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A Mean Value Theorem for Dirichlet Series and a General Divisor Problem

By

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Dedicated to Professor Takeshi Kano on the occasion of his sixtieth birthday

Abstract. We consider the mean value formula for general Dirichlet series by applying the approximate functional equation of Ramachandra's type, and derive the sum formula for its coefficients. The improvement of Landau's classical results is established for the general divisor function. We also obtain the asymptotic behaviour of the mean value when the real part of s is near the abscissa of absolute convergence.

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1. Introduction

In their paper [10], the first two authors studied the general divisor problem utilizing the mean value theorem for the general Dirichlet series and gave an improvement of Landau's [11] and Chandrasekharan-Narashimhan's [1] result in some cases. The mean value theorem itself is extensively studied by many authors. Here we only quote Perelli [17] and Matsumoto [14], where Perelli gave an upper bound for the mean square and zero-density theorem for his ‘‘general L-functions’’, and Matsumoto obtained the mean value theorem and universality of Rankin-Selberg L-series. For the proof of their mean value theorem, Perelli used Lavrik's approximate functional equation, while Matsumoto employed the reflection principle, originally due to Jutila. (See also Ivić [8].)

In this paper, we shall study the mean value theorem more closely in the critical strip, and obtain another kind of upper bound for the error term of the sum of general divisor function. As in [10], we shall follow Ramachandra [19]–[22] to prove a substitute for an approximate functional equation or what we call a finite modular relation in Lemma 3 below, which is a slight modification of Lemma 2.3 of [10]. This gives us a very simple and self-contained approach to the mean value formula of Dirichlet series. A special feature is this simplicity of the method that

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brings out a situation where the higher the power, the better the estimate. Thus, by a more elaborate argument than in [10] we improve Theorem 3 of [10] in our Theorem 1 (ii), which leads to an improvement of Theorem 1 of [10] as stated in our Theorem 2. We go on further to replace the estimate by an asymptotic formula (Theorems 3 and 4) when the real part is near the abscissa of absolute convergence.

We shall confine ourselves to the case where condition (1.7) is valid. Other cases will be considered subsequently.

We shall first fix some notation and make some assumptions.

1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers satisfying

$$a_n = O(n^{\alpha+\varepsilon}), \quad b_n = O(n^{\alpha+\varepsilon}) \tag{1.1}$$

for any $\varepsilon > 0$, where $\alpha \geqslant 0$ is a fixed real number.

2. Define the Dirichlet series

$$Z(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{and} \quad \tilde{Z}(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \tag{1.2}$$

which are absolutely convergent for $\Re s > 1 + \alpha$ by (1.1).

3. We suppose that $Z(s)$ can be continued to a meromorphic function in any finite strip $\sigma_1 \leqslant \Re s \leqslant \sigma_2$ such that $\sigma_2 \geqslant \alpha + 1$ with only real poles, and satisfies the following condition there:

$$Z(s) = O(e^{\gamma|\Im s|}) \tag{1.3}$$

for some constant $\gamma = \gamma(\sigma_1, \sigma_2) > 0$.

4. Suppose that $Z(s)$ and $\tilde{Z}(s)$ satisfy the functional equation

$$Z(s)\Delta(s) = A_1 A_2^{-s} \tilde{Z}(1 + \alpha - s)\tilde{\Delta}(-s), \tag{1.4}$$

where A_1 and A_2 are constants, $A_2 > 0$ and

$$\Delta(s) = \prod_{i=1}^{\mu} \Gamma(\alpha_i + \beta_i s) \quad \text{and} \quad \tilde{\Delta}(s) = \prod_{j=1}^{\nu} \Gamma(\gamma_j + \delta_j s)$$

are gamma factors with $\alpha_i \in \mathbb{R}$, $\beta_j > 0$ $(1 \leqslant i \leqslant \mu)$ and $\gamma_j \in \mathbb{R}$, $\delta_j > 0$ $(1 \leqslant j \leqslant \nu)$. We naturally assume that

$$\sum_{i=1}^{\mu} \beta_i = \sum_{j=1}^{\nu} \delta_j.$$

5. Let

$$H = 2\sum_{i=1}^{\mu} \beta_i \left(= 2\sum_{j=1}^{\nu} \delta_j \right) \tag{1.5}$$

and

$$\eta = \sum_{j=1}^{\nu} \gamma_j - \sum_{i=1}^{\mu} \alpha_i + \frac{\mu - \nu}{2}. \tag{1.6}$$

In this paper, we assume throughout that

$$\eta \geqslant 1+\alpha \quad \text{and} \quad H \geqslant \frac{\eta}{1+\alpha}+1. \tag{1.7}$$

6. For any fixed integer $k \geqslant 2$, we define the sequence $\{a_k(n)\}$ by

$$Z^k(s)=\sum_{n=1}^{\infty}\frac{a_k(n)}{n^s},$$

so that

$$a_k(n)=\sum_{n_1 n_2 \cdots n_k=n} a_{n_1}a_{n_2}\cdots a_{n_k}, \tag{1.8}$$

which we refer to as the general divisor function for $\{a_n\}$.

For many important Dirichlet series such as the Riemann zeta-function, the Dirichlet L-function or the Dirichlet series associated to a cusp form, the functional equation (1.4) holds true with $\tilde{\Delta}(-s)=\Delta(1+\alpha-s)$. In these cases we have $\eta=\frac{1+\alpha}{2}H$.

Now we are in a position to state the results of this paper. Unless otherwise specified, c with or without suffix will always denote a positive constant, and ε will denote arbitrarily small constant which is not necessarily the same at each occurrence.

Theorem 1. *Suppose that the conditions* 1, 3, 4 *and* (1.7) *of* 5 *hold. Then*

(i) *For* $0<\sigma<\frac{1}{2(H-\frac{\eta}{1+\alpha})}$, *we have*

$$\int_1^T |Z(\sigma+it)|^2 dt \ll T^{2\eta-2\sigma H+1+\varepsilon}.$$

(ii) *For* $\frac{1}{2(H-\frac{\eta}{1+\alpha})} \leqslant \sigma \leqslant (1+\alpha)\left(1-\frac{1}{2\eta}\right)$, *we have*

$$\int_1^T |Z(\sigma+it)|^2 dt \ll T^{2\eta(1-\frac{\sigma}{1+\alpha})+\varepsilon}.$$

(iii) *For* $(1+\alpha)\left(1-\frac{1}{2\eta}\right)<\sigma<1+\alpha$, *we have*

$$\int_1^T |Z(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}+O\left(T^{\eta(1-\frac{\sigma}{1+\alpha})+\frac{1}{2}+\varepsilon}\right).$$

The error term in (iii) is further improved for certain ranges of σ in Theorems 3 and 4 below, with Theorem 4 referring to the k-th power mean.

