Computing 74, 337–351 (2005)
Digital Object Identifier (DOI) 10.1007/s00607-004-0118-7

**Computing**
Printed in Austria

# Two-level Stabilized Finite Element Methods for the Steady Navier–Stokes Problem

**Yinnian He** and **Kaitai Li,** Xi'an

Received August 22, 2003; revised November 29, 2004
Published online: February 16, 2005


**Abstract**

In this article, the two-level stabilized finite element formulations of the two-dimensional steady Navier–Stokes problem are analyzed. A macroelement condition is introduced for constructing the local stabilized formulation of the steady Navier–Stokes problem. By satisfying this condition the stability of the $Q_1 - P_0$ quadrilateral element and the $P_1 - P_0$ triangular element are established. Moreover, the two-level stabilized finite element methods involve solving one small Navier–Stokes problem on a coarse mesh with mesh size $H$, a large Stokes problem for the simple two-level stabilized finite element method on a fine mesh with mesh size $h = O(H^2)$ or a large general Stokes problem for the Newton two-level stabilized finite element method on a fine mesh with mesh size $h = O(|\log h|^{1/2} H^3)$. The methods we study provide an approximate solution $(u^h, p^h)$ with the convergence rate of same order as the usual stabilized finite element solution, which involves solving one large Navier–Stokes problem on a fine mesh with mesh size $h$. Hence, our methods can save a large amount of computational time.

*AMS Subject Classifications:* 35L70, 65N30, 76D06.

*Keywords:* Navier–Stokes problem, stabilized finite element, two-level method, error estimate.

## 1. Introduction

The development of appropriate mixed finite element methods is a key component in the search for efficient techniques for solving the incompressible Navier–Stokes problem. Using a primitive variable formulation, the importance of ensuring the compatibility of the component approximations of velocity and pressure by satisfying the so-called inf-sup condition is widely understood. It is also well known that the simplest conforming low-order elements like the $P_1 - P_0$ (linear velocity, constant pressure) triangular element and $Q_1 - P_0$ (bilinear velocity, constant pressure) quadrilateral element are not stable. This impinges on efficiency, since the simple logic and regular data structure associated with low-order finite element methods make them particularly attractive on modern vector and parallel processing architectures.

The stability of the mixed approximations has become crucially important with the advent of "fast" iterative solution algorithms, for example, based on multigrid or preconditioned conjugate gradient iterations. Numerical experiments show that in the solution of the Stokes or Navier–Stokes problems, ensuring stability is essential if a reasonable rate of convegence of such iterations is to be achieved. For details, see the work of Verfürth [27], Bramble and Pasciak [5], and Kay and Silvester [16].

Recently, regularization of the discrete Stokes formulation have been developed as a means of overcoming the problem of incompatible mixed approximations. The idea of such a regularization was first proposed by Brezzi and Pitkäranta [6] in the context of the $P_1 - P_1$ triangular element. Subsequently, Hughes and Franca [13] derived a discrete Stokes formulation which ensured the stability of arbitrary mixed approximations. For a discontinuous pressure approximation, this stability is achieved by introducing a pressure jump operator into the discrete Stokes formulation. For low-order approximations, the only price to pay for having universal stability is that the jump operator must control pressure jumps across all internal interelement edges. This makes the Hughes and Franca formulation somewhat not efficient to implement since a nonstandard element assembly is required. We note that this limitation also applies to the absolutely stable formulation proposed by Douglas and Wang [8]. Furthermore, a general locally stabilized mixed finite element method was provided by Kechkar and Silvester in [17]. Recently, a fully discrete stabilized finite element method for the time-dependent Navier–Stokes problem was considered by He in [11].

Numerically, there is evidence (see, for example, Kay and Silvester [16], Silvester and Kechkar [24]) to suggest that a more robust way of stabilizing a mixed method based on discontinuous pressure is to modify the “global” jump operator of Hughes and Franca, so as to restrict the jumps in pressure in a “local” sense. In [17] and [24], Silvester and Kechkar refer to the original and the modified formulations as global jump and local jump stabilizations, respectively. The precise definitions were given in [17], [24] and recalled in Sect. 3. A key feature of a local jump stabilization is that a conventional macroelement implementation is possible, so that the modified methods can be directly implemented into element-by-element iterative solution techniques. Moreover, a posteriori error estimation, an adaptive refinement process and some numerical results of the steady-state Stokes problem were provided by Kay and Silvester [16]. These had confirmed the theoretical results and allowed a comparison of the efficiency and reliability of the indicators in an adaptive refinement setting.

The basic idea of two-level discretization method is to capture the “large eddies”, “low modes”, or “global solution envelope” by computing an initial approximation on a very coarse mesh (involving the solution of a very small number of nonlinear equations). The fine structures are captured by solving one linear system (linearized about the coarse mesh approximation) on a fine mesh. Some details of two-level can be founded in the works of Xu [28], [29], Layton [19], Layton and Lenferink [20], Ervin et al. [9], and Layton and Tobiska [21].

The method we study in this paper is to combine the stabilized finite element method with the two-level discretization for solving the two-dimensional steady Navier–Stokes problem under the assumptions of the uniqueness condition. The heart of the analysis is a local “macroelement condition” which is sufficient for overall stability of the method. The use of such a macroelement condition as a means of verifying the Babuška-Brezzi stability inequality is standard practice (see, for example, Girault and Raviart [10]). The basic idea was first introduced by Boland and Nicolaides [3], and independently by Stenberg [25]. For ease of notation and to keep the paper brief, we will confine our attention to $Q_1 - P_0$(bilinear velocity, constant pressure) quadrilateral and the $P_1 - P_0$ triangular element, where the domain $\Omega$ is assumed to be a polygon. We set $\tau_H$ is a triangulation of $\bar{\Omega}$ into triangles or quadrilaterals with mesh size $H$, assumed to be regular in the usual sense. The stabilized finite element space pair $(X_H, M_H)$ is constructed by $Q_1 - P_0$ quadrilateral element or $P_1 - P_0$ triangular element. Next, the fine element space pair $(X_h, M_h)$ can be thought of as generated from $(X_H, M_H)$ by a mesh refinement process and therefore nested, the finite element space pairs $(X_H, M_H)$ and $(X_h, M_h)$ do not posses the compatible properties of the so-called "inf-sup condition". The stabilized mixed finite element approximation under consideration is based on the combination of the standard variational formulation of the steady Navier–Stokes problem and the bilinear form related to the jump operator in pressure. The initial approximation $(u_H, p_H)$ of the steady Navier–Stokes problem is determined on the coarse mesh. Then the fine mesh approximation $(u^h, p^h)$ is obtained by solving a large Stokes problem for the simple two-level stabilized finite element method on a fine mesh with mesh size $h = O(H^2)$ or a large general Stokes problem for the Newton two-level stabilized finite element method on a fine mesh with mesh size $h = O(|\log h|^{1/2} H^3)$.

