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Sharp error bounds for piecewise linear interpolation of planar curves

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Abstract

Curves are commonly drawn by piecewise linear interpolation, but to worry about the error is rather seldom. In the present paper we give a strong mathematical error analysis for curve segments with bounded curvature and length. Though the result seems very clear, the proof turned out to be unexpectedly hard, comparable to that of the famous four vertex theorem.

AMS Subject Classifications: 53A04, 65D05, 41A05.

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1. Introduction

"Drawing a parametric curve" $\mathcal{C} \ldots\ \mathbf{x} : I \longrightarrow \mathbb{R}^2$ currently means the following procedure: Choose a sufficiently dense sequence of points on $\mathcal{C}$: $\mathbf{x}(t_0), \ldots, \mathbf{x}(t_n)$ and then draw the *polygon* with these points as vertices

Problem: *What is the sharp error bound for this kind of piecewise linear interpolation?* Or – equivalently – : *What is the maximal distance a point of the curve segment can have from the corresponding chord?*

Clearly, that distance from the chord can be arbitrarily large if the *length* L of the curve segment is unrestricted. On the other hand, there is a trivial error bound given by the height $h = \sqrt{(L')^2 - a^2}$ of the isosceles triangle, built with the chord (of length $2a$) and two legs of length $L' := L/2$. But the C^1-condition forbids that solution and an upper bound of the curvature should lead to a better error estimate. So our problem makes sense only for curve segments joining two fixed points $\mathbf{a}$ and $\mathbf{b}$ if restrictions on the length and the curvature are imposed. Even in that case there is a trivial solution if the curve segment is a so-called *spiral arc*:

Definition 1: *A compact C^1 curve, being piecewise C^2, is called a spiral arc if its curvature function* $\kappa : [0, L] \longrightarrow \mathbb{R}$ *is non-decreasing.*

There are many classical theorems on spiral arcs and questions around them. Besides the famous four vertex theorem (Muckhopadhaya [3]) we mention here only

Theorem 1 (Kneser [2]): *For any pair of parameter values $s_1 < s_2$ of a spiral arc the curvature circle at s_1 completely contains that one at s_2.*

(More about spiral arcs can be found in [1].) From Kneser's theorem we take immediately:

Theorem 2: *If the curve segnent $\mathcal{C}$ from* **a** *to* **b** *is a spiral arc with angle α of the tangent at* **a** *with the chord* **a b**, *then the maximal distance d of $\mathcal{C}$ from the chord satisfies*

$$d \le r \cdot (1 - \cos\alpha), \qquad \textit{with } r = 1/\kappa(0). \tag{1}$$

This error bound is *sharp* because $\mathcal{C}$ itself can be a circular arc.

The remainder of the present paper deals with *the general case* which turned out to be *incomparable more complicated*. The final result will be obtained by several steps. However, the strong limit of pages forces us to make simplifying assumptions and to omit some details of the proof.

2. Assumptions and the question of symmetry

Assumptions:

(1) $\mathcal{C}$ is a compact planar curve segment from **a** to **b** with length $L \le L_0$ (L_0 being a *given* quantity satisfying $L_0 > |\mathbf{ab}|$). $\mathcal{C}$ is assumed to be C^1 throughout and piecewise C^2. So there is a parametric representation with arc length as parameter:

$$\mathcal{C} \; \ldots \quad \mathbf{x} : [0, L] \longrightarrow \mathbb{R}^2 \;\text{ with }\; \mathbf{x}(0) = \mathbf{a},\; \mathbf{x}(L) = \mathbf{b}.$$

(2) The curvature function over $[0, L]$ is non-negative and bounded by κ_0.
(3) Denoting $r := 1/\kappa_0$ and $a := |\mathbf{ab}|/2$, the condition $r < a$ must hold.
(4) The (unoriented) tangent angles α, β at **a**, **b** resp. are restricted to $\alpha < \pi/2$, $\beta < \pi/2$.
(5) The curve $\mathcal{C}$ is *convex*. (So it has no other intersection points with the chord than **a** and **b** and it may wlog. be assumed to lie *below* the chord).

The set of curves satisfying these conditions is denoted by $\mathcal{A}$. Define, for $\mathcal{C} \in \mathcal{A}$, the *chord distance* by $d(\mathcal{C}) = \max\{d(\mathbf{x}(s), \mathbf{ab}) \mid s \in [0, L]\}$. Then, clearly, $d(\mathcal{C}) < L/2$ for all $\mathcal{C} \in \mathcal{A}$ and thus $d_0 = \sup\{d(\mathcal{C}) \mid \mathcal{C} \in \mathcal{A}\}$ exists.

