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Two effective measures of intuitionistic fuzzy entropy

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Abstract Based on the concept of fuzzy entropy, two effective measures of intuitionistic fuzzy entropy are proposed in intuitionistic fuzzy information, and then the essential properties of these measures are introduced. These measures are a generalized version of the fuzzy entropy and a complementarity of existing entropy for intuitionistic fuzzy sets. Based on this generalization, a connection between the concepts of the fuzzy entropy and the intuitionistic fuzzy entropy is established. Finally, a numeral example is given to show that the information measures of the proposed intuitionistic fuzzy entropy are reasonable and effective by the comparison of the proposed entropy and existing entropy.

Keywords Fuzzy set · Fuzzy entropy · Intuitionistic fuzzy set · Information measure · Intuitionistic fuzzy entropy

Mathematics Subject Classification (2000) 94 A17

1 Introduction

Entropy is a very important notion for measuring uncertain information. Fuzziness, a feature of imperfect information, results from the lack of crisp distinction between the elements belonging and not belonging to a set, i.e., the boundaries of the set under consideration are not sharply defined. A measure of fuzziness often used and cited in the literature is an entropy measure first mentioned in 1965 by Zadeh [16]. De Lica

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and Termini [5] proposed the axioms with which the fuzzy entropy should comply and defined the entropy of a fuzzy set based on Shannon's function. Kaufmann [7] proposed to measure the degree of fuzziness of any fuzzy set by a metric distance between its membership function and the membership function (characteristic function) of its nearest crisp set. Another way given by Yager [15] was to view the degree of fuzziness in terms of a lack of distinction between the fuzzy set and its complement. Kosko [8–10] investigated the fuzzy entropy in relation to a measure of subsethood (submessagehood) of one fuzzy set or message. Parkash et al. [11] have developed two new measures of weighted fuzzy entropy, the findings of which have been applied to study the principle of maximum weighted fuzzy entropy.

Atanassov [1,2] introduced the concept of intuitionistic fuzzy sets, which is a generalization of the concept of fuzzy sets. Later, Gau and Buehrer [6] introduced the concept of vague sets, which is another generalization of the concept of fuzzy sets. But Bustince and Burillo [4] point out that the notion of vague sets is the same as that of intuitionistic fuzzy sets. Burillo and Bustince [3] defined the entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets. Szmidt and Kacprzyk [12,13] proposed some entropy measures for intuitionistic fuzzy sets by employing a geometric interpretation of intuitionistic fuzzy sets, similarity measures of intuitionistic fuzzy sets, and some problems with entropy measures for the Atanassov's intuitionistic fuzzy sets. After that, Vlachos and Sergiagis [14] proposed another measure of intuitionistic fuzzy entropy and revealed an intuitive and mathematical connection between the notions of entropy for fuzzy sets and intuitionistic fuzzy sets. Furthermore, Zhang and Jiang [18] proposed non-probabilistic entropy of a vague set by means of the intersection and union of the membership degree and non-membership degree of the vague set.

Briefly motivated by the above-mentioned works, we will present two effective measures of intuitionistic fuzzy entropy, which are a generalized version of the fuzzy entropy in [11] and a complementarity of existing entropy for intuitionistic fuzzy sets. The rest of paper is organized as follows. In Sect. 2, some basic concepts related to fuzzy sets and intuitionistic fuzzy sets are briefly depicted. In Sect. 3, two new information measures of intuitionistic fuzzy entropy are proposed, and then to check their authenticity, the essential properties of these measures are studied. A numeral example is given to demonstrate the effectiveness by the comparison of the proposed entropy and existing entropy [12,14,18] in Sect. 4. Concluding remarks are drawn in Sect. 5.

2 Intuitionistic fuzzy sets

In this section we introduce some basic concepts related to fuzzy sets and intuitionistic fuzzy sets.

Definition 1 A fuzzy set $\tilde{A}$ defined on a universe of discourse X is given as Zadech [16]:

$$\tilde{A} = \left\{ \left\langle x, \mu_{\tilde{A}}(x) \right\rangle | x \in X \right\}, \tag{1}$$

where $\mu_{\tilde{A}} : X \to [0, 1]$ is the membership function of $\tilde{A}$. The membership value $\mu_{\tilde{A}}(x)$ describes the degree of belongingness of $x \in X$ in $\tilde{A}$.