For the usual stabilized finite element solution $(u_h, p_h)$, which involves solving one large Navier–Stokes problem on a fine mesh with mesh size $h$, we prove the following error estimate:

$$\|u - u_h\|_{H^1} + \|p - p_h\|_{L^2} \leq ch, \tag{1}$$

where $c$ denotes some generic constant which may stand for different values at its different occurrences. Furthermore, we prove that the simple two-level stabilized finite element solution $(u^h, p^h)$ is of the following error estimate:

$$\|u - u^h\|_{H^1} + \|p - p^h\|_{L^2} \leq c(h + H^2). \tag{2}$$

Also we prove that the Newton two-level stabilized finite element solution $(u^h, p^h)$ is of the following error estimate:

$$\|u - u^h\|_{H^1} + \|p - p^h\|_{L^2} \leq c(h + |\log h|^{1/2} H^3). \tag{3}$$

Hence, if we choose $H$ such that $h = O(H^2)$ for the simple two-level stabilized finite element solution and $h = O(|\log h|^{1/2} H^3)$ for the Newton two-level stabilized finite element solution, then the methods we study are of the convergence rate of same order as the usual stabilized finite element method. However, our method is more simple.

## 2. Functional Setting of the Navier–Stokes Problem

Let $\Omega$ be a bounded domain in $R^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$ and to satisfy a further condition stated in (Assumption 1) below. We consider the steady Navier–Stokes problem

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \ \text{div}\, u = 0 \quad x \in \Omega; \\ u|_{\partial\Omega} = 0, \end{cases} \tag{4}$$

where $u = (u_1(x), u_2(x))$ represents the velocity vector, $p = p(x)$ the pressure, $f = f(x)$ the prescribed body force, and $\nu > 0$ the viscosity.

For the mathematical setting of problem (4), we introduce the following Hilbert spaces

$$X = H_0^1(\Omega)^2, \ \ Y = L^2(\Omega)^2, \ \ M = L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q dx = 0\}.$$

The spaces $L^2(\Omega)^m, m = 1, 2, 4$ are endowed with the $L^2$-scalar product and $L^2$-norm denoted by $(\cdot,\cdot)$ and $|\cdot|$. The space $H_0^1(\Omega)$ and $X$ are equipped with their usual scalar product and norm

$$((u, v)) = (\nabla u, \nabla v), \quad \|u\| = ((u, u))^{1/2}.$$

As mentioned above, we need a further assumption on $\Omega$:

**Assumption 1:** Assume that the $\Omega$ is regular so that the unique solution $(v, q) \in (X, M)$ of the steady Stokes problem

$$-\Delta v + \nabla q = g, \ \text{div}\, v = 0 \text{ in } \Omega, \quad v|_{\partial\Omega} = 0,$$

for prescribed $g \in Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \le c|g|,$$

where $\|\cdot\|_i$ denotes the usual norm of Sobolev space $H^i(\Omega)$ or $H^i(\Omega)^2$ for $i = 1, 2$.

We also introduce the Laplace operator

$$Au = -\Delta u \quad \forall u \in D(A) = H^2(\Omega)^2 \cap X.$$

Moreover, we define a generalized bilinear form on $(X, M) \times (X, M)$ by

$$\mathcal{B}((u, p); (v, q)) = \nu((u, v)) - (\text{div}v, p) + (\text{div}u, q),$$

and a trilinear form on $X \times X \times X$ by

$$\begin{aligned} b(u, v, w) &= ((u\cdot\nabla)v, w) + \frac{1}{2}((\text{div}u)v, w) \\ &= \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v), \forall u, v, w \in X. \end{aligned}$$

We remark that the validity of Assumption 1 is known (see [14], [15]) if $\partial\Omega$ is of $C^2$, or if $\Omega$ is a two-dimensional convex polygon. From Assumption 1, it is easily shown [14] that

$$|v| \leq \gamma_0\|v\|, \quad \forall v \in X, \quad \|v\| \leq \gamma_0|Av|, \quad \|v\|_2 \leq \gamma_1|Av|, \quad \forall v \in D(A), \tag{5}$$

where $\gamma_0$ and $\gamma_1$ are positive constants depending only on $\Omega$.

It is easy to verify that $\mathcal{B}$ and $b$ satisfy the following important properties (see [1], [10], [14], [16], [17], [18], [26]): there hold

$$\begin{cases} \nu\|u\|^2 = \mathcal{B}((u,p);(u,p)), \\ |\mathcal{B}((u,p);(v,q))| \leq c(\|u\| + |p|)(\|v\| + |q|), \\ \alpha_0(\|u\| + |p|) \leq \sup\limits_{(v,q)\in(X,M)} \dfrac{\mathcal{B}((u,p);(v,q))}{\|v\| + |q|} \end{cases} \tag{6}$$

for all $(u, p),\ (v, q) \in (X, M)$ and constants $\gamma_2 > 0$ and $\alpha_0 > 0$,

$$b(u, v, w) = -b(u, w, v), \tag{7}$$

$$|b(u, v, w)| \leq \frac{1}{2}c_0|u|^{1/2}\|u\|^{1/2}(\|v\||w|^{1/2}\|w\|^{1/2} + |v|^{1/2}\|v\|^{1/2}\|w\|), \tag{8}$$

for all $u, v, w \in X$ and

$$|b(u, v, w)| + |b(v, u, w)| + |b(w, u, v)| \leq c\|u\||Av||w|, \tag{9}$$

for all $u \in X, v \in D(A), w \in Y$, where $\alpha_0$ and $c_0$ are positive constants depending on the domain $\Omega$.

