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# Analysis of an upwind finite-difference scheme for a system of coupled singularly perturbed convection-diffusion equations

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**Abstract**

We study a system of coupled convection-diffusion equations. The equations have diffusion parameters of different magnitudes associated with them which give rise to boundary layers at either boundary. An upwind finite difference scheme on arbitrary meshes is used to solve the system numerically. A general error estimate is derived that allows to immediately conclude robust convergence – w.r.t. the perturbation parameters – for certain layer-adapted meshes, thus improving and generalising previous results [4]. We present the results of numerical experiments to illustrate our theoretical findings.

*AMS Subject Classifications:* 65L10, 65L12, 65L60.

*Keywords:* Convection-diffusion, singular perturbation, layer-adapted mesh, systems of odes, derivative bounds.

## 1. Introduction

Interest in the numerical solution of convection-diffusion problems is currently at a high level. They may be regarded as linearized versions of the Navier-Stokes equations and provide an excellent paradigm for numerical techniques in computational fluid dynamics, see, e.g., [15], [18] for surveys. In this paper we study – inspired by a recent publication [4] – a simple upwind scheme for the following system of $\ell$ coupled singularly perturbed convection-diffusion equations: Find $\boldsymbol{u} = (u_1, \dots, u_\ell) \in \left(C^2(0,1) \cap C[0,1]\right)^\ell$ such that

$$\mathcal{L}_k u_k := -\varepsilon_k u_k'' + a_k u_k' + \sum_{m=1}^{\ell} b_{km} u_m = f_k \quad \text{in } (0,1), \quad u_k(0) = u_k(1) = 0, \qquad k = 1, \dots, \ell \quad (1)$$

(or short $\boldsymbol{\mathcal{L}u} = \boldsymbol{f}$), with small parameters $\varepsilon_k \in (0, 1]$, $k = 1, \dots, \ell$. The functions $a_k$, $b_{km}$ and $f_k$ are supposed to be continuous. Furthermore we shall assume that

$$\alpha_k := \min_{x \in [0,1]} |a_k(x)| > 0. \quad (2)$$

The solution of (1) exhibits layers, i.e., regions where the solution changes rapidly which causes problems in its accurate numerical treatment.

Cen [4] considers an upwind finite difference scheme on special layer-adapted piecewise-uniform meshes, so called Shishkin meshes, for a system of two equations of type (1). He shows that the error in the discrete maximum norm is bounded by $CN^{-1}\ln N$ with a constant $C$ that is independent of the perturbation parameters $\varepsilon_k$, and $N$ is the number of mesh points used.

The purpose of the present study is multifold. First, systems with an arbitrary number of equations are studied. Second, an analysis for arbitrary meshes is presented including standard layer-adapted meshes like the Shishkin mesh and the Bakhvalov mesh. Moreover the analysis is simpler than the one given in [4] and it requires less regularity of the solution. Last, but not least the assumptions on the $b_{km}$ will be weakend, see Sect. 2.

The paper is organized as follows: in Sect. 2, stability properties of the operator $\boldsymbol{\mathcal{L}}$ and the behaviour of the solution $\boldsymbol{u}$ of (1) are studied. In Sect. 3, we analyse the convergence properties of the upwind difference scheme. Finally, numerical results in Sect. 4 support our theory.

**Notation:** Throughout this paper we use $C$, sometimes subscripted, to denote a generic positive constant that is independent of both the perturbation parameters $\varepsilon_k$ and of $N$ the number of mesh intervals. We use $\|\cdot\|$ to denote the maximum norm, i.e.,

$$\|v\| := \max_{x\in[0,1]} \left|v(x)\right| \quad \text{and} \quad \|\boldsymbol{v}\| := \max_{k=1,\ldots,\ell} \|v_k\|$$

for vector-valued functions. By $\|\cdot\|_\omega$ we denote the analogous discrete maximum norms on a mesh $\omega : 0 = x_0 < x_1 < \ldots < x_N = 1$. For any function $g \in C[0,1]$ we set $g_i = g(x_i)$.

