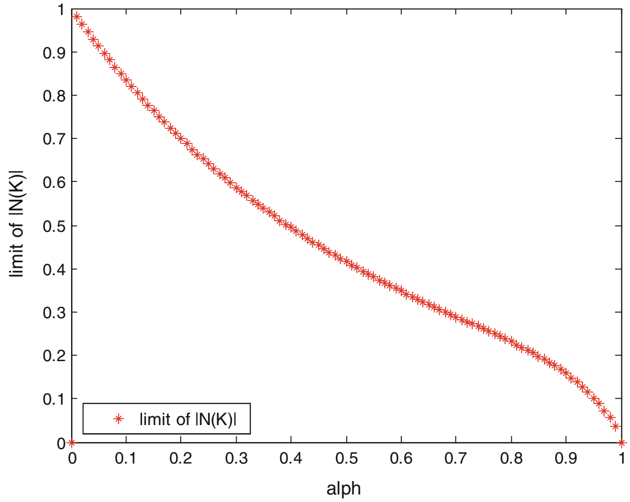

Fig. 1 Limit of $|N(K)|$ as a function of α

We define the discrete differential operator L_t^α by

$$L_t^\alpha u(x, t_{k+1}) := \frac{b_k}{\Gamma(\alpha+1)} \frac{u(x, t_1) - u(x, t_0)}{\Delta t^{1-\alpha}} + \frac{1}{2\Gamma(\alpha+1)} \sum_{j=0}^{k-1} b_j \frac{u(x, t_{k-j+1}) - u(x, t_{k-j-1})}{\Delta t^{1-\alpha}}.$$

Then (2.1) reads

$$\int_0^{t_{k+1}} \frac{(t_{k+1}-s)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{\partial u(x,s)}{\partial s} ds = L_t^\alpha u(x, t_{k+1}) + r_{\Delta t}^{k+1}.$$

Using $L_t^\alpha u(x, t_{k+1})$ as an approximation of $\int_0^{t_{k+1}} \frac{(t_{k+1}-s)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{\partial u(x,s)}{\partial s} ds$ leads to the following finite difference scheme to (1.1):

$$L_t^\alpha u(x, t_{k+1}) - u_{xx}(x, t_{k+1}) \cong f(x, t_{k+1}),$$

i.e.

$$\frac{b_k}{\Gamma(\alpha+1)} \frac{u(x, t_1) - u(x, t_0)}{\Delta t^{1-\alpha}} + \frac{1}{2\Gamma(\alpha+1)} \sum_{j=0}^{k-1} b_j \frac{u(x, t_{k+1-j}) - u(x, t_{k-j-1})}{\Delta t^{1-\alpha}} - \frac{\partial^2 u(x, t_{k+1})}{\partial x^2} \cong f(x, t_{k+1}). \tag{2.4}$$

Let us introduce the notations $u^{k+1}(x)$ is an approximation to $u(x, t_{k+1})$, scheme (2.4) can be rewritten into, with simplification by omitting the dependence of $u^{k+1}(x)$ on x:

$$b_0 u^{k+1} - 2\Gamma(\alpha+1)\Delta t^{1-\alpha}\frac{\partial^2 u^{k+1}}{\partial x^2} \\ = b_0 u^{k-1} - \sum_{j=1}^{k-1} b_j(u^{k+1-j} - u^{k-j-1}) - 2b_k(u^1 - u^0) + 2\Gamma(\alpha+1)\Delta t^{1-\alpha} f^{k+1}. \tag{2.5}$$

It is direct to check that

$$b_j > 0, \quad j = 0, 1, \ldots, k, \quad 1 = b_0 > b_1 > b_2 > \cdots > b_k, \quad b_k \to 0, \quad \text{as } k \to \infty.$$

Let us introduce the parameter $\alpha_0 : \alpha_0 = 2\Gamma(1+\alpha)\Delta t^{1-\alpha}$ and note that $b_0 = 1$, then by reformulating the right-hand side of (2.5), we obtain an equivalent form to scheme (2.5)

$$u^{k+1} - \alpha_0 \frac{\partial^2 u^{k+1}}{\partial x^2} = -b_1 u^k + \sum_{j=1}^{k-1}(b_{j-1} - b_{j+1})u^{k-j} - b_k u^1 \\ + (b_{k-1} + 2b_k)u^0 + \alpha_0 f^{k+1}, \quad k \geq 1 \tag{2.6}$$

Here again, when $k = 1$ scheme (2.6) becomes:

$$u^2 - \alpha_0 \frac{\partial^2 u^2}{\partial x^2} = -2b_1 u^1 + (1 + 2b_1)u^0 + \alpha_0 f^2.$$

For the special case $k = 0$, that is the first time step, the scheme simply reads

$$u^1 - \frac{1}{2}\alpha_0 \frac{\partial^2 u^1}{\partial x^2} = u^0 + \frac{1}{2}\alpha_0 f^1. \tag{2.7}$$

Equations (2.6) and (2.7), together with the boundary conditions

$$u^{k+1}(0) = u^{k+1}(1) = 0, \quad k \geq 0, \tag{2.8}$$

and the initial conditions

$$u^0(x) = \varphi(x), \quad 0 \leq x \leq 1, \tag{2.9}$$

form a complete set of the semi-discrete problem.