Definition 2 An intuitionistic fuzzy set A defined on a universe of discourse X is given by Atanassov [1]:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \, | x \in X\}, \tag{2}$$

where $\mu_A(x) : X \to [0, 1]$ and $\nu_A(x) : X \to [0, 1]$, with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to the set A.

Definition 3 For each intuitionistic fuzzy set A in X, if

$$m_A(x) = 1 - \mu_A(x) - \nu_A(x), \quad x \in X, \tag{3}$$

then $m_A(x)$ is called the intuitionistic index of the element x in the set A [1]. It is a hesitancy degree of x to A. It is obvious that $0 \leq m_A(x) \leq 1, x \in X$.

Fuzzy sets can also be represented using the notion of intuitionistic fuzzy sets. A fuzzy set $\tilde{A}$ defined on X can be represented as the following intuitionistic fuzzy set:

$$A = \left\{\left\langle x, \mu_{\tilde{A}}(x), 1 - \mu_{\tilde{A}}(x)\right\rangle | x \in X\right\}, \tag{4}$$

with $m_A(x) = 0$ for $x \in X$.

The complementary set A^c of A is defined as

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \, | x \in X\}. \tag{5}$$

3 Two effective measures of intuitionistic fuzzy entropy

In the following, we propose two new measures of intuitionistic fuzzy entropy based on a generalization of the fuzzy information entropy in [11].

Let A be an intuitionistic fuzzy set in the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$. The intuitionistic fuzzy set can be transformed into a fuzzy set to structure an entropy measure of the intuitionistic fuzzy set by means of $\mu_{\tilde{A}}(x_i) = (\mu_A(x_i) + 1 - \nu_A(x_i))/2$. Then, according to the definition of fuzzy information entropy [11], two entropy measures of the intuitionistic fuzzy set A are proposed as follows:

$$\begin{aligned} E_1(A) = \frac{1}{n}\sum_{i=1}^{n}\Bigg\{\Bigg\{\sin\frac{\pi \times [1 + \mu_A(x_i) - \nu_A(x_i)]}{4} \\ + \sin\frac{\pi \times [1 - \mu_A(x_i) + \nu_A(x_i)]}{4} - 1\Bigg\} \times \frac{1}{\sqrt{2} - 1}\Bigg\}, \end{aligned} \tag{6}$$

$$\begin{aligned} E_2(A) = \frac{1}{n}\sum_{i=1}^{n}\Bigg\{\Bigg\{\cos\frac{\pi \times [1 + \mu_A(x_i) - \nu_A(x_i)]}{4} \\ + \cos\frac{\pi \times [1 - \mu_A(x_i) + \nu_A(x_i)]}{4} - 1\Bigg\} \times \frac{1}{\sqrt{2} - 1}\Bigg\}. \end{aligned} \tag{7}$$

Theorem 1 *The above measures of the intuitionistic fuzzy entropy satisfy the following axiomatic requirements* [12,14,18]:

(P1) $E_1(A) = E_2(A) = 0$ *(minimum), iff A is a crisp set*;
(P2) $E_1(A) = E_2(A) = 1$ *(maximum), iff* $\mu_A(x_i) = \nu_A(x_i)$ *for any* $x_i \in X$;
(P3) $E_1(A) \leq E_1(B)$ *and* $E_2(A) \leq E_2(B)$ *if A is less fuzzy than B, i.e.,* $\mu_A(x_i) \leq \mu_B(x_i)$ *and* $\nu_A(x_i) \geq \nu_B(x_i)$ *for* $\mu_B(x_i) \leq \nu_B(x_i)$ *or* $\mu_A(x_i) \geq \mu_B(x_i)$ *and* $\nu_A(x_i) \leq \nu_B(x_i)$ *for* $\mu_B(x_i) \geq \nu_B(x_i)$ *and any* $x_i \in X$;
(P4) $E_1(A) = E_1(A^c)$ *and* $E_2(A) = E_2(A^c)$.