Our next theorem is concerned with the error term of $\sum_{n\leqslant x} a_k(n)$. Let $M_k(x)$ be the sum of residues of the function $\frac{Z^k(s)}{s}x^s$ at all positive real poles of $\frac{Z^k(s)}{s}$ in the strip $0<\sigma\leqslant 1+\alpha$. We put

$$R_k(x)=\sum_{n\leqslant x} a_k(n)-M_k(x) \tag{1.9}$$

and

$$\beta_k=\inf\left\{\lambda \middle| R_k(x)\ll x^{\lambda} \quad \text{as } x\to\infty\right\}$$

as usual. Landau's classical result applied to $Z^k(s)$ asserts that

$$\beta_k \leqslant (1+\alpha)\left(1-\frac{1}{k\eta+1/2}\right). \tag{1.10}$$

In the case $k\geqslant 2$, we can improve (1.10) as follows.

Theorem 2. *Let $k\geqslant 2$. In the above notation, we have*

$$\beta_k \leqslant (1+\alpha)\left(1-\frac{1}{k\eta}\right). \tag{1.11}$$

In connection with Theorem 2, we must also mention Lau's papers [12] and [13], where he studied the mean square of $R_1(x)$ very closely using the method of Meurman [15].

In the proof of Theorem 2, we use the estimate of $Z(s)$ obtained from the Phragmén-Lindelöf convexity principle. However, in certain important cases, we know better growth estimates of the Dirichlet series on its critical line. We will discuss such instances in the last section.

Using (1.10) and (1.11), we can improve the error term of (iii) in Theorem 1. The method of proof is based on that of Matsumoto [14].

Theorem 3. *Let $(1+\alpha)\left(1-\frac{1}{2\eta+1}\right)\leqslant\sigma<1+\alpha$, then we have*

$$\int_1^T |Z(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}} + O\left(T^{3\left(1-\frac{1}{(\eta+1/2)(1-\frac{\sigma}{1+\alpha})+1}\right)+\varepsilon}\right).$$

Theorem 4. *Let $k\geqslant 2$ and $(1+\alpha)\left(1-\frac{1}{2k\eta}\right)\leqslant\sigma<1+\alpha$, then we have*

$$\int_1^T |Z^k(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty}\frac{|a_k(n)|^2}{n^{2\sigma}} + O\left(T^{3\left(1-\frac{1}{k\eta(1-\frac{\sigma}{1+\alpha})+1}\right)+\varepsilon}\right).$$

Remark 1. In the course of the proofs of Theorems 3 and 4, we only use the pointwise estimate of $R_k(x)$ described in (1.10) and (1.11), whereas Matsumoto [14] employs the mean-square estimate of the corresponding error term for the Rankin-Selberg series. Since the latter gives a better estimate than the former on average, his result is better than the one our Theorem 3 provides in that special case. We return to this sort of incorporation of the mean value in the estimate in another occasion.

2. Preliminaries

We use complex variables $s=\sigma+it$, $w=u+iv$ with $|t|$ sufficiently large.
The Stirling formula (see *e.g.* Titchmarsh [24]) states that

$$|\Gamma(\sigma+it)| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|}\left(1+O\left(|t|^{-1}\right)\right) \tag{2.1}$$

as $|t|\to\infty$ in any fixed strip $\sigma_1\leqslant\sigma\leqslant\sigma_2$. If we write the functional equation (1.4) as

$$Z(s)=\chi(s)\tilde{Z}(\alpha+1-s), \tag{2.2}$$

then, from (2.1), we have

$$|\chi(s)| \asymp A_2^{-\sigma}|t|^{\eta-\sigma H} \tag{2.3}$$

where H and η are defined by (1.5) and (1.6) respectively. (The symbol $f \asymp g$ means that $f \gg g$ and $f \ll g$.)

Lemma 1. *We have*

$$Z(\sigma+it) \ll |t|^{\frac{(\eta+\varepsilon H)(1+\alpha+\varepsilon-\sigma)}{1+\alpha+2\varepsilon}} \tag{2.4}$$

uniformly in $-\varepsilon \leqslant \sigma \leqslant 1+\alpha+\varepsilon$.

Proof. In view of (1.3) we can apply the Phragmén-Lindelöf convexity principle (see *e.g.* [24]), and so the assertion (2.4) follows at once from (2.2) and (2.3).

Lemma 2 (Montgomery-Vaughan's mean value theorem [16]). *If* $\{h_n\}$ *is an infinite sequence of complex numbers such that* $\sum_{n=1}^{\infty} n|h_n|^2$ *is convergent, then*

$$\int_T^{T+H} \left|\sum_{n=1}^{\infty} h_n n^{-it}\right|^2 dt = \sum_{n=1}^{\infty} |h_n|^2 (H+O(n)),$$

where $c \leqslant H \leqslant T$.

Lemma 3 (Finite form of the modular relation). *Let* h *be a fixed positive constant such that* $1-\frac{1+\alpha}{h}>0$, *and let* Y *and* M *be positive parameters with* $Y \ll |t|^c$, $M \ll |t|^c$ *for some constant* c. *For* $s=\sigma+it$, $(0<\sigma<1+\alpha)$, *we have*

$$Z(s) = S - I_1 - I_2 + O(|t|^{-A}), \tag{2.5}$$

where

$$S = \sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{-(n/Y)^h}, \tag{2.6}$$

$$I_1 = \frac{1}{2\pi i}\int_{\substack{u=-\varepsilon \\ |v|\leqslant(\log|t|)^2}} \Gamma\Big(1+\frac{w}{h}\Big) Y^w \chi(s+w)\Big(\sum_{n\leqslant M} b_n n^{-1-\alpha+s+w}\Big)\frac{dw}{w}, \tag{2.7}$$

$$I_2 = \frac{1}{2\pi i}\int_{\substack{u=-\sigma-\varepsilon \\ |v|\leqslant(\log|t|)^2}} \Gamma\Big(1+\frac{w}{h}\Big) Y^w \chi(s+w)\Big(\sum_{n>M} b_n n^{-1-\alpha+s+w}\Big)\frac{dw}{w}, \tag{2.8}$$

and A *is an arbitrarily large constant.*

Proof. It is well-known that

$$e^{-x} = \frac{1}{2\pi i}\int_{(c)} \Gamma(1+z) x^{-z}\frac{dz}{z}, \tag{2.9}$$

where $c>0$, $x>0$, and the path of integration (c) means the vertical line from $c-i\infty$ to $c+i\infty$. Now suppose that $c>1+\alpha$. Putting $x=(n/Y)^h$, $z=w/h$ in (2.9), multiplying both sides by $a_n n^{-s}$ and summing over n, we have