Problem: Determine d_0 in terms of a, L_0 and κ_0, provide an answer to the question whether this maximal chord distance will be attained and, if so, describe that curve $\mathcal{C}_0$ (which will be called the *extremal arc* in the sequel).

Remarks:

(1) The assumptions are redundant! In particular, Condition 4 on the tangent angles is imposed only for simplicity.

(2) Condition 3 is imposed only to avoid cases where $\mathcal{A}$ is empty.
(3) The convexity condition may also be renounced, because it is *implied* by Lemma 1 (see below). The idea to prove convexity from it is to replace a bump by a bi-tangent leg.

Lemma 1: *An arc $\mathcal{C}$ with $L < L_0$ is not extremal.*

(Visually clear, exact proof omitted.)

Since the problem is symmetric wrt. the mid-perpendicular of the chord, one could expect that the solution is so, too. In the talk at the Dagstuhl Seminar we proved it by Steiner's symmetrisation (see [5]), however it was necessary to introduce an auxiliary assumption excluding the possibility that this procedure increases the curvature beyond its naximal value κ_0. Later on, all trials to prove symmetry otherwise (at this stage) failed. Though symmetry would considerably simplify the rest of the investigations, we renounce it here at all and will obtain it at the very end automatically.

3. The structure of the extremal arc

In the sequel, we divide the curve $\mathcal{C}$ into two "half-arcs" $\mathcal{B}_1, \mathcal{B}_2$, separated by the point $\mathbf{p}$ having maximal distance from the chord $\mathbf{ab}$. We think of the lengths L_1, L_2 and the corresponding parts a_1, a_2 of the chord to be *fixed and given* (not only the sums $L_1 + L_2 = 2L' = L_0,\ a_1 + a_2 = 2a$). Now we can treat both half-arcs separately.

We attach a coordinate system with origin at the minimum and x-axis parallel to the chord $\mathbf{ab}$. The (right) half-arc $\mathcal{B}_1$ starts at the origin with slope zero, has length L_1 and ends at the line $x = a_1$. The other half-arc $\mathcal{B}_2$ goes analogously to the left side with length L_2 and ends at $x = -a_2$. In the sequel, we focus on $\mathcal{B} := \mathcal{B}_1$.

By convexity and Assumption 4, $\mathcal{B}$ has a representation as a *function*

$$f : [0, a_1] \longrightarrow \mathbb{R}, \quad x \mapsto f(x), \quad x \in [0, a_1]. \tag{2}$$

The set of half-arcs satisfying these conditions (together with the other assumptions of Sect. 2) is denoted by $\mathcal{D}$. The ordinate $y(a)$ is called the "height"and denoted by $H(\mathcal{B})$.

The *problem* now reads as follows:

(1) Find $H_0 := \sup\{H(\mathcal{B}) \mid \mathcal{B} \in \mathcal{D}\}$ and (2) determine $\mathcal{B} \in \mathcal{D}$ with $H(\mathcal{B}) = H_0$ ("the extremal arc") and describe it geometrically.

Besides the functional representation (2) we use a *parametric representation of its slope angle* by arc length: $\phi : \quad [0, L_1] \longrightarrow \mathbb{R}$. By convexity, this function is monotonically increasing, continuous and piecewise C^1; in addition it satisfies the inequalities[1]

$$\phi(0) = 0, \quad \phi(L_1) \le \pi/2, \quad 0 \le \phi'(s) \le \kappa_0. \tag{3}$$

The curvature is given by $\phi'(s) = \kappa(s)$ (at break points one-sided limits) and one has

$$A := \int_0^{L_1} \cos(\phi(s)ds = a_1 \quad \text{and} \quad H(\mathcal{B}) = \int_0^{L_1} \sin(\phi(s)ds, \tag{4}$$

because the tangent unit vector is $(\cos(\phi), \sin(\phi))$.

So we seek among all piecewise C^1-functions satisfying the conditions (3), (4) that one with *maximal height* $H(\mathcal{B})$.

