Monatsh. Math. 133, 1–82 (2001)

Monatshefte für
Mathematik

Entropy Dissipation Methods for Degenerate Parabolic Problems and Generalized Sobolev Inequalities*

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(Received 11 October 2000; in revised form 13 March 2001)

Abstract. We analyse the large-time asymptotics of quasilinear (possibly) degenerate parabolic systems in three cases: 1) scalar problems with confinement by a uniformly convex potential, 2) unconfined scalar equations and 3) unconfined systems. In particular we are interested in the rate of decay to equilibrium or self-similar solutions. The main analytical tool is based on the analysis of the entropy dissipation. In the scalar case this is done by proving decay of the entropy dissipation rate and bootstrapping back to show convergence of the relative entropy to zero. As by-product, this approach gives generalized Sobolev-inequalities, which interpolate between the Gross logarithmic Sobolev inequality and the classical Sobolev inequality. The time decay of the solutions of the degenerate systems is analyzed by means of a generalisation of the Nash inequality. Porous media, fast diffusion, p-Laplace and energy transport systems are included in the considered class of problems. A generalized Csiszár–Kullback inequality allows for an estimation of the decay to equilibrium in terms of the relative entropy.

2000 Mathematics Subject Classification: 35B40, 35K55, 35K65, 35Q35, 35R45
Key words: Degenerate diffusion, asymptotic behavior, rates of decay, differential inequalities, degenerate parabolic systems, high-order diffusion

Contents

*The authors acknowledge support from the TMR-project “Asymptotic Methods in Kinetic Theory”, No. ERBFRMXCT 970157, funded by the EC. J.A.C. acknowledges support from DGES (Spain) PB98-1281 and PB98-1294 and DGESIC MEC-Spain Perfeccionamiento de Doctores en el Extranjero fellowship. A.J. was partially supported by the Gerhard–Hess Program of the Deutsche Forschungsgemeinschaft (DFG), No. JU359/3-1. J.A.C and P.A.M. acknowledge support from the Austrian-Spanish “Acciones Integradas”. J.A.C. and A.J. acknowledge support from the German-Spanish DAAD-“Acciones Integradas” Program. P.A.M. is also supported by his WITTGENSTEIN AWARD, funded by the Austrian Research Fund FWF.

1. Introduction

This paper is concerned with the large-time asymptotics of solutions of degenerate scalar parabolic convection-diffusion equations and of certain systems of degenerate parabolic equations. In the scalar case we consider the nonlinear Fokker–Planck equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(u\nabla V(x) + \nabla f(u)), \qquad (x \in \mathbb{R}^d, t > 0), \tag{1}$$

$$u(x, t = 0) = u_0(x) \geqslant 0, \qquad (x \in \mathbb{R}^d), \tag{2}$$

where the function f and the confining potential V satisfy suitable assumptions which guarantee the existence of a (unique) stationary solution u_∞. For parabolicity we require $f'(u) > 0$ for $u > 0$. We proceed, formally for the moment, to the analysis of the main properties of (1) imposing decay conditions at $|x| = +\infty$. The divergence form of (1) implies that the total mass is conserved at subsequent times,

$$\int_{\mathbb{R}^d} u(x,t)\, dx = \int_{\mathbb{R}^d} u_0(x)\, dx =: M.$$

In general, mass conservation is the only conservation law we can extract from (1). We now define the strictly convex function ϕ by

$$\Phi''(u) = \frac{f'(u)}{u}\,, \quad \Phi'(1) = 0\,, \quad \Phi(0) = 0.$$

If Vu and $\Phi(u)$ belong to $L^1(\mathbb{R}^d)$, a direct calculation shows that the entropy functional defined by

$$E(u(t)) := \int_{\mathbb{R}^d} (V(x)u(x,t) + \Phi(u(x,t)))\, dx \tag{3}$$

is nonincreasing in time when evaluated along a solution of (1), and

$$\frac{dE(u(t))}{dt} = -\int_{\mathbb{R}^d} u(x,t)\,|\nabla V + \Phi''(u)\nabla u|^2(x,t)\,dx. \tag{4}$$

Equation (4) measures the dissipation of entropy, and the functional

$$I(u(t)) := \int_{\mathbb{R}^d} u(x,t)\,|\nabla V + \Phi''(u)\nabla u|^2(x,t)\,dx \tag{5}$$

is usually referred to as the entropy production functional. Now, one can show that an equilibrium solution of the evolution problem (1) is the minimizer of the entropy functional E on the set of all admissible comparison functions

$$\mathscr{C} := \left\{ u \in L^1(\mathbb{R}^d) : u \geqslant 0, \int_{\mathbb{R}^d} u(x)\,dx = M \right\}.$$

Hence, the convergence of the solution of (1) towards the stationary solution can be seen as a consequence of the tendency of the system to evolve towards the state of minimal entropy.

The idea of using the time-monotonicity of the entropy of the system to measure the distance between the solution and the equilibrium density has been developed mainly in the framework of the kinetic theory of rarefied gases, with a detailed study of the spatially homogeneous Boltzmann equation [20], [21]. There, given an initial density of mass ρ, momentum v and temperature T, the stationary solution is the Maxwellian distribution function

$$M_{\rho,v,T}(x) = \rho(2\pi T)^{-d/2} \exp\left\{ -\frac{(x-v)^2}{2T} \right\} \tag{6}$$

and the corresponding entropy functional is the relative entropy well studied in information theory

$$E(u|M_{\rho,v,T}) = \int_{\mathbb{R}^d} u(x) \log \frac{u(x)}{M_{\rho,v,T}(x)}\,dx. \tag{7}$$

For this relative entropy (7), the classical Csiszar–Kullback inequality holds [22, 48],

$$\left\|u - M_{\rho,v,T}\right\|^2_{L^1(\mathbb{R}^d)} \leqslant 2E(u|M_{\rho,v,T}), \quad \text{if } \rho = \int_{\mathbb{R}^d} M_{\rho,v,T}(x)\,dx = \int_{\mathbb{R}^d} u(x)\,dx = M,$$

and convergence in L^1 of the solution to equilibrium follows from the convergence of the relative entropy to zero as $t \to \infty$.

Recently, this method has been successfully applied to general linear Fokker–Planck equations in [3]. There, the rate of decay in relative entropy has been obtained from the analysis of the time decay of the entropy production. The same procedure is at the basis of [17], [28], [56], where the rate of decay in L^1 of the solution of the porous medium/fast diffusion equation towards the self-similar solution has been investigated by studying an equivalent nonlinear Fokker–Planck equation of type (1) obtained through a suitable time rescaling.

The main tool of the entropy method is to analyze the relationship between the entropy production and its time derivative. In fact, if the system is such that, for some $\lambda > 0$

$$\frac{dI(u(t))}{dt} = -\lambda I(u(t)) - R(t), \tag{8}$$

where the remainder $R(t) \geqslant 0$ on $\mathbb{R}^d$, we obtain coupling equation (8) with (4)

$$\frac{dE(u(t))}{dt} = \frac{1}{\lambda}\left[\frac{dI(u(t))}{dt} + R(t)\right],$$

which implies at once

$$E(u_0) - E(u_\infty) \leqslant \frac{1}{\lambda} I(u_0) \tag{9}$$

and

$$\frac{d[E(u(t)) - E(u_\infty)]}{dt} \leqslant -\lambda[E(u(t)) - E(u_\infty)],$$

namely exponential convergence of the energy functional towards its minimum at a rate λ. Then, a Csiszar–Kullback type inequality gives exponential convergence in L^1 at a rate $\lambda/2$. It is interesting to remark that (9) is a generalized Sobolev type inequality. Thus, in addition to the exponential decay of the solution of the diffusion-advection equation, we obtain as a by-product a proof of differential inequalities which in several cases were unknown. A proof of the classical logarithmic Sobolev inequality originally obtained by Gross [35] by means of the study of the entropy decay of the solution to the linear Fokker–Planck equation can be found in [62].

In Section 2, we study the "time $t \to \infty$" behaviour of certain quasilinear systems of n ($\in \mathbb{N}$) degenerate parabolic equations in d ($\in \mathbb{N}$) space dimensions (see equations (10), (11) below for the precise form). We assume that the nonlinearities are such that $u = 0$ is a steady-state solution of the system (10).

Since we have no confinement potential in equation (10), we cannot expect to have exponential but only algebraic time decay of the solutions to the steady state $u = 0$. In order to derive the decay rate, we shall employ the entropy method in a somewhat simpler way than for the analysis of (1). More precisely, by taking the time derivative of the entropy, we derive an entropy inequality involving the entropy production. Then we relate the entropy production to (a power of) the entropy, obtaining a differential inequality which can be solved explicitly. This relation between the entropy production and the entropy is obtained by employing a generalized Nash inequality and making appropriate structural assumptions on the nonlinearities a and b.

We remark that these results are valid for a large class of scalar equations and systems, including fast diffusion, porous medium, p-Laplace equations and energy-transport systems. The decay rates for the solutions of the scalar equations are the same as those of the corresponding Barenblatt–Prattle (fundamental) solutions (see Remark 2).

In the second part of the paper, we shall perform rigorously the program we outlined above, finding conditions on f and V that imply exponential convergence in relative entropy. One of these conditions on the potential V concerns the growth at infinity. Indeed, we require uniform convexity. The case of a subquadratic growth of $V(x)$ (like $|x|^\alpha, \alpha < 2$) has been recently investigated in [63] for the linear case $f(u) = u$. A polynomial decay of the solution towards equilibrium, provided sufficiently many moments of the initial datum are finite, was shown there.

Section 4 deals with a proof of new generalized Csiszar–Kullback type inequalities, i.e. estimates for the L^1-distance of two functions in terms of their relative entropy. Classical entropies are defined in terms of a convex function ψ, with $\psi(1) = 0$, by the formula

$$L(u_1|u_2) = \int_{\mathbb{R}^n} \psi\left(\frac{u_1}{u_2}\right) u_2\, dx$$

where $0 \leqslant u_1, u_2 \in L^1$ and $\int u_1 = \int u_2$. For these entropies, the theory is well understood [3]. In our case, we will consider as relative entropy the difference $E(u) - E(u_\infty)$ between the entropies (3) of a general function $u \in \mathscr{C}$ and the function u_∞, minimizer of the entropy functional E on the set $\mathscr{C}$.

The analysis of Csiszár–Kullback inequalities allows us to obtain an explicit rate of decay in L^1 for all the aforementioned problems.

Finally, in Section 5 we introduce extensions of the entropy dissipation method to fourth-order parabolic equations. Here, two main examples will be discussed. The first one refers to a parabolic equation which arose initially as a scaling limit in the study of interface fluctuations in a certain spin system [14], while the second one is the surface-tension-dominated-equation of thin-films [11]. Open problems linked to this last equation are briefly discussed.

2. Degenerate Quasilinear Parabolic Systems Without Confinement

The goal of this section is to show the algebraic time decay of solutions of the following system of n $(\in \mathbb{N})$ degenerate parabolic equations in d $(\in \mathbb{N})$ space dimensions:

$$\partial_t b(u) - \operatorname{div} a(u, \nabla u) = f(u) \quad \text{in } \mathbb{R}^d \times (0, \infty), \tag{10}$$

$$b(u(\cdot, 0)) = b(u_0) \quad \text{in } \mathbb{R}^d. \tag{11}$$

Here $a(\cdot,\cdot)$ is a matrix-valued function with n rows and d columns, and ∇u stands for the Jacobian of the n-dimensional vector field u, i.e. $(\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j}$. The divergence of a matrix-field is defined in the usual way, i.e. it is the vector whose j-th component is the scalar divergence of the j-th matrix column.

Our assumptions on the nonlinearities are such that the trivial (zero) solution is a solution of the steady-state system. As already mentioned in the introduction, since we have no confinement potential in equation (10), we cannot expect to have exponential but only algebraic time decay of the solutions to 0. In this context we also refer to Section 3.2 of this paper, where connections between confined and unconfined problems are outlined.

Now we specify our assumptions on the nonlinear functions $b(u)$ and $a(u,z)$:

(HA1) The function $b:\mathbb{R}^n\to\mathbb{R}^n$ with $b(0)=0$ is strictly monotone and a gradient, i.e. there exists a function $\chi\in C^1(\mathbb{R}^n)$ with $b=\nabla\chi$, $\chi(0)=0$, and constants $\beta,\ B>0$, $m>0$ such that for all $u,\ v\in\mathbb{R}^n$,

$$\beta|u-v|^{1+1/m}\leqslant(b(u)-b(v))\cdot(u-v)\leqslant B|u-v|^{1+1/m}.$$

(HA2) The function $a:\mathbb{R}^n\times\mathbb{R}^{n\times d}\to\mathbb{R}^{n\times d}$ is continuous in $\mathbb{R}^n\times\mathbb{R}^{n\times d}$ and elliptic in the sense

$$(a(u,z_1)-a(u,z_2))\cdot(z_1-z_2)\geqslant\alpha|z_1-z_2|^p$$

for all $u\in\mathbb{R}^n, z_1,z_2\in\mathbb{R}^{n\times d}$, with constants $\alpha>0$ and $p\geqslant 2$.

(HA3) The function $f:\mathbb{R}^n\to\mathbb{R}^n$ satisfies

$$f(u)\cdot u\leqslant 0,\quad |f(u)|\leqslant Ce(b(u)),$$

for all $u\in\mathbb{R}^n$, where the function e is the Legendre transform of χ, i.e.

$$e(b(u))=b(u)\cdot u-\chi(u),\quad u\in\mathbb{R}^n. \tag{12}$$

The "$\cdot$" product of matrices in (HA2) is defined as sum over both indices of products of equally indexed matrix elements, i.e. $A\cdot B:=\mathrm{trace}(AB^T)$, where "$T$" stands for matrix transposition.

The initial datum satisfies

(HA4) $e(b(u_0))\in L^1(\mathbb{R}^d)$ with measurable u_0.

Systems of equations like (10)–(11) arise in a variety of physical situations. For example, they describe the evolution of a fluid in non-Newtonian filtration or the water flow through porous media (see [44] and the references therein). In this context, often *single* equations with $n=1$ are considered (see [47]). *Systems* of equations with $n>1$ arise, for instance, in non-equilibrium thermodynamics [23], semiconductor modeling [24], [37] and alloy solidification processes [36].

The porous medium equation $(m>1)$ or the fast diffusion equation $(0<m<1)$

$$\partial_t(u^{1/m})-\Delta u=0,\quad u>0,$$

are included in (10). Furthermore, the p-Laplace equation

$$\partial_t u-\mathrm{div}(|\nabla u|^{p-2}\nabla u)=0$$

is also included. Notice that the corresponding functions $b(u)$ and $a(u,z)$ satisfy the conditions (HA1) and (HA2).

The existence of (global) weak solutions of (10)–(11) in *bounded* domains subject to mixed Dirichlet–Neumann boundary conditions has been shown by Alt and Luckhaus in [2] (also see [42]). They obtained an existence result for elliptic-parabolic systems, that is, assuming the function b to be only monotone (instead of strictly monotone). This result has been extended in different directions by various authors, for instance under more general assumptions on $a(u,z)$ or $b(u_0)$ [1], [30], [43]. No existence result seems to be available for the whole space problem.

The uniqueness of weak solutions (always in bounded domains) in the case of a *single* equation has been first shown in [2] under the additional assumption $\partial_t b(u) \in L^1$. This condition could be removed by Otto in [55]. In the case of *systems* of equations, uniqueness results seem to be available only for functions $a(u,z) = Az + g(u)$ (see [2], [39]).

The plan of this section is as follows. In Section 2.2 we show the algebraic decay in time of the solutions. The main result of this chapter is Theorem 3. The proof is based on generalized Nash inequalities which we prove in Section 2.1. Section 2.3 is devoted to the time decay for the energy-transport system which arises in non-equilibrium thermodynamics and semiconductor modeling.

2.1. Generalized Nash inequalities. The classical Nash inequality reads as follows [4, 15, 53]: There exists a constant $\Gamma > 0$ such that for all $w \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$,

$$\|w\|_{L^2}^{1+2/d} \leqslant \Gamma \|w\|_{L^1}^{2/d} \|\nabla w\|_{L^2}. \tag{13}$$

For the degenerate parabolic system (10)–(11) under the assumption (HA1) however, it is more natural to work in the space $L^{1+1/m}$ instead of L^2. We shall call the corresponding inequality *generalized Nash inequality*:

Lemma 1. *Let $m > 1/2$, $d \in \mathbb{N}$ and $p \in [1,\infty)$ such that*

$$p > \frac{d(m+1)}{dm+m+1}.$$

Then there exists a constant $\Gamma > 0$ only depending on d, m and p such that for all $w \in W^{1,p}(\mathbb{R}^d)$ with $|w|^{1/m} \in L^1(\mathbb{R}^d)$:

$$\|w\|_{L^{1+1/m}}^{1+\sigma} \leqslant \Gamma \| \, |w|^{1/m} \|_{L^1}^{\sigma m} \|\nabla w\|_{L^p}, \tag{14}$$

where

$$\sigma = \frac{dpm + (m+1)(p-d)}{dpm^2} > 0.$$

The classical Nash inequality (13) is obtained for $m = 1$ and $p = 2$.

Proof. The generalized Nash inequality is a consequence of the Gagliardo–Nirenberg *and* the Hölder inequality. This is not very surprising since there are close relations between the Sobolev, the Gagliardo–Nirenberg and the Nash inequality [4].

First, let $w \in \mathscr{D}(\mathbb{R}^d)$ and $r \in (1,\infty)$ with $1/m < r < 1 + 1/m$. Then there exists a constant $G > 0$ only depending on d, p and r such that the Gagliardo–Nirenberg inequality holds:

$$\|w\|_{L^{1+1/m}} \leqslant G \|\nabla w\|_{L^p}^{\theta} \|w\|_{L^r}^{1-\theta}, \tag{15}$$

where

$$\theta = \frac{\dfrac{m}{m+1} - \dfrac{1}{r}}{\dfrac{1}{p} - \dfrac{1}{r} - \dfrac{1}{d}}.$$

It is easy to check that the inequality $p > d(m+1)/(dm+m+1)$ implies $0 < \theta < 1$.

For all $v \in L^1(\mathbb{R}^d) \cap L^{m+1}(\mathbb{R}^d)$, the Hölder inequality

$$\|v\|_{L^{rm}} \leqslant \|v\|_{L^1}^{\alpha} \|v\|_{L^{m+1}}^{1-\alpha}$$

holds, where

$$\alpha = \frac{m+1-rm}{rm^2}.$$

The inequalities $1/m < r < 1 + 1/m$ imply $0 < \alpha < 1$. Taking $v = |w|^{1/m}$ we obtain

$$\|w\|_{L^r} \leqslant \| \, |w|^{1/m} \|_{L^1}^{\alpha m} \|w\|_{L^{1+1/m}}^{1-\alpha}.$$

Substituting the L^r norm of w in (15), we conclude

$$\|w\|_{L^{1+1/m}}^{1/\theta-(1-\alpha)(1-\theta)/\theta} \leqslant G^{1/\theta} \| \, |w|^{1/m} \|_{L^1}^{\alpha m(1-\theta)/\theta} \|\nabla w\|_{L^p}.$$

Since

$$\frac{1}{\theta} - \frac{(1-\alpha)(1-\theta)}{\theta} = 1 + \alpha \frac{1-\theta}{\theta} = 1 + \frac{dpm + (m+1)(p-d)}{dpm^2} = 1 + \sigma,$$

we obtain the Nash inequality (14) for all $w \in \mathscr{D}(\mathbb{R}^d)$. The assertion then follows from a density argument. □

2.2. Long-time behavior of the solutions. We introduce our notion of weak solution of the system (10)–(11), see [2]. We call $u \in L^p(0,T; W^{1,p}(\mathbb{R}^d))$ a *weak solution* of (10)–(11) on the time interval $[0,T)$, if $b(u) \in L^\infty(0,T; L^1(\mathbb{R}^d))$, $\partial_t b(u) \in L^{p'}(0,T; W^{-1,p'}(\mathbb{R}^d))$, $a(u,\nabla u) \in L^{p'}((0,T) \times \mathbb{R}^d)$, u satisfies (10) in the distributional sense and the initial condition (11) is satisfied in the *weak sense*, i.e.

$$\int_0^T \langle \partial_t b(u), w \rangle dt + \int_0^T \int_{\mathbb{R}^d} (b(u) - b(u_0)) \cdot \partial_t w \, dx \, dt = 0$$

for all smooth function w such that $w(x,T) = 0$ for all $x \in \mathbb{R}^d$. Here, $p' = p/(p-1)$.

