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On the Box Dimension of Typical Measures

By

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Abstract. Let (X, ρ) be a complete metric space and let $\mathscr{M}$ be the space of all probability Borel measures on X. We give some estimations of the upper and lower box dimensions of the typical (in the sense of Baire category) measure in $\mathscr{M}$.

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1. Introduction

In the analytic theory of the dimension of sets and measures one of the most frequently used notions is the box dimension (called also Minkowski or entropy dimension). For some recent results concerning box dimension see [1, 5, 6–8, 13, 17]. The early bibliographical references can be found in the monographs [2, 9, 14]. In this note we study some typical properties of the box dimension of measures.

Recall that a subset of a metric space is called of the *first Baire category*, if it can be represented as a countable union of nowhere dense sets. A subset of a complete metric space $\mathscr{X}$ is called *residual*, if its complement is of the first Baire category. If the set of all elements of $\mathscr{X}$ satisfying some property P is residual in $\mathscr{X}$, then property P is called *generic* or *typical*. We also say that the typical elements of $\mathscr{X}$ has property P.

The typical properties of dimensions of sets have been investigated in [3, 5, 10, 11, 16]. Some typical (but in the sense of prevalence) properties of dimensions of measures were established in [15]. While, the typical properties of measures in the sense of Baire category were studied in [4]. More precisely, Genyuk proved that for a typical probability measure on a Polish space X, the lower local dimension is equal to 0 and the upper local dimension is equal to ∞, for all x except a set of first category. (Here by lower/upper local dimension of a measure μ we mean the lower/upper limit of $\log\mu(B(x,r))/\log r$ as $r \to 0$, see [12].) Further, she shows that the Hausdorff dimension of a typical measure is equal to 0. In this note we prove that for a typical probability measure defined on a complete metric space,

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the lower box dimension is equal to 0 whereas the upper box dimension is greater than or equal to the smallest local upper box dimension of X. Moreover, we show that for a typical probability measure on a complete separable metric space its support is equal to the whole space.

2. Notation

Let (X,ρ) be a complete metric space and let $B(x,r)$ denote the closed ball in X with centre at x and radius $r>0$. By $\mathscr{B}(X)$ we denote the σ-algebra of Borel subsets of X and by $\mathscr{M}$ we denote the set of all probability Borel measures on X. As usual, for $A\subset X$ the symbol $\overline{A}$ stands for the closure of A, the symbol card A stands for the cardinality of A and for $x\in X$ the symbol δ_x stands for the delta Dirac measure supported at x.

Throughout this paper we assume that the space $\mathscr{M}$ is endowed with the *Fortet-Mourier distance* denoted dist and given by the formulae

$$\operatorname{dist}(\mu_1,\mu_2)=\sup\left\{\left|\int_X f(x)\,\mu_1(dx)-\int_X f(x)\,\mu_2(dx)\right| : f\in\mathscr{L}\right\}, \tag{1}$$

for $\mu_1,\mu_2\in\mathscr{M}$. Here $\mathscr{L}$ is the subset of $C(X)$ which contains all the functions f such that

$$|f(x)|\leqslant 1 \quad \text{and} \quad |f(x)-f(y)|\leqslant\rho(x,y) \quad \text{for} \quad x,y\in X. \tag{2}$$

It can be proved that the sequence (μ_n), $\mu_n\in\mathscr{M}$, is weakly convergent to a measure $\mu\in\mathscr{M}$, if and only if

$$\lim_{n\to\infty}\operatorname{dist}(\mu_n,\mu)=0.$$

It is well known that the space $\mathscr{M}$ endowed with the metric *dist* is complete.

We recall that the *lower* and *upper box dimensions* of a set $E\subset X$ are defined, respectively, by the formulae

$$\underline{\dim}\,E=\liminf_{r\to 0^+}\frac{\log N(E,r)}{\log(1/r)}, \tag{3}$$

$$\overline{\dim}\,E=\limsup_{r\to 0^+}\frac{\log N(E,r)}{\log(1/r)}, \tag{4}$$

where $N(E,r)$ is the least number of closed balls of radius r which cover the set E. Note that if the closure of E is non-compact then $\overline{\dim}\,E=\underline{\dim}\,E=\infty$. Note also that lower and upper box dimensions of a set are invariant under topological closure.

Remark 1. In the definition of box dimensions we can replace the number $N(E,r)$ by $M(E,r)$ – the greatest possible number of disjoint closed balls of radius r that may be found with centres in E.