It will be useful to define the error term r^{k+1} by

$$r^{k+1} := \alpha_0 \left(\int_0^{t_{k+1}} \frac{(t_{k+1}-s)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{\partial u(x,s)}{\partial s} ds - L_t^{\alpha} u(x,t_{k+1}) \right) = \alpha_0 r_{\Delta t}^{k+1}. \tag{2.10}$$

Then we have from (2.1) and (2.3)

$$\left| r^{k+1} \right| = 2\Gamma(1+\alpha)\Delta t^{1-\alpha} \left| r_{\Delta t}^{k+1} \right| \le c_u \Delta t^2. \tag{2.11}$$

In order to introduce the variational formulation of the problem (2.6) and (2.7), we define some functional spaces endowed with standard norms and inner products that will be used hereafter.

$$H^1(\Lambda) := \left\{ v \in L^2(\Lambda), \frac{dv}{dx} \in L^2(\Lambda) \right\},$$
$$H_0^1(\Lambda) := \left\{ v \in H^1(\Lambda), v|_{\partial\Lambda} = 0 \right\},$$
$$H^m(\Lambda) := \left\{ v \in L^2(\Lambda), \frac{d^k v}{dx^k} \in L^2(\Lambda) \text{ for all positive integer } k \le m \right\},$$

where $L^2(\Lambda)$ is the space of measurable functions whose square is Lebesgue integrable in Λ. The inner products of $L^2(\Lambda)$ and $H^1(\Lambda)$ are defined, respectively, by

$$(u,v) = \int_\Lambda uv dx, \quad (u,v)_1 = (u,v) + \left(\frac{du}{dx}, \frac{dv}{dx} \right),$$

and the corresponding norms by

$$\|v\|_0 = (v,v)^{\frac{1}{2}}, \quad \|v\|_1 = (v,v)_1^{\frac{1}{2}}.$$

The norm $\|\cdot\|_m$of the space $H^m(\Lambda)$ is defined by

$$\|v\|_m = \left(\sum_{k=0}^m \left\| \frac{d^k v}{dx^k} \right\|_0^2 \right)^{\frac{1}{2}}.$$

In this paper, instead of using the above standard H^1-norm, we prefer to define $\|\cdot\|_1$by

$$\|v\|_1 = \left(\|v\|_0^2 + \alpha_0 \left\| \frac{du}{dx} \right\|_0^2 \right)^{\frac{1}{2}}. \tag{2.12}$$

It is well known that the standard H'-norm and the norm defined by (2.12) are equivalent; the latter will be used in what follows.

The variational (weak) formulation of Eq. (2.6) and (2.7) subject to the boundary condition (2.8) reads: find $u^{k+1} \in H_0^1(\Lambda)$, such that for $\forall v \in H_0^1(\Lambda)$:

$$(u^{k+1}, v)+\alpha_0\left(\frac{\partial u^{k+1}}{\partial x}, \frac{\partial v}{\partial x}\right) = -b_1(u^k, v)+\sum_{j=1}^{k-1}(b_{j-1}-b_{j+1})(u^{k-j}, v) - b_k(u^1, v) + (b_{k-1}+2b_k)(u^0, v)+\alpha_0(f^{k+1}, v), \quad k \geq 1, \tag{2.13a}$$

$$(u^1, v) + \frac{1}{2}\alpha_0\left(\frac{\partial u^1}{\partial x}, \frac{\partial v}{\partial x}\right) = (u^0, v) + \frac{1}{2}\alpha_0(f^1, v), \quad k = 0. \tag{2.13b}$$

We denote from now on by c a generic constant which may not be the same at different occurrences. In order to prove the following stability result, we introduce the Lemma 2.

Lemma 2 (Discrete Gronwall inequality) *If the sequences* $\{a_j\}$ *and* $\{z_j\}$, $j = 1, 2, \ldots, n$, *satisfy inequality*

$$z_j \leq \sum_{i=1}^{j-1} a_i z_i + b, \quad j = 1, 2, \ldots, n,$$

where $a_j \geq 0, b > 0$, *then the inequality*

$$z_j \leq b \cdot \exp\left(\sum_{i=1}^{j-1} a_i\right), \quad j = 1, 2, \ldots, n. \tag{2.14}$$

is true (*see* [31]).

For the weak semi-discrete problem, we have the following stability result.

Theorem 2.1 *The semi-discrete problem* (2.13) *is unconditionally stable in the sense that for all* $\Delta t > 0$, *it holds*

$$\left\|u^{k+1}\right\|_1 \leq c\left(\left\|u^0\right\|_0 + \alpha_0\left\|f^{k+1}\right\|_0\right), \quad k = 0, 1, \ldots, K-1. \tag{2.15}$$

Proof First when $k = 0$, we have

$$(u^1, v) + \frac{1}{2}\alpha_0\left(\frac{\partial u^1}{\partial x}, \frac{\partial v}{\partial x}\right) = (u^0, v) + \frac{1}{2}\alpha_0(f^1, v), \quad \forall v \in H_0^1(\Lambda).$$

Taking $v = u^1$ and using Schwarz inequality, we have

$$\left\|u^1\right\|_0^2 + \alpha_0\left\|\frac{\partial u^1}{\partial x}\right\|_0^2 \leq 2\left\|u^1\right\|_0^2 + \alpha_0\left\|\frac{\partial u^1}{\partial x}\right\|_0^2 \leq 2\left\|u^0\right\|_0\left\|u^1\right\|_0 + \alpha_0\left\|f^1\right\|_0\left\|u^1\right\|_0,$$

using the inequality $\|v\|_0 \le \|v\|_1$,we can get

$$\left\|u^1\right\|_1^2 \le 2\left\|u^0\right\|_0\left\|u^1\right\|_1 + \alpha_0\left\|f^1\right\|_0\left\|u^1\right\|_1,$$

obtain immediately

$$\left\|u^1\right\|_1 \le 2\left\|u^0\right\|_0 + \alpha_0\left\|f^1\right\|_0,$$

hence $\|u^1\|_1 \le c(\|u^0\|_0 + \alpha_0\|f^1\|_0)$.