To solve this extreme value problem with side conditions, we use methods (and denotations) of the calculus of variations. Denoting by ϕ_0 the function for the extremal arc, we consider a variation of ϕ_0 by ϵv, i.e., the function $\phi(s, \epsilon) = \phi_0(s) + \epsilon v(s), \quad s \in [0, L_1]$; then the extremal condition $\delta(H + \lambda A) = 0$ (with a Lagrangian multiplier λ), implies

$$\int_0^{L_1} v(s)\,(\cos(\phi_0(s)) - \lambda \sin(\phi_0(s)))\,ds = 0. \tag{5}$$

In contrast to the usual methods of the calculus of variations the integral does not depend on the derivative of ϕ_0; so no Euler equation can be expected. Furthermore, the perturbation function v can not be chosen arbitrarily, but the further side conditions $\phi(0, \epsilon) = 0$, and $0 \le \partial\phi(s, \epsilon)/\partial s \le \kappa_0$ have to be observed. The first one implies $v(0) = 0$. As to the second, we must distinguish between places s where $\phi_0'(s) = 0$ or $\phi_0'(s) = \kappa_0$ and all the others. At the first ones, ϕ_0 must be kept fixed, so $v(s) = 0$. Around the others, there are open intervals I where (5) implies $\cos(\phi_0(s)) - \lambda \sin(\phi_0(s)) = 0 \quad$ for all $s \in I$. Introducing α by $\lambda = \cos\alpha / \sin\alpha$ yields $\sin(\phi_0(s) - \alpha) = 0 \quad$ for all $s \in I$.

So the whole interval $[0, L_1]$ *is divided into subintervals I where either* $\phi_0(s) = \alpha = const \quad or \quad \phi_0'(s) = \kappa_0$. This proves

Theorem 3: *Existence provided, the extremal half-arc is composed by segments of straight lines and by circular arcs; all the latter ones have the same radius* $r = 1/\kappa_0$. *All these segments join with* C^1*-continuity.*

The next step is to show that the extremal half arc is composed by *exactly one circular arc and one segment of its tangent* at the endpoint (until the line $x = a_1$ is reached).

Again, we use variational methods to prove this seemingly easy task. If we had at least three segments or two with the circular arc as the last one, we consider the

[1] In contrast to Assumption 4, we allow here $\phi(L_1) = \pi/2$ for reasons of compactness.

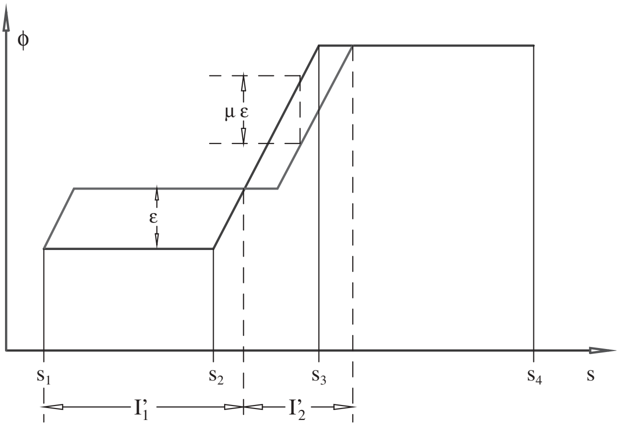

Fig. 1. Variation of ϕ_0

last pair of intervals, I_1 of the first kind ($\phi = const$), followed by I_2 being of the second kind (ϕ linear with slope κ_0). The latter may be the last one or there is still one interval of the first kind following it.

Let, as before, ϕ_0 be the extremal function. Then we add a (small) variation function $(\epsilon, s) \mapsto h(\epsilon, s), \quad s \in [0, L_1], \quad \epsilon \geq 0$ which increases the values of $\phi_0(s)$ in I_1' by ϵ and decreases them in I_2' by $\mu \cdot \epsilon$ (up to small transitions to keep the type of $\bar{\phi} := \phi_0 + h$ to be alternatively constant and linear with slope κ_0) as shown in Fig. 1.

In order to have still $\bar{\phi} := \phi_0 + h \in \mathcal{D}$, we must determine the factor μ such that the two corrections in I_1' and I_2', respectively. compensate with respect to the extend of the corresponding arc in x-direction: $A(\epsilon) := \int_0^{L_1} \cos(\phi_0 + h)ds = a_1$, thus $\delta A = 0$, i.e.,

$$\int_0^{L_1} \sin(\phi_0) \left.\frac{\partial h}{\partial \epsilon}\right|_{\epsilon=0} ds = \int_{I_1} \sin(\phi_0)ds - \mu \int_{I_2} \sin(\phi_0)ds = 0. \tag{6}$$

On the other hand we get

$$\delta H = \int_{I_1} \cos(\phi_0)ds - \mu \cdot \int_{I_2} \cos(\phi_0)ds. \tag{7}$$