Later we need an auxiliary result for integration by parts in time:

Lemma 2. *Let u be a weak solution of (10)–(11). Furthermore, let* (HA4) *hold. Then $e(b(u)) \in L^\infty(0,T; \; L^1(\mathbb{R}^d))$ and for almost all $t \in [0,T)$ the following formula holds:*

$$\int_{\mathbb{R}^d} e(b(u(t)))dx - \int_{\mathbb{R}^d} e(b(u_0))dx = \int_0^t \langle \partial_t b(u), u \rangle dt.$$

Here $\langle .,. \rangle$ denotes the duality bracket between $W^{1,p}(\mathbb{R}^d)$ and $W^{-1,p'}(\mathbb{R}^d)$.

The proof of this result is almost exactly as in [2, Lemma 1.5]. Since $\langle \partial_t b(u), u \rangle \in L^1(0,T)$ the *entropy*

$$H(t) = \int_{\mathbb{R}^d} e(b(u(x,t)))dx \tag{16}$$

is actually well defined for all $t \in [0,T]$ and absolutely continuous on $[0,T]$.

The main result of this subsection is the following theorem.

Theorem 3. *Let the hypotheses* (HA1)–(HA4) *hold and*

$$m > \frac{1}{2}, \qquad p > \frac{d(m+1)}{dm+1}.$$

Let u be a weak solution of the system (10)–(11) for $t \in [0,\infty)$ *with*

$$b(u) \in L^\infty(0,\infty; L^1(\mathbb{R}^d)).$$

Then there exist constants C_1, C_2, $C_3 > 0$ *only depending on* α, β, B, β_0, d, m, n, *and p with*

$$\beta_0 = \|b(u_0)\|_{L^\infty(0,\infty;L^1(\mathbb{R}^d))}$$

such that for almost all $t > 0$,

$$H(t) \leqslant (H(0)^{-\delta} + \delta C_1 t)^{-1/\delta}, \tag{17}$$

$$\|u(t)\|_{L^{1+1/m}} \leqslant C_2 (H(0)^{-\delta} + \delta C_1 t)^{-m/\delta(m+1)}, \tag{18}$$

and if $m > 1$,

$$\|u(t)\|_{L^1} \leqslant C_3 (H(0)^{-\delta} + \delta C_1 t)^{-(m-1)/\delta m}, \tag{19}$$

where

$$\delta = \frac{dm(p-1)+p-d}{dm} > 0. \tag{20}$$

Proof. The proof is divided into several steps.

Step 1: *Entropy inequality.* Using equation (10) and conditions (HA2)–(HA3), we obtain for $0 < s < t$ (see Lemma 2),

$$\begin{aligned} H(t) - H(s) &= \int_s^t \langle \partial_t b(u), u \rangle d\tau \\ &= -\int_s^t \int_{\mathbb{R}^d} a(u, \nabla u) \cdot \nabla u \, dx \, d\tau + \int_s^t \int_{\mathbb{R}^d} f(u) \cdot u \, dx \, d\tau \\ &\leqslant -\alpha \int_s^t \|\nabla u(\tau)\|_{L^p}^p d\tau. \end{aligned} \tag{21}$$

The condition (HA1) yields $b(u) \cdot u \geqslant \beta |u|^{1+1/m}$ for all $u \in \mathbb{R}^n$. Therefore, for all $i = 1, \ldots, n$,

$$\| \, |u_i(t)|^{1/m} \|_{L^1} \leqslant \| \, |u(t)|^{1/m} \|_{L^1} \leqslant (1/\beta) \|b(u(t))\|_{L^1} \leqslant b_0/\beta,$$

where $b_0 = \sup_{t>0}\|b(u(t))\|_{L^1(\mathbb{R}^d)}$. Since p satisfies the hypotheses of Lemma 1, we can apply the generalized Nash inequality (14):

$$\|u_i(t)\|_{L^{1+1/m}}^{1+\sigma} \leqslant \Gamma(b_0/\beta)^{\sigma m}\|\nabla u_i(t)\|_{L^p},$$

and hence

$$\begin{aligned}\|u(t)\|_{L^{1+1/m}}^{1+\sigma} &= \left(\sum_{i=1}^n \|u_i(t)\|_{L^{1+1/m}}\right)^{1+\sigma} \\ &\leqslant n^\sigma \Gamma(b_0/\beta)^{\sigma m}\sum_{i=1}^n \|\nabla u_i(t)\|_{L^p} = C_0\|\nabla u(t)\|_{L^p},\end{aligned}$$

where

$$C_0 = n^\sigma \Gamma(b_0/\beta)^{\sigma m}.$$

Employing the above inequality in (21) we obtain

$$H(t) - H(s) \leqslant -\alpha C_0^{-p}\int_s^t \|u(\tau)\|_{L^{1+1/m}}^{p(1+\sigma)} d\tau.$$

Step 2: *Relation between the entropy and* $\|u\|_{L^{1+1/m}}$. In order to relate the $L^{1+1/m}$ norm of $u(\tau)$ to $H(\tau)$ we use the condition (HA1). Then, for all $u \in \mathbb{R}^n$,

$$\begin{aligned}e(b(u)) &= \int_0^1 (b(u) - b(\sigma u))\cdot u\, d\sigma \\ &= \int_0^1 (b(u) - b(\sigma u))\cdot (u - \sigma u)\frac{d\sigma}{1-\sigma} \\ &\leqslant B\int_0^1 |u - \sigma u|^{1+1/m}\frac{d\sigma}{1-\sigma} = \frac{mB}{m+1}|u|^{1+1/m}.\end{aligned} \tag{22}$$

Therefore

$$H(\tau) \leqslant \frac{mB}{m+1}\|u(\tau)\|_{L^{1+1/m}}^{1+1/m},$$

which yields

$$H(t) - H(s) \leqslant -C_1\int_s^t H(\tau)^{mp(1+\sigma)/(1+m)} d\tau,$$

where

$$C_1 = \alpha C_0^{-p}\left(\frac{m+1}{mB}\right)^{mp(1+\sigma)/(m+1)}.$$

This implies

$$\frac{dH}{dt} \leqslant -C_1 H^{1+\delta}$$

for almost all $t > 0$, where $\delta > 0$ is given by (20). Notice that $\delta > 0$ if and only if $p > d(m+1)/(dm+1)$. The above differential inequality immediately implies (17). The decay (18) is obtained from condition (HA1) and (22):

$$H(t) \geqslant \frac{m\beta}{m+1} \|u(t)\|_{L^{1+1/m}}^{1+1/m} \qquad \text{for almost all } t > 0,$$

with $C_2 = ((m+1)/m\beta)^{m/(m+1)}$.

Step 3: *Decay rate in* L^1. In order to derive the decay rate (19), we employ the estimate (18) and the Hölder inequality

$$\|w\|_{L^m} \leqslant \|w\|_{L^{m+1}}^{1-1/m^2} \|w\|_{L^1}^{1/m^2},$$

applied to $w = |u_i(t)|^{1/m}$, to obtain

$$\begin{aligned} \|u(t)\|_{L^1} &= \sum_{i=1}^n \|u_i(t)\|_{L^1} \leqslant (b_0/\beta)^{1/m} \sum_{i=1}^n \|u_i(t)\|_{L^{1+1/m}}^{1-1/m^2} \\ &\leqslant n^{1/m^2} (b_0/\beta)^{1/m} \|u(t)\|_{L^{1+1/m}}^{1-1/m^2} \leqslant C_3 (H(0)^{-\delta} + \delta C_1 t)^{-(m-1)/\delta m}, \end{aligned}$$

where

$$C_3 = n^{1/m^2} (b_0/\beta)^{1/m} C_2^{1-1/m^2}. \qquad \square$$

This proves the theorem.

Remark 1. The most serious restriction of Theorem 3 is the uniform boundedness of $b(u(t))$ in $L^1(\mathbb{R}^d)$. In the following two important cases sufficient assumptions can be given:

(1) Let the solution $u(t) = (u_1(t), \ldots, u_n(t))$ of (10)–(11) satisfy $u_i(t) \geqslant 0$ for almost all $t > 0$, $i = 1, \ldots, n$, and assume

$$b_i(u) \geqslant 0, \quad \sum_{j=1}^n f_j(u) \leqslant 0 \text{ for all } u = (u_1, \ldots, u_n) \text{ with } u_k \geqslant 0,\ i,k = 1, \ldots, n$$

Also let $b(u_0) \in L^1(\mathbb{R}^d)$.

(2) $n = 1$ (scalar case) and $b(u_0) \in L^1(\mathbb{R}^d)$.

If (1) or (2) holds then $b(u) \in L^\infty(0,\infty; L^1(\mathbb{R}^d))$ for the solution $u = u(t)$ of (10)–(11). In the case (1) it is sufficient for the proof to add the rows of (10) and to integrate (formally) over $\mathbb{R}^d$:

$$\begin{aligned} \|b(u(t))\|_{L^1(\mathbb{R}^d)} &= \sum_{j=1}^n \int_{\mathbb{R}^d} b_j(u(t))dx \\ &= \sum_{j=1}^d \int_{\mathbb{R}^d} f(u(t))dx + \|b(u_0)\|_{L^1(\mathbb{R}^d)} \\ &\leqslant \|b(u_0)\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

To be more precise, use a regularization of the characteristic function on the ball $B_R(0)$ with center 0 and radius R as test function in the weak formulation of (10). It is not difficult to see that one obtains for $R \to \infty$:

$$\|b(u(t))\|_{L^1(\mathbb{R}^d)} = \lim_{R\to\infty} \|b(u(t))\|_{L^1(B_R(0))} \leqslant \|b(u_0)\|_{L^1(\mathbb{R}^d)}.$$

In the scalar case (2) we take a non-decreasing regularization S^γ of the sign function (with $\gamma > 0$ the regularization parameter) such that sign $-S^\gamma \to 0$ as $\gamma \to 0$ in $L^1(\mathbb{R}^d)$ and multiply Eq. (10) by $S^\gamma(b(u(t)))$. Integration by parts and the limit $\gamma \to 0$ give the desired result.

Remark 2. We consider examples for $n = 1$ (single equation) with $b(u) = |u|^{1/m-1}u$, $a(u,z) = |z|^{p-2}z$:

1. Heat equation ($m = 1, p = 2$): Let $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then

$$\|u(t)\|_{L^2} \sim t^{-d/4} \quad \text{as } t \to \infty.$$

More precisely, we have

$$\|u(t)\|_{L^2} \leqslant \frac{C_2\|u_0\|_{L^2}}{(1 + 2C_1\|u_0\|_{L^2}^{4/d}t)^{d/4}},$$

which is sharper for large t than the usual estimate $\|u(t)\|_{L^2} \leqslant \|u_0\|_{L^2}$ (see, for instance, [58]).

2. Porous medium equation ($m > 1, p = 2$): Let $u_0 \in L^1_+(\mathbb{R}^d) \cap L^{1+1/m}(\mathbb{R}^d)$. Then

$$\|u(t)\|_{L^1} \sim t^{-d(m-1)/(dm+2-d)} \quad \text{as } t \to \infty. \tag{23}$$

This estimate is sharp in the sense that the Barenblatt–Prattle solution has the same decay rate. Indeed, the Barenblatt–Prattle solution

$$V(t,x) = t^{-dk}\left(\left[C - \frac{m-1}{2m}\left(\frac{|x|}{t^k}\right)^2\right]_+\right)^{1/(m-1)} \tag{24}$$

with $k = 1/(2 + d(m-1))$ and $C > 0$ solves the equation

$$\partial_t V = \Delta V^m \quad \text{in } \mathbb{R}^d, \tag{25}$$

with $V(0,x) = D\,\delta(x)$, where D is a constant depending on C. Thus $U = V^m$ solves the equation (10) with the special choice of the nonlinear functions a and b given above. An easy calculation shows

$$\|U(t)\|_{L^1} \sim t^{-dk(m-1)} = t^{-d(m-1)/(dm+2-d)} \quad \text{as } t \to \infty.$$

We also refer to [16], [28] for related results.

3. Fast diffusion equation ($m < 1, p = 2$): Let $u_0 \in L^1_+(\mathbb{R}^d) \cap L^{1+1/m}(\mathbb{R}^d)$ and assume $m > \max(1/2, 1 - 2/d)$. Then

$$\|u(t)\|_{L^{1+1/m}} \sim t^{-dm^2/(dm+2-d)(m+1)} \quad \text{as } t \to \infty. \tag{26}$$

The Barenblatt–Prattle solution V (see (24)) solves the fast diffusion equation (25) for $m > 1 - 2/d$, and the function $U = V^m$ satisfies

$$\|U(t)\|_{L^{1+1/m}} \sim t^{-m^2dk/(m+1)} = t^{-dm^2/(dm+2-d)(m+1)} \quad \text{as } t \to \infty.$$

This decay rate is the same as derived above for the solution u (also see [16], [28]).

The condition $m > \max(1/2, 1 - 2/d)$ is weaker than the condition derived by Otto [56], i.e. $m > d/(d+2)$ and $m \geqslant 1 - 1/d$, if and only if $d \geqslant 3$. For $d = 2$, both conditions give the restriction $m > 1/2$.

4. p-Laplace equation ($m = 1, p \geqslant 2$): Let $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then

$$\|u(t)\|_{L^2} \sim t^{-d/(2d(p-2)+2p)} \quad \text{as } t \to \infty.$$

The function

$$U(t,x) = t^{-d\kappa}\left(\left[C - \frac{p-2}{p}\left(\frac{|x|}{t^\kappa}\right)^{p/(p-1)}\right]_+\right)^{(p-1)/(p-2)}$$

with $\kappa = 1/(d(p-2)+p)$ and $C > 0$ solves the p-Laplace equation with $U(0,x) = D\,\delta(x)$ where, again, D is a constant which depends on C. This function satisfies

$$\|U(t)\|_{L^2} \sim t^{-d\kappa/2} = t^{-d/(2d(p-2)+2p)} \quad \text{as } t \to \infty,$$

which is the same decay rate as above. For related results, see, e.g., [43].

Remark 3. The rates of decay of the solution $u(t)$ of the equation

$$\partial_t(u^{1/m}) = \Delta u \quad \text{in } \mathbb{R}^d$$

to the Barenblatt–Prattle solution $U(t)$ with the same mass in $L^1(\mathbb{R}^d)$ have been recently obtained in [17], [28], [56] by spatial-temporal rescaling techniques (cf. Section 3.2). For instance, from [17, Thm. 6.1] we have the estimate

$$\|u(t)^{1/m} - U(t)^{1/m}\|_{L^1} \sim t^{-1/((dm+2m-d)} \quad \text{as } t \to \infty,$$

for $m > 1$, whereas for $1 - 1/d < m < 1$ (and $d = 2, 3, 4$, $m \neq \frac{1}{2}$) [28, Thm. 1.2]:

$$\|u(t) - U(t)\|_{L^1} \sim t^{-(1-d(1-m))/(dm+2-d)} \quad \text{as } t \to \infty.$$

Using the triangle inequality and Remark 2 we can only conclude the same rate for $u(t) - U(t)$ as for $u(t)$ itself (i.e. the rate (23) in L^1 for $m > 1$ and the rate (26) in $L^{1+\frac{1}{m}}$ for $\max\left(\frac{1}{2}, 1 - \frac{2}{d}\right) < m < 1$). Clearly, these rates are not sharp.

We do not obtain the same results on the time decay of the difference $u(t) - U(t)$ as in [17], [28], [56] since we do not control the entropy dissipation rate. However, our method is simpler and valid for a very large class of problems.

2.3. Example: the energy-transport model. In this subsection we show how our methods can be applied to the energy-transport equations, which form a system of strongly coupled, quasilinear parabolic equations for a charged fluid or gas, exposed to an electric field. They arise originally from non-equilibrium thermodynamics and are used in many applications of charged particle transport, for instance in semiconductor theory [24], in electro-chemistry [26], and alloy solidification processes [36]. In order to simplify the presentation, we consider a gas consisting only of negatively charged particles with particle density ρ, internal energy density η, chemical potential μ, and temperature T. We assume that the self-consistent electric potential is negligible compared to the externally given potential $V = V(x)$. Setting

$$u = (u_1, u_2) = (\mu/T, -1/T) \quad \text{and} \quad b(u) = (b_1(u), b_2(u)) = (\rho, \eta),$$

the initial-value problem reads as follows [23]:

$$\partial_t b_1(u) - \operatorname{div} J_1 = 0 \qquad \text{in } \mathbb{R}^d, \tag{27}$$

$$J_1 = L_{11}(u)(\nabla u_1 + u_2 \nabla V) + L_{12}(u)\nabla u_2, \tag{28}$$

$$\partial_t b_2(u) - \operatorname{div} J_2 = -J_1 \cdot \nabla V, \tag{29}$$

$$J_2 = L_{21}(u)(\nabla u_1 + u_2 \nabla V) + L_{22}(u)\nabla u_2, \tag{30}$$

$$b(u(0)) = b(u_0) \qquad \text{in } \mathbb{R}^d. \tag{31}$$

The variables u_1, u_2 are called *entropy variables*, J_1 is the particle current density and J_2 is the energy current density.

We impose the following assumptions on the nonlinear functions:

(HB1) Let (HA1) with $n = 2$ and $m \geqslant 1$ hold.

(HB2) The matrix (L_{ij}) with $L_{ij} : \mathbb{R}^2 \to \mathbb{R}$ is symmetric, uniformly positive definite and bounded:

$$\sum_{i,j=1}^{2} L_{ij}(u) z_i z_j \geqslant \alpha |z|^2, \quad |L_{ij}(u)| \leqslant A,$$

for all $u, z \in \mathbb{R}^2$, for some $\alpha, A > 0$.

(HB3) $e(b(u_0)) \in L^1(\mathbb{R}^d)$ with u_0 measurable, and $b(u_0) \in L^1(\mathbb{R}^d)$, $V \in L^\infty(\mathbb{R}^d)$.

The functions e and H are defined as in the previous section (see (12) and (16)).

We refer to the paper [23] for a discussion of the above hypotheses. The existence of weak solutions to (27)–(31) subject to mixed Dirichlet–Neumann boundary conditions in bounded domains is shown in [23] under the assumption $m = 1$.

We obtain the following decay rate:

Theorem 4. *Let u be a global weak solution to (27)–(31) with $\rho(t) = b_1(u(t)) \geqslant 0$ and $\eta(t) = b_2(u(t)) \geqslant 0$ for a.e. $t > 0$ and assume* (HB1)–(HB3). *Then*

$$\|u(t)\|_{L^{1+1/m}} \leqslant C_4 t^{-dm^2/(dm+2-d)(m+1)} \qquad \textit{as } t \to \infty.$$

Remark 4. The non-negativity of the particle density $\rho(t)$ and of the internal energy density $\eta(t)$ is necessary for a physically reasonable solution.

Proof. We cannot apply Theorem 3 directly since the right-hand side of the energy equation (29) may not be dissipative. However, employing the *dual entropy variables*

$$w_1 = u_1 + u_2 V, \quad w_2 = u_2,$$

the system (27)–(30) can be written in symmetrized form:

$$\partial_t b_1(u) - \operatorname{div}[D_{11}\nabla w_1 + D_{12}\nabla w_2] = 0, \tag{32}$$

$$\partial_t (b_2(u) - b_1(u)V) - \operatorname{div}[D_{21}\nabla w_1 + D_{22}\nabla w_2] = 0, \tag{33}$$

where the new diffusion coefficients are given by

$$D_{11} = L_{11}, \quad D_{12} = D_{21} = L_{12} - VL_{11}, \quad D_{22} = L_{22} - 2VL_{12} + V^2 L_{11}.$$

The matrix (D_{ij}) is also uniformly positive definite. We immediately conclude the uniform L^1 bounds for b_1 and b_2:

$$\begin{aligned} \|b_1(u(t))\|_{L^1} &\leqslant \|b_1(u_0)\|_{L^1} < \infty, \\ \|b_2(u(t))\|_{L^1} &\leqslant \|V\|_{L^\infty}\|b_1(u(t))\|_{L^1} + \|b_2(u(t)) - b_1(u(t))V\|_{L^1} \\ &\leqslant \|V\|_{L^\infty}\|b_1(u_0)\|_{L^1} + \|b_2(u_0) - b_1(u_0)V\|_{L^1}. \end{aligned}$$

A calculation shows that for $t > s$,

$$\begin{aligned} H(t) - H(s) &= \int_s^t \langle \partial_t b(u), u\rangle d\tau \\ &= \int_s^t (\langle \partial_t b_1(u), w_1\rangle + \langle \partial_t (b_2(u) - b_1(u)V), w_2\rangle) d\tau \\ &= -\int_s^t \int_{\mathbb{R}^d} \sum_{i,j=1}^2 D_{ij}\nabla w_i \cdot \nabla w_j \, dx\, d\tau \leqslant -\lambda \int_s^t \|\nabla w\|_{L^2}^2 \, d\tau, \end{aligned}$$

where $\lambda > 0$ depends on $\|V\|_{L^\infty}$. Using the generalized Nash inequality (14) we obtain for $i = 1, 2$,

$$\|w_i\|_{L^{1+1/m}}^{1+\sigma} \leqslant \Gamma \| \, |w_i|^{1/m} \|_{L^1}^{m\sigma} \|\nabla w_i\|_{L^2} \leqslant C.$$

Here, $\sigma > 0$ and $\Gamma > 0$ are given by Lemma 1 and the L^1 norm of $|w_1|^{1/m}$ is uniformly bounded since, by assumption (HB1),

$$\begin{aligned} \| \, |w_1|^{1/m} \|_{L^1} &\leqslant c(\| \, |u_1|^{1/m} \|_{L^1} + \|V\|_{L^\infty}^{1/m} \| \, |u_2|^{1/m} \|_{L^1}) \\ &\leqslant c(\|b_1(u)\|_{L^1} + \|V\|_{L^\infty}^{1/m} \|b_2(u)\|_{L^1}) \leqslant c, \end{aligned}$$

where $c > 0$ denotes henceforth a constant independent of t. Therefore we obtain

$$\|w_i\|_{L^{1+1/m}}^{1+\sigma} \leqslant c\|\nabla w_i\|_{L^2}$$

and hence

$$H(t) - H(s) \leqslant -c \int_s^t \|w\|_{L^{1+1/m}}^{2(1+\sigma)}.$$

The assumption (HB1) and the definition of w yield

$$H(t) \leqslant c\|u(t)\|_{L^{1+1/m}}^{1+1/m} \leqslant c\|w(t)\|_{L^{1+1/m}}^{1+1/m}$$

and

$$H(t) \geqslant c\|u(t)\|_{L^{1+1/m}}^{1+1/m},$$

so that we can conclude

$$\frac{dH}{dt}(t) \leqslant -cH^{2m(1+\sigma)/(m+1)} \leqslant -c\|u(t)\|^{2(1+\sigma)}_{L^{1+1/m}}.$$

The assertion of the theorem follows.