For $0 < b_1 < b_0 = 1, 0 < 1 - b_1 < 1$, when $k \ge 1$, taking $v = u^{k+1}$, Eq. (2.13a) become the following form

$$\begin{aligned}&\left\|u^{k+1}\right\|_0^2 + \alpha_0\left\|\frac{du^{k+1}}{dx}\right\|_0^2\\&= (-b_1u^k, u^{k+1}) + \sum_{j=1}^{k-1}(b_{j-1} - b_{j+1})(u^{k-j}, u^{k+1}) - b_k(u^1, u^{k+1})\\&\quad +(b_{k-1} + 2b_k)(u^0, u^{k+1}) + \alpha_0(f^{k+1}, u^{k+1}).\end{aligned} \tag{2.16}$$

Using the inequality $\|v\|_0 \le \|v\|_1$ and Schwarz inequality, we obtain

$$\begin{aligned}&\left\|u^{k+1}\right\|_1^2\\&\le b_1\left\|u^k\right\|_0\left\|u^{k+1}\right\|_0 + \sum_{j=1}^{k-1}(b_{j-1} - b_{j+1})\left\|u^{k-j}\right\|_0\left\|u^{k+1}\right\|_0 + b_k\left\|u^1\right\|_0\left\|u^{k+1}\right\|_0\\&\quad +(b_{k-1} + 2b_k)\left\|u^0\right\|_0\left\|u^{k+1}\right\|_0 + \alpha_0\left\|f^{k+1}\right\|_0\left\|u^{k+1}\right\|_0\\&\le b_1\left\|u^k\right\|_1\left\|u^{k+1}\right\|_1 + \sum_{j=1}^{k-1}(b_{j-1} - b_{j+1})\left\|u^{k-j}\right\|_1\left\|u^{k+1}\right\|_1\\&\quad +b_k\left\|u^1\right\|_1\left\|u^{k+1}\right\|_1 + (b_{k-1} + 2b_k)\left\|u^0\right\|_0\left\|u^{k+1}\right\|_1 + \alpha_0\left\|f^{k+1}\right\|_0\left\|u^{k+1}\right\|_1,\end{aligned} \tag{2.17}$$

hence, (2.17) become the following form:

$$\begin{aligned}\left\|u^{k+1}\right\|_1 &\le b_1\left\|u^k\right\|_1 + \sum_{j=1}^{k-1}(b_{j-1} - b_{j+1})\left\|u^{k-j}\right\|_1 + b_k\left\|u^1\right\|_1\\&\quad +(2b_k + b_{k-1})\left\|u^0\right\|_0 + \alpha_0\left\|f^{k+1}\right\|_0\end{aligned}$$

by using (2.14), we have

$$\begin{aligned}\left\|u^{k+1}\right\|_1 &\leq \left((b_{k-1}+2b_k)\left\|u^0\right\|_0+\alpha_0\left\|f^{k+1}\right\|_0\right)\exp(b_1+(1-b_2)\\&\quad+\sum_{j=2}^{k-1}(b_{j-1}-b_{j+1})+b_k)\\&\leq\left(\left\|u^0\right\|_0+\alpha_0\left\|f^{k+1}\right\|_0\right)\exp(1+2b_1-b_{k-1})\\&\leq c\left(\left\|u^0\right\|_0+\alpha_0\left\|f^{k+1}\right\|_0\right),\quad(\forall k,k\to\infty,0<b_k<1,b_k\to 0).\end{aligned}$$

The proof is completed. □

Now we carry an error analysis for the solution of the semi-discrete problem.

Theorem 2.2 *Let u be the exact solution of (1.1)–(1.3), $\{u^k\}_{k=0}^K$ be the time-discrete solution of (2.6) and (2.7) with the initial condition $u^0(x) = u(x,0)$, then we have the following error estimates:*

$$\left\|u(t_k)-u^k\right\|_1 \leq c_u\Delta t^2,\quad k=1,2,\ldots,K. \tag{2.18}$$

Proof Let $e^k = u(x,t_k) - u^k(x)$, we have, for $k=1$, by combining (1.1), (2.7) and (2.10), the error equation

$$(e^1,v)+\frac{1}{2}\alpha_0\left(\frac{\partial e^1}{\partial x},\frac{\partial v}{\partial x}\right) = -b_1(e^0,v)+\left(\alpha_0 r^1_{\Delta t},v\right) = (r^1,v),\quad \forall v\in H_0^1(\Lambda). \tag{2.19}$$

Taking $v = e^1$, we have

$$\left\|e^1\right\|_0^2+\alpha_0\left\|\frac{\partial e^1}{\partial x}\right\|_0^2 \leq 2\left\|e^1\right\|_0^2+\alpha_0\left\|\frac{\partial e^1}{\partial x}\right\|_0^2 \leq 2\left\|r^1\right\|_0\left\|e^1\right\|_0,$$

and can get

$$\left\|e^1\right\|_1^2 \leq 2\left\|r^1\right\|_0\cdot\left\|e^1\right\|_0 \leq 2\left\|r^1\right\|_0\left\|e^1\right\|_1.$$

This, together with (2.11), we obtain

$$\left\|u(t_1)-u^1\right\|_1 \leq c_u\Delta t^2.$$

Therefore, (2.18) is proven for the case $k=1$.

For $k \geq 2$, by combining (1.1), (2.10) and (2.13), we derive

$$\begin{aligned}(e^{k+1}, v) + \alpha_0 \left(\frac{\partial e^{k+1}}{\partial x}, \frac{\partial v}{\partial x}\right) \\ = (-b_1 e^k, v) + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1})(e^{k-j}, v) - b_k (e^1, v) \\ +(2b_k + b_{k-1})(e^0, v) + (r^{k+1}, v), \quad \forall v \in H_0^1(\Lambda). \qquad (2.20)\end{aligned}$$