In order to be able to use (6) we insert the cotangens function and apply the second mean value theorem, getting

$$\begin{aligned} \delta H &= \int_{I_1} \cot(\phi_0)\sin(\phi_0)ds - \mu \int_{I_2} \cot(\phi_0)\sin(\phi_0)ds \\ &= c_1 \int_{I_1} \sin(\phi_0)ds - \mu \cdot c_2 \int_{I_2} \sin(\phi_0)ds, \end{aligned} \tag{8}$$

where $c_1 = \cot(\phi_0(\tilde{s}_1))$ and $c_2 = \cot(\phi_0(\tilde{s}_2)$ with certain mean values $\tilde{s}_1 \in I_1$ and $\tilde{s}_2 \in I_2$. Applying now (6) finally yields

$$\delta H = (c_1 - c_2) \int_{I_1} \sin(\phi_0) ds > 0 \tag{9}$$

since $\tilde{s}_1 \in I_1$ and $\tilde{s}_2 \in I_2$, hence $s_1 < s_2$ and ϕ_0 is monotonically increasing and the Cotangens function is monotonically decreasing.

This contradicts the extremal property of ϕ_0! So there is *exactly one* interval of each kind and that of the second kind must preceed that of the first one. This proves:

Theorem 4: *The slope function of the extremal half-arc is*

$$\phi_0(s) = \left\{ \begin{array}{ll} s \cdot \kappa_0 & \textit{for } s \in [0, s_1] \\ s_1 \cdot \kappa_0 & \textit{for } s \in [s_1, L_1] \end{array} \right\} \tag{10}$$

(*with some* s_1, $0 < s_1 < L_1$). □

4. Existence

First we observe that the investigations of the previous section remain true if we renounce the C^1-condition (but keeping C^0) and replace the curvature bounds by

$$0 \leq \frac{\phi(s_2) - \phi(s_1)}{s_2 - s_1} \leq \kappa_0 \tag{11}$$

for all pairs $s_1, s_2 \in [0, L_1]$ with $s_1 \neq s_2$. We denote this extended class of functions by $\hat{\mathcal{D}}$. Then *$\hat{\mathcal{D}}$ is equicontinuous and so, by the theorem of Arzelà-Ascoli, compact.*

By definition of the supremum, there is a sequence of functions $\phi_n \in \hat{\mathcal{D}}$ with

$$\lim_{n \to \infty} H(\phi_n) = \sup\{H(\phi) \mid \phi \in \hat{\mathcal{D}}\} =: H_0. \tag{12}$$

and, by compactnes, there is a subsequence of it, uniformly converging to a continuous function $\phi_0 \in \hat{\mathcal{D}}$ with $H(\phi_0) = H_0$. This function satisfies also (11) and is monotonically increasing in $[0, L_1]$ as all $\phi \in \hat{\mathcal{D}}$ do and thus solves the problem in that wider class of functions. It will also be called the *extremal function*.

If there exist any places $s_1, s_2, s_1 \neq s_2$ with $\phi(s_1) = \phi(s_2)$, *then ϕ is constant* in $[s_1, s_2]$, by monotonicity. For each occurence of such a pair, we take the *maximal interval* where ϕ stays constant. By continuity, such an interval is *closed.* In the sequel, it will be called an *interval of the first kind.* Thus the remaining subset of $[0, L_1]$, from which all intervals of the first kind are removed, is an *open subset* of $[0, L_1]$ (wrt. the relative topology), hence a union of open intervals each of which will be called an *interval of the second kind.*

The results of the previous section imply that any point in an interval of the second kind must be a limit of pairs $s_1,\ s_2 \in [0, L_1]$ where the divided difference attains its maximal value κ_0. By continuity we can conclude:

Lemma 2: *The extremal function is (automaticly) differentiable in each interval I of the second kind and satisfies (11), hence $\phi' = \kappa_0$ in I.*

Furthermore, since the breakpoints are endpoints of a pair of intervals, one of the first, the other of the second kind, these breakpoints are *isolated*. So there is at most a *finite number* of them, hence ϕ_0 is piecewise C^1 and thus belongs to $\mathcal{D}$. — So we have proved:

Theorem 5: *The extremal function exists in $\mathcal{D}$.* □

5. Global solution

Joining the two extremal half-arcs yields a curve $\mathcal{C}$ consisting of a circular arc with radius r and angle $\phi_1 + \phi_2$, tangent continuously continued by a leg at both ends with lengths l_1 and l_2, respectively (**p** lying on the circular arc, separating the angles ϕ_1 and ϕ_2 and with $l_i = L_i - r\phi_i$, $(i = 1, 2)$). Among the class of these curves we seek that one which solves the following

Optimization problem:

Determine the quantities $l_1,\ l_2,\ \phi_1,\ \phi_2$ such that the function

$$H = (l_1 cos(\phi_1) + l_2 cos(\phi_2) + r(2 - cos(\phi_1) - cos(\phi_2)))/2 \tag{13}$$

attains its maximal value under the conditions

- Total length:

$$l_1 + l_2 + r(\phi_1 + \phi_2) = L_0. \tag{14}$$

- Same hights at both sides:

$$l_1 sin(\phi_1) + r(1 - cos(\phi_1)) = l_2 sin(\phi_2) + r(1 - cos(\phi_2)). \tag{15}$$

- Chord length fitting:

$$l_1 cos(\phi_1) + l_2 cos(\phi_2) + r(sin(\phi_1 + sin(\phi_2)) = 2a. \tag{16}$$

Introducing the quantities

$$d := (1/2)(l_1 - l_2),\quad \phi := (1/2)(\phi_1 - \phi_2),\quad \delta := (1/2)(\phi_1 - \phi_2). \tag{17}$$

and eliminating $l_1,\ l_2$ from (14), (15) we get (recall $L' = L/2$)

$$l_1 = L' - r\phi + d, \quad l_1 = L' - r\phi - d \tag{18}$$

with

$$d = -((L' - r\phi)cos(\phi) + rsin(\phi))\frac{sin(\delta)}{sin(\phi)cos(\delta)}. \tag{19}$$

Inserting this into (13) and (16) we obtain

$$H = \frac{(L' - r\phi)\left(sin^2(\phi)cos^2(\delta) - cos^2(\phi)sin^2(\delta)\right)}{sin(\phi)cos(\delta)} + r\left(1 - \frac{cos(\phi)}{cos(\delta)}\right), \tag{20}$$

and the only remaining condition is

$$B := \frac{(L' - r\phi)cos(\phi) + rsin(\phi)}{cos(\delta)} - a = 0 \tag{21}$$

(both for the new unknowns $\phi,\ \delta$).

By Lagrange's method the extremal arc is obtained as the solution of

$$\frac{\partial H}{\partial \phi} - \lambda\frac{\partial B}{\partial \phi}, \quad \frac{\partial H}{\partial \delta} - \lambda\frac{\partial B}{\partial \delta} \tag{22}$$

and (21). Performing all these partial derivations, all the necessary substitutions and suitable simplifications (a huge amount of calculations, done with the aid of MAPLE) we finally obtain the following condition (after λ has been eliminated):

$$sin^3(\delta)\left[(L' - r\phi)cos(\phi) + rsin(\phi)\right] = 0. \tag{23}$$

Since the term in square brackets is non-zero, *the only solution of this condition is* $\delta = 0$. This means:

Theorem 6: *The extremal curve is symmetric wrt. the mid-perpendicular of the chord.*

With $\delta = 0$ (20) simplifies to

$$H = (L' - r\phi)sin(\phi) + r\cdot(1 - cos(\phi)) \tag{24}$$

and – according to (16) – the angle ϕ is determined by

$$(L' - r\phi)\cdot cos(\phi) + rsin(\phi) = a. \tag{25}$$

Summarising we have proved:

Theorem 7: *Both half-arcs of the extremal curve – being symmetric wrt. the mid-perpemdicular of the chord – consist of a circular arc with radius r and angle ϕ and a segment of the tangent to this circular arc at each of its endpoints until the line $x = \pm a$ resp. is reached. Furthermore, the angle ϕ is obtained as the solution of Eq. (25) and the maximal distance (the "height") is given by (24).*

So our problems are *completely solved*. The solution is shown in the following figure:

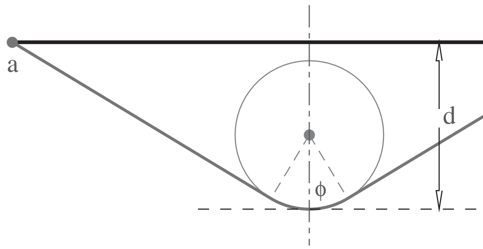

Fig. 2. The solution

6. Final remarks

- I attacked this problem already many years ago in a talk at a Geometry Symposium on Schloß Seggauberg, Austria, 1987. But there arose some difficulties, and so the preliminary results were not published other than in the conference proceedings.
- Theorem 3 can also be obtained using methods of control theory. The fact that the curvature of the extremal arc takes only the outermost values of the admissible interval confirms the so-called "bang-bang principle" of La Salle [4].
- Most of the results carry over to the case of a *positive* lower bound of the curvature. I acknowledge this remark to the kind anonymous referee. However I could not yet confirm that the extremal curve is also symmetric in this case.

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