## 2. Properties of the exact solution

Our analysis is based on the following stability property for scalar equations which can be verified using standard maximum principle techniques [17].

**Lemma 1:** *Let $u \in C^2(0,1) \cap C[0,1]$ be the solution of the scalar equation*

$$-\varepsilon u'' + au' + bu = f \quad \text{in} \quad (0,1), \qquad u(0) = u(1) = 0,$$

*with $\varepsilon > 0$ and $b(x) > 0$, $x \in [0,1]$. Then $\|u\| \le \|f/b\|$.*

The solution of (1) can be written as

$$-\varepsilon_k u_k'' + a_k u_k' + b_{kk} u_k = f_k - \sum_{\substack{m=1\\ m\neq k}}^{\ell} b_{km} u_m \quad \text{in } (0,1), \qquad u_k(0) = u_k(1) = 0.$$

Assuming that

$$b_{kk}(x) > 0 \quad \text{for } x \in [0,1] \quad \text{and} \quad k = 1, \ldots, \ell, \tag{3}$$

we can apply Lemma 1 to obtain

$$\|u_k\| - \sum_{\substack{m=1 \\ m \neq k}}^{\ell} \left\| \frac{b_{km}}{b_{kk}} \right\| \|u_m\| \le \left\| \frac{f_k}{b_{kk}} \right\|, \quad k = 1, \ldots, \ell.$$

Define the matrix $\Gamma$ with entries

$$\gamma_{km} := -\left\| \frac{b_{km}}{b_{kk}} \right\| \quad \text{for } k \neq m \quad \text{and} \quad \gamma_{kk} = 1.$$

Then

$$\|u_k\| + \sum_{\substack{m=1 \\ m \neq k}}^{\ell} \gamma_{km} \|u_m\| \le \left\| \frac{f_k}{b_{kk}} \right\|, \quad k = 1, \ldots, \ell.$$

Assume that $\Gamma$ is inverse monotone, i.e.,

$$\Gamma^{-1} \ge 0. \tag{4}$$

Note that $\Gamma$ is an $L_0$-matrix, i.e., it has positive diagonal entries and non-negative offdiagonal ones. Therefore the $M$-matrix criterion can be used to verify whether $\Gamma$ is inverse monoton. We get the following stability result for (1).

**Theorem 1:** *Suppose* $\boldsymbol{u}$ *solves* (1). *Assume the* $b_{km}$ *satisfy* (3) *and* (4). *Then* $\|\boldsymbol{u}\| \le C \|\boldsymbol{f}\|$.

**Remark 1:**

*(i)* *Theorem 1 means that the operator* $\boldsymbol{\mathcal{L}}$ *is* $(L_\infty, L_\infty)$*-stable although (in general) it does* not *satisfy a maximum principle.*

*(ii)* *The proof does not make use of the assumption* (2) *on the* $a_k$. *Thus it is valid for arbitrary* $a_k$. *The case of identically vanishing* $a_k$*'s is studied in [12].*

*(iii)* *In [4] it was assumed for a system of two equations the matrix* $\boldsymbol{B} = (b_{km})$ *is an* $L_0$*-matrix with*

$$\min\{b_{11}(x) + b_{12}(x), b_{21}(x) + b_{22}(x)\} > \beta > 0 \quad \textit{for } x \in [0,1].$$

*This implies* $\Gamma$ *is a strictly diagonally dominant* $L_0$*-matrix. Application of the* $M$*-matrix criterion with the test vector* $(1,1)^T$ *verifies* (4). *Therefore the present analysis is more general than the one in [4].*

By Theorem 1 the solution of (1) is bounded uniformly with respect to the perturbation parameters $\varepsilon_k$. However, for the error analysis in Sect. 3 bounds for the first-order derivatives of $\boldsymbol{u}$ are also required. These will be derived now.