Remark 5. We remark that the somewhat 'unphysical' result $T \to \infty$ stems from the assumption (HB1) on the constitutive relations $\rho = \rho(\mu, T)$, $\eta = \eta(\rho, T)$ (which itself is physically doubtful) and from the lack of temperature relaxation in our model.

3. Asymptotic Behavior of Degenerate Scalar Parabolic Equations and Generalized Sobolev Inequalities

The main objectives of this section are, first, to study the asymptotic behavior of certain degenerate scalar parabolic equations both in bounded domains with no-flux boundary conditions and in $\mathbb{R}^d$ (with confinement) and, second, to establish generalized Sobolev inequalities. Both aims will be accomplished using the entropy method. This method has been applied successfully in the linear case (general linear Fokker–Planck equations see [3], [16]) and in the porous medium/ fast diffusion cases [17], [28], [56]. In this section we extend these results to a more general class of parabolic equations.

Consider the Cauchy problem for the general nonlinear Fokker–Planck equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\,(u\nabla V(x) + \nabla f(u)), \qquad (x \in \Omega, t > 0), \tag{34}$$

$$u(x, t = 0) = u_0(x) \geqslant 0, \qquad (x \in \Omega) \tag{35}$$

supplemented by a decay condition at $|x| = \infty$ if $\Omega = \mathbb{R}^d$ or by a zero outflux condition on $\partial\Omega$ if Ω is bounded. We assume for the moment (further assumptions will be imposed in the course of the analysis)

(HD1) $\Omega \subseteq \mathbb{R}^d$ is either a smooth, bounded domain or $\Omega = \mathbb{R}^d$.

(HD2) $u_0 \in L^1(\Omega)$, $u_0 \geqslant 0$ and $\int_\Omega u_0(x)\,dx =: M \in (0, \infty)$.

and

(HV1) If $\Omega = \mathbb{R}^d$, then $V \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$, and if Ω is bounded, then $V \in W^{1,1}(\Omega)$.

(HV2) If $\Omega = \mathbb{R}^d$, then $\forall A \in \mathbb{R}$: $\{x | V(x) \leqslant A\}$ is bounded.

(HV3) $\inf_\Omega V = 0$.

as well as

(HF1) $f : \mathbb{R}^+_0 \to \mathbb{R}$ is continuous, strictly increasing and $f(0) = 0$.

(HF2) $f|_{\mathbb{R}^+} \in C^3(\mathbb{R}^+)$.

(HF3) The function h, defined by

$$h(u) := \int_1^u \frac{f'(s)}{s}\,ds, \quad u \in (0, \infty),$$

belongs to $L^1_{\mathrm{loc}}([0, \infty))$. Then

$$\Phi : [0, \infty) \to \mathbb{R}, \quad \Phi(u) = \int_0^u h(s)\,ds$$

is well-defined with $\Phi'(u) = h(u)$ for all $u \in \mathbb{R}^+$.

Remark 6. In case of $\Omega = \mathbb{R}^d$ assumption (HV2) is, e.g., satisfied for uniformly convex functions V which will be of distinctive importance later on. Furthermore (HV2) implies $V(x) \to \infty$ as $|x| \to \infty$.

Remark 7. a) Throughout this paper we shall be concerned with non-negative solutions u of (34), (35).

b) Canonical examples for f are $f(u) = u^m$ with $m \in (0, \infty)$.

c) The function h can be interpreted as the enthalpy function in semiconductor modeling or as the pressure in the study of the evolution of a gas density in a porous medium.

d) Since f is strictly increasing the enthalpy h is so, too. Therefore, h is a homeomorphism from $(0, \infty)$ onto the open interval $(\inf h, \sup h) = (h(0+), h(\infty))$ where in accordance with the definition of h the inequalities $-\infty \leqslant h(0+) < 0 < h(\infty) \leqslant \infty$ hold.

e) Since h is strictly increasing, the function Φ is strictly convex ($\Phi' = h$).

f) By a trivial calculation

$$\Phi(u) = u\,h(u) - f(u), \quad \text{for all } u \in \mathbb{R}^+.$$

g) It is easy to verify that $\min \Phi = \Phi(1) < 0$ and $\lim_{s\to\infty} \Phi(s) = \infty$. From the convexity of Φ we deduce: There is $s_\circ \in (1, \infty)$ such that Φ is decreasing and non-positive on $[0, 1]$, increasing and non-positive on $[1, s_\circ]$, and increasing and non-negative on $[s_\circ, \infty)$.

We shall be concerned with certain "generalized solutions" of (34), (35):

Definition 1. Assume (HD1), (HD2), (HV1)–(HV3), (HF1)–(HF3). Then, $u : [0, \infty) \times \Omega \longrightarrow \mathbb{R}$ is a generalized solution of (34)–(35) iff

1. $u \in L^\infty(\Omega \times (0, T))$ for any $T > 0$.
2. If $\Omega = \mathbb{R}^d$, then $\nabla f(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}_t : \mathbb{R}^d)$ and if Ω is bounded, then $(\nabla f(u))|_{\Omega\times(0,T)} \in L^1(\Omega \times (0, T) : \mathbb{R}^d)$, for any $T \in \mathbb{R}^+$.
3. For all test functions $\phi \in C_c^\infty(\mathbb{R}_x^d \times \mathbb{R}_t)$ with $\mathrm{supp}(\phi) \cap (\partial\Omega \times \{0\}) = \emptyset$ we have

$$-\int_\Omega u_0(x)\,\phi(x, 0)\,dx - \int_{\Omega\times\mathbb{R}_t} \left(u\,\frac{\partial\phi}{\partial t}\right)(x, \tau)\,dxd\tau + \int_{\Omega\times\mathbb{R}_t} (u\,\nabla V + \nabla f(u))\,(x, \tau)\,\nabla\phi(x, \tau)\,dxd\tau = 0.$$

Remark 8. If Ω is bounded, then the involved test functions correspond to zero out-flux boundary conditions

$$n(x) \cdot (u\nabla V + \nabla f(u))(x, t) = 0, \qquad (x \in \partial\Omega, \quad t > 0),$$

where $n(x)$ is the outward unit vector of $\partial\Omega$ at x.

This section is organized as follows. First, we study the stationary states of the equation (34). Then we analyze the relation with other nonlinear diffusion problems through time-dependent rescalings. Subsection 1.3 is devoted to the study of the asymptotic behavior for equation (34) in bounded domains with zero out-flux boundary conditions. Then we shall prove the main result of this section about generalized Sobolev inequalities. Finally, we deduce the existence of a weak solution and we analyze its asymptotic behavior for equation (34) in $\mathbb{R}^d$.

3.1. Equilibrium solutions. We consider stationary solutions of (34) in Ω which satisfy

$$u\nabla V(x) + \nabla f(u) = 0, \quad \int_\Omega u\,dx = M. \tag{36}$$

We wish to re-construct u from (36) now. The intuitive way to proceed is to replace $\nabla f(u)$ in (36) by $u\,\nabla h(u)$ and to cancel u from the resulting equation. One obtains $\nabla(V + h(u)) = 0$ such that

$$V(x) + h(u(x)) = C, \quad \forall x \in \Omega, \tag{37}$$

for some $C \in \mathbb{R}$.

This argumentation, however, has a gap which concerns the cancellation of u. For the sake of simplicity we only consider continuous u here.

1) The cancellation of u is rigorously justified only if $u > 0$ in Ω. As a consequence, equation (37) is a priori equivalent to equation (36) if and only if only *strictly positive* solutions are encountered. Indeed, if (36) has a solution u *which is strictly positive*, then u will satisfy (37) for some $C \in \mathbb{R}$. Furthermore if the L^1-norm of u solving (36) is prescribed, then C is – according to the fact that h is *strictly* increasing – unique. We conclude: (36) has at most one strictly positive solution of a prescribed L^1-norm.

2) In many cases also non strictly positive solutions of (36) are of importance. But for such u the open set $\{x : u(x) > 0\}$ may have several components. The derivation of (37) is valid on each of these components then and several, component-dependent constants $C \in \mathbb{R}$ may arise.

3) Let us now consider (37) for prescribed $C \in \mathbb{R}$. Then the value $u(x)$ is uniquely determined only for $C - V \in (h(0+), h(\infty))$. In this case, $u(x) = h^{-1}(C - V(x))$, where $h^{-1} : (h(0+), h(\infty)) \to (0, \infty)$ is the inverse function of h. No problems arise here if $h(0+) = -\infty$ and $h(\infty) = \infty$. However, if $h(0+) > -\infty$ or $h(\infty) < \infty$, the range of the function $C - V$ may exceed $(h(0+), h(\infty))$. In this case (37) makes no sense anymore.

The questions "Why are strictly positive solutions (if they exist) of (36) distinguished?" and "How to proceed if $C - V$ exceeds the range $(h(0+), h(\infty))$ of h?" can be settled at once by passing from equation (36) to a variational formulation. (This procedure has been succesfully employed for non-linear drift-diffusion models [50] where similiar difficulties with thermal equilibrium solutions arose [64], [65].)

We introduce the energy functional corresponding to the stationary problem, namely

Definition 2. Assume (HD1), (HF1)–(HF3). Let $V : \Omega \to \mathbb{R}$ be measurable and non-negative. Let

$$E : L^1_+(\Omega) \to \mathbb{R} \cup \{\infty\},$$

$$E(u) = \begin{cases} \int_\Omega (Vu + \Phi^+(u))(x)\ dx - \int_\Omega \Phi^-(u(x))\ dx, & \Phi^-(u) \in L^1(\Omega), \\ \infty, & \text{else}, \end{cases}$$

where $L^1_+(\Omega) = \{u \in L^1(\Omega) : u \geqslant 0\}$.

Remark 9. As usual, we denote the non-negative (non-positive) part of a real-valued function g by g^+ (g^-), i.e. $g = g^+ - g^-$ and $|g| = g^+ + g^-$.

Remark 10. Clearly, $E(u) < \infty$ iff $Vu \in L^1(\Omega)$ and $\Phi(u) \in L^1(\Omega)$. Furthermore, if $E(u) < \infty$, then

$$E(u) = \int_\Omega (Vu + \Phi(u))\, dx.$$

Next we introduce

Definition 3. Assume (HD1), (HV1)–(HV3), (HF1)–(HF3). A function $u_{\infty,M} \in L^1(\Omega)$ is an equilibrium solution of (34), (35) iff $u_{\infty,M}$ is a minimizer of E in

$$\mathscr{C} := \left\{ u \in L^1_+(\Omega) : \int_\Omega u(x)\, dx = M \right\}.$$

Remark 11. The Definition 3 of an equlilibrium solution explicitly refers to the mass M of the initial condition. This convention simplifies the forthcoming presentation.

As we shall see soon (Lemma 6) we have under rather natural additional assumptions existence and uniqueness of a minimizer of E in $\mathscr{C}$. Before entering the proof and the assumptions a few comments will clarify the situation in advance.

One can expect – and it will turn out that this is indeed justified – that the minimizer of E in $\mathscr{C}$ satisfies the corresponding Euler–Lagrange equations,

$$\begin{aligned} V(x) + h(U(x,C)) &= C, \quad \text{if } 0 < U(x,C), \\ V(x) + h(U(x,C)) &\geqslant C, \quad \text{if } U(x,C) = 0, \end{aligned} \tag{38}$$

i.e.

$$\min_{u \in \mathscr{C}} E(u) = E(U(.,C)),$$

where $C \in \mathbb{R}$ and $U(.,C) \in \mathscr{C}$. The first question is: Does $C \in \mathbb{R}$ exist such that $U(.,C) \in \mathscr{C}$, i.e. $\int_\Omega U(x,C)\, dx = M$ (non-negativity and measurability of $U(.,C)$ is obvious)? To answer this question let us remark that the (in)equality (38) allows for an explicit representation of $U(x,C)$,

$$U(x,C) := \bar{h}^{-1}(C - V(x)), \tag{39}$$

with the "generalized" inverse $\bar{h}^{-1}$

$$\bar{h}^{-1} : \mathbb{R} \to [0,\infty], \quad \bar{h}^{-1}(\sigma) = \begin{cases} 0, & \sigma \leqslant h(0+), \\ h^{-1}(\sigma), & h(0+) < \sigma < h(\infty), \\ \infty, & h(\infty) \leqslant \sigma. \end{cases}$$

One immediately verifies that – for all $C \in \mathbb{R}$ – the function $x \mapsto U(x,C)$ is measurable with

$$0 \leqslant U(x,C) \leqslant \infty.$$

Hence the quantity

$$\mathbf{M}(C) := \int_{\Omega} U(x,C)\,dx \in [0,\infty]$$

is well-defined for all $C \in \mathbb{R}$. We have to impose the additional assumption

(HV4) If $\Omega = \mathbb{R}^d$ and $h(0+) = -\infty$, then there is $C \in \mathbb{R}$ with $U(x,C) \in L^1(\mathbb{R}^d)$.

Remark 12. a) There is no necessity to impose a condition like (HV4) for bounded Ω or if $h(0+) > -\infty$, because in these cases the function $U(x,C)$ will be compactly supported (due to (HV3)) and bounded for any $C < h(\infty)$.

b) (HV4) is a condition on f as well as on V. If we, e.g., consider in case of $\Omega = \mathbb{R}^d$ a potential $V(x) = \alpha\,|x|^2 + \beta\,|x|^\kappa$ for some $\alpha, \beta, \kappa > 0$, and $f(u) = u^m$, $m > 0$, then (HV4) will be equivalent to the dimension-dependent restriction

$$\frac{d - \max\{2,\kappa\}}{d} < m.$$

Remark 13. A minimization procedure similiar to "$E \to \min$ in $\mathscr{C}$" can be found in [51]. In that paper no confining potential is involved and different interaction potentials are considered. The requirements on f are closely related to the ones presented here. In particular the need to impose an assumption like (HF4) arises.

The following continuity and monotonicity properties of the mapping $\mathbf{M} : C \mapsto \mathbf{M}(C)$ are of importance. Let us introduce

$$C^* := \sup\{C \in \mathbb{R} : U(x,C) \in L^1(\Omega)\}.$$

We have $C^* \in (h(0+), h(\infty)]$ and $C^* = h(\infty)$ if Ω is bounded or if $h(0+) > -\infty$.

a) $\mathbf{M}$ is increasing.

b) $\lim_{C\to-\infty}\mathbf{M}(C) = 0$, $\lim_{C\to\infty}\mathbf{M}(C) = \infty$.

c) $\mathbf{M}(C) = 0$ for all $C \in (-\infty, h(0+)]$.

d) $\mathbf{M}(C) = \infty$ for all $C \in (C^*, \infty)$.

e) $\mathbf{M}$ is continuous and strictly increasing on $(h(0+), C^*)$.

f) If $h(0+) \in \mathbb{R}$, then $\mathbf{M}$ is continuous at $h(0+)$ with $\mathbf{M}(h(0+)) = 0$.

We introduce

$$\bar{\mathbf{M}} = \lim_{C\to C^*}\mathbf{M}(C).$$

It is clear that $\bar{\mathbf{M}}$ may depend on Ω, we will eventually denote it by $\bar{\mathbf{M}}(\Omega)$. Let us distinguish several cases depending on $h(0+)$ and $h(\infty)$.

1. Case I: $h(0+) = -\infty$ and $h(\infty) = \infty$. (Example: $h(u) = u\log(u)$ which corresponds to $f(u) = u$.) In this case we have $\bar{h}^{-1} = h^{-1}$ and $U(x,C)$ solves (37). We deduce from the properties of $\mathbf{M}$ that $\bar{\mathbf{M}} = \infty$, i.e. there exists a unique $\bar{C} \in (h(0+), C^*)$ such that $\mathbf{M}(\bar{C}) = M$. We set $u_{\infty,M} := U(x,\bar{C})$.

2. Case II: $-\infty < h(0+)$ and $h(\infty) = \infty$. (Examples: $h(u) = (u^{m-1} - 1)/(1-(1/m))$, $1 < m$, which corresponds to $f(u) = u^m$.) We easily deduce $\bar{\mathbf{M}} = \infty$. Hence there is a unique $\bar{C} \in (h(0+), h(\infty))$ such that $\mathbf{M}(\bar{C}) = M$. As in Case I we set $u_{\infty,M} := U(x,\bar{C})$.

3. Case III: $h(0+) = -\infty$ and $h(\infty) < \infty$. (Examples: $h(u) = (u^{m-1} - 1)/(1 - (1/m))$, $0 < m < 1$, which corresponds to $f(u) = u^m$.) If $\bar{\mathbf{M}} > M$ (or, more specifically, $\bar{\mathbf{M}} = \infty$), then there is a unique $\bar{C} \in (-\infty, C^*)$ with $\mathbf{M}(\bar{C}) = M$. In case of existence we set $u_{\infty,M} = U(x, \bar{C})$.

4. Case IV: $-\infty < h(0+)$ and $h(\infty) < \infty$. Similar arguments as in the two previous cases ensure the existence of a unique equilibrium solution $u_{\infty,M}$ with mass M if $0 < M < \bar{\mathbf{M}} = \lim_{C \to C^*} \mathbf{M}(C)$.

Summarizing the discussion outlined so far we shall impose the additional assumption

(HV5) $M < \bar{\mathbf{M}}(\Omega) = \lim_{C \to C^*} \mathbf{M}(C)$.

Remark 14. If $h(0+) > -\infty$ one may work with a functional different from E. Consider $\tilde{\Phi}(u) = \Phi(u) - h(0+)u$. We observe that $\tilde{\Phi}(u) \geqslant 0$. Let us define $\tilde{E} : L^1_+(\Omega) \to \mathbb{R} \cup \{\infty\}$ by

$$\tilde{E}(u) = \int_\Omega (Vu + \tilde{\Phi}(u))(x)\, dx.$$

Of course, $\tilde{E}(u) - E(u) = -h(0+)M$.

It remains to verify that $u_{\infty,M}$ is indeed the unique minimizer of E in $\mathscr{C}$. The proof of this result heavily relies on the following inequality which involves the relative entropy functional

$$E(.|u_{\infty,M}) : L^1_+(\Omega) \to [0, \infty],$$

$$E(u|u_{\infty,M}) = \int_\Omega (\Phi(u) - \Phi(u_{\infty,M}) - \Phi'(u_{\infty,M})(u - u_{\infty,M}))(x)\, dx,$$

where we implicitly make use of the fact that $u_{\infty,M}(x)$ belongs for all $x \in \Omega$ to the domain of Φ' and due to convexity,

$$\Phi(u) - \Phi(u_{\infty,M}) - \Phi'(u_{\infty,M})(u - u_{\infty,M})(x) \geqslant 0$$

for all $x \in \Omega$ such that the integral in the definition of $E(.|u_{\infty,M})$ has a well-defined value in $[0, \infty]$.