Under the above notations, the variational formulation of problem (4) reads as: find $(u, p) \in (X, M)$ such that for all $(v, q) \in (X, M)$:

$$\mathcal{B}((u, p); (v, q)) + b(u, u, v) = (f, v). \tag{10}$$

The following existence and uniqueness results are classical (see [10], [26]).

**Theorem 2.1:** *Assume that* $\nu$ *and* $f \in Y$ *satisfy the following the uniqueness condition:*

$$1 - \frac{c_0\gamma_0^2}{\nu^2}|f| > 0. \tag{11}$$

*Then problem (10) admits a unique solution* $(u, p) \in (D(A), H^1(\Omega)\cap M)$ *with* $\operatorname{div} u = 0$ *such that*

$$\|u\| \leq \frac{\gamma_0}{\nu}|f|, |Au| + \|p\|_1 \leq c|f|. \tag{12}$$

## 3. Stabilized Finite Element Approximation

In this section, the stabilized finite element approximation for solving the steady Navier–Stokes problem is an extension of the ones provided by Kechkar and Silvester for solving the steady Stokes problem, where some statements and results are borrowed from Kechkar and Silvester's work [17].

Let $h > 0$ be a real positive parameter. Finite element subspace $(X_h, M_h)$ of $(X, M)$ is characterized by $\tau_h = \tau_h(\Omega)$, a partitioning of $\bar{\Omega}$ into triangles or quadrilaterals, assumed to be regular in the usual sense (see [7], [10], [16], [17]), i.e., for some $\sigma$ and $\omega$ with $\sigma > 1$ and $0 < \omega < 1$,

$$h_K \leq \sigma \rho_K \quad \forall K \in \tau_h, \tag{13}$$

$$|\cos \theta_{iK}| \leq \omega, i = 1, 2, 3, 4, \forall K \in \tau_h, \tag{14}$$

where $h_K$ is the diameter of element $K$, $\rho_K$ is the diameter of the inscribed circle of element $K$, and $\theta_{iK}$ are the angles of $K$ in the case of a quadrilateral partitioning. The mesh parameter $h$ is given by $h = \max\{h_K\}$, and the set of all interelement boundaries will be denoted by $\Gamma_h$.

The finite element subspaces of interest in this paper are defined by setting

$$R_1(K) = \begin{cases} P_1(K) \text{ if } K \text{ is triangular,} \\ Q_1(K) \text{ if } K \text{ is quadrilateral,} \end{cases} \tag{15}$$

giving the continuous piecewise (bi)linear velocity subspace

$$X_h = \{v \in X : \quad v_i|_K \in R_1(K), \quad i = 1, 2, \quad \forall K \in \tau_h\}$$

and the piecewise constant pressure subspace

$$M_h = \{q \in M : \quad q|_K \in P_0(K), \forall K \in \tau_h\}.$$

Note that neither of these methods are stable in standard Babuška-Brezzi sense; $P_1 - P_0$ triangle "locks" on regular grids (since there are more discrete incompressibility constraints than velocity degrees of freedom), and the $Q_1 - P_0$ quadrilateral is the most infamous example of an unstable mixed method, as elucidated by Sani et al. [23].

In order to define a locally stabilized formulation of the Navier–Stokes problem, we introduce *a macroelement partitioning* $\Lambda_h$ as follows: Given any subdivision $\tau_h$, a macroelement partitioning $\Lambda_h$ may be defined such that each macroelement $\mathcal{K}$ is connected set of adjoining elements from $\tau_h$. Every element $K$ must lie in exactly one macroelement, which implies that macroelements do not overlap. For each $\mathcal{K}$, the set of interelement edges which are strictly in the interior of $\mathcal{K}$ will be denoted by $\Gamma_{\mathcal{K}}$. The length of edge $e \in \Gamma_{\mathcal{K}}$ is denoted by $h_e$.

With these additional definitions a locally stabilized formulation of the Navier–Stokes problem (10) can be stated as follows.

**Definition 3.1** (Locally stabilized formulation): *Find* $(u_h, p_h) \in (X_h, M_h)$, *such that for all* $(v_h, q_h) \in (X_h, M_h)$

$$\mathcal{B}_h((u_h, p_h); (v_h, q_h)) + b(u_h, u_h, v) = (f, v), \tag{16}$$

*where*

$$\mathcal{B}_h((u_h, p_h); (v_h, q_h)) = \mathcal{B}((u_h, p_h); (v_h, q_h)) + \beta \mathcal{C}_h(p_h, q_h),$$

$$\mathcal{C}_h(p, q) = \sum_{\mathcal{K} \in \Lambda_h} \sum_{e \in \Gamma_{\mathcal{K}}} h_e \int_e [p]_e [q]_e ds,$$

*for all* $p,\ q$ *in the algebraic sum* $H^1(\Omega) + M_h$, $[\cdot]_e$ *is the jump operator across* $e \in \Gamma_{\mathcal{K}}$ *and* $\beta > 0$ *is the local stabilization parameter.*

A general framework for analyzing the locally stabilized formulation (16) can be developed using the notion of equivalent class of macroelements. As in Stenberg [25], each equivalence class, denoted by $\mathcal{E}_{\hat{\mathcal{K}}}$, contains macroelements which are topologically equivalent to a reference macroelement $\hat{\mathcal{K}}$. To illustrate the idea, two practical examples of locally stabilized mixed approximations are given below.