Let $\mu\in\mathscr{M}$. The quantities

$$\underline{\dim}\,\mu=\lim_{\kappa\to 0^+}\inf\{\underline{\dim}\,E : E\in\mathscr{B}(X),\ \mu(E)\geqslant 1-\kappa\} \tag{5}$$

and

$$\overline{\dim}\,\mu=\lim_{\kappa\to 0^+}\inf\{\overline{\dim}\,E : E\in\mathscr{B}(X),\ \mu(E)\geqslant 1-\kappa\} \tag{6}$$

are called the *lower* and *upper box dimension* of μ, respectively.

3. Results

Theorem 1. *The set*

$$\{\mu \in \mathscr{M} : \underline{\dim}\, \mu = 0\}$$

is residual in the space $\mathscr{M}$.

Proof. Denote by $\mathscr{F}$ the set of all measures $\mu \in \mathscr{M}$ with finite supports. For $\nu \in \mathscr{F}$ we define

$$\mathscr{D}_n(\nu) = \{\mu \in \mathscr{M} : \operatorname{dist}(\mu, \nu) < 3^{-k-n}\},$$

where $k = \operatorname{card}(\operatorname{supp} \nu)$.

Let

$$\mathscr{D} = \bigcap_{n=1}^{\infty} \mathscr{D}_n \quad \text{where} \quad \mathscr{D}_n = \bigcup_{\nu \in \mathscr{F}} \mathscr{D}_n(\nu).$$

Since for each $n \in \mathbb{N}$ the set $\mathscr{D}_n$ is open and dense in the space $\mathscr{M}$, the set $\mathscr{D}$ is residual. To complete the proof it is sufficient to show that $\underline{\dim}\, \mu = 0$ for every $\mu \in \mathscr{D}$. Let $\mu \in \mathscr{D}$ be given. Since $\mu \in \mathscr{D}$, for every $n \in \mathbb{N}$ there exists a measure $\nu_n \in \mathscr{F}$ such that

$$\operatorname{dist}(\mu, \nu_n) < 3^{-k(n)-n}, \tag{7}$$

where

$$k(n) = \operatorname{card} A_n \quad \text{and} \quad A_n = \operatorname{supp} \nu_n.$$

Now define

$$E_n = \{x \in X : \rho(x, A_n) \leqslant 2^{-k(n)-n}\}.$$

Observe that

$$\mu(E_n) > 1 - \left(\frac{2}{3}\right)^n. \tag{8}$$

Indeed, let $f(x) = \max\{2^{-k(n)-n} - \rho(x, A_n), 0\}$. Since $f \in \mathscr{L}$ we have

$$\begin{aligned} \operatorname{dist}(\mu, \nu_n) &\geqslant \int_X f(x)\, d\nu_n(x) - \int_X f(x)\, d\mu(x) \\ &\geqslant 2^{-k(n)-n} \nu_n(A_n) - 2^{-k(n)-n} \mu(E_n). \end{aligned} \tag{9}$$

By (9) and (7) we receive

$$\mu(E_n) \geqslant \nu_n(A_n) - 2^{k(n)+n} \operatorname{dist}(\mu, \nu_n) > 1 - \left(\frac{2}{3}\right)^{k(n)+n},$$

whence (8) follows immediately. Let $\kappa > 0$ be given. Using (8) one can easily verify that there exists $n_0 \in \mathbb{N}$ such that $\mu(E) \geqslant 1 - \kappa$, where $E = \bigcap_{n=n_0}^{\infty} E_n$. We check that $\underline{\dim}\, E = 0$. Indeed, set $\varepsilon_n = 2^{-k(n)-n}$. Let $A_n = \{x_1, \ldots, x_{k(n)}\}$. Since

$$E_n = \bigcup_{i=1}^{k(n)} B(x_i, 2^{-k(n)-n})$$

and $E \subset E_n$ for $n \geqslant n_0$, we have $N(E, \varepsilon_n) \leqslant k(n)$ for $n \geqslant n_0$. Thus we have

$$0 \leqslant \underline{\dim} E \leqslant \lim_{n\to\infty} \frac{\log N(E,\varepsilon_n)}{\log(1/\varepsilon_n)} \leqslant \lim_{n\to\infty} \frac{\log k(n)}{(n+k(n))\log 2} = 0.$$

From this and (5) it follows that $\underline{\dim}\,\mu = 0$. This completes the proof. □

Now we will formulate a result concerning the upper box dimension of the typical measure.