Let $v = e^{k+1}$ in (2.20), using the inequality $\|v\|_0 \leq \|v\|_1$ and Schwarz inequality, we obtain immediately

$$\begin{aligned}\left\|e^{k+1}\right\|_1^2 &\leq b_1 \left\|e^k\right\|_0 \left\|e^{k+1}\right\|_0 + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1}) \left\|e^{k-j}\right\|_0 \left\|e^{k+1}\right\|_0 + b_k \left\|e^1\right\|_0 \left\|e^{k+1}\right\|_0 \\ &\quad +(b_{k-1} + 2b_k) \left\|e^0\right\|_0 \left\|e^{k+1}\right\|_0 + \left\|r^{k+1}\right\|_0 \left\|e^{k+1}\right\|_0 \\ &\leq b_1 \left\|e^k\right\|_1 \left\|e^{k+1}\right\|_1 + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1}) \left\|e^{k-j}\right\|_1 \left\|e^{k+1}\right\|_1 + b_k \left\|e^1\right\|_1 \left\|e^{k+1}\right\|_1 \\ &\quad +(b_{k-1} + 2b_k) \left\|e^0\right\|_1 \left\|e^{k+1}\right\|_1 + \left\|r^{k+1}\right\|_0 \left\|e^{k+1}\right\|_1 .\end{aligned}$$

therefore, (2.20) become the following form

$$\left\|e^{k+1}\right\|_1 \leq b_1 \left\|e^k\right\|_1 + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1}) \left\|e^{k-j}\right\|_1 + b_k \left\|e^1\right\|_1 + \left\|r^{k+1}\right\|_0 .$$

By using (2.14) and (2.11), we thus obtain

$$\begin{aligned}\left\|e^{k+1}\right\|_1 &\leq \left\|r^{k+1}\right\|_0 \exp\left(b_1 + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1}) + b_k\right) \\ &\leq \left\|r^{k+1}\right\|_0 \exp(1 + 2b_1 - b_{k-1}) \\ &\leq c_u \Delta t^2.\end{aligned}$$

This proof is completed. □

3 Full discretization

In this section we shall consider the discretization in space of the initial-boundary value problem (1.1) by finite element method. Let thus $\{S_h\}$ be a family of finite-dimensional subspaces of $H_0^1(\Omega)$ with the approximation property $\inf_{x \in s_h} \{\|v - x\|_0 + h\|v - x\|_1\} \leq$

$ch^2\|v\|_2$, for $v \in H_0^1(\Omega)\cap H^2(\Omega)$, where $\|\cdot\|_2$ denotes the norm in $H^2(\Omega)$. The numerical solution is sought, for each $t \geq 0$, in a finite-dimensional space $S_h \subset H_0^1(\Lambda)$, depending on a small parameter h. In applications, h is typically the maximum diameter of a triangle in the triangulation underlying the definition of the finite element space S_h (cf., e.g. Ciarlet [2]).

Now we consider the finite element discretization to the (2.13) as follows: find $u_h^{k+1} \in S_h$, such that for all $v_h \in S_h$:

$$(u_h^{k+1}, v_h) + \alpha_0\left(\frac{\partial u_h^{k+1}}{\partial x}, \frac{\partial v_h}{\partial x}\right) = -b_1(u_h^k, v_h) + \sum_{j=1}^{k-1}(b_{j-1} - b_{j+1})(u_h^{k-j}, v_h)$$

$$-b_k(u_h^1, v_h) + (b_{k-1} + 2b_k)(u_h^0, v_h) + \alpha_0(f^{k+1}, v_h), \quad k \geq 1, \tag{3.1a}$$

$$(u_h^1, v_h) + \frac{1}{2}\alpha_0\left(\frac{\partial u_h^1}{\partial x}, \frac{\partial v_h}{\partial x}\right) = (u_h^0, v_h) + \frac{1}{2}\alpha_0(f^1, v_h), \quad k = 0. \tag{3.1b}$$

For the fully discrete problem, we have the following stability result.

Theorem 3.1 *The fully discrete problem* (3.1) *is unconditionally stable in the sense that for all* $\Delta t > 0$, *it holds*

$$\left\|u_h^{k+1}\right\|_1 \leq c\left\|u_h^0\right\|_0 + c\alpha_0\left\|f^{k+1}\right\|_0, \quad k = 0, 1, \ldots, K-1. \tag{3.2}$$

Proof The proof of Theorem 3.1 is similar to that for Theorem 2.1. So we omit the process of the proof here. □

Now we are interested in deriving error estimates for the fully discrete solution $\{u_h^k\}_{k=0}^K$.

Let R_h be the projection operator from $H_0^1(\Lambda)$ into S_h, that is, for all $\psi \in H_0^1(\Lambda)$, define $R_h\psi \in S_h$ such that

$$(R_h\psi, v_h) + \alpha_0\left(\frac{dR_h\psi}{dx}, \frac{dv_h}{dx}\right) = (\psi, v_h) + \alpha_0\left(\frac{d\psi}{dx}, \frac{dv_h}{dx}\right), \quad \forall v_h \in S_h. \tag{3.3}$$

In order to prove the following error estimates, we introduce the lemma.

Lemma 3 *Let* R_h *be the projection operator from* $H_0^1(\Lambda)$ *into* S_h, *satisfies* (3.3), *then the following projection estimate holds*:

$$\|\psi - R_h\psi\|_1 \leq ch^1\|\psi\|_2, \quad \textit{if } \psi \in H_0^1(\Lambda) \cap H^2(\Lambda).$$

Proof By the definition of R_h, we have

$$(R_h\psi - \psi, v_h) + \alpha_0\left(\frac{\partial}{\partial x}(R_h\psi - \psi), \frac{\partial}{\partial x}\psi_h\right) = 0. \tag{3.4}$$