The key estimate is

Proposition 5. *Assume* (HD1), (HV1)–(HV5), (HF1)–(HF3). *Furthermore, assume*

$$E(u_{\infty,M}) < \infty.$$

Then, for all $u \in \mathscr{C}$,

$$E(u) - E(u_{\infty,M}) \geqslant E(u|u_{\infty,M}), \tag{40}$$

where equality holds for all $u \in \mathscr{C}$ *iff*

$$V(x) + h(u_{\infty,M}(x)) = C, \quad \textit{for almost all } x \in \Omega.$$

Proof. Obviously there is nothing to prove in case $E(u) = \infty$. Hence assume $E(u) < \infty$. We observe

$$V(x) + h(u_{\infty,M}(x)) = C \quad \text{if } u_{\infty,M}(x) > 0,$$
$$V(x) + h(0+) \geqslant C \quad \text{if } u_{\infty,M}(x) = 0.$$

Furthermore, $E(u) < \infty$ implies $Vu, \Phi(u) \in L^1(\Omega)$. Since $E(u_{\infty,M}) < \infty$ is assumed, we also have $Vu_\infty, \Phi(u_{\infty,M}) \in L^1(\Omega)$. Let us now prove that these assumptions imply $E(u|u_{\infty,M}) < \infty$.

It sufficies to prove $\Phi'(u_{\infty,M})u, \Phi'(u_{\infty,M})u_{\infty,M} \in L^1(\Omega)$. This is trivial whenever $\Phi'(0+) = h(0+) > -\infty$. In case of $h(0+) = -\infty$ we have $V + \Phi'(u_{\infty,M}) = C$ such that $\Phi'(u_{\infty,M}) = C - V$ and $\Phi'(u_{\infty,M})u = Cu - Vu \in L^1(\Omega)$, $\Phi'(u_{\infty,M})u_{\infty,M} = Cu_{\infty,M} - Vu_{\infty,M} \in L^1(\Omega)$. This settles $E(u|u_{\infty,M}) < \infty$. Next we calculate

$$
\begin{aligned}
&E(u) - E(u_{\infty,M}) - E(u|u_{\infty,M}) \\
&= \int_\Omega \big(V(x)u(x) + \Phi(u(x)) - V(x)u_{\infty,M}(x) - \Phi(u_{\infty,M}(x)) \\
&\quad - \Phi(u(x)) + \Phi(u_{\infty,M}(x)) + \Phi'(u_{\infty,M}(x))(u(x) - u_{\infty,M}(x))\big)\, dx \\
&= \int_\Omega (V(x) + h(u_{\infty,M}(x)))(u(x) - u_{\infty,M}(x))dx \\
&= \int_{u_{\infty,M}>0} (V(x) + h(u_{\infty,M}(x)))(u(x) - u_{\infty,M}(x))dx \\
&\quad + \int_{u_{\infty,M}=0} (V(x) + h(u_{\infty,M}(x)))(u(x) - u_{\infty,M}(x))\, dx \\
&= \int_{u_{\infty,M}>0} C(u(x) - u_{\infty,M}(x))dx + \int_{u_{\infty,M}=0} (V(x) + h(u_{\infty,M}(x)))u(x)\, dx \\
&\geqslant C\int_{u_{\infty,M}>0} (u(x) - u_{\infty,M}(x))\, dx + C\int_{u_{\infty,M}=0} u(x)\, dx \\
&= C\int_{u_{\infty,M}>0} (u(x) - u_{\infty,M}(x))\, dx + C\int_{u_{\infty,M}=0} (u(x) - u_{\infty,M}(x))\, dx \\
&= C\int_\Omega (u(x) - u_{\infty,M}(x))\, dx = 0.
\end{aligned}
$$

The verification of the statement concerning equality is left to the reader. □

Remark 15. If $h(0+) = -\infty$, then one has for all $x \in \Omega$ the identity $V(x) + h(u_{\infty,M}(x)) = C$. Hence in this case the inequality of Proposition 5 is an equality.

Remark 16. a) The assumption $E(u_{\infty,M}) < \infty$ is essential for the forthcoming analysis which concerns to a large extent the convergence of $E(u(t))$ to $E(u_{\infty,M})$ as $t \to \infty$.

b) In case of bounded Ω we trivially have $E(u_{\infty,M}) < \infty$.

d) For $\Omega = \mathbb{R}^d$, $E(u_{\infty,M}) < \infty$ can be seen as a condition on the growth of $u_{\infty,M}$ locally and at $|x| = \infty$.

e) If $\Omega = \mathbb{R}^d$, if $V(x) = \alpha\,|x|^2 + \beta\,|x|^\kappa$, $\alpha, \beta, \kappa > 0$, and if $f(u) = u^m$, $m > 0$, then $E(u_{\infty,M}) < \infty$ iff

$$\frac{d}{d + \max\{2, \kappa\}} < m.$$

We shall henceforth assume

(HV6) If $\Omega = \mathbb{R}^d$, then $E(u_{\infty,\eta}) < \infty$ for $\eta \in (0, \bar{\mathbf{M}})$.

Proposition 5 contains the essential information to prove

Lemma 6. *Assume* (HD1), (HV1)–(HV6), (HF1)–(HF3). *Then $u_{\infty,M}$ is the unique minimizer of E in $\mathscr{C}$ and, by definition, the unique equilibrium solution of* (34), (35).

Proof. Since E is strictly convex there is at most one minimizer of E in $\mathscr{C}$. Hence it sufficies to prove $E(u_{\infty,M}) \leqslant E(u)$ for all $u \in \mathscr{C}$. This inequality is trivial whenever $E(u) = \infty$. Therefore we assume $E(u) < \infty$ henceforth. In this case we obtain from (40) the estimate $E(u) \geqslant E(u_{\infty,M}) + E(u|u_{\infty,M})$ which gives due to the non-negativity of $E(u|u_{\infty,M})$ the required estimate $E(u) \geqslant E(u_{\infty,M})$ □

For later reference we need two monotonicity properties of the functional E:

Lemma 7. *Assume* (HD1), (HF1)–(HF3). *Let $V : \Omega \to \mathbb{R}$ be measurable and non-negative. Assume furthermore*

- $u_0 \in L^1_+(\Omega)$ *and* $(u_n)_{n\in\mathbb{N}}$ *is a sequence in* $L^1_+(\Omega)$.
- $E(u_0) < \infty$.
- $u_n \leqslant u_0$, *for all* $n \in \mathbb{N}$.

Then $E(u_n) < \infty$ for all $n \in \mathbb{N}$, and if $u_n(x) \to v(x)$ as $n \to \infty$ for almost all $x \in \Omega$, then $v \in L^1_+(\Omega)$, $E(v) < \infty$ and $E(v) = \lim_{n\to\infty} E(u_n)$.

Proof. Due to the positivity of V we have $Vu_n \leqslant Vu_0 \in L^1(\Omega)$ for all $n \in \mathbb{N}$. Hence $\int_\Omega (Vu_n)(x)\,dx \leqslant \int_\Omega (Vu_0)(x)\,dx$. We recall that Φ is decreasing and non-positive on $[0,1]$, increasing and non-positive on $[1, s_\circ]$ and increasing and non-negative on $[s_\circ, \infty)$. Hence, $|\Phi|$ is increasing on $[0,1] \cup [s_\circ, \infty)$. We set

$$\Omega_0^1 := \{u_0 \leqslant 1\},\ \Omega_0^2 := \{1 < u_0 < s_\circ\},\ \Omega_0^3 := \{s_\circ \leqslant u_0\},$$

and obtain for each $n \in \mathbb{N}$:

If $x \in \Omega_0^1$, then $|\Phi(u_n(x))| \leqslant |\Phi(u_0(x))|$;

If $x \in \Omega_0^2$, then $|\Phi(u_n(x))| \leqslant |\Phi(1)|$;

If $x \in \Omega_0^3$, then $|\Phi(u_n(x))| \leqslant \max\{|\Phi(u_0(x))|, |\Phi(1)|\}$.

Furthermore, both Ω_0^2 and Ω_0^3 have finite measure such that we can conclude

$$|\Phi(u_n)| \leqslant |\Phi(u_0)| \mathrm{ind}_{\Omega_0^1} + |\Phi(1)|\,\mathrm{ind}_{\Omega_0^2} + \max\{|\Phi(u_0)|, |\Phi(1)|\}\,\mathrm{ind}_{\Omega_0^3} \in L^1(\Omega). \tag{41}$$

Hence $|\Phi(u_n)| \in L^1(\Omega)$ and therefore $E(u_n) < \infty$.

Now assume $u_n(x) \to v(x)$ as $n \to \infty$ for almost all $x \in \Omega$. By Lebesgue's dominated convergence theorem we obtain $\int_\Omega (Vv)(x)\,dx = \lim_{n\to\infty} \int_\Omega (Vu_n)(x)\,dx \leqslant \int_\Omega (Vu_0)(x)\,dx < \infty$. We obtain $E(v) < \infty$ and $E(v) = \lim_{n\to\infty} E(u_n)$ from Lebesgue's dominated convergence theorem. □

Lemma 8. *Assume* (HD1), (HF1)–(HF3). *Let $V : \Omega \to \mathbb{R}$ be measurable and non-negative. Assume furthermore*

- $u_0 \in L^1_+(\Omega)$ *and* $(u_n)_{n\in\mathbb{N}}$ *is a sequence in* $L^1_+(\Omega)$.
- $u_n \leqslant u_{n+1} \leqslant u_0$, *for all* $n \in \mathbb{N}$.
- $\lim_{n\to\infty} u_n(x) = u_0(x)$ *for almost all* $x \in \Omega$.

Then

a) $\lim_{n\to\infty}\int_\Omega (Vu_n)(x)\,dx = \int_\Omega (Vu_0)(x)\,dx \in \mathbb{R}^+ \cup \{\infty\}$.
b) $\lim_{n\to\infty}\int_\Omega \Phi^+(u_n)(x)\,dx = \int_\Omega \Phi^+(u_0)(x)\,dx \in \mathbb{R}^+ \cup \{\infty\}$.
c) $\lim_{n\to\infty}\int_\Omega \Phi^-(u_n)(x)\,dx = \int_\Omega \Phi^-(u_0)(x)\,dx \in \mathbb{R}^+ \cup \{\infty\}$.
d) *If* $\int_\Omega \Phi^-(u_0)(x)\,dx < \infty$, *then* $\int_\Omega \Phi^-(u_n)(x)\,dx < \infty$ *for all* $n \in \mathbb{N}$, *and*

$$\lim_{n\to\infty} E(u_n) = E(u_0).$$

Proof. a) follows from the non-negativity of V and from the monotone convergence theorem.

b) We observe: Φ^+ is increasing. Hence b) follows from the monotone convergence theorem.

c) We set as in the proof of Lemma 7

$$\Omega_0^1 := \{u_0 \leqslant 1\},\ \Omega_0^2 := \{1 < u_0 < s_\circ\},\ \Omega_0^3 := \{s_\circ \leqslant u_0\}.$$

Since Ω_2^0 has bounded measure and since $\Phi^-(u_0)$ is bounded on Ω_0^2, we obtain

$$\lim_{n\to\infty}\int_{\Omega_0^2} \Phi^-(u_n)(x)\,dx = \int_{\Omega_0^2} \Phi^-(u_0)(x)\,dx$$

from Lebesgue's dominated convergence theorem. Clearly,

$$\lim_{n\to\infty}\int_{\Omega_0^3} \Phi^-(u_n)(x)\,dx = \int_{\Omega_0^3} \Phi^-(u_0)(x)\,dx = 0, \quad \forall n \in \mathbb{N}.$$

Since Φ^- is increasing on $[0,1]$ we obtain

$$\lim_{n\to\infty}\int_{\Omega_0^1} \Phi^-(u_n)(x)\,dx = \int_{\Omega_0^1} \Phi^-(u_0)(x)\,dx$$

from the monotone convergence theorem.

d) We have for all $n \in \mathbb{N}$,

$$\int_{\Omega_0^2} \Phi^-(u_n)(x)\,dx \leqslant |\Phi(1)|\,\mathrm{meas}(\Omega_0^2) < \infty,$$

$$\int_{\Omega_0^1} \Phi^-(u_n)(x)\,dx \leqslant \int_{\Omega_0^1} \Phi^-(u_0)(x)\,dx < \infty.$$

$\lim_{n\to\infty} E(u_n) = E(u_0)$ follows from a), b), c). □

3.2. Time dependent scalings, $\Omega = \mathbb{R}^d$. Time dependent scalings have been studied in [17], [28] establishing a connection between the asymptotic behavior of equations (34) and the general filtration equation

$$\frac{\partial v}{\partial t} = \Delta f(v), \qquad (x \in \mathbb{R}^d, t > 0). \tag{42}$$

In fact, the results in [17] show that if f' is homogeneous of degree r with $dr + 2 > 0$, there exists a time dependent scaling

$$v(x,t) = \alpha(t)^d u(\alpha(t)x, \beta(t)) \tag{43}$$

with $\alpha(0) = 1$, $\beta(0) = 0$ and $\beta(t) \to \infty$ as $t \to \infty$, such that u is a solution of (34) with $V(x) = |x|^2/2$ if and only if v is a solution of (42). Thus, this scaling is valid for $f(u) = u^m$ with $d(m-1)+2 > 0$ for $d \geqslant 1$ or $f(u) = \log(u)$ for $d = 1$.

The time dependent scaling is very useful since results on the asymptotic behavior of equations of the type (34) with $V(x) = |x|^2/2$ and $f(u) = u^m$ ($d(m-1)+2 > 0$) translate into results of the asymptotic behavior of the equations (42) with $f(u) = u^m$ ($d(m-1)+2 > 0$). This includes the porous medium equation and the fast diffusion equation with $m \geqslant \frac{d-2}{d}$.

Moreover, the stationary solutions for (34) obtained in the previous subsection correspond to the Barenblatt–Prattle self-similar solutions for the filtration equation (42) through the scaling (43) upto a time translation.

If f is not a power function, there are no self-similar solutions of the corresponding filtration equation (42) and then the time dependent scaling does not work. Nevertheless, one can show very easily that the scaling (43) with $\alpha(t) = (2t+1)^{-1/2}$ and $\beta(t) = -\log \alpha(t)$ translates (34) with $V(x) = |x|^2/2$ into

$$\frac{\partial v}{\partial t} = \alpha(t)^{d-2}\beta'(t)\Delta f(\alpha(t)^{-d} v), \qquad (x \in \mathbb{R}^d, t > 0). \tag{44}$$

and choosing $w = \alpha(t)^{-d} v$ we have

$$\frac{\partial w}{\partial t} = \frac{d}{2t+1} w + \Delta f(w), \qquad (x \in \mathbb{R}^d, t > 0). \tag{45}$$

Therefore, we have found a time dependent scaling translating (34) with $V(x) = |x|^2/2$ into the filtration equation with sources (45).

Again, the stationary solutions of (34) with $V(x) = |x|^2/2$ correspond to self-similar solutions of (45) upto a time translation and the asymptotic behavior of (45) is reduced to the asymptotic behavior of (34) with $V(x) = |x|^2/2$.

We conclude: From the analysis of the asymptotic behavior of (35), we will obtain (as a by-product) results on the asymptotic behavior of (35) and (45) using different time dependent scalings (43). The interested reader can produce them without any difficulty.

Let us also mention that related results were obtained by several authors in different settings and particular cases [31], [32], [45], [46], [54], [59] by proving the decay to zero of solutions of nonlinear diffusion equations without confinement (42). Also they proved the convergence to a self-similar profile without rate in L^1 and the Liapunov functional introduced in previous subsection was used in the unconfined case to prove this convergence without rate in L^1.

3.3 Exponential decay of the entropy, Ω bounded. In this section we consider the generalized Fokker–Planck equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(u\nabla V(x) + \nabla f(u)), \qquad (x \in \Omega, t > 0), \tag{46}$$

with initial condition

$$u(x, t = 0) = u_0(x) \geqslant 0, \qquad (x \in \Omega), \tag{47}$$

and zero-outflux boundary condition

$$u\frac{\partial V(x)}{\partial n} + \frac{\partial f(u)}{\partial n} = 0, \qquad (x \in \partial\Omega, t > 0) \tag{48}$$

assuming (HD1), (HD2), (HV1)–(HV6), (HF1)–(HF3) *and* bounded Ω.

The aim of this subsection is to apply the entropy dissipation (production) method to prove exponential decay of the Liapunov-type relative entropy

$$RE(u(t)|u_{\infty,M}) = E(u(t)) - E(u_{\infty,M}), \quad \text{as } t \to \infty.$$

The strategy of the proof is the following.
First of all, smooth solutions for appropriately modified data (eventually u_0 and f have to be mollified) are considered. In this case the relative entropy $RE(u(t)|u_{\infty,M})$ is (twice) differentiable in time. The crucial step is the verification of the exponential decay of the entropy production

$$I(u(t)) := -\frac{d}{dt}(RE(u(t)|u_{\infty,M})).$$

From this decay rate it is easy to deduce the corresponding exponential decay of $RE(u(t)|u_{\infty,M})$.

In a second step, the generalized solution u of the original system (46), (47), (48) (which is generally not smooth enough to apply the techniques of the first step) is approximated by solutions u_ε of appropriately defined "mollified" systems. The limit $\varepsilon \to 0$ means to go back to the original system. Several ε-independent estimates and a lower semi-continuity argument allow for the verification of the exponential decay of the relative entropy. Several difficulties arise from the fact that one might loose the differentiability of the relative entropy in the limit $\varepsilon = 0$.

The entropy production method requires additional assumptions both on f and on V. As it will become clear in the proof of Theorem 11, we have to require

(HV7) Ω is convex.

(HV8) $V = W|_\Omega$, where $W \in C^2(\mathbb{R}^d, \mathbb{R})$ is uniformly convex, that is, there is $\alpha_1 > 0$ such that

$$\xi \cdot \text{Hess}(W(x)) \cdot \xi^T \ \geqslant\ \alpha_1|\xi|^2$$

for any $x, \xi \in \mathbb{R}^d$.

(HF4) $f(u) \leqslant \dfrac{d}{d-1}uf'(u)$, for all $u > 0$.

Remark 17. a) Obviously assumption (HV8) implies (HV1) and (HV2).

b) Assumption (HF4) restricts the possible values for m if $f(u) = u^m$, $m > 0$, namely $m \geqslant \frac{d-1}{d}$. Thus, (HF4) is a restriction of the velocity of the diffusion, that is, the diffusion cannot be very fast.

Remark 18. In the sequel assumption (HF4) will be of distinctive importance. It seems appropriate to add a few remarks whose verifications are left to the reader. We assume that f satisfies (HF1)–(HF3).

a) (HF4) is equivalent to

- There is a non-decreasing, non-negative function $a \in C^3(\mathbb{R}^+)$ with

$$0 < a(u) \leqslant 1, \quad \text{if } 0 < u < 1, \quad \text{and} \quad 1 \leqslant a(u), \quad \text{if } 1 < u, \quad \text{and}$$
$$f(u) = f(1)\, a(u)\, u^{\frac{d-1}{d}}, \quad u > 0,$$

i.e.

$$a(u) = \exp\left(\int_1^u \gamma(s)\, ds\right), \quad u > 0,$$

with non-negative $\gamma \in C^\infty(\mathbb{R}^+)$.

b) In the sequel we use approximations f_ε, $\varepsilon \in (0,\infty)$, of f such that each f_ε satisfies (HF1)–(HF4) with $f_\varepsilon'(0+) > 0$, $f_\varepsilon' \geqslant f'$, $f_\varepsilon' \to f'$ uniformly on compact subsets of $(0,\infty)$ and $f_\varepsilon \to f$ uniformly on compact subsets of $[0,\infty)$. Indeed, if $f'(0+) > 0$, then one may take $f_\varepsilon = f$, so it remains to consider the case $f'(0+) = 0$. We choose $\gamma \in C^\infty(\mathbb{R}^+)$ such that $\mathrm{supp}(\gamma) \subseteq [0,2]$, $\gamma \geqslant 0$ and $\gamma(u) = \frac{1}{d\,u}$ for $u \in (0,1)$. Then we define for $\varepsilon \in (0,\infty)$ the function $f_\varepsilon : [0,\infty) \to \mathbb{R}$, where for $u \geqslant 0$,

$$f_\varepsilon(u) := f(u) + \varepsilon \exp\left(\int_1^u \gamma(s)\, ds\right) u^{\frac{d-1}{d}}.$$

We obtain: $f_\varepsilon(u) = \varepsilon\, u + f(u)$ for $0 < u < 1$ and f_ε trivially satisfies (HF1)–(HF4) and the required approximation properties as $\varepsilon \to 0$.