**Example 3.1:** *The first example is the standard* $Q_1 - P_0$ *approximation pair. A locally stabilized formulation (16) can be constructed in this case, if* $\tau_h$ *is such that the elements* $K$ *can be grouped into* $2 \times 2$ *macroelements* $\mathcal{K} = \{K_1,\ K_2,\ K_3,\ K_4\}$, *with the reference macroelement*

$$\hat{\mathcal{K}} = \{\hat{K}_1,\ \hat{K}_2,\ \hat{K}_3,\ \hat{K}_4\}.$$

*An obvious way of constructing such a partitioning in practice is to form the grid* $\tau_h$ *by uniformly refining a coarse grid* $\Lambda_h$, *for example, by joining the mid-edge points.*

**Example 3.2:** *The triangle approximation pair* $P_1 - P_0$ *can similarly be established if the partitioning* $\tau_h$ *is constructed such that the elements can be grouped into disjoint macroelements, all consisting of four elements.*

*For the above finite element spaces* $X_h$ *and* $M_h$, *it is well-known that the following approximation estimates*

$$|v - I_h v| + h\|v - I_h v\| \le ch^2|Av|, \quad \forall v \in D(A), \tag{17}$$

$$|q - J_h q| \le ch\|q\|_1, \quad \forall q \in H^1(\Omega) \cap M, \tag{18}$$

*and the inverse inequality*

$$\|v_h\| \le ch^{-1}|v_h|, \quad \forall v_h \in X_h, \tag{19}$$

*hold (see [1], [7], [10]), where* $(I_h, J_h) : (D(A), H^1(\Omega) \cap M) \to (X_h, M_h)$ *is the interpolation operator.*

*The following stability results and the continuous properties of these mixed methods for the macroelement partitioning defined above were established by Kay and Silvester [16], Kechkar and Silvester [17], and Babuška, Osborn and Pitkäranta [2].*

**Theorem 3.2:** *Given a stabilization parameter $\beta \geq \beta_0 > 0$, suppose that every macroelement $\mathcal{K} \in \Lambda_h$ belongs to one of the equivalence classes $\mathcal{E}_{\hat{\mathcal{K}}}$, and that the following macroelement connectivity condition is valid: for any two neighboring macroelements $\mathcal{K}_1$ and $\mathcal{K}_2$ with $\int_{\mathcal{K}_1 \cap \mathcal{K}_2} ds \neq 0$ there exists $v \in X_h$ such that*

$$\mathit{supp} v \subset \mathcal{K}_1 \cup \mathcal{K}_2 \text{ and } \int_{\mathcal{K}_1 \cap \mathcal{K}_2} v \cdot n ds \neq 0. \tag{20}$$

*Then,*

$$|\mathcal{C}_h(p,q)| \leq c \sum_{K \in \tau_h} (\int_K (|p|^2 + h^2|\nabla p|^2)dx)^{1/2} (\int_K (|q|^2 + h^2|\nabla q|^2)dx)^{1/2}, \tag{21}$$

*for all $p,\ q \in H^1(\Omega) + M_h$, and*

$$\alpha(\|u_h\| + |p_h|) \leq \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\mathcal{B}_h((u_h,p_h);(v_h,q_h))}{\|v_h\| + |q_h|}, \tag{22}$$

*for all $(u_h, p_h) \in (X_h, M_h)$, and*

$$\mathcal{C}_h(p,q_h) = 0,\ \mathcal{C}_h(p_h,q) = 0,\ \mathcal{C}_h(p,q) = 0,\ \forall p,q \in H^1(\Omega),\ \forall p_h,q_h \in M_h, \tag{23}$$

*where $\alpha > 0$ is a constant independent of $h$ and $\beta$, and $\beta_0$ is some fixed positive constant and $n$ is the outnormal vector.*

*Finally, we shall provide the existence and uniqueness of the solution $(u_h, p_h)$ for problem (16).*

**Theorem 3.3:** *Under the assumptions of Theorem 2.1 and Theorem 3.2, problem (16) admits a unique solution $(u_h, p_h) \in (X_h, M_h)$ satisfying*

$$\|u_h\| \leq \frac{\gamma_0}{\nu}|f|,\ \ |p_h| \leq \alpha^{-1}(c_0\nu^{-2}\gamma_0^3|f|^2 + \gamma_0|f|). \tag{24}$$

*Proof:* Let Hilbert space $H_h = (X_h, M_h)$ be with the scalar product and norm:

$$((v,q);(w,r))_{H_h} = ((v,w)) + (q,r),\ \|(v,q)\|_{H_h}^2 = \|v\|^2 + |q|^2,$$

and $K_h$ be a non-void, convex and compact subset of $H_h$ defined by

$$K_h = \left\{(v,q) \in H_h :\ \ \|v\| \leq \frac{\gamma_0}{\nu}|f|,\ \ |q| \leq \frac{c_0\gamma_0^3}{\alpha\nu^2}|f|^2 + \frac{\gamma_0}{\alpha}|f|\right\}.$$

We now define a continuous mapping from $K_h$ into $H_h$ as follows: Given $(\bar{v}, \bar{q}) \in K_h$ find $(v,q) = F(\bar{v},\bar{q})$ such that for all $(w,r) \in H_h$

$$\mathcal{B}_h((v,q);(w,r)) + b(\bar{v},v,w) = (f,w). \tag{25}$$

Taking $(w,r) = (v,q)$ in (25) and using (5)–(7) yields

$$\nu\|v\|^2 \leq \gamma_0|f|\|v\|;$$

and using (5), (7) and (21), (22), we obtain

$$\alpha(\|v\| + |q|) \leq \gamma_0|f| + c_0\gamma_0\|\bar{v}\|\|v\| \leq \gamma_0|f| + c_0\nu^{-2}\gamma_0^3|f|^2.$$

Hence, the two estimates imply $(v, q) = F(\bar{v}, \bar{q}) \in K_h$. By the fixed point theorem (see [10]), the mapping $(v, q) = F(\bar{v}, \bar{q})$ has at least one fixed point $(u_h, p_h) \in K_h$, namely, $(u_h, p_h) \in K_h$ is a stabilized finite element solution of problem (16).