For the purpose of the proof of Lemma 3, we introduce the L^2- projection operator $P_h : H_0^1(\Lambda) \to S_h$, and satisfy

$$(P_h\Psi - \Psi, v_h) = 0, \quad \forall v_h \in S_h, \quad \text{for} \quad \Psi \in H_0^1(\Lambda), \tag{3.5}$$

so that, we have the inequality holds, for $v \in H^2 \cap H_1^0$,

$$\|P_h v - v\|_1 \le ch \|v\|_2 . \tag{3.6}$$

(cf., e.g., Chen [37] pp.374 [11.1.4]) and hence by (3.4)

$$\begin{aligned}
&(R_h\Psi - P_h\Psi + P_h\Psi - \Psi, v_h) + \alpha_0 \left(\frac{\partial}{\partial x} \left[(R_h\Psi - P_h\Psi) + (P_h\Psi - \Psi) \right], \frac{\partial}{\partial x} v_h \right) \\
&= (R_h\Psi - P_h\Psi, v_h) + \alpha_0 \left(\frac{\partial}{\partial x}(R_h\Psi - P_h\Psi), \frac{\partial}{\partial x} v_h \right) \\
&\quad + \alpha_0 \left(\frac{\partial}{\partial x}(P_h\Psi - \Psi), \frac{\partial}{\partial x} v_h \right) \\
&= 0,
\end{aligned} \tag{3.7}$$

which is equivalent to

$$(R_h\Psi - P_h\Psi, v_h) + \alpha_0 \left(\frac{\partial}{\partial x}(R_h\Psi - P_h\Psi), \frac{\partial}{\partial x} v_h \right) = -\alpha_0 \left(\frac{\partial}{\partial x}(P_h\Psi - \Psi), \frac{\partial}{\partial x} v_h \right). \tag{3.8}$$

Taking $v_h = R_h\Psi - P_h\Psi$ into (3.8) results in

$$\begin{aligned}
&\|R_h\Psi - P_h\Psi\|_0^2 + \alpha_0 \left\| \frac{\partial}{\partial x}(R_h\Psi - P_h\Psi) \right\|_0^2 \\
&\le \alpha_0 \left\| \frac{\partial}{\partial x}(\Psi - P_h\Psi) \right\|_0 \left\| \frac{\partial}{\partial x}(R_h\Psi - P_h\Psi) \right\|_0 \\
&\le \alpha_0 \cdot ch \|\Psi\|_2 \left\| \frac{\partial}{\partial x}(R_h\Psi - P_h\Psi) \right\|_0 ,
\end{aligned} \tag{3.9}$$

here the first term of the left-side is nonnegative, we obtain

$$\alpha_0 \left\| \frac{\partial}{\partial x}(R_h\Psi - P_h\Psi) \right\|_0^2 \le \alpha_0 \cdot ch \|\Psi\|_2 \left\| \frac{\partial}{\partial x}(R_h\Psi - P_h\Psi) \right\|_0 ,$$

hence

$$\left\| \frac{\partial}{\partial x}(R_h\Psi - P_h\Psi) \right\|_0 \le ch \|\Psi\|_2 . \tag{3.10}$$

It is well known that Friedrichs' inequality holds

$$\|\Psi - R_h\Psi\|_1 \le c \left\| \frac{\partial}{\partial x}(\Psi - R_h\Psi) \right\|_0, \quad R_h\Psi \in S_h \subset H_0^1(\Omega). \tag{3.11}$$

(Friedrichs' lemma, see e.g. [2] or [15]) and hence, using (3.11) together with (3.10) and (3.6)

$$\begin{aligned} \|\Psi - R_h\Psi\|_1 &\le c \left\| \frac{\partial}{\partial x}(\Psi - R_h\Psi) \right\|_0 \\ &\le \left\| \frac{\partial}{\partial x}(\Psi - P_h\Psi) \right\|_0 + \left\| \frac{\partial}{\partial x}(P_h\Psi - R_h\Psi) \right\|_0 \\ &\le ch\,\|\Psi\|_2 + ch\,\|\Psi\|_2 \\ &= ch\,\|\Psi\|_2\,. \end{aligned}$$

Which is completed the proof of the Lemma 3. □

Theorem 3.2 *Let $\{u_h^k\}_{k=0}^K$ is the solution of the problem (3.1) with the initial condition u_h^0 taken to be $R_h u^0$, $\{u^k\}_{k=0}^K$ the solution of the problem (2.13) such that $u^k \in H^2(\Lambda) \cap H_0^1(\Lambda)$, then*

$$\left\| u^k - u_h^k \right\|_1 \le ch^1 \cdot \max_{0\le j\le k} \left\| u^j \right\|_2, \quad k = 1, 2, \ldots, K.$$

Proof By the definition of R_h, (3.3), we have, for the solution u^{k+1} of (2.13), for $\forall v_h \in S_h$

$$(R_h u^{k+1}, v_h) + \alpha_0 \left(\frac{\partial}{\partial x} R_h u^{k+1}, \frac{\partial v_h}{\partial x} \right) = -b_1(u^k, v_h) + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1})(u^{k-j}, v_h)$$
$$-b_k(u^1, v_h) + (b_{k-1} + 2b_k)(u^0, v_h) + \alpha_0(f^{k+1}, v_h), \quad k \ge 1, \tag{3.12a}$$
$$(R_h u^1, v_h) + \frac{1}{2}\alpha_0 \left(\frac{\partial}{\partial x} R_h u^1, \frac{\partial v_h}{\partial x} \right) = \left(u_h^0, v_h\right) + \frac{1}{2}\alpha_0(f^1, v_h), \quad k = 0. \tag{3.12b}$$