Thus, $E_1(A)$ *and* $E_2(A)$ *are two measures of intuitionistic fuzzy entropy.*

Proof Since (6) and (7) are the summation of terms, Let us now consider $E_{1i}(A)$ and $E_{2i}(A)$:

$$E_{1i}(A) = \left\{\sin\frac{\pi \times [1 + \mu_A(x_i) - \nu_A(x_i)]}{4} + \sin\frac{\pi \times [1 - \mu_A(x_i) + \nu_A(x_i)]}{4} - 1\right\} \times \frac{1}{\sqrt{2} - 1}, \tag{8}$$

$$E_{2i}(A) = \left\{\cos\frac{\pi \times [1 + \mu_A(x_i) - \nu_A(x_i)]}{4} + \cos\frac{\pi \times [1 - \mu_A(x_i) + \nu_A(x_i)]}{4} - 1\right\} \times \frac{1}{\sqrt{2} - 1}. \tag{9}$$

Since for $x_i \in X = \{x_1, x_2, \ldots, x_n\}$, then we have $0 \leq E_{1i}(A) \leq 1$ and $0 \leq E_{2i}(A) \leq 1$.

(P1) When A is a crisp set, i.e., $\mu_A(x_i) = 0$, $\nu_A(x_i) = 1$ or $\mu_A(x_i) = 1$, $\nu_A(x_i) = 0$ for any $x_i \in X$. No matter in which case, we have $E_{1i}(A) = E_{2i}(A) = 0$. Hence, $E_1(A) = E_2(A) = 0$.

(P2) Let $\mu_A(x_i) = \nu_A(x_i)$ for $x_i \in X$. From (8) and (9) we obtain $E_{1i}(A) = E_{2i}(A) = 1$. Then, by applying (6) and (7), we get $E_1(A) = E_2(A) = 1$.

(P3) In order to show that (6) and (7) fulfill the requirement of (P3), they suffice to prove that the functions:

$$F_1(x, y) = \left\{\sin\frac{\pi \times [1 + x - y]}{4} + \sin\frac{\pi \times [1 - x + y]}{4} - 1\right\} \times \frac{1}{\sqrt{2} - 1}, \tag{10}$$

$$F_2(x, y) = \left\{\cos\frac{\pi \times [1 + x - y]}{4} + \cos\frac{\pi \times [1 - x + y]}{4} - 1\right\} \times \frac{1}{\sqrt{2} - 1}, \tag{11}$$

where $x, y \in [0, 1]$ and two functions F_1 and F_2 are increasing with respect to its first argument x and decreasing for y. Taking the partial derivative of F_1 and F_2 with respect to x and y, respectively, yields

$$\frac{\partial F_1(x, y)}{\partial x} = \frac{\pi}{4(\sqrt{2} - 1)}\left[\cos\frac{\pi(1 + x - y)}{4} - \cos\frac{\pi(1 - x + y)}{4}\right], \tag{12}$$

$$\frac{\partial F_1(x, y)}{\partial y} = \frac{\pi}{4(\sqrt{2} - 1)}\left[\cos\frac{\pi(1 - x + y)}{4} - \cos\frac{\pi(1 + x - y)}{4}\right], \tag{13}$$

$$\frac{\partial F_2(x,y)}{\partial x} = \frac{\pi}{4(\sqrt{2}-1)}\left[\sin\frac{\pi(1-x+y)}{4} - \sin\frac{\pi(1+x-y)}{4}\right], \tag{14}$$

$$\frac{\partial F_2(x,y)}{\partial y} = \frac{\pi}{4(\sqrt{2}-1)}\left[\sin\frac{\pi(1+x-y)}{4} - \sin\frac{\pi(1-x+y)}{4}\right]. \tag{15}$$