Before we develop the entropy production method for sufficiently smooth solutions $u(t)$ we cite an auxiliary result which will be of great importance in the sequel

Lemma 9 [56]. *Given any non negative $u \in L^1_{loc}(\mathbb{R}^d)$ and vector valued $A \in L^1_{loc}(\mathbb{R}^d, \mathbb{R}^d)$:*

$$\frac{1}{2}\int_{\mathbb{R}^d} \mathrm{frac}\left\{|A|^2(x) : u(x)\right\} dx$$
$$= \sup\left\{\int_{\mathbb{R}^d} A \cdot \xi\, dx - \frac{1}{2}\int_{\mathbb{R}^d} u|\xi|^2\, dx : \xi \in C_c^\infty(\mathbb{R}^d : \mathbb{R}^d)\right\},$$

where the "generalized fraction" frac is

$$\mathrm{frac} : [0,\infty) \times [0,\infty) \to \mathbb{R} \cup \{\infty\}, \quad \mathrm{frac}\{w : z\} = \begin{cases} w/z, & z \neq 0, \\ 0, & w = z = 0, \\ \infty, & w \neq 0,\ z = 0. \end{cases}$$

The representation of the integral $\frac{1}{2}\int_{\mathbb{R}^d} \mathrm{frac}\{|A|^2(x) : u(x)\}\, dx$ as supremum of a (nonlinear) functional allows for the verification of a lower semi-continuity property similiar to the lower semi-continuity of the norm.

To fix ideas let us give

Definition 4. Assume (HD1), (HV1). Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be measurable. Then

$$K_f : L^1_+(\Omega) \to \mathbb{R} \cup \{+\infty\},$$

$$K_f(u) = \begin{cases} \int_\Omega \operatorname{frac}\Big\{|(u\nabla V + \nabla f(u))(x)|^2 : u(x)\Big\}\, dx, & u \in \mathscr{D}_f, \\ \infty, & \mathit{else}, \end{cases}$$

where

$$\mathscr{D}_f := \{u \in L^1_+(\Omega) : f(u) \in L^1_{\mathrm{loc}}(\Omega),\ \nabla f(u) \in L^1_{\mathrm{loc}}(\Omega : \mathbb{R}^d)\}.$$

Then we have

Lemma 10. *Assume* (HD1), (HV8). *Let* $f : \mathbb{R}_0^+ \to \mathbb{R}$ *be measurable. For* $n \in \mathbb{N}$ *let* $f_n : \mathbb{R}_0^+ \to \mathbb{R}$ *be measurable and let* $u_n \in \mathscr{D}_{f_n}$. *Assume furthermore*

a) $u_n \rightharpoonup u$ *as* $n \to \infty$, *weakly in* $L^1(\Omega)$.

b) $\nabla f_n(u_n) \rightharpoonup \nabla f(u)$ *as* $n \to \infty$, *weakly in* $L^1(\Omega : \mathbb{R}^d)$.

Then, with the notations of Definition 4,

1) $K_f(u) \leqslant \liminf_{n\to\infty} K_{f_n}(u_n)$.

2) If $\liminf_{n\to\infty} K_{f_n}(u_n) < \infty$, *then* $u \in \mathscr{D}_f$ *and* $K_f(u) < \infty$.

Proof. Obviously, we can restrict ourselves to the case $\liminf_{n\to\infty} K_{f_n}(u_n) < \infty$. We wish to apply Lemma 9 [56] with

$$A : \mathbb{R}^d \to \mathbb{R}^d, \quad A(x) = \begin{cases} (u\nabla V + \nabla f(u))(x), & x \in \Omega, \\ 0, & \text{else}, \end{cases}$$

where we observe that due to assumptions a) and b) we have $u \in L^1_+(\Omega)$, $f(u) \in L^1(\Omega : \mathbb{R}^d)$. Furthermore, we infer $A \in L^1_{\mathrm{loc}}(\mathbb{R}^d : \mathbb{R}^d)$ from a), b), (HV8). We also introduce the trivial extension

$$u^{\mathrm{ext}} : \mathbb{R}^d \to \mathbb{R}^d, \quad u^{\mathrm{ext}}(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & \text{else}. \end{cases}$$

of u. Then we have

$$K_f(u) = \int_{\mathbb{R}^d} \operatorname{frac}\Big\{|A|^2(x) : u^{\mathrm{ext}}(x)\Big\}\, dx. \tag{49}$$

Proceeding in analogy we obtain

$$K_{f_n}(u_n) = \int_{\mathbb{R}^d} \operatorname{frac}\Big\{|A_n|^2(x) : u_n^{\mathrm{ext}}(x)\Big\}\, dx, \quad n \in \mathbb{N}. \tag{50}$$

According to assumptions a), b), (HV8) we have for each $\xi \in C_c^\infty(\mathbb{R}^d : \mathbb{R}^d)$

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} A_n \cdot \xi\, dx = \int_{\mathbb{R}^d} A \cdot \xi\, dx, \quad \lim_{n\to\infty} \int_{\mathbb{R}^d} u_n^{\mathrm{ext}} |\xi|^2\, dx = \int_{\mathbb{R}^d} u^{\mathrm{ext}} |\xi|^2\, dx.$$

On the other hand we have for each such ξ and each $n \in \mathbb{N}$ due to Lemma 9 the estimate

$$\int_{\mathbb{R}^d} \operatorname{frac}\Big\{|A_n|^2(x) : u_n^{\mathrm{ext}}(x)\Big\}\, dx \geqslant 2 \int_{\mathbb{R}^d} A_n \cdot \xi\, dx - \int_{\mathbb{R}^d} u_n^{\mathrm{ext}} |\xi|^2\, dx,$$

such that we obtain

$$
\begin{aligned}
2\int_{\mathbb{R}^d} A\cdot\xi\,dx - \int_{\mathbb{R}^d} u^{\mathrm{ext}}|\xi|^2\,dx
&= \lim_{n\to\infty} 2\int_{\mathbb{R}^d} A_n\cdot\xi\,dx - \int_{\mathbb{R}^d} u_n^{\mathrm{ext}}|\xi|^2\,dx\\
&\leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^d} \mathrm{frac}\Big\{|A_n|^2(x) : u_n^{\mathrm{ext}}(x)\Big\}\,dx,
\end{aligned}
$$

and we obtain (again due to Lemma 9) the estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{frac}\Big\{|A|^2(x) : u^{\mathrm{ext}}(x)\Big\}\,dx\\
&\quad = \sup\Big\{2\int_{\mathbb{R}^d} A\cdot\xi\,dx - \int_{\mathbb{R}^d} u^{\mathrm{ext}}|\xi|^2\,dx : \xi\in C_0^\infty(\mathbb{R}^d:\mathbb{R}^d)\Big\}\\
&\quad \leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^d} \mathrm{frac}\Big\{|A_n|^2(x) : u_n^{\mathrm{ext}}(x)\Big\}\,dx.
\end{aligned}
$$

Propositions 1) and 2) follow from (49) and (50) now. □

For later use we define

Definition 5. Assume (HD1), (HV1). Let $h:\mathbb{R}^+\to\mathbb{R}$ be measurable. Then

$$
J_h : L^1_+(\Omega)\to\mathbb{R}\cup\{+\infty\},
$$

$$
J_h(u) = \begin{cases} \int_{u>0}\Big(u\,|\nabla V + \nabla h(u))|^2\Big)(x)\,dx, & u\in\mathscr{D}_h,\\ \infty, & \text{else},\end{cases}
$$

where $\mathscr{D}_h$ is as in definition 4 with "h" replacing "f".

We introduce for any $T>0$, $Q_T := (0,T)\times\Omega$.

Theorem 11. *Assume* (HD1), (HD2), (HV3)–(HV8), (HF1)–(HF4) *with bounded* Ω *and let* u *be a generalized solution of* (46), (47), (48). *Assume furthermore*

(A0) For each $t\in\mathbb{R}_0^+$, $\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} = M$.

(A1) $u\in C^{1,2}(\bar{Q}_T)$, *for each* $T\geqslant 0$.

(A2) There is $\rho_0\in\mathbb{R}^+$ *with* $u_0\geqslant\rho_0$.

(A3) There is $K\in\mathbb{R}^+$ *such that* $\|u(t)\|_{L^\infty(\Omega)}\leqslant K$, *for all* $t\in\mathbb{R}_0^+$.

(A4) $h(0+) = -\infty$.

(A5) Each sequence $(u_k)_{k\in\mathbb{N}}$ *in* $C^{1,2}(\bar{Q}_T)$ *of solutions of* (46), (48), which is uniformly bounded in $L^\infty(Q_T)$, has a subsequence $(u_{\nu(k)})_{k\in\mathbb{N}}$ such that $\{u_{\nu(k)} : k\in\mathbb{N}\}$ is equicontinuous.

Then

a) The functions $t\mapsto RE(u(t)|u_{\infty,M})$ *and* $t\to E(u(t))$ *belong to* $C^2(\mathbb{R}_0^+)$ *with*

$$
RE(u(t)|u_{\infty,M}) \leqslant RE(u(t_0)|u_{\infty,M})\,e^{-2\alpha_1(t-t_0)},\quad t\geqslant t_0\geqslant 0. \tag{51}
$$

b) The entropy production rate

$$I(u(t)) := -\frac{d}{dt}E(u(t)) = -\frac{d}{dt}RE(u(t)|u_{\infty,M}) \tag{52}$$

satisfies

$$I(u(t)) \leqslant I(u(t_0))\, e^{-2\,\alpha_1\,(t-t_0)}. \tag{53}$$

c) $RE(u(t)|u_{\infty,M})$ and $I(u(t))$ are related via

$$0 \leqslant RE(u(t)|u_{\infty,M}) \leqslant \frac{1}{2\,\alpha_1} I(u(t)), \quad t \geqslant 0. \tag{54}$$

Remark 19. a) The requirement (A1) implies that $u(x,t)$ satisfies the differential equation (46) pointwise and the zero outflux boundary condition (48).

b) Due to (A1) the initial function u_0 belongs to $C^2(\bar{\Omega})$ and satisfies the no-flux boundary condition (48).

c) (A5) can be viewed as requirement on f, [27]. We shall discuss this topic later on.

d) The main ideas of this theorem appeared previously in [56] (Proposition 1) with somewhat different hypotheses, although some of the computations (mainly Steps 2–4) are almost coincident. We have included it here for the sake of the reader and completeness of the proof.

Proof. The proof is divided into several steps.

Step 0: $u(t) \geqslant \rho_\infty$, $\rho_\infty \in \mathbb{R}^+$, for all $t \in \mathbb{R}^+$. By assumption we have $u_0 \geqslant \rho_0$ for some $\rho_0 \in \mathbb{R}^+$. Since $h(0+) = -\infty$ we have for each $C \in \mathbb{R}$: $U(x,C) = \bar{h}^{-1}(C - V(x)) = h^{-1}(C - V(x))$, such that $V + h(U(.,C)) = C$. We immediately obtain: $U(.,C)$ is a generalized solution of

$$\frac{\partial u}{\partial t} = \operatorname{div}(u\nabla V + \nabla f(u)), \quad u(t=0) = U(.,C),$$

subject to no-flux boundary conditions. Furthermore, $U(.,C) \leqslant h^{-1}(C)$. Since $\lim_{C\to-\infty} h^{-1}(C) = 0$, we can take $C_\infty \in \mathbb{R}$ with $u_0 \geqslant \rho_0 > h^{-1}(C_\infty) \geqslant U(.,C_\infty)$. Hence by the comparison principle for strictly positive generalized solutions [49] we obtain

$$u(t) \geqslant U(.,C_\infty) \geqslant \inf_{x\in\Omega} U(x,C_\infty) =: \rho_\infty,$$

where due to the boundedness of V on Ω, $\rho_\infty \in \mathbb{R}^+$.

Step 1: $RE(u(t)|u_{\infty,M})$ is decreasing with $\lim_{t\to\infty} RE(u(t)|u_{\infty,M}) = 0$. Since $u(t) \geqslant \rho_\infty$ independently of t and since for all $T > 0$, $u \in C^{1,2}(\bar{Q}_T)$, we can interchange integration with the derivatives and we obtain

$$RE(u(t)|u_{\infty,M}) \in C^2(\mathbb{R}_0^+)$$

with

$$-I(u(t)) = \frac{d}{dt}RE(u(t)|u_{\infty,M}) = \int_\Omega (V + h(u))\left(\frac{\partial u}{\partial t}\right)(x,t)\,dx, \tag{55}$$

where we made use of $\Phi'(u) = h(u)$ for all $u \in \mathbb{R}^+$. Since u is a strong solution of (47) we can replace $\partial u/\partial t$ by $\operatorname{div}(u\nabla V(x) + \nabla f(u))$, and obtain after an integration by parts by using the no-flux boundary condition

$$I(u(t)) = \int_\Omega u(x,t)\,|y|^2(x,t)\,dx,$$

where $y(x,t) = (\nabla V + \nabla h(u))(x,t)$. Hence the function $t \mapsto RE(u(t)|u_{\infty,M})$, $t \in \mathbb{R}_0^+$, is decreasing. On the other hand, we trivially have $RE(u(t)|u_{\infty,M}) \geqslant 0$ such that $RE(u(t)|u_{\infty,M})$ is bounded below. We obtain: $\lim_{t\to\infty} RE(u(t)|u_{\infty,M}) =: H^*$ exists and $RE(u_0|u_{\infty,M}) \geqslant H^* \geqslant 0$. We furthermore have for all $t \in \mathbb{R}^+$ the estimate

$$\infty > RE(u_0|u_{\infty,M}) \geqslant RE(u_0|u_{\infty,M}) - RE(u(t)|u_{\infty,M}) = \int_0^t I(u(s))\,ds,$$

such that we obtain due to the non-negativity of $I(u(.))$, $I(u(.)) \in L^1(\mathbb{R}_0^+)$ and

$$0 \leqslant \int_0^\infty I(u(s))\,ds \leqslant RE(u_0|u_{\infty,M}).$$

Hence there is a sequence $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^+$ with $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} I(u(t_k)) = 0$. We observe: The set $\{u(.+t_k) : [0,T] \to \Omega : k \in \mathbb{N}\}$ consists of $C^{1,2}(\bar{Q}_T)$-solutions of (47) and is uniformly bounded in $L^\infty(Q_T)$. Hence by assumption there is a subsequence – again denoted by $(t_k)_{k\in\mathbb{N}}$ – and $\hat{u} \in C(\bar{Q}_T)$ such that $u(.+t_k) \to \hat{u}$ uniformly on Q_T. We especially have $u(t_k) \to g$ uniformly on Ω. Furthermore, due to $\|u(t_k)\|_{L^\infty(\Omega)} \leqslant K \in (0,\infty)$, K independent of $k \in \mathbb{N}$, and due to $u_k \geqslant \rho_\infty$, we have

$$\int_\Omega (u(t_k)\nabla V + \nabla f(u(t_k)))^2(x)\,dx \leqslant \int_\Omega \frac{K}{u(t_k)}(u(t_k)\nabla V + \nabla f(u(t_k)))^2(x)\,dx$$
$$= K\int_\Omega u(t_k)(\nabla V + \nabla h(u(t_k)))^2(x)\,dx = K\,I(u(t_k)) \leqslant K_1,$$

where $K_1 \in (0,\infty)$ is independent of $k \in \mathbb{N}$. This estimate implies: Since the L^2-norms of $u(t_k)\nabla V$ are uniformly bounded, the L^2-norms of $\nabla f(u(t_k))$ are uniformly bounded, too. Furthermore the L^∞-norms of $f(u(t_k))$ are uniformly bounded. We obtain due to the boundedness of Ω and possibly after extraction of a subsequence (which is again denoted $(t_k)_{k\in\mathbb{N}}$),

$$f(u(t_k)) \rightharpoonup f^*, \quad \text{weakly in } H^1(\Omega) \text{ as } k \to \infty.$$

On the other hand we have $f(u(t_k)) \to f(g)$ strongly in $L^2(\Omega)$ as $k \to \infty$. Hence $f^* = f(g) \in H^1(\Omega)$.

The convergences $u(t_k) \to g$ uniformly on Ω and $\nabla f(u(t_k)) \rightharpoonup \nabla f(g)$ weakly in $H^1(\Omega)$, as $n \to \infty$, allow for an application of Lemma 10:

$$\begin{aligned} K_f(g) &= \int_\Omega \mathrm{frac}\Big\{|g\nabla V + \nabla f(g)|^2(x) : g(x)\Big\}\,dx \\ &\leqslant \liminf_{k\to\infty} K_f(u(t_k)) \\ &= \liminf_{k\to\infty} \int_\Omega \frac{1}{u(t_k)}\,|u(t_k)\,\nabla V + \nabla f(u(t_k))|^2\,dx \\ &= -\liminf_{k\to\infty} I(u(t_k)) = 0, \end{aligned}$$

where we made use of the fact that – due to $u(t_k) \geqslant \rho_\infty$ on Ω – $K_f(u(t_k)) = \int_\Omega \frac{1}{u(t_k)} |u(t_k)\, \nabla V + \nabla f(u(t_k))|^2\, dx$. Hence

$$\int_\Omega \mathrm{frac}\Big\{ |g\nabla V + \nabla f(g)|^2(x) : g(x) \Big\}\, dx = 0,$$

from which we readily deduce from the definition of frac,

$$\int_{g>0} \frac{1}{g} |g\nabla V + \nabla f(g)|^2\, dx = 0. \tag{56}$$

On the other hand we have due to uniform convergence, $g \geqslant \rho_\infty$. Hence $\Omega = \{g > 0\}$ and we deduce from (56)

$$I(g) = \int_\Omega g|\nabla V + \nabla h(g))|^2(x)\, dx = 0,$$

as well as $g = h^{-1}(C - V)$ for some $C \in \mathbb{R}$. Furthermore, $\int_\Omega g(x)\, dx = M$. Hence $g = u_{\infty,M}$.

We summarize: $u(t_k) \to u_{\infty,M} = g$ uniformly on Ω as $k \to \infty$. This convergence is sufficient to obtain

$$\lim_{k\to\infty} E(u(t_k)) = \lim_{k\to\infty} \int_\Omega (Vu(t_k) + \Phi(u(t_k)))(x)\, dx = E(u_{\infty,M}),$$

such that $\lim_{k\to\infty} RE(u(t_k)|u_{\infty,M}) = 0$. Since $RE(u(t)|u_{\infty,M})$ is decreasing in t, we finally obtain $H^* = \lim_{t\to\infty} RE(u(t)|u_{\infty,M}) = 0$.