Let $\overset{=}{e}{}_h^{k+1} = R_h u^{k+1} - u_h^{k+1}$, $e_h^{k+1} = u^{k+1} - u_h^{k+1}$, by subtracting (3.1) from (3.12) we obtain

$$\begin{aligned} &\left(R_h u^{k+1} - u_h^{k+1}, v_h\right) + \alpha_0 \left(\frac{\partial}{\partial x}\left(R_h u^{k+1} - u_h^{k+1}\right), \frac{\partial}{\partial x} v_h \right) \\ &= \left(\overset{=}{e}{}_h^{k+1}, v_h\right) + \alpha_0 \left(\frac{\partial}{\partial x} \overset{=}{e}{}_h^{k+1}, \frac{\partial v_h}{\partial x} \right) \\ &= \left(-b_1(u^k - u_h^k), v_h\right) + \left(\sum_{j=1}^{k-1} (b_{j-1} - b_{j+1}) \left(u^{k-j} - u_h^{k-j}\right), v_h \right) \end{aligned}$$

$$-b_k\left((u^1-u_h^1),v_h\right)+\left((b_{k-1}+2b_k)\left(u^0-u_h^0\right),v_h\right)$$
$$=\left(-b_1e_h^k+\sum_{j=1}^{k-1}(b_{j-1}-b_{j+1})e_h^{k-j}-b_ke_h^1+(b_{k-1}+2b_k)e_h^0,v_h\right),\quad k\geq 1, \tag{3.13a}$$

$$(R_hu^1-u_h^1,v_h)+\frac{1}{2}\alpha_0\left(\frac{\partial}{\partial x}(R_hu^1-u_h^1),\frac{\partial}{\partial x}v_h\right)$$
$$=(\overset{=1}{e_h},v_h)+\frac{1}{2}\alpha_0\left(\frac{\partial}{\partial x}\overset{=1}{e_h},\frac{\partial v_h}{\partial x}\right)=(u^0-u_h^0,v_h),\quad k=0, \tag{3.13b}$$

Taking $v_h=\overset{=k+1}{e_h}$ in (3.13) results in

$$\left\|e_h^{k+1}\right\|_1\leq b_1\left\|e_h^k\right\|_1+\sum_{j=1}^{k-1}(b_{j-1}-b_{j+1})\left\|e_h^{k-j}\right\|_1-b_k\left\|e_h^1\right\|_1$$
$$+(b_{k-1}+2b_k)\left\|e_h^0\right\|_1,\quad k\geq 1, \tag{3.14a}$$
$$\left\|\overset{1}{e_h}\right\|_1\leq 2\left\|e_h^0\right\|_1,\quad k=0, \tag{3.14b}$$

then using the triangular inequality

$$\left\|e_h^{k+1}\right\|_1\leq\left\|\overset{=k+1}{e_h}\right\|_1+\left\|R_hu^{k+1}-u^{k+1}\right\|_1. \tag{3.15}$$

Now, by applying a similar argument as in Theorem 2.1 and using (3.15) together with (3.14) and Lemma 3, we obtain

$$\left\|e_h^{k+1}\right\|_1\leq b_1\left\|e_h^k\right\|_1+\sum_{j=1}^{k-1}(b_{j-1}-b_{j+1})\left\|e_h^{k-j}\right\|_1+b_k\left\|e_h^1\right\|_1+(b_{k-1}+2b_k)\left\|e_h^0\right\|_1$$
$$+\left\|R_hu^{k+1}-u^{k+1}\right\|_1\leq ch^1\left\|u^{k+1}\right\|_2.\exp(1+2b_1+2b_k)$$
$$\leq ch^1\max_{0\leq j\leq k+1}\left\|u^j\right\|_2,\quad k=1,2,\ldots,K-1. \tag{3.16a}$$
$$\left\|e_h^1\right\|_1\leq 2\left\|e_h^0\right\|_1+\left\|R_hu^1-u^1\right\|_1\leq ch^1\max_{0\leq j\leq 1}\left\|u^j\right\|_2,\quad k=0. \tag{3.16b}$$

This finished the proof of theorem 3.2. □

Now we aim at deriving an estimate for $\|u(t_k)-u_h^k\|_1$, which is given in the following theorem.

Theorem 3.3 *Let u be the exact solution of* (1.1)–(1.3)*, $\{u_h^k\}_{k=0}^K$ is the solution of the problem* (3.1) *with the initial condition $u_h^0=R_hu^0$, such that for all: $u\in$*

$H^1([0, T], H^2(\Lambda) \cap H_0^1(\Lambda))$, *then we have*

$$\left\| u(t_k) - u_h^k \right\|_1 \leq c_u \Delta t^2 + ch^1 \max_{0 \leq j \leq k} \left\| u^j \right\|_2, \quad k = 1, 2, \ldots, K. \tag{3.17}$$

Proof Since the proof follows a standard procedure as above, we omit the details by giving only the sketch.

From (2.10), $\{u(t_j)\}_{j=1}^K$ satisfy $\forall v \in H_0^1(\Lambda)$,

$$\begin{aligned} &(u(t_{k+1}), v) + \alpha_0 \left(\frac{\partial}{\partial x} u(t_{k+1}), \frac{\partial v}{\partial x} \right) \\ &\quad = (-b_1 u(t_k), v) + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1})(u(t_{k-j}), v) - b_k(u(t_1), v) \\ &\qquad + (b_{k-1} + 2b_k)\,(u(t_0), v) + \alpha_0(f^{k+1}, v) + (r^{k+1}, v), \quad k \geq 1, \end{aligned} \tag{3.18a}$$

$$\begin{aligned} &(u(t_1), v) + \frac{1}{2}\alpha_0 \left(\frac{\partial u(t_1)}{\partial x}, \frac{\partial v}{\partial x} \right) \\ &\quad = (u(t_0), v) + \frac{1}{2}\alpha_0(f^1, v) + (r^1, v), \quad k = 0. \end{aligned} \tag{3.18b}$$

By projection $u(t_{k+1})$ into $R_h u(t_{k+1}) \in S_h$, and using (3.3), we have for all $v_h \in S_h$,

$$\begin{aligned} &(R_h u(t_{k+1}), v_h) + \alpha_0 \left(\frac{\partial}{\partial x} R_h u(t_{k+1}), \frac{\partial v_h}{\partial x} \right) \\ &\quad = (-b_1 u(t_k), v_h) + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1})(u(t_{k-j}), v_h) - b_k(u(t_1), v_h) \\ &\qquad + (b_{k-1} + 2b_k)(u(t_0), v_h) + \alpha_0(f^{k+1}, v) + (r^{k+1}, v_h), \quad k \geq 1, \end{aligned} \tag{3.19a}$$

$$\begin{aligned} &(R_h u(t_1), v) + \frac{1}{2}\alpha_0 \left(\frac{\partial R_h u(t_1)}{\partial x}, \frac{\partial v}{\partial x} \right) \\ &\quad = (u(t_0), v) + \frac{1}{2}\alpha_0(f^1, v) + (r^1, v), \quad k = 0. \end{aligned} \tag{3.19b}$$