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}\exp\big(-(n/Y)^h\big) = \frac{1}{2\pi i}\int_{(c)} \Gamma\Big(1+\frac{w}{h}\Big) Y^w Z(s+w)\frac{dw}{w}. \tag{2.10}$$

We truncate the line of integration at $|v| = (\log|t|)^2$. By using the Stirling formula (2.1), the integrals along the half lines with $|v| > (\log|t|)^2$ are estimated as $O(|t|^{-A})$ for any large positive constant A. In the remaining integral, we move the line of integration to $u = -\sigma - \varepsilon$, encountering the simple pole of the integrand at $w = 0$ with residue $Z(s)$. Note that because of the assumption on h and σ, we can take a small number $\varepsilon > 0$ such that $\Re(1 + \frac{w}{h}) = 1 - \frac{\sigma+\varepsilon}{h} > 0$, so that there are no poles of I other than 0 in the rectangle in question, and that since the height of this rectangle is $\log^2|t|$, no poles of $Z(s + w)$ are inside thereof. The integral on the horizontal segment $-\sigma - \varepsilon \leqslant u \leqslant c$, $|v| = (\log|t|)^2$ is also estimated by $O(|t|^{-A})$. Hence we get

$$S = Z(s) + I + O(|t|^{-A}),$$

where

$$I = \frac{1}{2\pi i}\int_{\substack{u=-\sigma-\varepsilon \\ |v|\leqslant(\log|t|)^2}} \Gamma\left(1 + \frac{w}{h}\right) Y^w Z(s + w)\frac{dw}{w}.$$

To transform the integral I, we substitute the functional equation (2.2) for $Z(s)$. Noting that the resulting series $\sum_{n=1}^{\infty} b_n n^{-(1+\alpha-s-w)}$ is absolutely convergent and dividing this series into two parts at $n = M$, we get

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{\substack{u=-\sigma-\varepsilon \\ |v|\leqslant(\log|t|)^2}} \Gamma\left(1 + \frac{w}{h}\right) Y^w \chi(s + w)\left(\sum_{n\leqslant M} b_n n^{-1-\alpha+s+w}\right)\frac{dw}{w} \\ &\quad + \frac{1}{2\pi i}\int_{\substack{u=-\sigma-\varepsilon \\ |v|\leqslant(\log|t|)^2}} \Gamma\left(1 + \frac{w}{h}\right) Y^w \chi(s + w)\left(\sum_{n>M} b_n n^{-1-\alpha+s+w}\right)\frac{dw}{w} \\ &=: I_1' + I_2', \end{aligned}$$

say. The second integral I_2' equals I_2 itself in our assertion. For I_1' we move the line of integration back to $u = -\varepsilon$, the integral on the horizontal lines being estimated by $O(|t|^{-A})$. Hence $I_1' = I_1 + O(|t|^{-A})$. This completes the proof of Lemma 3. □

3. Proof of Theorem 1

We consider the upper bounds of the mean square of $Z(s)$ in the first place. Since

$$\int_1^T |Z(\sigma + it)|^2 dt \ll \int_1^T \left(|S|^2 + |I_1|^2 + |I_2|^2\right) dt + 1, \tag{3.1}$$

it is enough to compute the mean squares of S, I_1 and I_2 over the segment $T \leqslant t \leqslant 2T$ on account of the well-known device of dividing the interval $[1, T]$ in the union of subintervals $[2^{-k}T, 2^{-k+1}T]$.

Applying Lemma 2 to the expression (2.6) of S, we get

$$\begin{aligned} \int_T^{2T} |S|^2 dt &= \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} \exp\left(-2(n/Y)^h\right)\left(T + O(n)\right) \\ &= \sum_{n\leqslant Y} + \sum_{n>Y} \\ &=: S_1 + S_2, \end{aligned}$$

say. Since $\exp(-(n/Y)^h) = O(1)$ for $n \leqslant Y$, and $\exp(-(n/Y)^h) = O((Y/n)^{lh})$ for $n > Y$ where l is any fixed positive integer, we have

$$S_1 \ll \sum_{n \leqslant Y} |a_n|^2 n^{-2\sigma}(T+n) \ll \sum_{n \leqslant Y} n^{2\alpha-2\sigma+\varepsilon}(T+n)$$
$$\ll \begin{cases} TY^{1+2\alpha-2\sigma+\varepsilon} + Y^{2+2\alpha-2\sigma+\varepsilon} & \text{if} \quad 0 < \sigma \leqslant \frac{1}{2}+\alpha, \\ T + Y^{2+2\alpha-2\sigma+\varepsilon} & \text{if} \quad \frac{1}{2}+\alpha < \sigma < 1+\alpha, \end{cases}$$

and

$$S_2 \ll \sum_{n>Y} n^{2\alpha-2\sigma+\varepsilon}\left(\frac{Y}{n}\right)^{hl}(T+n)$$
$$\ll TY^{1+2\alpha-2\sigma+\varepsilon} + Y^{2+2\alpha-2\sigma+\varepsilon}.$$

These estimates imply that

$$\int_T^{2T} |S|^2 dt \ll \begin{cases} TY^{1+2\alpha-2\sigma+\varepsilon} + Y^{2+2\alpha-2\sigma+\varepsilon} & \text{if} \quad 0 < \sigma \leqslant \frac{1}{2}+\alpha, \\ T + Y^{2+2\alpha-2\sigma+\varepsilon} & \text{if} \quad \frac{1}{2}+\alpha < \sigma < 1+\alpha. \end{cases} \tag{3.2}$$

Next we consider the mean square of I_1. By (2.7) and the Cauchy-Schwarz inequality, we infer that

$$|I_1|^2 \ll \left(\int_{\substack{u=-\varepsilon \\ |v| \leqslant (\log|t|)^2}} \left|\Gamma\left(1+\frac{w}{h}\right)Y^w \chi(s+w)\right|^2 \left|\frac{dw}{w}\right|\right)$$
$$\times \left(\int_{\substack{u=-\varepsilon \\ |v| \leqslant (\log|t|)^2}} \left|\sum_{n \leqslant M} b_n n^{-1-\alpha+s+w}\right|^2 |dw|\right),$$

hence we have

$$\int_T^{2T} |I_1|^2 dt \ll Y^{-2\varepsilon} T^{2\eta-2(\sigma-\varepsilon)H} \int_{\substack{u=-\varepsilon \\ |v| \leqslant (\log|t|)^2}} \int_T^{2T} \left|\sum_{n \leqslant M} b_n n^{-1-\alpha+s+w}\right|^2 dt\, dv.$$