In order to find the critical point of F_1 and F_2, we set $\frac{\partial F_1(x,y)}{\partial x} = 0$, $\frac{\partial F_1(x,y)}{\partial y} = 0$, $\frac{\partial F_2(x,y)}{\partial x} = 0$, and $\frac{\partial F_2(x,y)}{\partial y} = 0$. By solving the critical point x_{cp}, we obtain

$$x_{cp} = y. \tag{16}$$

From (12), (14), and (16), we have

$$\frac{\partial F_1(x,y)}{\partial x} \geq 0 \quad \text{for } x \leq y \quad \text{and} \quad \frac{\partial F_1(x,y)}{\partial x} \leq 0 \quad \text{for } x \geq y, \tag{17}$$

$$\frac{\partial F_2(x,y)}{\partial x} \geq 0 \quad \text{for } x \leq y \quad \text{and} \quad \frac{\partial F_2(x,y)}{\partial x} \leq 0 \quad \text{for } x \geq y. \tag{18}$$

For any $x, y \in [0, 1]$, F_1 and F_2 are increasing with respect to x for $x \leq y$ and decreasing when $x \geq y$. Similarly, we obtain

$$\frac{\partial F_1(x,y)}{\partial y} \leq 0 \quad \text{for } x \leq y \quad \text{and} \quad \frac{\partial F_1(x,y)}{\partial y} \geq 0 \quad \text{for } x \geq y, \tag{19}$$

$$\frac{\partial F_2(x,y)}{\partial y} \leq 0 \quad \text{for } x \leq y \quad \text{and} \quad \frac{\partial F_2(x,y)}{\partial y} \geq 0 \quad \text{for } x \geq y. \tag{20}$$

Let us now consider (6) and (7) with $A \leq B$. Assume that the finite universe of discourse X is partitioned into two disjoint sets X_1 and X_2 with $X_1 \cup X_2 = X$. Let us further suppose that for $x_i \in X_1, \mu_A(x_i) \leq \mu_B(x_i) \leq \nu_B(x_i) \leq \nu_A(x_i)$, while for $x_i \in X_2, \mu_A(x_i) \geq \mu_B(x_i) \geq \nu_B(x_i) \geq \nu_A(x_i)$. Then, from the monotonicity of F_1, F_2, (6) and (7), we obtain that $E_1(A) \leq E_1(B)$ and $E_2(A) \leq E_2(B)$ when $A \leq B$.

From the proposed intuitionistic fuzzy entropy, the nearer $\mu_A(x_i)$ to $\nu_A(x_i)$ for $x_i \in X$, the greater the intuitionistic fuzzy entropy of $E_1(A)$ and $E_2(A)$, and then when $\mu_A(x_i)$ is equal to $\nu_A(x_i)$ for $x_i \in X$, the intuitionistic fuzzy entropy reach the maximum entropy, i.e., $E_1(A) = E_2(A) = 1$.

(P4) It is clear that $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}$ for $x_i \in X$, i.e., $\mu_{A^c}(x_i) = \nu_A(x_i)$ and $\nu_{A^c}(x_i) = \mu_A(x_i)$. By applying (8) and (9), we have $E_{1i}(A) = E_{1i}(A^c)$ and $E_{2i}(A) = E_{2i}(A^c)$. Hence, $E_1(A) = E_1(A^c)$ and $E_2(A) = E_2(A^c)$ are obtained. $\square$

Moreover, from the two measures of the intuitionistic fuzzy entropy, one can see that if an intuitionistic fuzzy set A is an ordinary fuzzy set, i.e., for any $x_i \in X$, $\mu_A(x_i) = 1 - \nu_A(x_i)$, then the intuitionistic fuzzy entropy becomes the fuzzy information entropy as proposed in [11].

4 Numeric example

In the following, a numerical example will demonstrate the effectiveness by the comparison of the proposed entropy and the entropy in [12,14,18].

Let A be an intuitionistic fuzzy set in the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$. Szmidt and Kacprzyk [12] introduced a measure of intuitionistic fuzzy entropy E_F. Vlachos and Sergiagis [14] suggested a measure of intuitionistic entropy E_{ln}. Zhang and Jiang [18] defined a measure of intuitionistic fuzzy entropy (vague entropy) H. Now we list these measures of corresponding entropy for intuitionistic fuzzy sets as follows:

$$E_F(A) = \frac{1}{n}\sum_{i=1}^{n} \frac{\max\ count(A(x_i) \cap A^c(x_i))}{\max\ count(A(x_i) \cup A^c(x_i))}, \tag{21}$$

$$\begin{aligned} E_{\ln} = -\frac{1}{n\ln 2}\sum [&\mu_A(x_i)\ln\mu_A(x_i) + \nu_A(x_i)\ln\nu_A(x_i) \\ &-(1-m_A(x_i))\ln(1-m_A(x_i)) - m_A(x_i)\ln 2], \end{aligned} \tag{22}$$

$$H(A) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_A(x_i) \cap \nu_A(x_i)}{\mu_A(x_i) \cup \nu_A(x_i)}. \tag{23}$$