Next, we shall prove that problem (16) has a unique solution $(u_h, p_h)$. In fact, if $(v_h, q_h)$ also satisfies formulation (16), then for all $(w, r) \in (X_h, M_h)$

$$\mathcal{B}_h((u_h - v_h, p_h - q_h); (w, r)) = b(v_h - u_h, u_h, w) + b(v_h, v_h - u_h, w). \quad (26)$$

Taking $(w, r) = (u_h - v_h, p_h - q_h)$ in (26) and using (5)–(8), it follows that

$$\nu\|u_h - v_h\|^2 \leq c_0\gamma_0\|u_h\|\|u_h - v_h\|^2 \leq c_0\frac{\gamma_0^2}{\nu}|f|\|u_h - v_h\|^2,$$

which together with the fact

$$\nu - c_0\frac{\gamma_0^2}{\nu}|f| = \nu(1 - c_0\frac{\gamma_0^2}{\nu^2}|f|) > 0,$$

gives $u_h = v_h$. Using again (26), (22) and (8), we obtain $\alpha|p_h - q_h| \leq 0$ which implies $p_h = q_h$. □

## 4. Error Estimates

In order to derive error estimates of the stabilized finite element solution $(u_h, p_h)$, we also need the Galerkin projection $(R_h, Q_h) : (X, M) \rightarrow (X_h, M_h)$ defined by

$$\mathcal{B}_h((R_h(v, q), Q_h(v, q)); (v_h, q_h)) = \mathcal{B}((v, q); (v_h, q_h)), \quad (27)$$

for each $(v, q) \in (X, M)$ and all $(v_h, q_h) \in (X_h, M_h)$. Note that, due to Theorem 3.2, $(R_h, Q_h)$ is well defined. Due to (23) in Theorem 3.2, there holds

$$\mathcal{B}_h((R_h(v, q), Q_h(v, q)); (v_h, q_h)) = \mathcal{B}_h((v, q); (v_h, q_h)), \quad (28)$$

for each $(v, q) \in (D(A), H^1(\Omega) \cap M)$ and $(v_h, q_h) \in (X_h, M_h)$.

By using an argument similar to the ones used by Layton and Tobiska in [21], the following approximate properties can be obtained.

**Lemma 4.1:** *Under the assumptions of Theorem 3.2, the projection $(R_h, Q_h)$ satisfies*

$$\|v - R_h(v, q)\| + |q - Q_h(v, q)| \leq c(\|v\| + |q|), \quad (29)$$

*for all $(v, q) \in (X, M)$ and*

$$|v - R_h(v, q)| + h\|v - R_h(v, q)\| + h|q - Q_h(v, q)| \leq ch^2(|Av| + \|q\|_1), \quad (30)$$

*for all $(v, q) \in (D(A), H^1(\Omega) \cap M)$.*

*Proof:* The stability of the projection follows simply by Theorem 3.2 and (6), namely

$$\begin{aligned}
&\|R_h(v,q)\| + |Q_h(v,q)| \\
&\leq \alpha^{-1} \sup_{(w_h,r_h)\in(X_h,M_h)} \frac{\mathcal{B}_h((R_h(v,q),\, Q_h(v,q));\, (w_h,r_h))}{\|w_h\| + |r_h|} \\
&\leq \alpha^{-1} \sup_{(w_h,r_h)\in(X_h,M_h)} \frac{\mathcal{B}((v,q);\, (w_h,r_h))}{\|w_h\| + |r_h|} \\
&\leq c(\|v\| + |q|),\ \forall (v,q) \in (X,M).
\end{aligned} \tag{31}$$

Now the triangle inequality gives

$$\|v - R_h(v,q)\| + |q - Q_h(v,q)| \leq c(\|v\| + |q|),\ \forall (v,q) \in (X,M), \tag{32}$$

which is (29). □

Next, let $(v,q) \in (D(A), H^1(\Omega) \cap M)$ and introduce the dual Stokes problem: find $(\Phi, \Psi) \in (X, M)$ such that

$$\mathcal{B}((w,r);(\Phi,\Psi)) = (w, v - R_h(v,q)),\ \ \forall (w,r) \in (X,M).$$

Using the the regularity Assumption 1, there holds

$$\|\Phi\|_2 + \|\Psi\|_1 \leq c|v - R_h(v,q)|. \tag{33}$$

Now, setting $w = v - R_h(v,q),\ r = q - Q_h(v,q)$ and using (6), (17), (18), (21), (23), (28) and (33), we obtain that for $(\Phi_h, \Psi_h) = (I_h\Phi, J_h\Psi) \in (X_h, M_h)$,

$$\begin{aligned}
|v - R_h(v,q)|^2 &= \mathcal{B}((v - R_h(v,q), q - Q_h(v,q)); (\Phi,\Psi)) \\
&= \mathcal{B}_h((v - R_h(v,q), q - Q_h(v,q)); (\Phi,\Psi)) \\
&= 7\mathcal{B}_h((v - R_h(v,q), q - Q_h(v,q)); (\Phi - \Phi_h, \Psi - \Psi_h)) \\
&\leq c(\|\Phi - \Phi_h\| + |\Psi - \Psi_h| + h|\nabla\Psi|) \\
&\quad \times(\|v - R_h(v,q)\| + |q - Q_h(v,q)| + h|\nabla q|) \\
&\leq ch(\|v - R_h(v,q)\| + |q - Q_h(v,q)| + h|\nabla q|)(\|\Phi\|_2 + \|\Psi\|_1) \\
&\leq ch(\|v - R_h(v,q)\| + |q - Q_h(v,q)| + h|\nabla q|)|v - R_h(v,q)|.
\end{aligned} \tag{34}$$

Then, using the standard interpolation $(I_h v, J_h p) \in (X_h, M_h)$, we deduce from Theorem 3.2 and (28) that,

$$\begin{aligned}
&\|I_h v - R_h(v,q)\| + |J_h q - Q_h(v,q)| \\
&\leq \alpha^{-1} \sup_{(w_h,r_h)\in(X_h,M_h)} \frac{\mathcal{B}_h((I_h v - R_h(v,q),\, J_h q - Q_h(v,q));\, (w_h,r_h))}{\|w_h\| + |r_h|} \\
&\leq \alpha^{-1} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\mathcal{B}_h((I_h v - v,\, J_h q - q);\, (w_h,r_h))}{\|w_h\| + |r_h|} \\
&\leq c\|I_h v - v\| + c|J_h q - q|.
\end{aligned}$$

Thus the triangles inequality and approximate properties (17), (18) give

$$\|v - R_h(v,q)\| + |q - Q_h(v,q)| + h|\nabla q| \leq ch(|Av| + \|q\|_1). \tag{35}$$

It now follows from (34) and (35) that

$$|v - R_h(v,q)| \leq ch^2(|Av| + \|q\|_1). \tag{36}$$

Thus, (35) and (36) imply (30).