Step 2: *Calculation of* $\frac{d}{dt} I(u(t))$. Equation (55) relates the relative entropy to the entropy production. From this relation we just concluded the convergence to zero of the solution in relative entropy, without any rate. To find the rate of convergence requires a further step. Therefore let us calculate the time evolution of $I(u(t))$,

$$\begin{aligned} \frac{d}{dt} I(u(t)) &= \int_\Omega \left(\frac{\partial u}{\partial t} |y|^2 \right)(x,t)\, dx + 2 \int_\Omega \left(uy \cdot \frac{\partial y}{\partial t} \right)(x,t)\, dx \\ &=: I_1(t) + I_2(t), \end{aligned}$$

where – according to $u \in C^{1,2}(\bar{Q}_T)$ with $u \geqslant c_T > 0$ such that $h(u) \in C^{1,2}(\bar{Q}_T)$ as well – we interchanged integration with differentiation. The second term, taking into account the boundary conditions (48), can be written as

$$I_2(t) = -2 \int_\Omega \left(\mathrm{div}\,(uy) \frac{\partial}{\partial t} h(u) \right)(x,t)\, dx = -2 \int_\Omega \left(h'(u)(\mathrm{div}\,(uy))^2 \right)(x,t)\, dx,$$

and the first term can be written as (using again (48) and the assumed smoothness of the involved functions)

$$I_1(t) = -2 \int_\Omega \left(u(y \cdot \mathrm{Jacob}(y) \cdot y^T) \right)(x,t)\, dx\ .$$

Since $\mathrm{Jacob}(y) = \mathrm{Hess}(V) + \mathrm{Hess}(h(u))$ we have

$$I_1(t) = -2\int_\Omega \big(u(y\cdot \mathrm{Hess}(V)\cdot y^T)\big)(x,t)\,dx \\ -2\int_\Omega \big(u(y\cdot \mathrm{Hess}(h(u))\cdot y^T\big)(x,t)\,dx$$

Now, the last integral reads

$$\int_\Omega \big(u(y\cdot \mathrm{Hess}(h(u))\cdot y^T)\big)(x,t)\,dx = \sum_{i,j=1}^d \int_\Omega \left(uy_iy_j \frac{\partial^2 h(u)}{\partial x_i \partial x_j}\right)(x,t)\,dx \\ = \sum_{i,j=1}^d \int_\Omega \left(\frac{1}{u}(uy_i)(uy_j)\frac{\partial^2 h(u)}{\partial x_i \partial x_j}\right)(x,t)\,dx.$$

Using the divergence theorem, the smoothness of the involved functions and taking into account the boundary conditions (46), it is straightforward to check that

$$\sum_{i,j=1}^d \int_\Omega \left(\frac{1}{u}(uy_i)(uy_j)\frac{\partial^2 h(u)}{\partial x_i \partial x_j}\right)(x,t)\,dx \\ = -\sum_{i,j=1}^d \int_\Omega \left(\frac{\partial h(u)}{\partial x_i}\left(-\frac{1}{u^2}(uy_i)(uy_j)\frac{\partial u}{\partial x_j} + \frac{1}{u}\frac{\partial[(uy_i)(uy_j)]}{\partial x_j}\right)\right)(x,t)\,dx \\ = \sum_{i,j=1}^d \int_\Omega \left(h'(u)y_iy_j\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)(x,t)\,dx \\ -\sum_{i,j=1}^d \int_\Omega \left(h'(u)\frac{\partial u}{\partial x_i}\left(y_i\frac{\partial (uy_j)}{\partial x_j} + y_j\frac{\partial (uy_i)}{\partial x_j}\right)\right)dx \\ = \sum_{i,j=1}^d \int_\Omega \left(h'(u)y_iy_j\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)(x,t)\,dx - 2\sum_{i,j=1}^d \int_\Omega \left(h'(u)y_i\frac{\partial u}{\partial x_i}\frac{\partial (uy_j)}{\partial x_j}\right)(x,t)\,dx \\ + \sum_{i,j=1}^d \int_\Omega \left(h'(u)\frac{\partial u}{\partial x_i}\left(y_i\frac{\partial (uy_j)}{\partial x_j} - y_j\frac{\partial (uy_i)}{\partial x_j}\right)\right)(x,t)\,dx.$$

Since $\dfrac{\partial y_i}{\partial x_j} = \dfrac{\partial y_j}{\partial x_i}$, we obtain

$$\sum_{i,j=1}^d \frac{\partial u}{\partial x_i}\left(y_i\frac{\partial (uy_j)}{\partial x_j} - y_j\frac{\partial (uy_i)}{\partial x_j}\right) \\ = \sum_{i,j=1}^d u\left(y_i\frac{\partial u}{\partial x_i}\frac{\partial y_j}{\partial x_j} - y_j\frac{\partial u}{\partial x_i}\frac{\partial y_i}{\partial x_j}\right) \\ = u\sum_{i,j=1}^d \left(y_i\frac{\partial u}{\partial x_i}\frac{\partial y_j}{\partial x_j} - \frac{1}{2}\frac{\partial u}{\partial x_i}\frac{\partial y_j^2}{\partial x_i}\right) \\ = u\left((y\cdot\nabla u)\mathrm{div}\, y - \frac{1}{2}\nabla|y|^2\cdot\nabla u\right).$$

Thus, simplifying and collecting terms we obtain

$$
\begin{aligned}
&I_1(t) + I_2(t) \\
&\quad = -2\int_\Omega \big(u(y\cdot \mathrm{Hess}(V)\cdot y^T)\big)(x,t)\,dx - 2\int_\Omega \Big(f'(u)u(\operatorname{div} y)^2\Big)(x,t)\,dx \\
&\qquad - 2\int_\Omega \left(f'(u)\left((y\cdot\nabla u)\operatorname{div} y - \frac{1}{2}\nabla|y|^2\cdot\nabla u\right)\right)(x,t)\,dx.
\end{aligned}
$$

that we can write as

$$
\begin{aligned}
&I_1(t) + I_2(t) \\
&\quad = -2\int_\Omega \big(u(y\cdot \mathrm{Hess}(V)\cdot y^T)\big)(x,t)\,dx - 2\int_\Omega \Big(f'(u)u\{\operatorname{div} y\}^2\Big)(x,t)\,dx \\
&\qquad - 2\int_\Omega \left(\left\{(y\cdot\nabla f(u))\operatorname{div} y - \frac{1}{2}\nabla|y|^2\cdot\nabla f(u)\right\}\right)dx.
\end{aligned}
$$

Applying the divergence theorem in the last two terms and taking into account that

$$
\operatorname{div}(y\operatorname{div} y) = (\operatorname{div} y)^2 + y\cdot\nabla(\operatorname{div} y)
$$

we deduce

$$
\begin{aligned}
&\frac{d}{dt}I(u(t)) \\
&\quad = -2\int_\Omega \big(u(y\cdot \mathrm{Hess}(V)\cdot y^T)\big)(x,t)\,dx - 2\int_\Omega \Big((f'(u)u - f(u))\{\operatorname{div} y\}^2\Big)(x,t)\,dx \\
&\qquad - 2\int_\Omega \left(f(u)\left(\frac{1}{2}\Delta|y|^2 - (y\cdot\nabla(\operatorname{div} y))\right)\right)(x,t)\,dx \\
&\qquad + 2\int_{\partial\Omega} \big(f(u)(y\cdot \mathrm{Jacob}(y)\cdot n^T)\big)(x,t)\,dS.
\end{aligned}
$$

Using that $\dfrac{\partial y_i}{\partial x_j} = \dfrac{\partial y_j}{\partial x_i}$, we obtain that

$$
\frac{1}{2}\Delta|y|^2 - (y\cdot\nabla(\operatorname{div} y)) = \sum_{i,j=1}^{d}\left(\frac{\partial y_i}{\partial x_j}\right)^2
$$

and as a consequence

$$
\begin{aligned}
&\frac{d}{dt}I(u(t)|u_{\infty,M}) \\
&\quad = -2\int_\Omega \big(u(y\cdot \mathrm{Hess}(V)\cdot y^T)\big)(x,t)\,dx - 2\int_\Omega \Big((f'(u)u - f(u))(\operatorname{div} y)^2\Big)(x,t)\,dx \\
&\qquad - 2\int_\Omega \left(f(u)\left(\sum_{i,j=1}^{d}\left(\frac{\partial y_i}{\partial x_j}\right)^2\right)\right)(x,t)\,dx + 2\int_{\partial\Omega} f(u)(y\cdot \mathrm{Jacob}(y)\cdot n^T)\,dS \\
&\quad \leqslant -2\,\alpha_1\, I(u(t)) - R(t), \qquad (57)
\end{aligned}
$$

where we made use of the uniform convexity of V and we put

$$
\begin{aligned}
R(t) := {} & 2\int_\Omega \Big((f'(u)u - f(u))(\operatorname{div} y)^2 \Big)(x,t)\,dx \\
& + 2\int_\Omega \left(f(u) \left(\sum_{i,j=1}^{d} \left(\frac{\partial y_i}{\partial x_j} \right)^2 \right) \right)(x,t)\,dx \\
& - 2\int_{\partial\Omega} \big(f(u)(y \cdot \text{Jacob}\,(y) \cdot n^T) \big) dS.
\end{aligned}
$$

Step 3: *Exponential decay of* $I(u(t))$. With respect to (57) it sufficies to prove $R(t) \geqslant 0$. This estimate is a consequence of the convexity of Ω and the imposed assumptions on f:

Let us first show that the boundary term of $R(t)$ is negative for convex domains [56]. Consider the function $\Psi : \partial\Omega \to \mathbb{R}$ defined by $\Psi(x) = y(x) \cdot n(x)$. Now, since $y(x)$ is a vector in the tangent space to $\partial\Omega$ in x due to the boundary condition (48), take a smooth curve $\alpha(t)$ such that $\alpha(0) = x$ and $\alpha'(0) = y(x)$. Using basic differential geometry calculus we have

$$
\frac{d}{dt}\Psi(\alpha(t)) = \alpha'(t)^T \text{Jacob}(y)(\alpha(t))\, n(\alpha(t)) + y(\alpha(t))^T I\!I_{\alpha(t)}\, \alpha'(t)
$$

where $I\!I_{\alpha(t)}$ is the second fundamental form of the manifold $\partial\Omega$. Since $\Psi = 0$ because of boundary condition (48), we have

$$
\frac{d}{dt}\Psi(\alpha(t))\|_{t=0} = y(x)^T \text{Jacob}(y)(x)\, n(x) + y(x)^T\ I\!I_x\, y(x) = 0.
$$

Due to the convexity of Ω we have $y(x)^T \text{Jacob}(y)(x)\, n(x) = -y(x)^T I\!I_x\, y(x) \leqslant 0$ on the boundary $\partial\Omega$.

We therefore have $R(t) \geqslant R_\circ(t)$ with

$$
\begin{aligned}
R_\circ(t) := {} & 2\int_\Omega \Big((f'(u)u - f(u)(\operatorname{div} y)^2 \Big)(x,t)\,dx \\
& + 2\int_\Omega \left(f(u) \left(\sum_{i,j=1}^{d} \left(\frac{\partial y_i}{\partial x_j} \right)^2 \right) \right)(x,t)\,dx,
\end{aligned}
$$

which gives

$$
\frac{d}{dt} I(u(t)) \leqslant \, - 2\alpha_1 I(u(t)) - R_\circ(t). \tag{58}
$$

Let us set $Z(x,t) = \text{Jacob}(y)(x,t)$. We can write $R_\circ(t)$ as

$$
R_\circ(t) = 2\int_\Omega T(u)(x,t)\,dx
$$

with

$$
T(u)(x,t) = \Big((a_2 - a_1)(\text{trace}(Z))^2 + a_1 \text{trace}(Z^2) \Big)(x,t)
$$

where $a_1(x,t) = f(u)(x,t)$ and $a_2(x,t) = (f'(u)u)(x,t)$. Taking into account that Z is symmetric it is easy to show that

$$(\text{trace}(Z))^2 \leqslant d\ \text{trace}(Z^2), \quad \text{i.e.} \quad (\text{trace}(Z))^2(x,t) = \theta(x,t)\, d\ \text{trace}(Z^2)(x,t),$$

with $\theta(x,t) \in [0,1]$. Hence

$$T(u)(x,t) = ((a_2 - a_1)\theta\, d + a_1)(x,t)\ \text{trace}(Z^2)(x,t)$$

which is non-negative if

$$f(u) \leqslant \frac{d}{d-1} uf'(u) \text{ for any } u > 0.$$

Therefore, $T(u) \geqslant 0$ which proves $R_\circ(t) \geqslant 0$ as well as $R(t) \geqslant 0$. Now we infer from (57)

$$I(u(t)) \leqslant I(u(t_0))\, e^{-2\,\alpha_1\,(t-t_0)}. \tag{59}$$

Step 4: *Exponential decay of* $RE(u(t)|u_{\infty,M})$. We recover $I(u(t))$ from (58), and we plug it into (55):

$$2\alpha_1 \frac{d}{dt} RE(u(t)|u_{\infty,M}) \geqslant \frac{d}{dt} I(u(t)) + R_\circ(t).$$

Using $\lim_{t\to\infty} RE(u(t)|u_{\infty,M}) = 0$ (see Step 1) we integrate between $t \geqslant 0$ and $+\infty$ and obtain

$$0 \leqslant RE(u(t)|u_{\infty,M}) \leqslant \frac{1}{2\,\alpha_1} I(u(t)), \quad t \geqslant 0. \tag{60}$$

Finally, we use inequality (60) in (55) to conclude

$$RE(u(t)|u_{\infty,M}) \leqslant RE(u(t_0)|u_{\infty,M})\, e^{-2\,\alpha_1\,(t-t_0)}, \quad t \geqslant t_0 \geqslant 0. \tag{61}$$

□

The next step in the analysis is to get rid of the assumptions (A0)–(A5) by approximation arguments.

Let us recall the available L^1 theory which was developed by M. Bertsch and D. Hilhorst in [13] for $h(0+) > -\infty$ (plus additional assumptions, see below). However, it is not difficult to check the proofs in [13], [27] to conclude that these results hold for the cases $h(0+) = -\infty$ (see sections 5,7,8 in [13], Theorems 6.2 and 7.1 in [27] and [49]) as well. Hence we can make use of the next existence result.

Theorem 12 *[13]. Assume* (HD1), (HD2), (HV3)–(HV8), (HF1)–(HF4) *with* Ω *bounded. Furthermore, assume*

(HD3) $u_0 \in L^\infty(\Omega)$.

(HF5) *If* $h(0+) > -\infty$, *then there is* $s_0 \in \mathbb{R}^+$ *with* $f''(u) \geqslant 0$ *for all* $u \in (0, s_0)$. *Then:*

a) (46), (47), (48) *has a unique non-negative, preserving-mass solution* u, *i.e.* $\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} = M$, $t > 0$.

b) $u \in C(\mathbb{R}^+ \times \bar\Omega)$.

c) If $u_0 \in C(\bar\Omega)$, *then* $u \in C(\mathbb{R}_0^+ \times \bar\Omega)$.

d) $u(t) \to u_{\infty,M}$ *in* $C(\bar\Omega)$ *as* $t \to \infty$.

Remark 20. In [13] it is additionally assumed that $f'(0) = 0$. This is, however, a direct consequence of the continuity of f at 0, of the assumed monotonicity of f', locally around 0, and of the assumption

$$\lim_{u\to 0}\left|\int_u^1 \frac{f'(s)}{s}\,ds\right| = |h(0+)| < \infty.$$

Let us also point out that (HF5) is only assumed in the $h(0+) > -\infty$ case. In this case this hypothesis is used in the proof of the regularization procedure they perform. If $h(0+) = -\infty$ we do not need to regularize the nonlinearity but only the initial data and then standard existence and regularity + equicontinuity results in [49] and Theorems 6.2 and 7.1 in [27] respectively, are directly applicable.

Now it is our aim to obtain exponential decay of the entropy (and of the entropy production) under the assumptions of Theorem 12. The strategy is to make use of a "parabolic approximation of the nonlinearity f" as already specified in Remark 18:

Proposition 13. *Assume that* $f : \mathbb{R}_0^+ \to \mathbb{R}$ *satisfies* (HF1)–(HF5). *Then there is for any* $\epsilon > 0$ *a function* $f_\epsilon : \mathbb{R}_0^+ \to \mathbb{R}$ *satisfiying* (HF1)–(HF5) *(with* $h_\varepsilon(u) := \int_1^u f_\varepsilon'(s)\,s^{-1}\,ds,\ u > 0$*) and*

1. $0 < c_1(\epsilon) \leqslant f_\epsilon'(u)$ *(hence* $h_\varepsilon(0+) = -\infty$ *and* $h_\varepsilon(\infty) = \infty$*) and* $f'(u) \leqslant f_\epsilon'(u)$ *for any* $u > 0$*.*

2. $f_\epsilon \to f$ *uniformly in compacts of* $\mathbb{R}_0^+$*, and* $f_\epsilon' \to f'$ *uniformly in compacts of* $\mathbb{R}^+$ *as* $\epsilon \to 0$*.*

3. $h_\varepsilon \to h$ *as* $\varepsilon \to 0$ *uniformly in compacts of* $\mathbb{R}^+$*.*

4. $\bar{h}_\varepsilon^{-1} \to \bar{h}^{-1}$ *uniformly on each half-infinite interval* $(-\infty, c]$*,* $c < h(\infty)$*.*

5. $\Phi_\varepsilon \to \Phi$ *uniformly on compacts in* $\mathbb{R}_0^+$*, where* $\Phi_\varepsilon : \mathbb{R}_0^+ \to \mathbb{R}$*,* $u \mapsto \Phi_\varepsilon(u) := \int_0^u h_\varepsilon(s)\,ds$*, i.e.* $\Phi_\varepsilon(u) = u\,h_\varepsilon(u) - f_\varepsilon(u)$ *for* $u > 0$ *and* $\Phi_\varepsilon(0) = 0$*.*

We recall: The function f needs to be regularized only if $h(0^+) = \underline{h} > -\infty$. For the other cases we can keep the original f.

We refer to [13], [27], [49] for the following approximation result:

Theorem 14. *Assume* (HD1)–(HD3), (HV3)–(HV8), (HF1)–(HF5). *For* $\varepsilon \in \mathbb{R}^+$ *let* f_ε *as specified in remark 18 and proposition 13, respectively. Then, there exists for each* $\varepsilon \in \mathbb{R}^+$ *a function* $u_0^\varepsilon \in C^2(\bar{\Omega})$ *such that*

1) $0 < c_2(\epsilon) \leqslant u_0^\epsilon \leqslant \|u_0\|_{L^\infty(\Omega)}$*,*

2) $u_0^\epsilon \to u_0$ *in* $L^2(\Omega)$ *as* $\epsilon \to 0$*,*

3) the no-flux regularized boundary condition is satisfied, that is,

$$u_0^\epsilon \frac{\partial V(x)}{\partial n} + \frac{\partial f_\epsilon(u_0^\epsilon)}{\partial n} = 0, \qquad (x \in \partial\Omega),$$

and the regularized system

$$\frac{\partial u^\epsilon}{\partial t} = \operatorname{div}\left(u^\epsilon \nabla V(x) + \nabla f_\epsilon(u^\epsilon)\right), \qquad (x \in \Omega, t > 0), \tag{62}$$

with initial condition

$$u^\epsilon(x, t = 0) = u_0^\epsilon(x), \qquad (x \in \Omega) \tag{63}$$

and no-flux boundary condition

$$u^\epsilon \frac{\partial V(x)}{\partial n} + \frac{\partial f_\epsilon(u^\epsilon)}{\partial n} = 0, \qquad (x \in \partial\Omega, t > 0). \tag{64}$$

satisfies

(A5) Each sequence $(u_k^\varepsilon)_{k\in\mathbb{N}}$ *in* $C^{1,2}(\bar{Q}_T)$ *of solutions of* (62), (64), *which is uniformly bounded in* $L^\infty(Q_T)$*, has a subsequence* $(u_{\nu(k)}^\varepsilon)_{k\in\mathbb{N}}$ *such that* $\{u_{\nu(k)}^\varepsilon : k \in \mathbb{N}\}$ *is equicontinuous,*

and for each $\varepsilon \in \mathbb{R}^+$*, the system* (62), (63), (64) *has a unique generalized solution* u^ϵ *which satisfies*

(A0) for each $t \in \mathbb{R}_0^+$, $\|u^\varepsilon(t)\|_{L^1(\Omega)} = \|u_0^\varepsilon\|_{L^1(\Omega)} =: M_\varepsilon$, $0 < M_\varepsilon < \bar{\mathbf{M}}(\Omega)$.

(A1) $u^\varepsilon \in C^{1,2}(\bar{Q}_T)$*, for each* $T \geqslant 0$.

(A3) there is $K \in \mathbb{R}^+$ *such that* $\|u^\varepsilon(t)\|_{L^\infty(\Omega)} \leqslant K$*, for all* $t \in \mathbb{R}_0^+$.

Furthermore,

4) $u^\epsilon \geqslant c_3(\epsilon) > 0$,

and for all $T > 0$,

5) $u^\epsilon \to u$ *uniformly in all compacts* $[\tau, T] \times \bar{\Omega}$ *with* $0 < \tau \leqslant T$,

6) $u^\epsilon \to u$ *strongly in* $L^2(Q_T)$ *and almost everywhere,*

where u is the unique non-negative, mass-preserving solution of (46), (47), (48) *(see Theorem 12).*

Let us discuss the stationary states for the regularized problem now. We adopt the notations of the subsection "Stationary Solutions" and assume that $\varepsilon \in (0, \infty)$ is fixed. We distinguish two cases.

I) $h(0+) = -\infty$. In this case we have $f_\varepsilon = f$. We particularly have the unique existence of $C_\varepsilon \in (-\infty, h(\infty))$ with $\mathbf{M}(C_\varepsilon) = M_\varepsilon < \bar{\mathbf{M}}(\Omega)$. Hence the corresponding stationary solution $u_{\infty,M_\varepsilon}^\varepsilon := u_{\infty,M_\varepsilon}$ is well-defined.

II) If $h(0+) > -\infty$, then the parabolic regularization f_ε is constructed such that $h_\varepsilon(0+) = -\infty$ and $h_\varepsilon(\infty) = \infty$. As discussed in the subsection "Stationary Solutions" there is a unique $C_\varepsilon \in \mathbb{R}$ with $\mathbf{M}_\varepsilon(C_\varepsilon) = M_\varepsilon$, where $\mathbf{M}_\varepsilon$ is defined in analogy to $\mathbf{M}$ by replacing "$\bar{h}^{-1}$" by "$\bar{h}_\varepsilon^{-1}$". We set $u_{\infty,M_\varepsilon}^\varepsilon := \bar{h}_\varepsilon^{-1}(C_\varepsilon - V)$.