**Theorem 2:** *Let* $\boldsymbol{u}$ *be the solution of* (1)*. Suppose* (2)*,* (3) *and* (4) *hold true. Then, for* $x \in [0,1]$ *and* $n = 0, 1$*,*

$$\left|u_k^{(n)}(x)\right| \le \begin{cases} C\left[1+\varepsilon_k^{-n}\exp\left(-\frac{\alpha_k(1-x)}{\varepsilon_k}\right)\right] & \text{if } a_k \ge \alpha_k, \\ C\left[1+\varepsilon_k^{-n}\exp\left(-\frac{\alpha_k x}{\varepsilon_k}\right)\right] & \text{if } a_k \le -\alpha_k. \end{cases}$$

*Proof:* A single equation of (1) can be written as

$$-\varepsilon_k u_k'' + a_k u_k' = g_k := f_k - \sum_{m=1}^{\ell} b_{km} u_m \quad \text{in } (0,1), \quad u_k(0) = u_k(1) = 0.$$

By Theorem 1 we have $g_k \le C$ and application of the technique from [5] (see also Lemma I.1.6 in [18]) yields the desired bounds. □

**Remark 2:** *The analysis in [4] for a system of two equations with negative* $a_k$ *provides the bounds*

$$\left|u_1'(x)\right| \le C\left[1+\varepsilon_1^{-1}\exp\left(-\frac{\alpha_1 x}{\varepsilon_1}\right)+\varepsilon_2^{-1}\exp\left(-\frac{\alpha_2 x}{\varepsilon_2}\right)\right]$$

*and*

$$\left|u_2'(x)\right| \le C\left[1+\varepsilon_2^{-1}\exp\left(-\frac{\alpha_2 x}{\varepsilon_2}\right)\right]$$

*if* $\varepsilon_1/\alpha_1 \le \varepsilon_2/\alpha_2$*. These bounds suggest a strong impact of the second equation on the first, however the sharper bounds of Theorem 2 show that there can only be a weak interaction.*

*This is in sharp contrast to systems of reaction-diffusion equations, i.e.,* $a_k \equiv 0$ *in* (1)*, where strong interactions between the various equations are observed, see* [12], [13].

## 3. Discretization

We discretize (1) by means of the simple upwind scheme:

Find $\boldsymbol{U} = (U_1, \dots, U_\ell)$, $U_k \in \mathbb{R}_0^{N+1} := \left\{v \in \mathbb{R}^{N+1} : v_0 = v_n = 0\right\}$ such that

$$\left.\begin{aligned} [L_k \boldsymbol{U}]_i := [\Lambda_k U_k]_i + \sum_{\substack{m=1\\ m\ne k}}^{\ell} b_{km;i} U_{m;i} = f_{k;i}, \quad i = 1, \dots, N-1, \\ U_{k;0} = U_{k;N} = 0, \end{aligned}\right\} \quad k = 1, \dots, \ell, \tag{5}$$

where

$$[\Lambda_k v]_i := \begin{cases} -\varepsilon_k v_{x\bar{x};i} + a_{k;i} v_{\bar{x};i} + b_{kk;i} v_i & \text{if } a_k \text{ is positive,} \\ -\varepsilon_k v_{\bar{x}x;i} + a_{k;i} v_{x;i} + b_{kk;i} v_i & \text{if } a_k \text{ is negative} \end{cases}$$

and

$$v_{x;i} := \frac{v_{i+1} - v_i}{h_{i+1}}, \quad v_{\bar{x};i} := \frac{v_i - v_{i-1}}{h_i} \quad \text{and} \quad h_i := x_i - x_{i-1}.$$

We have the following discrete counterpart of Lemma 1.

**Lemma 2:** *Suppose* (3) *holds. Fix* $k \in \{1, \ldots, \ell\}$. *Then*

$$\|v\|_\omega \le \left\| \frac{\Lambda_k v}{b_{kk}} \right\|_\omega \quad \textit{for any} \quad v \in \mathbb{R}_0^{N+1}.$$

*Proof:* Define the operator $M$ by $[Mv]_i := [\Lambda_k v]_i / b_{kk;i}$. The matrix associated with $M$ is an $L_0$-matrix with row sum 1. The proposition of the lemma follows. □