Next, we will derive the following error estimates of the finite element solution $(u_h, p_h)$ defined in Sect. 3.

**Theorem 4.2:** *Assume that the assumptions of Theorem 2.1 and Theorem 3.2 hold. Then the stabilized finite element solution $(u_h, p_h)$ satisfies the error estimates:*

$$|u - u_h| + h(\|u - u_h\| + |p - p_h|) \leq ch^2. \tag{37}$$

*Proof:* From Theorem 2.1, we know $(u, p) \in (D(A) \cap V, H^1(\Omega) \cap M)$. Hence, we derive from (10), (16) and (28) that for all $(v_h, q_h) \in (X_h, M_h)$

$$\mathcal{B}_h((e_h, \eta_h); (v_h, q_h)) + b(u - u^h, u, v_h) + b(u_h, u - u^h, v_h) = 0, \tag{38}$$

where $e_h = R_h(u,p) - u_h$ and $\eta_h = Q_h(u,p) - p_h$. Taking $(v,q) = (e_h, \eta_h)$ in (38) and using (7), we obtain

$$\begin{aligned} &\nu\|e_h\|^2 + \beta_0 \mathcal{C}_h(\eta_h, \eta_h) + b(e_h, u, e_h) \\ &\quad \leq |b(u - R_h(u,p), u, e_h)| + |b(u_h, u - R_h(u,p), e_h)|. \end{aligned} \tag{39}$$

We find from (8), (12), (24) and (30) that

$$\begin{aligned} \nu\|e_h\|^2 - |b(e_h, u, e_h)| &\geq \nu\|e_h\|^2 - c_0\gamma_0\|u\|\|e_h\|^2 \\ &\geq \nu(1 - c_0\gamma_0|f|\nu^{-2})\|e_h\|^2, \end{aligned} \tag{40}$$

$$\begin{aligned} &|b(u_h, u - R_h(u,p), e_h)| + |b(u - R_h(u,p), u, e_h)| \\ &\quad \leq c_0\gamma_0(\|u\| + \|u_h\|)\|e_h\|\|u - R_h(u,p)\| \leq ch\|e_h\|. \end{aligned} \tag{41}$$

Combining the above estimates with (39) and using the uniqueness condition (11) yields

$$\|e_h\| \leq ch. \tag{42}$$

Moreover, by using (8), (9), (12), (30) and (42), we have

$$\begin{aligned} &|b(u_h, u - R_h(u,p), e_h)| + |b(u - R_h(u,p), u, e_h)| \\ &\quad \leq |b(u, u - R_h(u,p), e_h)| + |b(u - R_h(u,p), u, e_h)| \\ &\qquad + |b(u - R_h(u,p), u - R_h(u,p), e_h)| + |b(e_h, u - R_h(u,p), e_h)| \\ &\quad \leq c_1|Au||u - R_h(u,p)|\|e_h\| \\ &\qquad + c_0\gamma_0(\|u - R_h(u,p)\| + \|e_h\|)\|u - R_h(u,p)\|\|e_h\| \leq ch^2\|e_h\|. \end{aligned} \tag{43}$$

Combining (39, (40) with (43) gives

$$\|e_h\| \le ch^2, \tag{44}$$

Finally, one finds from (22), (30), (38), (44) and (12) that

$$\begin{aligned} |u-u_h| &\le |e_h| + |u - R_h(u,p)| \le \gamma_0\|e_h\| + ch^2(|Au| + \|p\|_1) \le ch^2,\\ \|u-u_h\| &\le \|e_h\| + \|u - R_h(u,p)\| \le ch^2 + ch(|Au| + \|p\|_1) \le ch,\\ |p-p_h| &\le |p - Q_h(u,p)| + |\eta_h|\\ &\le ch(|Au| + \|p\|_1) + \alpha^{-1}c(\|e_h\| + \|u - R_h(u,p)\|) \le ch. \end{aligned}$$

Hence, (37) follows. □

## 5. Two-level Stabilized Finite Element Approximations

From now on, $H$ and $h \ll H$ will be two real positive parameter tending to 0. Also, a coarse mesh triangulation of $\tau_H(\Omega)$ of $\Omega$ is made as like in Sect. 3 such that the macroelement connectivity condition stated in Theorem 3.2 is valid. And a fine mesh triangulation $\tau_h(\Omega)$ is generated by a mesh refinement process to $\tau_H(\Omega)$. The conforming finite element space pairs $(X_h, M_h)$ and $(X_H, M_H) \subset (X_h, M_h)$ based on the triangulations $\tau_h(\Omega)$ and $\tau_H(\Omega)$, respectively, are constructed as like in Sect. 3. With the above finite element space pair, we will consider the following two-level stabilized finite element methods.