We introduce for $\varepsilon \in (0, \infty)$ the functionals

$$E_\varepsilon : \mathscr{C}_\varepsilon \to \mathbb{R} \cup \{\infty\},$$

$$E_\varepsilon(u) = \begin{cases} \int_\Omega (V(x)u(x) + \Phi_\varepsilon(u(x)))\, dx, & Vu, \Phi_\varepsilon(u) \in L^1(\Omega), \\ \infty, & \text{else}, \end{cases}$$

where

$$\mathscr{C}_\varepsilon := \left\{ u \in L^1(\Omega) : u \geqslant 0, \int_\Omega u(x)\, dx = M_\varepsilon \right\},$$

$$RE_\varepsilon(.|u_{\infty,M_\varepsilon}^\varepsilon) : \mathscr{C}_\varepsilon \to \mathbb{R} \cup \{\infty\}, \quad RE_\varepsilon(u|u_{\infty,M_\varepsilon}^\varepsilon) = E_\varepsilon(u) - E_\varepsilon(u_{\infty,M_\varepsilon}^\varepsilon),$$

and

$$E_\varepsilon(.|u_{\infty,M_\varepsilon}) : \mathscr{C}_\varepsilon \to [0, \infty],$$

$$E(u|u_{\infty,M_\varepsilon}) = \int_\Omega (\Phi_\varepsilon(u) - \Phi_\varepsilon(u_{\infty,M_\varepsilon}^\varepsilon) - \Phi_\varepsilon'(u_{\infty,M_\varepsilon}^\varepsilon)(u - u_{\infty,M_\varepsilon}^\varepsilon))(x)\, dx.$$

Now it is a straight-forward task to verify that the assumptions of Theorem 14 except (HF5) (which is needed for the limit $\varepsilon \to 0$) are sufficient to ensure that system (62), (63), (64) satisfies all assumptions of Theorem 11:

Theorem 15. *Assume* (HD1)–(HD3), (HV3)–(HV8), (HF1)–(HF5). *For* $\varepsilon \in \mathbb{R}^+$ *let* f_ε *as specified in Remark 18 and Proposition 13, respectively, and let* u_0^ε *as specified in Theorem 14. Let* u^ε *be the unique, non-negative mass-preserving solution of* (46), (47), (48) *(see Theorem 14).*

Then

a) The function $t \mapsto RE_\varepsilon(u^\varepsilon(t)|u^\varepsilon_{\infty,M_\varepsilon})$ *belongs to* $C^2(\mathbb{R}_0^+)$ *with*

$$RE_\varepsilon(u^\varepsilon(t)|u^\varepsilon_{\infty,M_\varepsilon}) \leqslant RE_\varepsilon(u^\varepsilon(t_0)|u^\varepsilon_{\infty,M_\varepsilon})\, e^{-2\,\alpha_1\,(t-t_0)}, \quad t \geqslant t_0 \geqslant 0. \tag{65}$$

b) The entropy production rate

$$I_\varepsilon(u^\varepsilon(t)) := -\frac{d}{dt}E_\varepsilon(u^\varepsilon(t)) = -\frac{d}{dt}RE_\varepsilon(u^\varepsilon(t)|u^\varepsilon_{\infty,M_\varepsilon}) \tag{66}$$

satisfies

$$I_\varepsilon(u^\varepsilon(t)) \leqslant I_\varepsilon(u^\varepsilon(t_0))\, e^{-2\,\alpha_1\,(t-t_0)}, \quad t \geqslant t_0 \geqslant 0. \tag{67}$$

c) $RE_\varepsilon(u^\varepsilon(t)|u^\varepsilon_{\infty,M_\varepsilon})$ *and* $I_\varepsilon(u^\varepsilon(t))$ *are related via*

$$0 \leqslant RE_\varepsilon(u^\varepsilon(t)|u^\varepsilon_{\infty,M_\varepsilon}) \leqslant \frac{1}{2\,\alpha_1} I_\varepsilon(u^\varepsilon(t)), \quad t \geqslant 0. \tag{68}$$

Now we want to make use of approximation arguments to pass from the propositions of Theorem 15 – valid for regularized data and smooth solutions – to corresponding results for generalized solutions of (46), (47), (48). The involved approximation arguments require a careful handling of the relative entropy and the entropy production rate. It is therefore apropriate to give a few remarks without refering explicitly to several additional assumptions which will be specified later on.

a) It turns out that the relative entropy $RE_\varepsilon(u^\varepsilon(t)|u^\varepsilon_{\infty,M_\varepsilon})$ converges for all $t \geqslant 0$ to the entropy $RE(u(t)|u_{\infty,M})$. Hence part a) of Theorem 11 (exponential decay of the relative entropy as $t \to \infty$) will hold for generalized solutions as well.

b) A much more delicate question concerns the differentiability of the function $t \to RE(u(t)|u_{\infty,M})$. This is by no means a trivial problem. Indeed Theorem 15 is a consequence of Theorem 11 whose proof makes use of the equalities

$$I(u(t)) = -J_h(u(t)) = -K_f(u(t)), \tag{69}$$

and of

$$J_h(u(t)) = \int_\Omega \Big(u\,|\nabla V + \nabla h(u)|^2\Big)(x,t)\,dx, \tag{70}$$

and of

$$K_f(u(t)) = \int_\Omega \left(\frac{1}{u}\,|u\,\nabla V + \nabla f(u)|^2\right)(x,t)\,dx. \tag{71}$$

When passing from the regularized problem to the original system one loses the assumptions which are required to verify (69), (70), (71) and – due to the possible lack of differentiability of f and h at 0 – the function $I(.)$ may not be the classical derivative of the entropy anymore.

All in all it is not clear whether after passing to the limit $\varepsilon \to 0$,

1) the entropy $E(.)$ is strongly differentiable, i.e. whether the entropy production rate can be defined as classical derivative of $E(.)$;

2) $J_h(.)$ equals $K_f(.)$;

3) the distributional derivative of $E(.)$ equals $J_h(.)$ or $K_f(.)$;

4) by which quantity the function $I_\varepsilon(u^\varepsilon(t))$ in b) and c) of Theorem 15 has to be replaced;

5) which modifications of b) and c) of Theorem 15 are necessary.

It turns out – and this is according to the lower semi-continuity of the functional $K_f(.)$, see definition 4 and lemma 10 not entirely surprising – that the exponential decay in time of $K_f(u(t))$ is not affected by the limiting procedure $\varepsilon \to 0$.

Theorem 16. *Assume* (HD1)–(HD3), (HV3)–(HV8), (HF1)–(HF5) *with* Ω *bounded. Let u be the unique, non-negative mass-preserving solution of* (46), (47), (48) *(see Theorem 12). Then*

a) $RE(u(t)|u_{\infty,M}) \leqslant RE(u(t_0)|u_{\infty,M})\, e^{-2\,\alpha_1\,(t-t_0)}, \quad t \geqslant t_0 \geqslant 0.$

b) For all $t \geqslant 0$,

$$K_f(u(t)) \leqslant J_h(u_0)\, e^{-2\,\alpha_1\,t}.$$

c) For all $t \geqslant 0$,

$$RE(u(t)|u_{\infty,M}) \leqslant \frac{e^{-2\,\alpha_1\,t}}{2\,\alpha_1} J_h(u_0).$$

d) If $J_h(u_0) < \infty$, *then the* distributional *derivative*

$$I := -\frac{d}{dt}E(u(.)) = -\frac{d}{dt}RE(u(.)|u_{\infty,M})$$

satisfies

$$I \geqslant K_f(u(.)), \quad \textit{in the sense of distributions}.$$

Proof. In the sequel let u^ε, $\varepsilon \in \mathbb{R}^+$, be the unique, non-negative mass-preserving solution of (46), (47), (48) as defined in Theorem 14. Eventually we shall replace the parameter ε by the term $\varepsilon(n)$, where we tacitly assume that $(\varepsilon(n))_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^+$ with $\lim_{n\to\infty}\varepsilon(n) = 0$. Occasionally we shall pass to a subsequence of $(\varepsilon(n))_{n\in\mathbb{N}}$ and we do this – by a slight abuse of notation – without changing notations.

Step 1: $\lim_{\varepsilon\to 0} u^\varepsilon_{\infty,M_\varepsilon} = u_{\infty,M}$, *uniformly on* Ω. We observe $\lim_{\varepsilon\to 0} M_\varepsilon = M$. Let us consider the net $(C_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$, where

$$u^\varepsilon_{\infty,M_\varepsilon} = \bar{h}_\varepsilon^{-1}(C_\varepsilon - V).$$

We wish to prove: $\lim_{\varepsilon\to 0} C_\varepsilon = C$, where $u_{\infty,M} = \bar{h}^{-1}(C - V)$. We deduce from $\lim_{\varepsilon\to 0} M_\varepsilon = M$ the estimates $h(0+) < \liminf_{\varepsilon\to 0} C_\varepsilon \leqslant \limsup_{\varepsilon\to 0} C_\varepsilon < h(\infty)$.

Indirect. Assume there is a sequence $(\eta(k))_{k\in\mathbb{N}}$, $\eta(k)\in\mathbb{R}^+$, $\lim_{k\to\infty}\eta(k)=0$, such that $(C_{\eta(k)})_{k\in\mathbb{N}}$ *does not* converge to C. Then – eventually by passing to a subsequence, but without changing notations – we have $|C-C_{\eta(k)}|\geqslant\rho_0>0$, for all $k\in\mathbb{N}$. Since $h(0+)<\liminf_{k\to\infty}C_{\eta(k)}\leqslant\limsup_{k\to\infty}C_{\eta(k)}<h(\infty)$, the sequence $(C_{\eta(k)})_{k\in\mathbb{N}}$ is bounded. Hence – after eventually passing to asubsequence, but without changing notations – we have $\lim_{k\to\infty}C_{\eta(k)}=C^*\in(h(0+),h(\infty))$ and $h(0+)<a\leqslant C_{\eta(k)}\leqslant b<h(\infty)$ for all $k\in\mathbb{N}$. We recall: $\bar{h}_\varepsilon^{-1}\to\bar{h}^{-1}$ uniformly on the half-infinite interval $(-\infty,b]$. Hence $\bar{h}_{\eta(k)}^{-1}(C_{\eta(k)}-V)-\bar{h}^{-1}(C_\varepsilon-V)\to 0$ uniformly on Ω. Furthermore, $\bar{h}^{-1}$ is uniformly continuous on $(-\infty,b]$. Hence $\bar{h}^{-1}(C_\varepsilon-V)-\bar{h}^{-1}(C^*-V)\to 0$ uniformly on Ω as well. Due to the boundedness of Ω these two convergence statements are sufficient to deduce

$$\begin{aligned}\lim_{k\to\infty}M_{\eta(k)}&=\int_\Omega\bar{h}_{\eta(k)}^{-1}(C_{\eta(k)}-V)(x)\,dx\\&=\int_\Omega\bar{h}^{-1}(C^*-V)(x)\,dx=M(C^*)=M,\end{aligned}$$

such that $C^*=C$.

We obtain: $\lim_{\varepsilon\to 0}C_\varepsilon=C$. Mimicking the argumentation carried out right now, we easily deduce that

$$u^\varepsilon_{\infty,M_\varepsilon}\to u_{\infty,M}\quad\text{uniformly on }\Omega\text{ as }\varepsilon\to 0.$$

Step 2: $\lim_{\varepsilon\to 0}RE_\varepsilon(u^\varepsilon(t)|u^\varepsilon_{\infty,M_\varepsilon})=RE(u(t)|u_{\infty,M})$, *for all* $t\geqslant 0$. We recall: $\Phi_\varepsilon\to\Phi$ on compact subsets of $\mathbb{R}_0^+$. Furthermore, according to Theorem 14, we have for all $t\in\mathbb{R}^+$, $u^\varepsilon(t)\to u(t)$ uniformly on Ω as $\varepsilon\to 0$. We also have $u^\varepsilon_{\infty,M_\varepsilon}\to u_{\infty,M}$ uniformly on Ω as $\varepsilon\to 0$. Due to the boundedness of Ω, these convergence statements are sufficient to prove

$$\lim_{\varepsilon\to 0}RE_\varepsilon(u^\varepsilon(t)|u^\varepsilon_{\infty,M_\varepsilon})=RE(u(t)|u_{\infty,M}),\quad\text{for all }t>0.$$

Furthermore, we have $u_0^\varepsilon\to u_0$ strongly in $L^2(\Omega)$ as $\varepsilon\to 0$ with $\|u_0^\varepsilon\|_{L^\infty(\Omega)}\leqslant\|u_0\|_{L^\infty(\Omega)}$. Together with the available uniform convergence of $\Phi_\varepsilon\to\Phi$ on compact subsets of $\mathbb{R}_0^+$ we also have

$$\lim_{\varepsilon\to 0}RE_\varepsilon(u_0^\varepsilon|u^\varepsilon_{\infty,M_\varepsilon})=RE(u_0|u_{\infty,M}).$$

Step 3: $RE(u(t)|u_{\infty,M})\leqslant RE(u(t_0)|u_{\infty,M})\,e^{-2\alpha_1(t-t_0)}$, $t\geqslant t_0\geqslant 0$. We obtain from Step 2 for all $0\leqslant t_0\leqslant t$,

$$\begin{aligned}RE(u(t)|u_{\infty,M})&=\lim_{\varepsilon\to 0}RE_\varepsilon(u^\varepsilon(t)|u^\varepsilon_{\infty,M_\varepsilon})\\&\leqslant\lim_{\varepsilon\to 0}RE_\varepsilon(u^\varepsilon(t_0)|u^\varepsilon_{\infty,M_\varepsilon})\,e^{-2\alpha_1(t-t_0)}\\&=RE(u(t_0)|u_{\infty,M})\,e^{-2\alpha_1(t-t_0).}\end{aligned}$$

Step 4: In case of $J_h(u_0)=+\infty$ nothing remains to be shown. Hence we may assume $J_h(u_0)<\infty$ henceforth. We observe: According to $J_h(u_0)<\infty$ we may assume without loss of generality that the sequence $(u_0^{\varepsilon(n)})_{n\in\mathbb{N}}$ in $L^\infty(\Omega)$ has the additional property

$$J_{h_{\varepsilon(n)}}(u_0^{\varepsilon(n)}) = \int_\Omega (u_0^{\varepsilon(n)} \, |\nabla V + \nabla h_{\varepsilon(n)}(u_0^{\varepsilon(n)})|^2)(x)\, dx \to J_h(u_0) \quad \text{as } n \to \infty,$$

especially

$$\int_\Omega \left(u^{\varepsilon(n)}(t) \, |\nabla V + \nabla h_{\varepsilon(n)}(u^{\varepsilon(n)}(t))|^2\right)(x)\, dx \leqslant K_0,$$

where $K_0 \in \mathbb{R}_0^+$ is independent of $t \in \mathbb{R}_0^+$, $n \in \mathbb{N}$.

Step 5: $K_f(u(t)) \leqslant \liminf_{n\to\infty} K_{f_{\varepsilon(n)}}(u^{\varepsilon(n)}(t))$ *for all* $t \in \mathbb{R}^+$. Let $t \in \mathbb{R}^+$. We set $\gamma(t) := \liminf_{n\to\infty} K_{f_{\varepsilon(n)}}(u^{\varepsilon(n)})$. By passing (if necessary) to a subsequence (which may depend on t) we have $\lim_{n\to\infty} K_{f_{\varepsilon(n)}}(u^{\varepsilon(n)}) = \gamma(t)$. We recall: $u^{\varepsilon(n)}(t) \to u(t)$ uniformly on Ω as $n \to \infty$. (In particular: $\|u^{\varepsilon(n)}(t)\|_{L^\infty(\Omega)} \leqslant K_1(t) \in \mathbb{R}^+$, independently of $n \in \mathbb{N}$ with $K_1(t) \in \mathbb{R}^+$.) Furthermore, $f_{\varepsilon(n)} \to f$ uniformly on compact subsets of $\mathbb{R}_0^+$ as $n \to \infty$. Hence $f_{\varepsilon(n)}(u^{\varepsilon(n)}(t)) \to f(u(t))$ uniformly on Ω as $n \to \infty$. In particular we obtain $f_{\varepsilon(n)}(u^{\varepsilon(n)}(t)) \rightharpoonup f(u(t))$ in the sense of distributions as $n \to \infty$. From the estimate of Step 4 it is easy to deduce

$$\int_\Omega \left|u^{\varepsilon(n)}(t)\nabla V + \nabla f_{\varepsilon(n)}(u^{\varepsilon(n)}(t))\right|^2(x)\, dx \leqslant K_1(t)\, K_0.$$

We obtain: $(f_{\varepsilon(n)}(u^{\varepsilon(n)}(t)))_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\Omega)$. We can extract a subsequence (without changing notations) such that

$$f_{\varepsilon(n)}(u^{\varepsilon(n)}(t)) \rightharpoonup g(t), \quad \text{weakly in } H^1 \ (\Omega) \text{ as } n \to \infty.$$

Since $f_{\varepsilon(n)}(u^{\varepsilon(n)}(t)) \rightharpoonup f(u(t))$ in the sense of distributions as $n \to \infty$, we obtain $g(t) = f(u(t))$. Now we can apply Lemma 10 to obtain

$$K_f(u(t)) \leqslant \gamma(t) = \liminf_{n\to\infty} K_{f_{\varepsilon(n)}}(u^{\varepsilon(n)}(t)).$$

Step 6: *Proof of b*). We have for each $t \in \mathbb{R}_0^+$ and for each $n \in \mathbb{N}$ the estimate

$$I_{\varepsilon(n)}(u^{\varepsilon(n)}(t)) = K_{f_{\varepsilon(n)}}(u^{\varepsilon(n)}(t)) \leqslant I_{\varepsilon(n)}(u_0^{\varepsilon(n)})\, e^{-2\,\alpha_1\, t} = J_{h_{\varepsilon(n)}}(u_0^{\varepsilon(n)})\, e^{-2\,\alpha_1\, t},$$

such that we get from Step 4 and Step 5

$$\begin{aligned} K_f(u(t)) &\leqslant \liminf_{n\to\infty} K_{f_{\varepsilon(n)}}(u^{\varepsilon(n)}(t)) \\ &\leqslant \liminf_{n\to\infty} J_{h_{\varepsilon(n)}}(u_0^{\varepsilon(n)})\, e^{-2\,\alpha_1\, t} \\ &= \lim_{n\to\infty} J_{h_{\varepsilon(n)}}(u_0^{\varepsilon(n)})\, e^{-2\,\alpha_1\, t} \\ &= J_h(u_0)\, e^{-2\,\alpha_1\, t}, \end{aligned}$$

for all $t > 0$.

Step 7: Proof of c). We deduce c) from a) and Step 4.

Step 8: Proof of d). We have for all $T > 0$, for all $\xi \in C_0^\infty(\mathbb{R}_0^+)$, and for all $n \in \mathbb{N}$

$$\int_0^T I_{\varepsilon(n)}(u^{\varepsilon(n)}(s))\,\xi(s)\,ds = \int_0^T \left(-\frac{d}{dt} RE_{\varepsilon(n)}(u^{\varepsilon(n)}(t)|u_{\infty,M_{\varepsilon(n)}})\right)(s)\,\xi(s)\,ds$$
$$= \int_0^T RE_{\varepsilon(n)}(u^{\varepsilon(n)}(t)|u_{\infty,M_{\varepsilon(n)}})(s)\,\frac{d}{dt}\xi(s)\,ds.$$

We easily obtain from Step 2

$$\lim_{n\to\infty}\int_0^T RE_{\varepsilon(n)}(u^{\varepsilon(n)}(t)|u_{\infty,M_{\varepsilon(n)}})(s)\,\frac{d}{dt}\xi(s)\,ds = \int_0^T RE(u(s)|u_{\infty,M})\,\frac{d}{dt}\xi(s)\,ds,$$

such that in the sense of distributions,

$$I_{\varepsilon(n)}(u^{\varepsilon(n)}) \rightharpoonup I(.) = -\frac{d}{dt} RE(u(.)|u_{\infty,M}), \text{ as } n\to\infty.$$

On the other hand, we have for each $n \in \mathbb{N}$ and each $t \in \mathbb{R}_0^+$

$$I_{\varepsilon(n)}(u^{\varepsilon(n)}(t)) = K_{f_{\varepsilon(n)}}(u^{\varepsilon(n)}(t)),$$

such that we obtain from Fatou's lemma with the aid of b) for each non-negative $\xi \in C_c^\infty(\mathbb{R}_0^+)$

$$\int_0^T I(s)\,\xi(s)\,ds = \lim_{n\to\infty}\int_0^T I_{\varepsilon(n)}(u^{\varepsilon(n)}(s))\,\xi(s)\,ds$$
$$= \lim_{n\to\infty}\int_0^T K_{f_{\varepsilon(n)}}(u^{\varepsilon(n)}(s))\,\xi(s)\,ds$$
$$= \liminf_{n\to\infty}\int_0^T K_{f_{\varepsilon(n)}}(u^{\varepsilon(n)}(s))\,\xi(s)\,ds$$
$$\geqslant \int_0^T K_f(u(s))\,\xi(s)\,ds,$$

which settles d). □

Remark 21. Under the assumptions of Theorem 16 one may rather wish to deal with the functional J_h than with K_f. When are they equal ? A sufficient condition is certainly $u(t) > 0$. In this case we can supplement statements b) and d) as follows:

b. supp) For all $t \geqslant 0$: If $u(t) > 0$, then $J_h(u(t)) \leqslant J_h(u_0)\,e^{-2\alpha_1 t}$.

d. supp) If $J_h(u_0) < \infty$, and if $u(.) > 0$ on a measurable set $\tau \subseteq \mathbb{R}_0^+$, then $I \geqslant J_h(u(.))$ on τ in the sense of distributions.