By Lemma 2, we can see that the inner integral $\int_T^{2T}$ is bounded from above by

$$\ll \sum_{n \leqslant M} |b_n|^2 n^{-2-2\alpha+2\sigma-2\varepsilon}(T+n)$$
$$\ll \begin{cases} T + M^{2\sigma-\varepsilon} & \text{if} \quad 0 < \sigma \leqslant \frac{1}{2}, \\ \left(\frac{T}{M}+1\right)M^{2\sigma-\varepsilon} & \text{if} \quad \frac{1}{2} < \sigma < 1+\alpha. \end{cases}$$

Hence we have

$$\int_T^{2T} |I_1|^2 dt \ll T^{2\eta-2\sigma H+\varepsilon} \times \begin{cases} (T + M^{2\sigma}) & \text{if} \quad 0 < \sigma \leqslant \frac{1}{2}, \\ \left(\frac{T}{M}+1\right)M^{2\sigma} & \text{if} \quad \frac{1}{2} < \sigma < 1+\alpha. \end{cases} \tag{3.3}$$

As for the mean square of I_2, we can similarly derive that

$$\int_T^{2T} |I_2|^2 dt \ll Y^{-2\sigma-2\varepsilon} T^{2\eta+2\varepsilon H} \int_{\substack{u=-\sigma-\varepsilon \\ |v|\leqslant(\log|t|)^2}} \int_T^{2T} \left| \sum_{n>M} b_n n^{-1-\alpha+s+w} \right|^2 dt\, dv$$
$$\ll Y^{-2\sigma} T^{2\eta+\varepsilon} \left(\frac{T}{M} + 1 \right).$$

Here we distinguish three cases. In the first case $0 < \sigma \leqslant \frac{1}{2}$, choosing $M = T^{\frac{1}{2\sigma}}$ and $Y = T^{\frac{\eta}{1+\alpha}}$, we have

$$\int_T^{2T} |Z(\sigma + it)|^2 dt \ll T^{2\eta(1-\frac{\sigma}{1+\alpha})+\varepsilon} + T^{2\eta-2\sigma H+1+\varepsilon}. \tag{3.4}$$

In the second case $\frac{1}{2} < \sigma \leqslant \frac{1}{2} + \alpha$, we choose $Y = T^{\frac{\eta}{1+\alpha}}$, $M = T^{H-\frac{\eta}{1+\alpha}}$, and get

$$\int_T^{2T} |Z(\sigma + it)|^2 dt \ll T^{2\eta(1-\frac{\sigma}{1+\alpha})+\varepsilon}. \tag{3.5}$$

In the third case $\frac{1}{2} + \alpha < \sigma < 1 + \alpha$, we take Y and M in the same way as in the second case. This gives us

$$\int_T^{2T} |Z(\sigma + it)|^2 dt \ll T + T^{2\eta(1-\frac{\sigma}{1+\alpha})+\varepsilon}. \tag{3.6}$$

Thus the assertions (i) and (ii) of Theorem 1 follow immediately from (3.4), (3.5) and (3.6).

Now we turn to the proof of (iii) of Theorem 1. We follow the method of Ivić [8]. (See also [14].)

Suppose that

$$(1+\alpha)\left(1 - \frac{1}{2\eta}\right) < \sigma < 1 + \alpha. \tag{3.7}$$

Let

$$f(s) = Z(s)^2 - \left(\sum_{n \leqslant L} \frac{a_n}{n^s} \right)^2,$$

where L is a parameter which is chosen later. It is easily seen that

$$\int_2^T |Z(\sigma + it)|^2 dt = \int_2^T \left| \sum_{n \leqslant L} \frac{a_n}{n^{\sigma+it}} \right|^2 dt + O\left(\int_2^T |f(\sigma + it)| dt \right). \tag{3.8}$$

By Lemma 2, the first term on the right-hand side of (3.8) is evaluated as

$$\int_2^T \left| \sum_{n \leqslant L} \frac{a_n}{n^{\sigma+it}} \right|^2 dt = T \sum_{n \leqslant L} \frac{|a_n|^2}{n^{2\sigma}} + O\left(\sum_{n \leqslant L} |a_n|^2 n^{1-2\sigma} \right)$$
$$= T \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} + O\big(TL^{1+2\alpha-2\sigma+\varepsilon}\big) + O\big(L^{2+2\alpha-2\sigma+\varepsilon}\big).$$

We consider the second term on the right-hand side of (3.8). For that purpose, we apply Lemma 8.3 of [8] to get

$$\int_2^T |f(\sigma+it)|dt \ll \left(\int_1^{2T} \left|f\left((1+\alpha)\left(1-\frac{1}{2\eta}\right)+it\right)\right|dt+1\right)^{\frac{1+\alpha+\delta-\sigma}{1+\alpha+\delta-(1+\alpha)(1-\frac{1}{2\eta})}}$$
$$\times\left(\int_1^{2T} \left|f(1+\alpha+\delta+it)\right|dt+1\right)^{\frac{\sigma-(1+\alpha)(1-\frac{1}{2\eta})}{1+\alpha+\delta-(1+\alpha)(1-\frac{1}{2\eta})}}$$

for any $\delta>0$. By using (ii) of Theorem 1 and Lemma 2 again, we see that the first integral on the right-hand side is estimated as

$$\ll T^{1+\varepsilon}+TL^{\varepsilon}+L^{\frac{1}{\eta}(1+\alpha)+\varepsilon}.$$

On the other hand, noting that $f(1+\alpha+\delta+it)=\sum_{n>L}h(n)n^{-(1+\alpha+\delta+it)}$ with $h(n)=O(n^{\alpha+\varepsilon})$, and using the Cauchy-Schwarz inequality and Lemma 2, we see that the second integral on the right-hand side is estimated as

$$\ll T^{\frac{1}{2}}\left((T/L)^{\frac{1}{2}}+1\right)L^{-\varepsilon}.$$

Hence we have, by taking $L=T$,

$$\int_2^T |f(\sigma+it)|dt \ll T^{\eta(1-\frac{\sigma}{1+\alpha})+\frac{1}{2}+\varepsilon}.$$

Since $2+2\alpha-2\sigma\leqslant\eta(1-\frac{\sigma}{1+\alpha})+\frac{1}{2}$ under the condition (3.7), we conclude that

$$\int_2^T |Z(\sigma+it)|^2dt = T\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}+O(T^{\eta(1-\frac{\sigma}{1+\alpha})+\frac{1}{2}+\varepsilon})$$

for σ in this range. This completes the proof of (iii) of Theorem 1.