The error $\boldsymbol{e} := \boldsymbol{U} - \boldsymbol{u}$ is split into two parts $\boldsymbol{e} = \boldsymbol{\eta} + \boldsymbol{\psi}$ with

$$\left[\Lambda_k \eta_k\right]_i = [L_k(\boldsymbol{U} - \boldsymbol{u})]_i, \quad i = 1, \ldots, N-1, \quad \eta_{k,0} = \eta_{k,N} = 0, \quad i = 1, \ldots, \ell \tag{6}$$

and

$$\left[\Lambda_k \psi_k\right]_i = -\sum_{\substack{m=1 \\ m \ne k}}^{\ell} b_{km;i} e_{m,i}, \quad i = 1, \ldots, N-1, \quad \psi_{k,0} = \psi_{k,N} = 0, \quad k = 1, \ldots, \ell.$$

A triangle inequality and Lemma 2 yield

$$\|e_k\|_\omega \le \|\eta_k\|_\omega + \|\psi_k\|_\omega \le \|\eta_k\|_\omega + \sum_{\substack{m=1 \\ m \ne k}}^{\ell} \left\| \frac{b_{km}}{b_{kk}} \right\|_\omega \|e_m\|_\omega, \quad k = 1, \ldots, \ell.$$

Assuming that (4) holds true, we obtain

$$\|\boldsymbol{U} - \boldsymbol{u}\|_\omega \le C \, \|\boldsymbol{\eta}\|_\omega, \tag{7}$$

and we are left with bounding the $\eta_k$'s.

For the mere sake of simplicity in the presentation we study only the case when $a_k$ is negative. The case when $a_k$ is positive is dealt with analogously. We adapt the technique from [1], [2]; see also the monograph [11]. Subsequently, assume that $a_k, b_{km}, f_k \in C^1[0, 1]$, for $k, m = 1, \ldots, \ell$.

We start from the stability inequality

$$\|v\|_\omega \le C \min_{V: V_x = \Lambda_k v} \|V\|_\omega \quad \text{for all } v \in \mathbb{R}_0^{N+1}, \quad \text{see [2].} \tag{8}$$

Introduce the continuous and discrete operators and functions

$$\mathcal{A}_k \boldsymbol{v} := \varepsilon_k v_k' - a_k v_k - \int_{\cdot}^{1} \left(a_k' v_k\right)(s) ds + \int_{\cdot}^{1} \sum_{m=1}^{\ell} \left(b_{km} v_m\right)(s) ds, \quad \mathcal{F}_k := \int_{\cdot}^{1} f_k(s) ds$$

and

$$A_k \boldsymbol{v} := \varepsilon_k v_{\bar{x}} - a_k v_k - \sum_{j=\cdot}^{N-1} h_{j+1} a_{x;j} v_{k;j+1} + \sum_{j=\cdot}^{N-1} h_{j+1} \sum_{m=1}^{\ell} b_{km;j} v_{m;j},$$

$$F_k := \sum_{k=\cdot}^{N-1} h_{j+1} f_{k;j}.$$

Note that $\mathcal{L}_k \boldsymbol{v} = -(\mathcal{A}_k \boldsymbol{v})'$ and $f_k = -\mathcal{F}_k'$ on $(0, 1)$, and $L_k \boldsymbol{v} = -(A_k \boldsymbol{v})_x$ and $f_k = -F_{k,x}$ on $\omega$. Thus

$$\mathcal{A}_k \boldsymbol{u} - \mathcal{F}_k \equiv \alpha \quad \text{on} \quad (0, 1) \quad \text{and} \quad A_k \boldsymbol{U} - F_k \equiv a \quad \text{on} \quad \omega \tag{9}$$

with constants $\alpha$ and $a$.