### *5.1. Simple Two-level Stabilized Finite Element Approximation*

Step I: Solve the Navier–Stokes problem on a coarse mesh, i.e., find $(u_H, p_H) \in (X_H, M_H)$ such that for all $(v_H, q_H) \in (X_H, M_H)$

$$\mathcal{B}_H((u_H, p_H); (v_H, q_H)) + b(u_H, u_H, v_H) = (f, v_H). \tag{45}$$

Step II: Solve the Stokes problem on a fine mesh, i.e., find $(u^h, p^h) \in (X_h, M_h)$ such that for all $(v_h, q_h) \in (X_h, M_h)$

$$\mathcal{B}_h((u^h, p^h); (v_h, q_h)) + b(u_H, u_H, v_h) = (f, v_h). \tag{46}$$

Next, will study the convergence of $(u^h, p^h)$ to $(u, p)$ in some norms. To do this, let us set $e_h = R_h(u,p) - u^h$, $\eta_h = Q_h(u,p) - p^h$. Then we see from (10), (46) and (28) that $(e_h, \eta_h)$ satisfies for all $(v_h, q_h) \in (X_h, M_h)$

$$\begin{aligned} &\mathcal{B}_h((e_h, \eta_h); (v_h, q_h)) + b(u - u_H, u, v_h) + b(u, u - u_H, v_h)\\ &\quad + b(u_H - u, u - u_H, v_h) = 0. \end{aligned} \tag{47}$$

**Theorem 5.1:** *Under the assumptions of Theorem 2.1 and Theorem 3.2 for $H$ and $h$, the simple two-level stabilized finite element solution $(u^h, p^h)$ satisfies the following error estimates*

$$\|u - u^h\| + |p - p^h| \le c(h + H^2). \tag{48}$$

*Proof:* By using (8), (9), (12), (22) and (37), it follows from (47) that

$$\alpha(\|e_h\| + |\eta_h|) \leq \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\mathcal{B}_h((e_h,\eta_h);(v_h,q_h))}{\|v_h\| + |q_h|}$$
$$\leq c|Au||u - u_H| + c\|u - u_H\|^2 \leq cH^2. \tag{49}$$

Thanks to (30) and (49), one finds

$$\|u - u^h\| + |p - p^h| \leq \|u - R_h(u,p)\| + |p - Q_h(u,p)| + \|e_h\| + |\eta_h|$$
$$\leq c(h + H^2). \tag{50}$$

### 5.2. Newton Two-level Stabilized Finite Element Approximation

Step I: Solve the Navier–Stokes problem on a coarse mesh, i.e., find $(u_H, p_H) \in (X_H, M_H)$ by (45).

Step II: Solve the general Stokes problem on a fine mesh, i.e., apply one Newton step to find $(u^h, p^h) \in (X_h, M_h)$ such that for all $(v_h, q_h) \in (X_h, M_h)$

$$\mathcal{B}_h((u^h,p^h);(v_h,q_h)) + b(u_H, u^h, v_h) + b(u^h, u_H, v_h)$$
$$= (f, v_h) + b(u_H, u_H, v_h). \tag{51}$$

Next, we will study the convergence of the Newton two-level stabilized finite element solution $(u^h, p^h)$ to $(u, p)$ in some norms. To do this, let us set $e_h = R_h(u,p) - u^h$, $\eta_h = Q_h(u,p) - p^h$. Then we see from (10), (51) and (28) that $(e_h, \eta_h)$ satisfies for all $(v_h, q_h) \in (X_h, M_h)$

$$\mathcal{B}_h((e_h,\eta_h);(v_h,q_h)) = -b(u - u^h, u, v_h) - b(u^h, u - u^h, v_h)$$
$$-b(u^h - u_H, u^h - u_H, v_h). \tag{52}$$

**Theorem 5.2:** *Under the assumptions of Theorem 2.1 and Theorem 3.2 for $H$ and $h$, the Newton two-level stabilized finite element solution $(u^h, p^h)$ satisfies the following error estimates*

$$\|u - u^h\| + |p - p^h| \leq c(h + |\log h|^{1/2} H^3). \tag{53}$$

*Proof:* Taking $(v_h, q_h) = (e_h, \eta_h)$ in (52) and using (7), we obtain

$$\nu\|e_h\|^2 = -b(u - u^h, u, e_h) - b(R_h(u,p), u - R_h(u,p), e_h)$$
$$+b(e_h, u - R_h(u,p), e_h) - b(u^h - u_H, R_h(u,p) - u_H, e_h)$$
$$= -b(u - R_h(u,p), u, e_h) - b(R_h(u,p), u - R_h(u,p), e_h)$$
$$-b(e_h, u_H, e_h) + b(R_h(u,p) - u_H, e_h, R_h(u,p) - u_H). \tag{54}$$

From Ref. [11], there holds

$$|b(u_h, v_h, w_h)| \leq c|\log h|^{1/2}\|u_h\|\|v_h\||w_h|, \quad \forall u_h, v_h, w_h \in X_h. \tag{55}$$

Hence, by using (8), (9), (12), (24), (29) and (55), we have

$$|b(e_h, u_H, e_h)| \le c_0\gamma_0|f|\nu^{-1}\|e_h\|^2,$$

$$|b(u - R_h u, u, e_h)| + |b(R_h(u,p), u - R_h(u,p), e_h)| \tag{56}$$

$$\le c\|u - R_h(u,p)\|(\|u\| + \|R_h(u,p)\|)\|e_h\|$$
$$\le c\|u - R_h(u,p)\|\|e_h\|, \tag{57}$$

$$|b(R_h(u,p) - u_H, e_h, R_h(u,p) - u_H)|$$
$$\le c|\log h|^{1/2}\|R_h(u,p) - u_H\||R_h(u,p) - u_H|\|e_h\|. \tag{58}$$

Combining (54) with (56)–(58) and using the uniqueness condition (11), one finds

$$\|e_h\| \le c\|u - R_h(u,p)\| + c|\log h|^{1/2}\|R_h(u,p) - u_H\||R_h(u,p) - u_H|. \tag{59}$$

Combining (59) with (30) and using (37) and using (12) yields

$$\begin{aligned}\|e_h\| &\le c\|u - R_h(u,p)\| + c|\log h|^{1/2}\|R_h(u,p) - u_H\||R_h(u,p) - u_H| \\ &\le ch + c|\log h|^{1/2}H^3.\end{aligned} \tag{60}$$

Finally, by applying (22) to (52) and using (5), (8), (12), (24), (55), (60) and (30) yields

$$\begin{aligned}|\eta_h| &\le c(\|u\| + \|e_h\| + \|R_h u^h\|)(\|u - R_h(u,p)\| + \|e_h\|) \\ &\quad + c(\|e_h\| + \|R_h(u,p) - u_H\|)\|e_h\| \\ &\quad + c|\log h|^{1/2}\|R_h(u,p) - u_H\||R_h(u,p) - u_H| \\ &\le ch + c|\log h|^{1/2}H^3.\end{aligned} \tag{61}$$

Combining (60), (61) with (30) and using (12) yields (53).