4. Proof of Theorem 2

Let $c_0=1+\alpha+\varepsilon$. We make use of the standard truncated Perron formula, which asserts that for $0<T\leqslant x$,

$$\sum_{n\leqslant x}a_k(n)=\frac{1}{2\pi i}\int_{c_0-iT}^{c_0+iT}Z^k(s)\frac{x^s}{s}ds+O\left(x^{c_0}T^{-1}\right)+O(x^{\varepsilon}). \tag{4.1}$$

We move the line of integration in the above integral to some $\sigma=\sigma_0$, which is not a pole of $Z(s)$ and $\frac{1}{2(H-\frac{\eta}{1+\alpha})}\leqslant\sigma_0\leqslant(1+\alpha)\left(1-\frac{1}{2\eta}\right)$. By the theorem of residues we obtain

$$\sum_{n\leqslant x}a_k(n)=M_k'(x)+I_h+I_v+O(x^{c_0}T^{-1})+O(x^{\varepsilon}) \tag{4.2}$$

where $M_k'(x)$ is the sum of residues of the function $\frac{Z^k(s)}{s}x^s$ at all possible poles of $\frac{Z^k(s)}{s}$ in the interval $\sigma_0<\sigma\leqslant 1+\alpha$, and I_h (resp. I_v) denotes the horizontal (resp.

vertical) integral. For these integrals we have

$$\begin{aligned} I_h =& \frac{1}{2\pi i}\int_{\sigma_0+iT}^{c_0+iT} Z^k(s)\frac{x^s}{s}ds + \frac{1}{2\pi i}\int_{c_0-iT}^{\sigma_0-iT} Z^k(s)\frac{x^s}{s}ds \\ =& O(x^{c_0}T^{-1}) + O\left(x^{\sigma_0}T^{k\eta(1-\frac{\sigma_0}{1+\alpha})-1+\varepsilon}\right) \end{aligned} \tag{4.3}$$

by (2.4), while

$$\begin{aligned} I_v &= \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} Z^k(s)\frac{x^s}{s}ds \\ &= O(x^{\sigma_0}) + O\left(x^{\sigma_0}\int_1^T |Z(\sigma_0+it)|^k\frac{dt}{t}\right). \end{aligned} \tag{4.4}$$

In the integral of the error term for I_v we factor the integrand $|Z|^k$ into $|Z|^{k-2}$ and $|Z|^2$ to which we apply (2.4) and (ii) in Theorem 1 respectively. Thus we get

$$I_v = O\left(x^{\sigma_0}T^{k\eta(1-\frac{\sigma_0}{1+\alpha})-1+\varepsilon}\right). \tag{4.5}$$

We may duly write $M_k(x)$ instead of $M_k'(x)$, since the contribution from possible poles in the interval $0<\sigma\leqslant\sigma_0$ is $O(x^{\sigma_0})$.

Hence from (4.1)–(4.3), (4.5) and the above remark we conclude that

$$\sum_{n\leqslant x} a_k(n) = M_k(x) + O(x^{c_0}T^{-1}) + O\left(x^{\sigma_0}T^{k\eta(1-\frac{\sigma_0}{1+\alpha})-1+\varepsilon}\right). \tag{4.6}$$

Taking $T = x^{\frac{1+\alpha}{k\eta}}$, we get

$$\sum_{n\leqslant x} a_k(n) = M_k(x) + O\left(x^{(1+\alpha)(1-\frac{1}{k\eta})+\varepsilon}\right),$$

whence we conclude the assertion of Theorem 2.

Example 1 (Rankin-Selberg series). Let $f(z)=\sum_{n=1}^{\infty}c(n)e^{2\pi inz}$ be a cusp form of weight κ for the full modular group, and let $Z(s)$ be the Rankin-Selberg series defined by

$$Z(s) = \zeta(2s)\sum_{n=1}^{\infty}\frac{|c(n)|^2}{n^{s+\kappa-1}} = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

for $\Re s>1$, where the coefficients a_n are given by

$$a_n = n^{-(\kappa-1)}\sum_{m^2|n} m^{2(\kappa-1)}|c(n/m^2)|^2.$$

In [23], Rankin proved that $Z(s)$ can be continued as a meromorphic function over the whole plane with a simple pole at $s=1$, satisfying the functional equation

$$\Delta(s)Z(s) = (2\pi)^{4s-2}\Delta(1-s)Z(1-s)$$

with

$$\Delta(s) = \Gamma(s)\Gamma(s+\kappa-1).$$

In our notation, we have $\alpha = 0$, $H = 4$ and $\eta = 2$. Using the Landau estimate (1.10), Rankin derived his famous result:

$$\sum_{n \leqslant x} a_n = Cx + O(x^{3/5})$$

where C is some constant. Regarding the general divisor problem for a_n, Theorem 2 gives us

$$\sum_{n \leqslant x} a_k(n) = M_k(x) + O(x^{1-\frac{1}{2k}})$$

for $k \geqslant 2$.

Example 2. Let $\alpha > 0$ and let $Z(s) = \zeta(s)\zeta(s-\alpha) = \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^s}$, $(\Re s > 1+\alpha)$, where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ is a sum of α-th powers of positive divisors of n. From the functional equation of the Riemann zeta-function, we see that $Z(s)$ satisfies

$$\Delta(s)Z(s) = \pi^{2s-\alpha-1}\Delta(1+\alpha-s)Z(1+\alpha-s)$$

with $\Delta(s) = \Gamma(s/2)\Gamma((s-\alpha)/2)$, whence we have $H = 2$ and $\eta = 1+\alpha$ in this case.

Let $a_k(n) = \sum_{n_1 \cdots n_k = n} \sigma_\alpha(n_1) \cdots \sigma_\alpha(n_k)$ denote the general divisor function for $\sigma_\alpha(n)$ and let $M_k(x)$ the sum of residues of the function $\frac{Z(s)^k}{s} x^s$ at $s = 1$ and $s = 1+\alpha$. We note that $M_1(x) = \frac{\zeta(1+\alpha)}{1+\alpha} x^{1+\alpha} + \zeta(1-\alpha)x$. Then Landau's result (1.10) and our Theorem 2 give

$$\sum_{n \leqslant x} \sigma_\alpha(n) = M_1(x) + O\left(x^{\alpha + \frac{1}{2\alpha+3}}\right), \tag{4.7}$$

and

$$\sum_{n \leqslant x} a_k(n) = M_k(x) + O\left(x^{\alpha+1-\frac{1}{k}}\right), \tag{4.8}$$

for $k \geqslant 2$ (see also (4.2) of [1]). These bounds are the same as in [10] since $\eta = 1+\alpha$ in this case.