Applying the stability inequality (8) to (6), we have

$$\|\eta_k\|_\omega \le C \min_{c \in \mathbb{R}} \|A_k(\boldsymbol{u} - \boldsymbol{U}) + c\|_\omega .$$

Taking $c = a - \alpha$, where $a$ and $\alpha$ are the constants from (9), we get

$$\|\eta_k\|_\omega \le C \, \|A_k \boldsymbol{u} - \mathcal{A}_k \boldsymbol{u} - F_k + \mathcal{F}_k\|_\omega . \tag{10}$$

Furthermore

$$(A_k \boldsymbol{u} - \mathcal{A}_k \boldsymbol{u} - F_k + \mathcal{F}_k)_i = \varepsilon \left(u_{k_1 \bar{x}} - u_k'\right)_i + \int_{x_i}^{1} g_k(s) ds - \sum_{j=i}^{N-1} h_{j+1} g_{k;j}$$
$$+ \int_{x_i}^{x_N} \left(a_k' u_k\right)(s) ds - \sum_{j=i}^{N-1} h_{j+1} a_{k,x;j} u_{k;j+1}. \tag{11}$$

Use Taylor expansions with the integral form of the remainder in order to obtain

$$\left| \int_{x_j}^{x_{j+1}} g_k(s) ds - h_{j+1} g_{k;j} \right| \le h_{j+1} \int_{x_j}^{x_{j+1}} \left|g_k'(s)\right| ds,$$

$$\left| \int_{x_j}^{x_{j+1}} \left(a_k' u_k\right)(x) dx - h_{j+1} a_{k,x;j} u_{k;j+1} \right| \le \|a_k'\| \int_{x_j}^{x_{j+1}} \left|u_k'(s)\right| ds$$

and

$$\varepsilon \left| \left( u_{k_1 \bar{x}} - u_k' \right)_j \right| \le \varepsilon \int\limits_{x_{j-1}}^{x_j} \left| u_k''(s) \right| ds \le \int\limits_{x_{j-1}}^{x_j} \left| \left( g_k - a_k u_k' \right)(s) \right| ds$$

by (1). Combine these bounds with (10) and (11):

$$\|\eta_k\|_\omega \le C \max_{j=1,\dots,N} \int\limits_{x_{j-1}}^{x_j} \left\{ 1 + \sum_{m=1}^{\ell} \left| u_m'(s) \right| \right\} ds,$$

where we have also used Theorem 1 to bound the $u_m$ by a constant. Finally apply (7) to get the main result of this paper.

**Theorem 3:** *Let* $\boldsymbol{u}$ *be the solution of* (1) *and* $\boldsymbol{U}$ *the finite difference approximation obtained by* (5)*. Suppose the data* $a_k, b_{km}, f_k \in C^1[0,1]$*,* $k, m = 1, \dots, \ell$*, satisfies* (3) *and* (4)*. Then*

$$\|\boldsymbol{U} - \boldsymbol{u}\|_\omega \le C \max_{j=1,\dots,N} \int\limits_{x_{j-1}}^{x_j} \left\{ 1 + \sum_{m=1}^{\ell} \left| u_m'(s) \right| \right\} ds.$$

The a priori bounds on the $u_m'$ of Theorem 2 can be used to derive more explicit error bounds. Let $I$ denote the set of indices $k \in \{1, \dots, \ell\}$ for which $a_k$ is negative and $I^*$ its complement. Then

$$\|\boldsymbol{U} - \boldsymbol{u}\|_\omega \le C \max_{j=1,\dots,N} \int\limits_{x_{j-1}}^{x_j} \left\{ 1 + \sum_{m \in I} \varepsilon_m^{-1} \exp\left( -\frac{\alpha_m s}{\varepsilon_m} \right) \right. \left. + \sum_{m \in I^*} \varepsilon_m^{-1} \exp\left( -\frac{\alpha_m (1-s)}{\varepsilon_m} \right) \right\} ds.$$

From this error estimates for special layer-adapted meshes can be obtained:

$$\|\boldsymbol{U} - \boldsymbol{u}\|_\omega \le \begin{cases} C N^{-1} \ln N & \text{for Shishkin meshes [14],} \\ C N^{-1} & \text{for Bakhvalov meshes [3],} \end{cases} \tag{12}$$

cf. [8], [10].