Let λ_0 denote the smallest local upper box dimension of X, i.e. the quantity given by the formula

$$\lambda_0 = \lambda_0(X) = \inf\{\overline{\dim}\, B(x,r) : x \in X,\ r > 0\}. \tag{10}$$

Theorem 2. *The set*

$$\{\mu \in \mathcal{M} : \inf\{\overline{\dim}\, E : \mu(E) > 0\} \geqslant \lambda_0\}.$$

is residual in $\mathcal{M}$.

Proof. Without any loss of generality we can assume that $\lambda_0 > 0$. Fix $\lambda \in (0, \lambda_0)$ and $\kappa > 0$. Let $x \in X$ and $r, \varepsilon > 0$. By $M(x, r; \varepsilon)$ we denote the greatest possible number of disjoint closed balls of radius ε that may be found with centres in the ball $B(x, r)$. Since $\overline{\dim}\, B(x, r) > \lambda$, from (3) and Remark 1 it follows that there is $\varepsilon \in (0, 1)$ such that

$$M(x, r; \varepsilon) \geqslant \varepsilon^{-\lambda}. \tag{11}$$

For every $x \in X$ and $r > 0$ we fix an ε satisfying (11) and we will denote it by $\varepsilon(x, r)$. It is obvious that $\varepsilon(x, r) < 2r$. Now, in the ball $B(x, r)$ we choose the points $y_1, \ldots, y_M$, where $M = M(x, r; \varepsilon(x, r))$, such that

$$B(y_i, \varepsilon(x,r)) \cap B(y_j, \varepsilon(x,r)) = \emptyset \quad \text{for} \quad i, j \in \{1, \ldots, M\},\ i \neq j$$

and we define

$$\mu_{x,r} = \frac{1}{M}(\delta_{y_1} + \cdots + \delta_{y_M}).$$

Clearly $\mu \in \mathcal{M}$. By **Fin** we denote the family of all finite subsets of X. For given $A \in \mathbf{Fin}$ and $r > 0$, we denote by $k(A)$ the number of elements of A and we define

$$\mu_{A,r} = \frac{1}{k(A)} \sum_{x\in A} \mu_{x,r},$$

$$\alpha(A, r) = \frac{\kappa}{6} \min_{x\in A} \varepsilon(x, r),$$

$$\mathcal{M}(A, r) = \{\mu \in \mathcal{M} : \operatorname{dist}(\mu, \mu_{A,r}) < \alpha(A, r)\}.$$

Let

$$\mathcal{M}(r) = \bigcup_{A\in\mathbf{Fin}} \mathcal{M}(A, r) \quad \text{and} \quad \mathcal{G}_m = \bigcup_{n=m}^{\infty} \mathcal{M}\left(\frac{1}{n}\right).$$

Since for all n the sets $\mathscr{M}\left(\frac{1}{n}\right)$ are open, also the sets $\mathscr{G}_m$ are open. Moreover the sets $\mathscr{G}_m$ are dense in $\mathscr{M}$. Indeed, since $\lambda_0 > 0$ the space X has no isolated points. This implies that the set of all measures of the form $\mu_A = \frac{1}{k(A)}\sum_{x\in A}\delta_x$, where $A\in\mathbf{Fin}$, is dense in $\mathscr{M}$. Since $\mathrm{dist}(\mu_A, \mu_{A,\frac{1}{n}}) < \frac{1}{n}$ the set $\mathscr{F}_m = \bigcup_{n=m}^{\infty}\{\mu_{A,\frac{1}{n}} : A\in\mathbf{Fin}\}$ is also dense in $\mathscr{M}$. As $\mathscr{F}_m \subset \mathscr{G}_m$, the set $\mathscr{G}_m$ is dense in $\mathscr{M}$. Recall that $\mu_{A,r}$ and, consequently, $\mathscr{M}(r)$ and $\mathscr{G}_m$ were constructed for fixed $\lambda\in(0,\lambda_0)$ and $\kappa > 0$. Now, for such λ and κ we define

$$\mathscr{G}(\lambda,\kappa) = \bigcap_{m=1}^{\infty}\mathscr{G}_m. \tag{12}$$

The set $\mathscr{G}(\lambda,\kappa)$ is residual in the space $\mathscr{M}$, because for each $m\in\mathbb{N}$ the set $\mathscr{G}_m$ is open and dense in $\mathscr{M}$.