For a better result than (4.7), confer Pétermann [18]. See also Walfisz [25] for the case $\alpha = 1$.

Example 3. Landau considers the shifted divisor problem in the last section of [11]. While Landau restricts to the *positive* values of parameters, we allow the parameters assume all integer values in order to stress the usage of Theorem 2. For that purpose we must deal with the more general Dirichlet series $Z(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$ and $\tilde{Z}(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s}$ which are connected by the functional equation (1.4). We note that by a slight modification of arguments the assertion of Theorem 2 holds true if we assume $n^{1-\varepsilon} \ll \lambda_n \ll n^{1+\varepsilon}$ and $n^{1-\varepsilon} \ll \mu_n \ll n^{1+\varepsilon}$.

Let $Z_0(s; z, h)$ be the function defined by

$$Z_0(s; z, h) = \sum_{\substack{v=-\infty \\ v+z \neq 0}}^{\infty} \frac{e^{2\pi i h v}}{|v+z|^s}$$

for $\Re s > 1$. The function $Z_0(s; z, h)$ can be continued to the whole s plane as a holomorphic function when h is not an integer, and as a meromorphic function with a simple pole at $s = 1$ when h is an integer, respectively. Moreover it is known that $Z_0(s; z, h)$ satisfies the functional equation

$$\Gamma\left(\frac{s}{2}\right)Z_0(s; z, h) = e^{-2\pi izh}\pi^{-\frac{1}{2}+s}\Gamma\left(\frac{1-s}{2}\right)Z_0(1-s, h, -z).$$

Let $m \geqslant 2$ be an integer and $z_1, \dots, z_m, h_1, \dots, h_m$ be real numbers. We set

$$Z(s) = Z_0(s; z_1, h_1)Z_0(s; z_2, h_2)\cdots Z_0(s; z_m, h_m) = \sum_{n=1}^{\infty}\frac{c_n}{\lambda_n}.$$

In this case we have $\alpha = 0$, $\eta = \frac{m}{2}$, $H = m$. Let $M_k(x)$ denote the residue of $\frac{Z^k(s)}{s}x^s$ at $s = 1$. The Landau's result asserts that

$$\begin{aligned}\sum_{\lambda_n \leqslant x} c_n &= \sum_{|(v_1+z_1)\cdots(v_m+z_m)| \leqslant x} e^{2\pi i(h_1 v_1 + \cdots + h_m v_m)} \\ &= M_1(x) + O(x^{\frac{m-1}{m+1}}\log^{m-1} x),\end{aligned} \tag{4.9}$$

while our Theorem 2 asserts that

$$\begin{aligned}\sum_{\lambda_{n_1}\cdots\lambda_{n_k} \leqslant x} c_{n_1}\cdots c_{n_k} &= \sum_{\left|\prod_{j=1}^{k}\prod_{i=1}^{m}(v_{ij}+z_i)\right| \leqslant x} e^{2\pi i\sum_{j=1}^{k}\sum_{i=1}^{m} h_i v_{ij}} \\ &= M_k(x) + O(x^{1-\frac{2}{km}+\varepsilon})\end{aligned} \tag{4.10}$$

for $k \geqslant 2$, which gives a little sharper estimate when we apply (4.9) replaced m with km. In other words, if we consider the parameters

$$\underbrace{z_1, \dots, z_1}_{k}, \underbrace{z_2, \dots, z_2}_{k}, \dots, \underbrace{z_m, \dots, z_m}_{k}$$

for $k \geqslant 2$ from a given set of parameters $z_1, z_2, \dots, z_m$, our error estimate supersedes Landau's results in 1915.

We also refer to Chandrasekharan-Narashimhan [4] for the related topics.

5. Proof of Theorems 3 and 4

We follow the method of [14]. Recalling the situation in §1, we see that the partial sum of a_n can be written as

$$\sum_{n \leqslant x} a_n = \sum_j x^{\rho_j}P_j(\log x) + R(x)$$

with

$$R(x) \ll x^{\beta+\varepsilon}, \tag{5.1}$$

where ρ_j are the poles of $\frac{Z(s)}{s}$ in $0 < \sigma \leqslant 1 + \alpha$ and $P_j(x)$ are certain polynomials of x. We assume for the present that

$$\frac{1}{2} + \alpha < \beta < 1 + \alpha. \tag{5.2}$$

Let

$$F_X(s) = Z(s) - \sum_{n \leqslant X} \frac{a_n}{n^s},$$

for $\sigma > 1 + \alpha$ with $X > 0$ to be chosen later. We have by partial summation

$$\begin{aligned} F_X(s) &= \lim_{Y\to\infty} \left\{ \left(\sum_{X<n\leqslant Y} a_n \right) Y^{-s} - \int_X^Y \left(\sum_{X<n\leqslant \xi} a_n \right) \frac{d}{d\xi} \xi^{-s} d\xi \right\} \\ &= s \int_X^\infty \left(\sum_j \xi^{\rho_j} P_j(\log \xi) - \sum_j X^{\rho_j} P_j(\log X) + R(\xi) - R(X) \right) \xi^{-s-1} d\xi \\ &= -\sum_j \frac{s}{\rho_j - s} P_j(\log X) X^{\rho_j - s} + \sum_j \frac{s}{(\rho_j - s)^2} Q_j(\log X) X^{\rho_j - s} \\ &\quad - \sum_j P_j(\log X) X^{\rho_j - s} - R(X) X^{-s} + s \int_X^\infty R(\xi) \xi^{-s-1} d\xi, \end{aligned}$$

where $Q_j(x)$ are certain polynomials of x. From the inequality (5.1) and the assumption (5.2), we can see that the above expression of $F_X(s)$ is valid in the wider region $\sigma > \beta$. Therefore, for $\beta < \sigma < 1 + \alpha$, we have

$$F_X(s) = s \int_X^\infty R(\xi) \xi^{-s-1} d\xi + O(X^{\beta - \sigma + \varepsilon}) + O\big(|s|^{-1} X^{1+\alpha-\sigma+\varepsilon}\big),$$

hence

$$\int_T^{2T} |F_X(\sigma + it)|^2 dt \ll T^{-1} X^{2(1+\alpha-\sigma)+\varepsilon} + T X^{2(\beta-\sigma)+\varepsilon} + J$$

with

$$J = \int_X^\infty \int_X^\infty R(\xi) \overline{R(\eta)} (\xi\eta)^{-\sigma-1} \int_T^{2T} (\sigma^2 + t^2)(\eta/\xi)^{it} dt\, d\xi\, d\eta.$$