**Remark 3:** *For Shishkin meshes we have recovered Cen's result [4], but due to the use of the strong stability inequality* (8) *less smoothness of the solution of* (1) *is required. Moreover no comparison principle for* (1) *is needed which allowed us to weaken the assumptions on the reaction coefficients* $b_{km}$.

**Remark 4:** *A posteriori error bounds can be obtained by adapting the technique from [6]. Let $\boldsymbol{U^I}$ denote the piecewise linear nodal interpolant to $\boldsymbol{U}$ on $\omega$. Then*

$$\left\| \boldsymbol{u} - \boldsymbol{U^I} \right\| \leq C \max_{j=1,\dots,N} h_j \left\{ 1 + \sum_{m=1}^{\ell} \left| U_{m,x,j} \right| \right\}.$$

*The argument mimics the above a priori analysis on a continuous level. For details the reader is refered to* [6], [11]. *This result can be employed in an adaptive procedure in the style of [7].*

## 4. Numerical results

**Example 1:** We consider the following example of (1) with two overlapping layers near $x = 1$:

$$\begin{aligned} -\varepsilon_1 u_1'' - u_1' + 2u_1 - u_2 &= \mathrm{e}^x, \qquad u_1(0) = u_1(1) = 0, \\ -\varepsilon_2 u_2'' - 2u_2' - u_1 + 4u_2 &= \cos x, \qquad u_2(0) = u_2(1) = 0. \end{aligned}$$

The exact solution to the test problem is not available, so we estimate the accuracy of the numerical solution by comparing it to the numerical solution of the Richardson extrapolation method, which is of higher order: Let $\boldsymbol{U}_{\boldsymbol{\varepsilon}}^N$ be the solution of the difference scheme on the original mesh and $\tilde{\boldsymbol{U}}_{\boldsymbol{\varepsilon}}^{2N}$ that on the mesh obtained by uniformly bisecting the original mesh. Then the extrapolated solution is

$$\boldsymbol{U}_{\boldsymbol{\varepsilon}}^{R,N} = 2\tilde{\boldsymbol{U}}_{\boldsymbol{\varepsilon}}^{2N} - \boldsymbol{U}_{\boldsymbol{\varepsilon}}^N.$$

This method of estimating the errors is motivated by [9], [16], where it was shown that extrapolation on layer-adapted meshes for scalar convection-diffusion problems yields higher-order accuracy. And a similar behaviour can be expected for systems too. We estimate the error for fixed $N$ and $\boldsymbol{\varepsilon}$

$$\left\| \boldsymbol{u} - \boldsymbol{U}^N \right\|_\omega \approx \eta_{\boldsymbol{\varepsilon}}^N := \left\| \boldsymbol{U}_{\boldsymbol{\varepsilon}}^N - \boldsymbol{U}_{\boldsymbol{\varepsilon}}^{R,N} \right\|_\omega = 2 \left\| \boldsymbol{U}_{\boldsymbol{\varepsilon}}^N - \tilde{\boldsymbol{U}}_{\boldsymbol{\varepsilon}}^{2N} \right\|_\omega.$$

The uniform errors are estimated by

$$\eta^N := \max_{\mu,\nu=0,-1,\dots,-8} \eta_{(10^\mu, 10^\nu)}^N.$$

The numerical rates of convergence are computed using the standard formula

$$r^N = \log_2 \left( \eta^N / \eta^{2N} \right).$$

We also compute the constants in the error estimate, i. e., if we have the theoretical error bound $\left\| \boldsymbol{u} - \boldsymbol{U}^N \right\|_\omega \leq C\lambda(N)$ then we compute the quantity $C^N = \eta^N / \lambda(N)$.

In our test we take Shishkin and Bakhvalov meshes with a third of the mesh points used to resolve each of the two boundary layers. The results of our test computations are given in Table 1. The numbers are in perfect agreement with Theorem 3 and (12): For increasing $N$ the quantities $C^N$ settle to a fixed value.