Let $\bar{\varepsilon}_h^{k+1} = R_h u(t_{k+1}) - u_h^{k+1}$, $\varepsilon_h^{k+1} = u(t_{k+1}) - u_h^{k+1}$, by subtracting (3.1) from (3.19), we obtain, for $\forall v_h \in S_h$,

$$\begin{aligned} &(\bar{\varepsilon}_h^{k+1}, v_h) + \alpha_0 \left(\frac{\partial}{\partial x} \bar{\varepsilon}_h^{k+1}, \frac{\partial v_h}{\partial x} \right) = (-b_1 \varepsilon_h^k, v_h) + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1})(\varepsilon_h^{k-j}, v_h) \\ &\quad - b_k(\varepsilon_h^1, v_h) + (b_{k-1} + 2b_k)\,(\varepsilon_h^0, v_h) + (r^{k+1}, v_h), \quad k \geq 1, \end{aligned} \tag{3.20a}$$

$$(\bar{\varepsilon}_h^1, v_h) + \frac{1}{2}\alpha_0 \left(\frac{\partial}{\partial x}\bar{\varepsilon}_h^1, \frac{\partial v_h}{\partial x} \right) = (\varepsilon_h^0, v_h) + (r^1, v_h), \quad k = 0. \tag{3.20b}$$

Taking $v_h = \bar{\varepsilon}_h^{k+1}$ in (3.20) and using the triangular inequality

$$\left\| \varepsilon_h^{k+1} \right\|_1 \leq \left\| \bar{\varepsilon}_h^{k+1} \right\|_1 + \| u(t_{k+1}) - P_h u(t_{k+1}) \|_1 ,$$

we have

$$\begin{aligned}
\left\| \varepsilon_h^{k+1} \right\|_1 &\leq b_1 \left\| \varepsilon_h^k \right\|_0 + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1}) \left\| \varepsilon_h^{k-j} \right\|_0 \\
&\quad + b_k \left\| \varepsilon_h^1 \right\|_0 + (b_{k-1} + 2b_k) \left\| \varepsilon_h^0 \right\|_0 + \left\| r^{k+1} \right\|_0 \\
&\quad + \| u(t_{k+1}) - R_h u(t_{k+1}) \|_1 \\
&\leq b_1 \left\| \varepsilon_h^k \right\|_0 + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1}) \left\| \varepsilon_h^{k-j} \right\|_0 + b_k \left\| \varepsilon_h^1 \right\|_0 + (b_{k-1} + 2b_k) \left\| \varepsilon_h^0 \right\|_0 \\
&\quad + c_u \Delta t^2 + \| u(t_{k+1}) - R_h u(t_{k+1}) \|_1 , \quad k \geq 1, \\
\left\| \varepsilon_h^1 \right\|_1 &\leq 2 \left\| \varepsilon_h^0 \right\|_0 + 2 \left\| r^1 \right\|_0 + \| u(t_1) - R_h u(t_1) \|_1 \\
&\leq 2 \left\| \varepsilon_h^0 \right\|_0 + c_u \Delta t^2 + \| u(t_1) - R_h u(t_1) \|_1 \quad k = 0.
\end{aligned}$$

Follow the same lines as in Theorem 2.1 to obtain

$$\begin{aligned}
\left\| \varepsilon_h^{k+1} \right\|_1 &\leq c \left\| u(t_0) - u_h^0 \right\|_0 + c_u \Delta t^2 + \| u(t_{k+1}) - R_h u(t_{k+1}) \|_1 \\
&\leq c \left\| u(t_0) - R_h u^0 \right\|_1 + \| u(t_{k+1}) - R_h u(t_{k+1}) \|_1 + c_u \Delta t^2 \\
&\leq c h^1 \max_{0 \leq j \leq k+1} \left\| u(t_j) \right\|_2 + c_u \Delta t^2, \quad k = 0, 1, 2, \ldots, K-1.
\end{aligned}$$

The proof of the theorem 3.3 is completed. □

In fact, it is also adapted to treat the equation in high-degree finite element methods, that is, the numerical solution converges to the exact one with order $O(\Delta t^2 + h^l)$, where $l \geq 1$ is the polynomial degree.

4 Numerical experiment

In this section, we describe briefly the computation and present some results to confirm our theoretical statements. The main purpose is to check the convergence behavior of the numerical solution with respect to the time step Δt used in the calculation. We compute the problem (1.1)–(1.3) with an exact analytical solution

$$u(x, t) = t^2 \sin(2\pi x),$$

Table 2 Temporal approximation order for several K

k	Error	Order
4	0.0002	
8	0.0008	2.0000
16	0.0033	2.0444
32	0.0130	1.9780
64	0.0518	1.9944
40	0.0002	
80	0.0008	2.0000
160	0.0031	1.9542
320	0.0124	2.0000
640	0.0493	1.9912
400	0.0002	
800	0.0007	1.8074
1,600	0.0029	2.0506
3,200	0.0117	2.0247
6,400	0.0466	1.9815

the corresponding forcing term and the initial condition are respectively

$$f(x,t) = \frac{2}{\Gamma(1+\alpha)} t^{1+\alpha} \sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x), \quad \varphi(x) = 0.$$