Claim. *If $\mu\in\mathscr{G}(\lambda,\kappa)$ and $E\in\mathscr{B}(X)$ be such that $\mu(E)\geqslant\kappa$ then $\overline{\dim}\,E\geqslant\lambda$.*

Indeed, let $\mu\in\mathscr{G}(\lambda,\kappa)$ and $E\in\mathscr{B}(X)$. Since $\overline{\dim}\,E = \overline{\dim}\,\overline{E}$, where $\overline{E}$ is the closure of E, we can assume that E is a closed set. From (12) it follows that for every $m\in\mathbb{N}$ there exist an integer number $n\geqslant m$ and a set $A\in\mathbf{Fin}$ such that $\mathrm{dist}(\mu,\mu_{A,1/n}) < \alpha(A,\frac{1}{n})$.

Define

$$E_c = \{x\in X : \rho(x,E)\leqslant c\}, \quad \text{where} \quad c = \frac{2}{\kappa}\alpha\left(A,\frac{1}{n}\right).$$

Let $f(x) = \max\{c-\rho(x,E),0\}$. Since $f\in\mathscr{L}$ we have

$$\mathrm{dist}(\mu,\mu_{A,1/n}) \geqslant \int_X f(x)\,d\mu(x) - \int_X f(x)\,d\mu_{A,1/n}(x) \geqslant c\mu(E) - c\mu_{A,1/n}(E_c).$$

This implies that

$$\mu_{A,1/n}(E_c) \geqslant \mu(E) - \frac{1}{c}\mathrm{dist}(\mu,\mu_{A,1/n}) > \kappa - \frac{1}{c}\alpha\left(A,\frac{1}{n}\right) \geqslant \frac{1}{2}\kappa.$$

From the definition of the measure $\mu_{A,r}$ it follows that there exists $x\in A$ such that $\mu_{x,1/n}(E_c) > \frac{1}{2}\kappa$. In turn, this implies that at least $\frac{1}{2}\kappa M$ points from the set $\{y_1,\ldots,y_M\}$ belongs to E_c. Consequently

$$M\left(E_c,\varepsilon\left(x,\frac{1}{n}\right)\right) \geqslant \frac{1}{2}\kappa M\left(x,\frac{1}{n};\,\varepsilon\left(x,\frac{1}{n}\right)\right)$$

and by (11) we obtain

$$M\left(E_c,\varepsilon\left(x,\frac{1}{n}\right)\right) \geqslant \frac{1}{2}\kappa\left(\varepsilon\left(x,\frac{1}{n}\right)\right)^{-\lambda}.$$

Since $c\leqslant\frac{1}{3}\varepsilon\left(x,\frac{1}{n}\right)$, we receive

$$M\left(E,\frac{1}{3}\varepsilon\left(x,\frac{1}{n}\right)\right) \geqslant \frac{1}{2}\kappa\left(\varepsilon\left(x,\frac{1}{n}\right)\right)^{-\lambda}. \tag{13}$$

Since $\varepsilon\big(x,\frac{1}{n}\big)$ in (13) can be arbitrary small, we have

$$\overline{\dim}\,E=\limsup_{\varepsilon\to 0^+}\frac{\log M(E,\varepsilon)}{-\log(\varepsilon)}\geqslant\lim_{\varepsilon\to 0^+}\frac{\log(\frac{1}{2}\kappa\varepsilon^{-\lambda})}{-\log(\varepsilon/3)}=\lambda, \tag{14}$$

which completes the proof of the Claim.

Now, let (λ_n) be an increasing sequence of positive numbers convergent to λ_0 and let (κ_n) be a decreasing sequence of positive numbers convergent to 0. Then the set

$$\mathscr{G}=\bigcap_{n=1}^{\infty}\mathscr{G}(\lambda_n,\kappa_n)$$

is residual in $\mathscr{M}$ and

$$\inf\{\overline{\dim}\,E:\mu(E)>0\}\geqslant\lambda_0$$

for $\mu\in\mathscr{G}$. This completes the proof. □

From Theorem 2 follows immediately

Corollary 1. *For the typical measure μ in $\mathscr{M}$ we have $\overline{\dim}\,\mu\geqslant\lambda_0$, where λ_0 is given by* (10).