Since the innermost integral in J is $O(\min(T^3, \frac{T^2}{|\log(\xi/\eta)|}))$, we see that

$$\begin{aligned} J \ll & T^3 \int_X^\infty |R(\eta)| \eta^{-\sigma-1} \int_{\eta-\eta/T}^{\eta+\eta/T} |R(\xi)| \xi^{-\sigma-1} d\xi\, d\eta \\ & + T^2 \int_X^\infty |R(\eta)| \eta^{-\sigma-1} \int_{\substack{\xi \geqslant X \\ |\xi-\eta| \geqslant \eta/T}} |R(\xi)| \xi^{-\sigma-1} \frac{d\xi\, d\eta}{|\log(\xi/\eta)|} \\ \ll & T^2 X^{2\beta - 2\sigma + \varepsilon} \log T. \end{aligned}$$

Hence we have

$$\int_T^{2T} |F_X(\sigma + it)|^2 dt \ll T^{-1} X^{2(1+\alpha-\sigma)+\varepsilon} + T^{2+\varepsilon} X^{2(\beta-\sigma)+\varepsilon} \tag{5.3}$$

in the range $\beta < \sigma < 1 + \alpha$.

By applying Lemma 2 to the sum $\sum_{n\leqslant X} a_n n^{-s}$, we also have

$$\int_T^{2T}\left|\sum_{n\leqslant X} a_n n^{-\sigma-it}\right|^2 dt = T\sum_{n\leqslant X}|a_n|^2 n^{-2\sigma} + O\left(\sum_{n\leqslant X}|a_n|^2 n^{1-2\sigma}\right)$$
$$= T\sum_{n=1}^{\infty}|a_n|^2 n^{-2\sigma} + O(TX^{2\alpha-2\sigma+1+\varepsilon} + X^{2+2\alpha-2\sigma+\varepsilon}). \tag{5.4}$$

From (5.3), (5.4) and the Cauchy-Schwarz inequality, we finally obtain

$$\begin{aligned}\int_T^{2T}|Z(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty}|a_n|^2 n^{-2\sigma} &+ O\big(TX^{2\alpha-2\sigma+1+\varepsilon} + X^{2+2\alpha-2\sigma+\varepsilon}\big)\\ &+ O\big(T^{-1}X^{2(1+\alpha-\sigma)+\varepsilon} + T^{2+\varepsilon}X^{2(\beta-\sigma)+\varepsilon}\big)\\ &+ O\Big(\big(T + X^{2(1+\alpha-\sigma)+\varepsilon}\big)^{\frac{1}{2}}\\ &\quad\times\big(T^{-1}X^{2(1+\alpha-\sigma)+\varepsilon} + T^{2+\varepsilon}X^{2(\beta-\sigma)+\varepsilon}\big)^{\frac{1}{2}}\Big).\end{aligned} \tag{5.5}$$

Under the conditions $T \ll X$, $T \ll X^{1+\alpha-\beta}$ and $T^3 \ll X^{2(2+2\alpha-\beta-\sigma)}$, the error terms on the right-hand side of (5.5) become $O(X^{2(1+\alpha-\sigma)+\varepsilon})$.

Now we restrict the range of σ to $(1+\alpha+\beta)/2 \leqslant \sigma < 1+\alpha$ and choose

$$X = T^{\frac{3}{2(2+2\alpha-\beta-\sigma)}}.$$

Thus from (5.5), we have

$$\int_T^{2T}|Z(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty}|a_n|^2 n^{-2\sigma} + O\Big(T^{\frac{3(1+\alpha-\sigma)}{2+2\alpha-\beta-\sigma}+\varepsilon}\Big).$$

We can take $\beta = (1+\alpha)(1-\frac{1}{\eta+1/2})$ from Landau's result (1.10). This gives the assertion of Theorem 3.

Under the assumption of Theorem 4, we can take $\beta = (1+\alpha)(1-\frac{1}{k\eta})$ by Theorem 2, which gives the assertion of Theorem 4.

(We note that the above choice of β satisfies the condition (5.2) in both cases.)

Remark 2. As remarked in the Introduction, we only used the pointwise estimate of $R_k(x)$ here. Of course, the mean square estimate for $R_k(x)$ is known for the Riemann zeta-function and other important Dirichlet series (see *e.g.* Chandrasekharan-Narashimhan [3]). However, it is not known for the most general Dirichlet series at present.

6. Supplemental Discussion and Other Remarks

In the proof of Theorem 2, we have used the growth estimates of $Z(s)$ obtained from Phragmén-Lindelöf's convexity principle. We have no other tools to improve (2.4) at present in the most general case. But in some of the important cases, we know better growth estimates for $Z(s)$ on the critical line. Thus it is possible to get better result than Theorem 2 in these cases. In fact, this problem had been discussed under the assumptions of growth condition and certain upper bound for higher power moments (see Theorem 3 and Section 5 in [10] for details).

6.1. Incorporating Better Growth Estimates. Here we shall assume only the growth condition and restrict ourselves to the case $k \geqslant 2$. Let σ_0 be a real number such that

$$\frac{1}{2(H - \frac{\eta}{1+\alpha})} \leqslant \sigma_0 \leqslant (1+\alpha)\left(1 - \frac{1}{2\eta}\right),$$

and λ_0 be a positive number such that

$$Z(\sigma_0 + it) \ll |t|^{\lambda_0 + \varepsilon} \tag{6.1}$$

as $|t| \to \infty$. According to Lemma 1, we can assume $\lambda_0 \leqslant \eta(1 - \frac{\sigma_0}{1+\alpha})$.

Proposition 1. *Let* $R_k(x)$ *be the function defined by* (1.9). *Under the conditon* (6.1), *we have*

$$R_k(x) \ll x^{(1+\alpha)\left(1 - \frac{1}{(k-2)\lambda_0\left(1+\frac{\sigma_0}{1+\alpha-\sigma_0}\right)+2\eta} + \varepsilon\right)} \tag{6.2}$$

for $k \geqslant 2$.