**Table 1.** The upwind scheme for Example 1

| $N$ | Shishkin mesh | | | Bakhvalov mesh | | |
|---|---|---|---|---|---|---|
| | $\eta^N$ | $r^N$ | $C^N$ | $\eta^N$ | $r^N$ | $C^N$ |
| 144 | 1.938e-2 | 0.84 | 0.56 | 1.572e-2 | 0.96 | 2.26 |
| 288 | 1.081e-2 | 0.86 | 0.55 | 8.078e-3 | 0.97 | 2.33 |
| 576 | 5.945e-3 | 0.88 | 0.54 | 4.112e-3 | 0.98 | 2.37 |
| 1152 | 3.233e-3 | 0.89 | 0.53 | 2.081e-3 | 0.99 | 2.40 |
| 2304 | 1.743e-3 | 0.90 | 0.52 | 1.049e-3 | 0.99 | 2.42 |
| 4608 | 9.341e-4 | 0.91 | 0.51 | 5.276e-4 | 0.99 | 2.43 |
| 9216 | 4.979e-4 | 0.91 | 0.50 | 2.648e-4 | 1.00 | 2.44 |
| 18432 | 2.643e-4 | 0.92 | 0.50 | 1.328e-4 | 1.00 | 2.45 |
| 36864 | 1.398e-4 | 0.92 | 0.49 | 6.652e-5 | 1.00 | 2.45 |
| 73728 | 7.370e-5 | 0.93 | 0.48 | 3.330e-5 | 1.00 | 2.46 |
| 147456 | 3.881e-5 | – | 0.48 | 1.667e-5 | – | 2.46 |

**Table 2.** The upwind scheme for Example 2

| $N$ | Shishkin mesh | | | Bakhvalov mesh | | |
|---|---|---|---|---|---|---|
| | $\eta^N$ | $r^N$ | $C^N$ | $\eta^N$ | $r^N$ | $C^N$ |
| 128 | 1.737e-2 | 0.75 | 0.46 | 9.094e-3 | 0.91 | 1.16 |
| 256 | 1.037e-2 | 0.79 | 0.48 | 4.843e-3 | 0.96 | 1.24 |
| 512 | 5.981e-3 | 0.83 | 0.49 | 2.484e-3 | 0.97 | 1.27 |
| 1024 | 3.372e-3 | 0.85 | 0.50 | 1.272e-3 | 0.98 | 1.30 |
| 2048 | 1.869e-3 | 0.87 | 0.50 | 6.444e-4 | 0.99 | 1.32 |
| 4096 | 1.024e-3 | 0.88 | 0.50 | 3.254e-4 | 0.99 | 1.33 |
| 8192 | 5.554e-4 | 0.89 | 0.50 | 1.640e-4 | 0.99 | 1.34 |
| 16384 | 2.992e-4 | 0.90 | 0.51 | 8.248e-5 | 1.00 | 1.35 |
| 32768 | 1.604e-4 | 0.91 | 0.51 | 4.133e-5 | 0.99 | 1.35 |
| 65536 | 8.560e-5 | 0.91 | 0.51 | 2.074e-5 | 1.00 | 1.36 |
| 131072 | 4.549e-5 | – | 0.51 | 1.039e-5 | – | 1.36 |

**Example 2:** The second test problem is the system of three equations:

$$\begin{aligned} -\varepsilon_1 u_1'' - (2x+1)u_1' + 3xu_1 - xu_2 + xu_3 &= \mathrm{e}^x, \qquad & u_1(0) = u_1(1) = 0, \\ -\varepsilon_2 u_2'' - 3u_2' + u_1 + 4u_2 + 2u_3 &= \cos x, \qquad & u_2(0) = u_2(1) = 0, \\ -\varepsilon_3 u_3'' + (2-x)u_3' - x^2 u_1 + (1+x)u_3 &= \sinh x, \qquad & u_3(0) = u_3(1) = 0. \end{aligned}$$

It exhibits three boundary layers: two overlapping layers at $x = 0$ and a single layer at $x = 1$. This time we use Shishkin and Bakhvalov meshes with a quarter of the mesh points for any of the three layers. The results of our test computations are documented in Table 2.

Again the numbers are in agreement with our theoretical findings.

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