Remark 2. A result similar to Theorem 2 concerning the box and packing dimensions of the typical compact subsets of a complete metric space X were obtained in [11]. It was proved that if the space of all nonempty compact subsets of X is equipped with the Hausdorff metric then the typical compact set has the upper box and packing dimensions at least λ_0, where λ_0 is given by (10).

Theorem 3. *Assume that (X,ρ) is a complete separable metric space. Then the set*

$$\{\mu\in\mathscr{M}:\operatorname{supp}\mu=X\}$$

is residual in $\mathscr{M}$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X such that for each n_0 the subsequence $(x_n)_{n\geqslant n_0}$ is dense in X. Let $\mathbf{P}$ be the set of all real sequences $\mathbf{p}=(p_n)_{n\in\mathbb{N}}$ such that $p_n>0$ for all $n\in\mathbb{N}$ and $\sum_{n=1}^{\infty}p_n=1$. For each $\mathbf{p}\in\mathbf{P}$ we define a measure $\mu_{\mathbf{p}}$ by $\mu_{\mathbf{p}}=\sum_{n=1}^{\infty}p_n\delta_{x_n}$. For $\mathbf{p}\in\mathbf{P}$ and $n\in\mathbb{N}$ we define

$$\alpha(\mathbf{p},n)=\frac{1}{n}\min_{1\leqslant i\leqslant n}p_i,$$

$$\mathscr{A}_n(\mathbf{p})=\{\mu\in\mathscr{M}:\operatorname{dist}(\mu,\mu_{\mathbf{p}})<\alpha(\mathbf{p},n)\}.$$

Let

$$\mathscr{A}_n=\bigcup_{\mathbf{p}\in\mathbf{P}}\mathscr{A}_n(\mathbf{p})\quad\text{and}\quad\mathscr{A}=\bigcap_{n=1}^{\infty}\mathscr{A}_n.$$

Since for each $n\in\mathbb{N}$ the set $\mathscr{A}_n$ is dense and open in the space $\mathscr{M}$, the set $\mathscr{A}$ is residual in $\mathscr{M}$. Let $\mu\in\mathscr{A}$. We check that $\operatorname{supp}\mu=X$. Fix a point $y\in X$ and a number $r \in (0,1)$. Let i and n be integers such that $\rho(x_i, y) < r/2$ and $n \geqslant \max\{i, 2/r\}$. Take a measure $\mu_{\mathbf{p}} \in \mathscr{A}_n$ such that

$$\mathrm{dist}(\mu, \mu_{\mathbf{p}}) < \alpha(\mathbf{p}, n). \tag{15}$$

Since the function $f(x) = \max\{r/2 - \rho(x, x_i), 0\}$ belongs to the set $\mathscr{L}$, we have

$$\mathrm{dist}(\mu, \mu_{\mathbf{p}}) \geqslant \int_X f(x)\, d\mu_{\mathbf{p}}(x) - \int_X f(x)\, d\mu(x) \geqslant \frac{r}{2}\mu_{\mathbf{p}}(x_i) - \frac{r}{2}\mu\left(B\left(x_i, \frac{r}{2}\right)\right).$$

As $\mu_{\mathbf{p}}(x_i) = p_i$, the last inequality implies

$$\mu\left(B\left(x_i, \frac{r}{2}\right)\right) \geqslant p_i - \frac{2}{r}\mathrm{dist}(\mu, \mu_{\mathbf{p}}). \tag{16}$$

Since $\alpha(\mathbf{p}, n) \leqslant \frac{1}{n}p_i \leqslant \frac{r}{2}p_i$, inequalities (15) and (16) give

$$\mu\left(B\left(x_i, \frac{r}{2}\right)\right) \geqslant p_i - \frac{2}{r}\mathrm{dist}(\mu, \mu_{\mathbf{p}}) > p_i - \frac{2}{r}\alpha(\mathbf{p}, n) \geqslant 0.$$

From this and inequality $\rho(x_i, y) < r/2$ it follows that $\mu(B(y, r)) > 0$. Since $y \in X$ and $r > 0$ were arbitrary, this shows that $\mathrm{supp}\,\mu = X$. The proof is completed. □

From Theorem 3 it follows immediately

Corollary 2. *Let (X, ρ) be a complete separable metric space. Then for a typical measure μ in $\mathscr{M}$ we have*

$$\underline{\dim}\, E = \underline{\dim}\, X \quad \textit{and} \quad \overline{\dim}\, E = \overline{\dim}\, X,$$

for every $E \in \mathscr{B}(X)$ such that $\mu(E) = 1$.

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