We take $t_k := k\Delta t, k = 0, 1, \ldots, K$, $\Delta t := \frac{T}{K}$ is the time step, set $T = 1$, $0 = x_1 < x_2 < \cdots < x_N = 1$, $h = 1/N$, $\Lambda = (0, 1)$. we set

$$u_h^{k+1} = \sum_{i=0}^{N} u_i^{k+1} \phi_i(x), \quad k = 0, 1, \ldots, K, \tag{4.1}$$

where $\phi(x)$ is the basic function:

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h}, & x_{j-1} \le x \le x_j \\ \frac{x_{j+1} - x}{h}, & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases},$$

$$\phi_0(x) = \begin{cases} \frac{x_1 - x}{h}, & x_0 \le x \le x_1 \\ 0, & \text{otherwise} \end{cases},$$

$$\phi_N(x) = \begin{cases} \frac{x - x_{N-1}}{h}, & x_{N-1} \le x \le x_N \\ 0, & \text{otherwise} \end{cases},$$

By bringing (4.1) into (3.1), and taking into account the homogeneous Dirichlet boundary condition (i.e. $u_0^{k+1} = u_N^{k+1} = 0$), choosing each function $v = \phi_j(x)$, $j = 1, 2, \ldots, N-1$ and using the definition of the inner product$(\cdot, \cdot)$, thus, we arrive at the following matrix statement of problem (3.1)

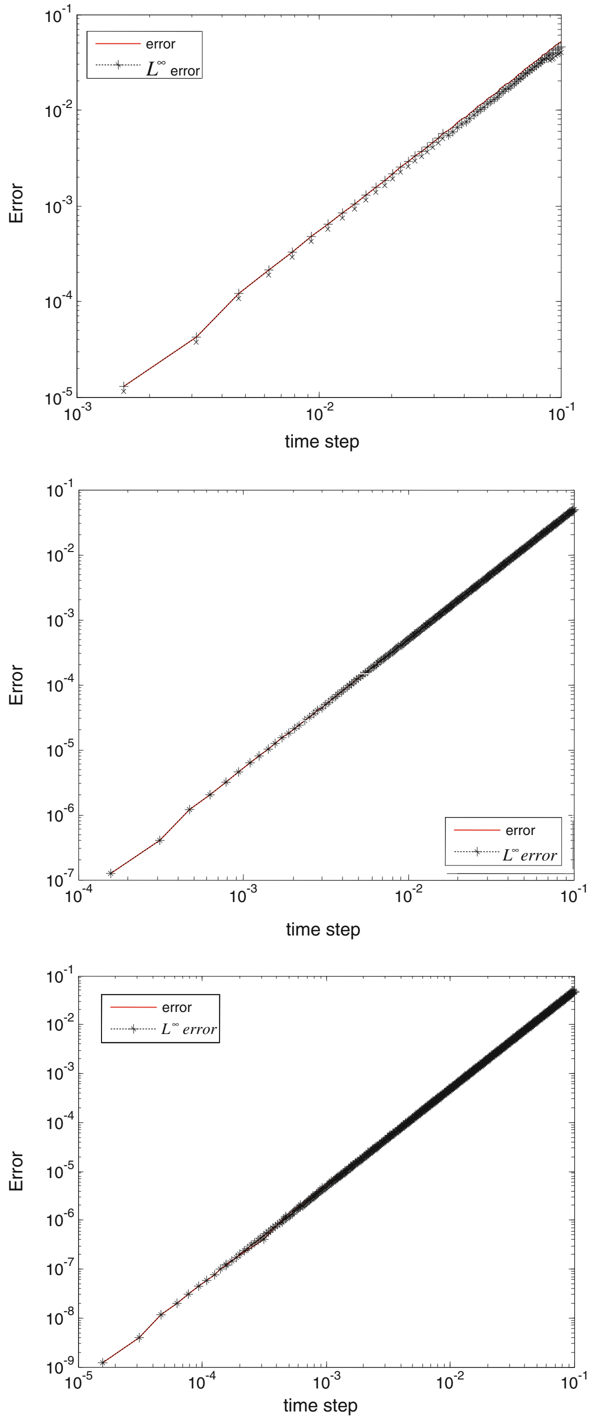

Fig. 2 Errors as a function of the time step Δt for several K

$$CU^{k+1} = F^{k+1}, \quad k \geq 1, \tag{4.2a}$$

$$\left(A + \frac{1}{2}\alpha_0 B\right) U^1 = \frac{1}{2}\alpha_0 \overline{F}^1, \quad k = 0, \tag{4.2b}$$

where for all $i, j = 1, 2, \ldots, N-1$,

$$U^{k+1} = \left(u_1^{k+1}, u_2^{k+1}, u_3^{k+1}, \ldots, u_{N-1}^{k+1}\right)^T, \quad C = A + \alpha_0 B, \quad A = (a_{ij}),$$

$$B = (b_{ij}), \quad a_{ij} = (\phi_i, \phi_j), \quad b_{ij} = \left(\frac{\partial \phi_i}{\partial x}, \frac{\partial \phi_j}{\partial x}\right),$$

$$F^{k+1} = -b_1 AU^k + \sum_{j=1}^{k-1} (b_{j-1} - b_{j+1}) AU^{k-j} - b_k AU^1$$

$$+ (b_{k-1} + 2b_k) AU^0 + \alpha_0 \overline{F}^{k+1}$$

$$\overline{F}^{k+1} = (f_1^{k+1}, f_2^{k+1}, \ldots, f_{N-1}^{k+1})^T, \quad f_j^{k+1} = (f^{k+1}, \phi_j).$$

We investigate the temporal convergence rate. In Table 2, we figure out the temporal convergence order according to the result of errors. In addition, In Fig. 2, we plot the errors and the errors in the L^∞ norms as a function of the time step sizes. A logarithmic scale has been used for both the time step Δt-axis and error-axis in these figures. From Table 2 and Fig. 2, we can see clearly that, as predicted by the theoretical estimates, the finite difference yields a temporal approximation order close to 2. So numerical experiments support the theoretical error estimates.

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