Proof. Now we imitate the proof of Theorem 2. Indeed, we use (6.1) instead of (2.4) in the evaluation of the integral on the horizontal and vertical line (*cf.* (4.3), (4.4)). Thus we get

$$I_h \ll x^{c_0} T^{-1} + x^{\sigma_0} T^{k\lambda_0 - 1} \tag{6.3}$$

and

$$I_v \ll x^{\sigma_0} + x^{\sigma_0} T^{(k-2)\lambda_0 + 2\eta(1 - \frac{\sigma_0}{1+\alpha}) - 1 + \varepsilon}. \tag{6.4}$$

Hence we have

$$R_k(x) \ll x^{1+\alpha+\varepsilon} T^{-1} + x^{\varepsilon} + x^{\sigma_0} T^{(k-2)\lambda_0 + 2\eta(1 - \frac{\sigma_0}{1+\alpha}) - 1 + \varepsilon}.$$

The choice $T = x^{(1+\alpha-\sigma_0)/((k-2)\lambda_0 + 2\eta(1 - \frac{\sigma_0}{1+\alpha}))}$ gives the desired result. □

Here are some examples.

Example 4 (The L-function associated with a cusp form). Let $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$, ($\Im z > 0$) be a holomorphic cusp form of weight κ for the full modular group. Define the (normalized) Dirichlet series attached to f by

$$L_f(s) = \sum_{n=1}^{\infty} \frac{\tilde{a}(n)}{n^s},$$

where $\tilde{a}(n) = a(n) n^{-(\kappa-1)/2}$. The Dirichlet series $L_f(s)$ satisfy the functional equation

$$\Gamma\left(\frac{\kappa-1}{2} + s\right) L_f(s) = (-1)^{\kappa/2} (2\pi)^{2s-1} \Gamma\left(\frac{\kappa+1}{2} - s\right) L_f(1-s),$$

so that we have $\alpha = 0$, $\eta = 1$ and $H = 2$. It is known that

$$L_f(1/2 + it) \ll |t|^{1/3} (\log|t|)^{5/6}$$

(see Good [5]), while the Phragmén-Lindelöf convexity principle gives us the bound $\ll |t|^{1/2+\varepsilon}$. We can take $\sigma_0 = 1/2$ and $\lambda_0 = 1/3$ in Proposition 1 and get

$$R_k(x) \ll x^{1-\frac{3}{2k+2}}$$

for $k \geqslant 2$.

Example 5 (The Dedekind zeta-function of a number field). Let K be an algebraic number field of degree $n \geqslant 2$. The Dedekind zeta-function is defined by

$$\zeta_K(s) = \sum_A \frac{1}{N(A)^s} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad (\Re s > 1)$$

where the summation runs over all non-zero integral ideals in K and a_n is the number of integral ideals with norm n. Let r_1 and r_2 be the number of real and complex places respectively, so that $n = r_1 + 2r_2$. Let D be the discriminant of K. The functional equation for $\zeta_K(s)$ may be written as

$$\Delta(s)\zeta_K(s) = A^{2s-1}\Delta(1-s)\zeta_K(1-s)$$

where

$$\Delta(s) = \Gamma(s/2)^{r_1}\Gamma(s)^{r_2}$$

and

$$A = 2^{r_2}\pi^{n/2}|D|^{-1/2}.$$

In our notation, we have $\alpha = 0$, $\eta = n/2$ and $H = n$. Heath-Brown [6] proved that

$$\zeta_K(1/2+it) \ll |t|^{n/6+\varepsilon} \quad (t \geqslant 1)$$

for any fixed $\varepsilon > 0$. Thus we can apply Proposition 1 with $\sigma_0 = 1/2$ and $\lambda_0 = n/6$ to get

$$R_k(x) \ll x^{1-\frac{3}{n(k+1)}}$$

for $k \geqslant 2$.

6.2. Incorporating Better Mean Value Results. We dealt with the quite general Dirichlet series in Theorems 3 and 4. In certain special cases, better estimates for the mean value are also known. For example, we take $Z(s) = \zeta(s)^2$. It is known that

$$\int_0^T |\zeta(\sigma+it)|^4 dt = \frac{\zeta^2(2\sigma)}{\zeta(4\sigma)} T + O(T^{2-2\sigma}\log^3 T)$$

for $1/2 < \sigma < 1$ (see Ivić [19]), which is better than Theorem 3.

Next we consider the higher power moments of the Riemann zeta-function. Let $k \geqslant 2$. From Theorem 4, we have

$$\int_1^T |\zeta(\sigma+it)|^{4k} dt = T\sum_{n=1}^{\infty} \frac{d_{2k}(n)^2}{n^{2\sigma}} + O(T^{3k(1-\sigma)/(k+1-k\sigma)+\varepsilon})$$

for $1-\frac{1}{2k}<\sigma<1$ where $d_k(n)$ is the so-called generalized divisor function. In certain range of σ, this gives us a better estimate than Theorem 8.5 in Ivić [8]. For example, it is stated in that theorem that

$$\int_1^T |\zeta(\sigma+it)|^8 dt = T\sum_{n=1}^{\infty}\frac{d_4(n)^2}{n^{2\sigma}} + O(T^{(11-8\sigma)/6+\varepsilon})$$

for $\frac{5}{8}<\sigma<1$. So our result is better in $(5+\sqrt{73})/16<\sigma<1$.

Recently, A. Ivić pointed out the possibilities of improving our results for the higher power moments of the Riemann zeta-function (see his forthcoming paper).

Chandrasekharan-Narashimhan [2] obtained the mean square estimate for the Dedekind zeta function of an algebraic number field K. In the notation in Example 6 above, they showed

$$\int_1^T |\zeta_K(\sigma+it)|^2 dt = \begin{cases} T\sum_{m=1}^{\infty}\frac{a_m^2}{m^{2\sigma}} + O(T^{\frac{n(1-\sigma)}{2}+\frac{1}{2}}\log^{\frac{n}{2}} T), & \text{if } \sigma>1-\frac{1}{\mathrm{n}}, \\ O(T\log^n T), & \text{if } \sigma=1-\frac{1}{n}, \\ O(T^{n(1-\sigma)}\log^n T), & \text{if } \frac{1}{2}\leqslant\sigma<1-\frac{1}{n}. \end{cases}$$

On the other hand, our Theorem 3 gives us

$$\int_1^T |\zeta_K(\sigma+it)|^2 dt = T\sum_{m=1}^{\infty}\frac{a_m^2}{m^{2\sigma}} + O\left(T^{\frac{3(n+1)(1-\sigma)}{(n+1)(1-\sigma)+2}}\right)$$

for $1-\frac{1}{n+1}<\sigma<1$. Hence our theorem is superior to theirs in the range $\frac{2n^2-n-5+\sqrt{n^2+22n+25}}{2n(n+1)}<\sigma<1$.

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