## Acknowledgments

This work is supported by the special funds for Major State Basic Research Projects G1999032801-07, NSF of China 10371095. The authors would like to thank the referees for their criticism, valuable comments and suggestions which helped to improve the results of this paper.

## References

[1] Ait Ou Ammi, A., Marion, M.: Nonlinear Galerkin methods and mixed finite elements: Two-grid algorithms for the Navier–Stokes equations. Numer. Math. *68*, 189–213 (1994).
[2] Babuška, I., Osborn, J., Pitkäranta, J.: Analysis of mixed methods using mesh dependent norms. Math. Comp. *35*, 1039–1062 (1980).
[3] Boland, J., Nicolaides, R. A.: Stability of finite elements under divergence constraints. SIAM J. Numer. Anal. *20*, 722–731 (1983).
[4] Bercovier, J., Pironneau, O.: Error estimates for finite element solution of the Stokes problem in the primitive variables. Numer. Math. *33*, 211–224 (1979).
[5] Bramble, J. H., Pasciak, J. E.: A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. Math. Comp. *50*, 1–17 (1988).
[6] Brezzi, F., Pitkäranta, J.: On the stabilisation of finite element approximations of the Stokes problems. Efficient solutions of elliptic systems (Hackbusch, W., ed.), pp. 11–19. Notes on Numerical Fluid Mechanics, vol. 10. Braunschweig: Vieweg 1984.

[7] Ciarlet, P. G.: The finite element method for elliptic problems. Amsterdam: North-Holland 1978.
[8] Douglas, Jr. J., Wang, J.: A absolutely stabilized finite element method for the Stokes problem. Math. Comp. *52*, 495–508 (1989).
[9] Ervin, V., Layton, W., Maubach, J.: A posteriori error estimators for a two-levelfinite element method for the Navier–Stokes equations. Numer. Meth. PDEs *12*, 333–346 (1996).
[10] Girault, V., Raviart, P. A.: Finite element method for Navier–Stokes equations: theory and algorithms. Berlin Heidelberg: Springer 1986.
[11] Yinnian He.: A fully discrete stabilized finite-element method for the time-dependent Navier–Stokes problem. IMA J. Numer. Anal. *23*, 665–691 (2003).
[12] Yinnian He., Kaitai Li.: Convergence and stability of finite element nonlinear Galerkin method for the Navier–Stokes equations. Numer. Math. *79*, 77–107 (1998).
[13] Hughes, T. J. R., Franca, L. P.: A new finite element formulation for CFD: VII. The Stokes problem with various well-posed boundary conditions: Symmetric formulations that converge for all velocity/pressure spaces. Comp. Meth. Appl. Mech. Engng. *65*, 85–97 (1987).
[14] Heywood, J. G., Rannacher, R.: Finite element approximation of the nonstationary Navier–Stokes problem I: Regularity of solutions and second-order error estimates for spatial discretization. SIAM J. Numer. Anal. *19*, 275–311 (1982).
[15] Kellogg, R. B., Osborn, J. E.: A regularity result for the Stokes problem in a convex polygon. J. Funct. Anal. *21*, 397–431 (1976).
[16] Kay, D., Silvester, D.: A posteriori error estimation for stabilized mixed approximations of the Stokes equations. SIAM J. Sci. Comput. *21*, 1321–1337 (2000).
[17] Kechkar, N., Silvester, D.: Analysis of locally stabilized mixed finite element methods for the Stokes problem. Math. Comp. *58*, 1–10 (1992).
[18] Ladyzenskaja, O. A.: Mathematical questions of viscous non-compressible fluid dynamics. M. Nauka, 1970, pp. 288.
[19] Layton, W.: A two level discretization method for the Navier–Stokes equations. Comput. Math. Appl. *26*, 33–38 (1993).
[20] Layton, W., Lenferink, W.: Two-level Picard, defect correction for the Navier–Stokes equations. Appl. Math. Comput. *80*, 1–12 (1995).
[21] Layton, W., Tobiska, L.: A two-level method with backtracking for the Navier–Stokes equations. SIAM J. Numer. Anal. *35*, 2035–2054 (1998).
[22] Marion, M., Xu, J.: Error estimates on a new nonlinear Galerkin method based on two-grid finite elements. SIAM J. Numer. Anal. *32*, 1170–1184 (1995).
[23] Sani, R. L., Gresho, P. M., Lee, R. L., Griffiths, D. F.: The cause and cure(?) of the spurious pressures generated by certain finite element method solutions of the incompressible Navier–Stokes equations. Parts 1 and 2. Int. J. Numer. Meth. Fluids *1*, 17–43; 171–204 (1981).
[24] Silvester, D. J., Kechkar, N.: Stabilized bilinear-constant velocity-pressure finite elements for the conjugate gradient solution of the Stokes problem. Comp. Meth. Appl. Mech. Engng. *79*, 71–87 (1990).
[25] Stenberg, R.: Analysis of mixed finite elements for the Stokes problem: A unified approach. Math. Comp. *42*, 9–23 (1984).
[26] Temam, R.: Navier–Stokes equations, theory and numerical analysis, 3rd ed. Amsterdam: North-Holland 1983.
[27] Verfürth, R.: A multilevel algorithm for mixed problems. SIAM J. Numer. Anal. *21*, 264–271 (1984).
[28] Xu, J.: A novel two-grid method for semilinear elliptic equations. SIAM J. Sci. Comput. *15*, 231–237 (1994).
[29] Xu, J.: Two-grid discretization techniques for linear and nonlinear PDEs. SIAM J. Numer. Anal. *33*, 1759–1778 (1996).

Yinnian He
Faculty of Science
Xi'an Jiaotong University
Xi'an 710049
P. R. China
e-mail: heyn@mail.xjtu.edu.cn

Kaitai Li
Faculty of Science
Xi'an Jiaotong University
Xi'an 710049
P. R. China
e-mail: ktli@mail.xjtu.edu.cn