Example Let us calculate the entropy for intuitionistic fuzzy sets:

$$A_1 = \langle 0.2, 0.5\rangle,\ A_2 = \langle 0.3, 0.5\rangle,\ \text{and}\ A_3 = \langle 0.5, 0.5\rangle.$$

Applying (6) and (7), we obtain

$$\begin{aligned} E_1(A_1) &= \left\{\sin\frac{\pi\times[1+\mu_A(x_i)-\nu_A(x_i)]}{4} + \sin\frac{\pi\times[1-\mu_A(x_i)+\nu_A(x_i)]}{4} - 1\right\} \\ &\quad\times\frac{1}{\sqrt{2}-1} \\ &= \left\{\sin\frac{3.1416\times[1+0.2-0.5]}{4} + \sin\frac{3.1416\times[1-0.2+0.5]}{4} - 1\right\} \\ &\quad\times\frac{1}{\sqrt{2}-1} = 0.9057, \\ E_2(A_1) &= \left\{\cos\frac{\pi\times[1+\mu_A(x_i)-\nu_A(x_i)]}{4} + \cos\frac{\pi\times[1-\mu_A(x_i)+\nu_A(x_i)]}{4} - 1\right\} \\ &\quad\times\frac{1}{\sqrt{2}-1} \\ &= \left\{\cos\frac{3.1416\times[1+0.2-0.5]}{4} + \cos\frac{3.1416\times[1-0.2+0.5]}{4} - 1\right\} \\ &\quad\times\frac{1}{\sqrt{2}-1} = 0.9057. \end{aligned}$$

Table 1 Comparison of the different entropy measures under A_1, A_2, and A_3

IFSs	E_1	E_2	E_F	H	E_{ln}
A_1	0.9057	0.9057	0.6250	0.4000	0.9042
A_2	0.9580	0.9580	0.7143	0.6000	0.9635
A_3	1.0000	1.0000	1.0000	1.0000	1.0000

Similarly, we obtain $E_1(A_2) = E_2(A_2) = 0.9580$ and $E_1(A_3) = E_2(A_3) = 1$. From these results, we can obtain the same values from (6) and (7), and then the nearer a membership degree to a non-membership degree, the greater a intuitionistic fuzzy entropy ($E_1(A_3) \geq E_1(A_2) \geq E_1(A_1)$ and $E_2(A_3) \geq E_2(A_2) \geq E_2(A_1)$). Especially, when the membership degree is equal to the non-membership degree, i.e., $A_3 = \langle 0.5, 0.5 \rangle$, the intuitionistic fuzzy entropy reaches the maximum entropy, i.e., $E_1(A_3) = E_2(A_3) = 1$.

We use these intuitionistic fuzzy sets (IFSs) to compare the entropy measures $E_1, E_2, E_F, H,$ and E_{ln}, respectively. The results are presented in Table 1.

From the results of Table 1, the measure values of the intuitionistic fuzzy entropy have the following order:

$$H(A_3) \geq H(A_2) \geq H(A_1).$$

According to the results in Table 1, we see that two measures of the intuitionistic fuzzy entropy demonstrate their effectiveness.

5 Concluding remarks

Though many information measures have been developed, still there is scope that better measures can be developed, which will find applications in a variety of fields. In this work, we have proposed two new information measures of entropy for intuitionistic fuzzy sets. These measures are a generalized version of the fuzzy entropy [11] and a complementarity of existing entropy for intuitionistic fuzzy sets. Based on this generalization, a connection between the concepts of the fuzzy entropy and the intuitionistic fuzzy entropy was established. A numerical example was given to show that two measures of the proposed entropy are reasonable and effective.

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