We deduce as a byproduct from Theorem 16 c) for $t = 0$ a generalized Sobolev inequality in bounded domains.

Corollary 1. *Assume* (HD1)–(HD3), (HV3)–(HV8), (HF1)–(HF5), *in particular: $u_0(x) \geqslant 0$ belongs to $L^\infty(\Omega)$ with mass M. Then,*

$$RE(u_0|u_{\infty,M}) \leqslant \frac{1}{2\alpha_1} J_h(u_0). \qquad (72)$$

3.4 Generalized Sobolev inequalities, $\Omega = \mathbb{R}^d$. The aim of this section is to prove generalized Sobolev inequalities on $\mathbb{R}^d$. The procedure is as follows. Let us take a positive function $u_0 \in L^1(\mathbb{R}^d)$ with mass M such that the entropy production $J_h(u_0)$ in $\mathbb{R}^d$ is finite. We can approximate it by functions defined on Euclidean balls B_n of radius n. For these approximations the generalized Sobolev inequality (72) is available. Letting tend $n \to \infty$ finishes the proof.

Theorem 17. *Let $\Omega = \mathbb{R}^d$ and assume that u_0 satisfies*

$$u_0 \in L^1(\mathbb{R}^d), u_0 \geqslant 0 \text{ and } \int_{\mathbb{R}^d} u_0(x)\,dx = M.$$

Furthermore, assume (HF1)–(HF5), (HV3)–(HV8). *For $n \in \mathbb{N}$ let B_n be the Euclidean ball of radius n centered at the origin. Assume that*

(U1) *there exists $g \in L^1(\mathbb{R}^d)$ and $n_4 \in \mathbb{N}$ such that*

$$n\,|\nabla V|^2\,\mathrm{ind}_{\{x \in B_n : u_0(x) > n\}} \leqslant g, \quad \forall n \geqslant n_4,$$

where $\mathrm{ind}_{\{x \in B_n : u_0(x) > n\}}$ is the indicator function of $\{x \in B_n : u_0(x) > n\}$.

(U2) $\displaystyle\int_{\mathbb{R}^d} \Phi^-(u_0)(x)\,dx < \infty.$

Then

$$RE(u_0|u_{\infty,M}) \leqslant \frac{1}{2\alpha_1} J_h(u_0). \tag{73}$$

Remark 22. a) According to (HV6), the relative entropy $RE(u_0|u_{\infty,M})$ has a well-defined value in $\mathbb{R} \cup \{+\infty\}$.

b) u_0 may be unbounded.

c) (U1) is trivially satisfied if $u_0 \in L^\infty(\mathbb{R}^d)$ or if $u_0|\nabla V|^2 \in L^1(\mathbb{R}^d)$ or if there is $A \in \mathbb{R}^+$ such that the set $\{x : u_0(x) > A\}$ is bounded.

d) (U2) holds if $E(u_0) < \infty$ or if $h(0+) > -\infty$. However it is worth noting that (U2) is weaker than $E(u_0) < \infty$.

Proof. We observe: There is nothing to prove in case $J_h(u_0) = \infty$. Hence assume $J_h(u_0) < \infty$ henceforth. We introduce for $n \in \mathbb{N}$ the function

$$u_0^n := \min\{n, u_0\}|_{B_n}.$$

We observe: $u_0^n \in L^\infty(B_n)$, for all $n \in \mathbb{N}$. We set for $n \in \mathbb{N}$,

$$M_n := \int_{B_n} u_0^n(x)\,dx.$$

It is easy to see that $M_n \leqslant M$ for all $n \in \mathbb{N}$ and $M_n \to M$ as $n \to \infty$. Let us put for $n \in \mathbb{N}$,

$$\bar{\mathbf{M}}_n := \sup\left\{\int_{B_n} \bar{h}^{-1}(C - V)\,dx : \bar{h}^{-1}(C - V)|_{B_n} \in L^1(B_n)\right\}.$$

It is left to the reader to deduce from the boundedness of B_n for $n \in \mathbb{N}$, that

$$\bar{\mathbf{M}}_n = \lim_{C \to h(\infty)} \int_{B_n} \bar{h}^{-1}(C - V)\,dx.$$

Since $\lim_{n\to\infty}\int_{B_n}\bar{h}^{-1}(C-V)\,dx = \int_{\mathbb{R}^d}\bar{h}^{-1}(C-V)\,dx$ for any $C<h(\infty)$, we obtain

$$\bar{\mathbf{M}} \leqslant \liminf_{n\to\infty} \bar{\mathbf{M}}_n.$$

Taking into account (HV5), there is $n_0 \in \mathbb{N}$ with

$$0 < M_n < \min\{\bar{\mathbf{M}}, \bar{\mathbf{M}}_n\}, \quad n_0 \leqslant n, n \in \mathbb{N}. \tag{74}$$

This observation allows for the unambiguous introduction of

$$u^n_{\infty,M_n} := \bar{h}^{-1}(C_n - V^n), \quad C_n \in \mathbb{R} \text{ such that } \int_{B_n} \bar{h}^{-1}(C_n - V^n(x))\,dx = M_n,$$

where $n_0 \leqslant n \in \mathbb{N}$. Now we introduce for $n_0 \leqslant n \in \mathbb{N}$ the functional

$$E^n : \mathscr{C}_n \to \mathbb{R}\cup\{\infty\}$$
$$u \mapsto \begin{cases} \int_{B_n} (V(x)u(x) + \Phi(u(x)))\,dx, & Vu, \Phi(u) \in L^1(B_n), \\ \infty, & \text{else}, \end{cases}$$

with

$$\mathscr{C}_n := \left\{ u \in L^1(B_n) : u \geqslant 0, \int_{B_n} u(x)\,dx = M \right\},$$

and the functional

$$L^n_h : L^1_+(B_n) \to \mathbb{R}\cup\{+\infty\},$$
$$L^n_h(u) = \begin{cases} \int_{u>0} \Big(u\,|\nabla V + \nabla h(u))|^2 \Big)(x)\,dx, & u \in \mathscr{D}^n_h, \\ \infty, & \text{else}, \end{cases}$$

where

$$\mathscr{D}^n_h := \{u \in L^1_+(B_n) : h(u) \in L^1_{\mathrm{loc}}(B_n),\ \nabla h(u) \in L^1_{\mathrm{loc}}(B_n : \mathbb{R}^d)\}.$$

Step 1: $E^n(u^n_{\infty,M_n}) < \infty$, for all $n_0 \leqslant n \in \mathbb{N}$. This is obvious, because u^n_{∞,M_n} is bounded on the bounded set B_n. Hence we can define for each $n_0 \leqslant n \in \mathbb{N}$,

$$RE^n(u^n_0 | u^n_{\infty,M_n}) := E^n(u^n_0) - E^n(u^n_{\infty,M_n}).$$

Step 2: There is $n_1 \in \mathbb{N}$ with $n_0 \leqslant n_1$ and

$$RE^n(u^n_0 | u^n_{\infty,M_n}) \leqslant \frac{1}{2\,\alpha_1} J^n_h(u^n_0), \quad n_1 \leqslant n \in \mathbb{N}. \tag{75}$$

We wish to apply (72) of Corollary 1. We have to check the assumptions imposed there for $\Omega = B_n$, u^n_0, M_n and V^n, $n \in \mathbb{N}$ sufficiently large.

(HD1) holds for $\Omega = B_n$, for each $n \in \mathbb{N}$.

(HD2) holds for “M_n” replacing “M” for all $n_0 \leqslant n \in \mathbb{N}$.

(HV8) V^n is the restriction of V to B_n, i.e. (HV8) holds for all $n \in \mathbb{N}$.

(HV7) is satisfied for all $n \in \mathbb{N}$.

(HV6) is of no relevance.

(HV5) holds due to (74) for all $n \in \mathbb{N}$ with $n_0 \leqslant n$.

(HV4) is of no relevance.

(HV3) holds due to the assumed uniform convexity of V and $\inf_{\mathbb{R}^d} V = 0$ for all sufficiently large $n \in \mathbb{N}$, say, for $n_3 \leqslant n \in \mathbb{N}$ with $n_3 \in \mathbb{N}$.

(HD3) holds due to the definition of u_0^n for all $n \in \mathbb{N}$.

(HF1)–(HF5) are not affected by restricting u_0 and V to B_n.

Summarizing the discussion, we obtain (75) from Corollary 1 with $n_1 := \max\{n_0, n_3\}$.

Step 3: $\lim_{n\to\infty} J_h^n(u_0^n) = J_h(u_0)$. We introduce for $n_1 \leqslant n \in \mathbb{N}$,

$$\mathbf{a}_n : \mathbb{R}^d \to \mathbb{R},$$

$$\mathbf{a}_n(x) = \begin{cases} u_0(x)\,|\nabla V + \nabla h(u_0)|^2(x), & x \in B_n, 0 < u_0(x) \leqslant n, \\ n\,|\nabla V|^2(x), & x \in B_n, u_0(x) > n, \\ 0, & \text{else}, \end{cases}$$

where we tacitly make use of the fact that both $h(u_0)$ and $\nabla h(u_0)$ are locally integrable. We obviously have

$$J_h^n(u_0^n) = \int_{\mathbb{R}^d} \mathbf{a}_n(x)\, dx,$$

and for all sufficiently large $n \in \mathbb{N}$ (see (U1)),

$$0 \leqslant \mathbf{a}_n \leqslant u_0|\nabla V + \nabla h(u_0)^2 + g \in L^1(\mathbb{R}^d),$$

and clearly $\lim_{n\to\infty} \mathbf{a}_n(x) = (u_0|\nabla V + \nabla h(u_0)|^2)\,\mathrm{ind}_{\{u_0>0\}}(x)$ for almost all $x \in \mathbb{R}^d$. Hence $\lim_{n\to\infty} J_h^n(u_0^n) = J_h(u_0)$ by Lebesgue's dominated convergence theorem.

Step 4: $\lim_{n\to\infty} E^n(u_{\infty,M_n}^n) = E(u_{\infty,M})$. We have $u_{\infty,M} = \bar{h}^{-1}(C - V)$. Since $M < \bar{\mathbf{M}}$ there is $M^+ \in (M, \bar{\mathbf{M}})$ and $C^+ \in (C, C^*)$ with $u_{\infty,M^+} = \bar{h}^{-1}(C_+ - V)$. Furthermore, for all sufficiently large $n \in \mathbb{N}$,

$$u_{\infty,M_n}^n = \bar{h}^{-1}(C_n - V^n), \quad C_n \in \mathbb{R}.$$

Since $M_n \to M$ as $n \to \infty$ we obtain: $\lim_{n\to\infty} C_n = C$. Due to the growth property of V it is easy to deduce that $u_{\infty,M_n}^n \to u_{\infty,M}$ uniformly on $\mathbb{R}^d$. We furthermore have $C < C_+$. Due to $\lim_{n\to\infty} C_n = C$ we have $C_n < C_+$ for all sufficiently large n – say, $n \geqslant n_5$. Hence

$$U_n := \bar{h}^{-1}(C_n - V) \leqslant u_{\infty,M^+}, \quad n \geqslant n_5.$$

Let us denote by $[u_{\infty,M_n}^n]^{\text{ext}}$ the trivial extension of u_{∞,M_n}^n to $\mathbb{R}^d$. By (HV6) we have $E(u_{\infty,M^+}) < \infty$. Since $[u_{\infty,M_n}^n]^{\text{ext}} \leqslant U_n \leqslant u_{\infty,M^+}$ and $E([u_{\infty,M_n}^n]^{\text{ext}}) = E^n(u_{\infty,M_n}^n)$ for all $n \geqslant n_5$, we obtain $\lim_{n\to\infty} E^n(u_{\infty,M_n}^n) = E(u_{\infty,M})$ from Lemma 7.

Step 5: $\lim_{n\to\infty} E^n(u_0^n) = E(u_0)$. If we denote the trivial extension of u_0^n to $\mathbb{R}^d$ by $[u_0^n]^{\text{ext}}$, then we will obtain $[u_0^n]^{\text{ext}} \leqslant [u_0^{n+1}]^{\text{ext}} \leqslant u_0$. Hence $\lim_{n\to\infty} E^n(u_0^n) = E(u_0)$ by Lemma 8.

Step 6: Finishing the proof. By Step 4 and Step 5 we have $\lim_{n\to\infty} RE^n(u_0^n|u_{\infty,M_n}^n) = RE(u_0|u_{\infty,M})$, by Step 3 we have $\lim_{n\to\infty} J_h^n(u_0^n) = J_h(u_0)$. Hence it is possible to pass to the limit $n \to \infty$ in (75) yielding (73). □

Equation (73) is a generalized Sobolev inequality. Indeed after some elementary manipulations we obtain in particular for $f(u) = u^m$ and $V(x) = |x|^2/2$ the following inequality (see [17], [28]).

Corollary 2. *Let $u_0(x) \geqslant 0$ belong to $L^1(\mathbb{R}^d)$ and $u_0^m \in L^1(\mathbb{R}^d)$ with mass M and $m \geqslant \max\left(\frac{d-1}{d}, \frac{d}{d+2}\right)$, $m \neq 1, \frac{1}{2}$ such that the distributional gradient of $u_0^{m-1/2}$ is square integrable, then*

$$\left(d + \frac{1}{m-1}\right) \int_{\mathbb{R}^d} u_0^m \, dx \leqslant \frac{1}{2} \left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla u_0^{m-1/2}|^2 \, dx + A_m(\|u_0\|_1) \tag{76}$$

where

$$A_m(M) = \int_{\mathbb{R}^d} \left[\frac{|x|^2}{2} u_{\infty,M} + \frac{1}{m-1} u_{\infty,M}^m\right] dx.$$

Remark 23. Naturally we derive inequality (76) from Theorem 17 only for functions u_0 which satisfy (U1), (U2). We observe: (U2) holds due to $u_0^m \in L^1(\mathbb{R}^d)$ and if assumption (U1) does not hold, then we can approximate u_0 by $L^\infty(\mathbb{R}^d)$ functions and pass to the limit in inequality (76).

In a similar way we can proceed if $m = 1$ to recover the logarithmic Sobolev inequality (see [35], [3]).

Corollary 3. *Let $u_0(x) \geqslant 0$ belong to $L^1(\mathbb{R}^d)$ with mass M, such that the distributional gradient of $\sqrt{u_0}$ is square integrable, then*

$$\int_{\mathbb{R}^d} u_0 \log(u_0) \, dx \leqslant 2 \int_{\mathbb{R}^d} |\nabla \sqrt{u_0}|^2 \, dx + A_1(\|u_0\|_1), \tag{77}$$

where

$$A_1(M) = d + M \log\left(\frac{M}{(2\pi)^{d/2}}\right).$$

3.5. Existence of solution and exponential decay of the entropy, $\Omega = \mathbb{R}^d$. The main problem to attack the large-time asymptotics in the $\mathbb{R}^d$ case is the lack of existence and uniqueness results for problem (34)–(35). We show the existence of a generalized solution (Definition 1) verifiying the exponential convergence of the entropy and the entropy production. Let us denote by $F(u)$ a primitive of the nonlinearity $f(u)$ with $F(0) = 0$ and $G(u) = uf(u) - F(u)$ such that $G'(u) = uf'(u)$.

Theorem 18. *Assume* (HD1)–(HD2), (HV3)–(HV8), (HF1)–(HF5). *In addition, we assume* (U1),(U2),

(HE1) $F(u_0) \in L^1(\mathbb{R}^d)$.

(HE2) *There exists $A > 0$ such that $G(u) \leqslant AF(u)$ for $u > 0$.*

(HE3) $\Delta V \in L^\infty(\mathbb{R}^d)$.

(HE4) $u_0 \in L^\infty(\mathbb{R}^d)$.

(HF6) *Either f is convex on $[0,\infty)$ or f^{-1} is globally Hölder continuous.*

(HF7) *If* $h(0+) = -\infty$ *we assume there exists* $0 < s_1 < 1$ *such that*

$$b := \sup\left\{\frac{\Phi(u)}{uh(u)}, 0 < u < s_1\right\} < +\infty.$$

Then, there exists a generalized solution of problem (34)–(35) *satisfying*

a) If $h(0+) = -\infty$*, then*

$$RE(u(t)|u_{\infty,M}) \leqslant RE(u_0|u_{\infty,M})\, e^{-2\,\alpha_1 t}, \quad a.e.\ t > 0.$$

If $h(0+) > -\infty$*, then*

$$\tilde{RE}(u(t)|u_{\infty,M}) = \tilde{E}(u(t)) - \tilde{E}(u_{\infty,M}) \leqslant RE(u_0|u_{\infty,M})\, e^{-2\,\alpha_1 t}, \quad a.e.\ t > 0.$$

b) For a.e. $t > 0$,

$$K_f(u(t)) \leqslant J_h(u_0)\, e^{-2\,\alpha_1 t}.$$

c) For a.e. $t > 0$,

$$\|u(t)\|_{L^1(\mathbb{R}^d)} = M.$$

Remark 24. a) (HE1)–(HE3) imply an energy estimate needed for the compactness argument. In order to apply this argument we need condition (HF6). Note that $f(u) = u^m$ for all $m \geqslant 1$ satisfies that f is convex and $f(u) = u^m$ for all $0 < m \leqslant 1$ satisfies that f^{-1} is globally Hölder continuous, and thus, (HF6) is satisfied for $f(u) = u^m$ for all $m > 0$.

b) Assumption (HE2) is implied by the condition: there exists $A > 0$ such that

$$f(u) \geqslant \frac{1}{A} uf'(u) \text{ for } u > 0.$$

This condition is verified by $f(u) = u^m$ for any m.

c) We do not know the uniqueness of the generalized solution for problem (34)–(35). Under more restrictive assumptions (strong L^1-solutions) one may produce a standard uniqueness result, we refer to [2], [66].

d) (HE3)–(HE4) can be substituted by alternative hypotheses; we come back to this point after the proof of this theorem.

e) We recall that $\tilde{E}$ is defined in Remark 14. $\tilde{RE}$ coincides with RE provided u is a mass-preserving solution.

f) Condition (HF7) is needed for the mass conservation.

Proof. Step 1: Sequence of approximate problems. Consider the sequence of subsets Ω_n defined by

$$\Omega_n = \{x \in \mathbb{R}^d \text{ such that } V(x) \leqslant n\}$$

for any $n \geqslant 1$. Using (HV3)–(HV8) Ω_n is an increasing sequence of convex bounded smooth domains covering $\mathbb{R}^d$ as $n \to \infty$. Let us define

$$u_0^n := u_0|_{\Omega_n}.$$

We observe: $u_0^n \in L^\infty(\Omega_n)$, for all $n \in \mathbb{N}$. We set for $n \in \mathbb{N}$,

$$M_n := \int_{B_n} u_0^n(x)\, dx.$$

We follow the same notations as in the proof of Theorem 17 by substituting B_n by Ω_n. It is obvious that (U1), (U2) and (HV3)–(HV8) together with the same arguments in the proof of Theorem 17 imply that for $n \geqslant n_o$

1. $M_n \leqslant M$ for all $n \in \mathbb{N}$, $M_n \to M$ as $n \to \infty$, $0 < M_n < \min\{\bar{\mathbf{M}}(\Omega_n), \bar{\mathbf{M}}\}$.
2. $E^n(u_0^n) \to E(u_0)$ as $n \to \infty$.
3. $J_h^n(u_0^n) \to J_h(u_0)$ as $n \to \infty$.
4. $E^n(u_{\infty,M_n}^n) \to E(u_{\infty,M})$ as $n \to \infty$.
5. $$\int_{\Omega_n} F(u_0^n)\,dx \;\to\; \int_{\mathbb{R}^d} F(u_0)\,dx \text{ as } n \to \infty.$$

Taking now u_0^n as the initial data for the problem (34)–(35) in the bounded domain Ω_n and applying Theorem 16, we deduce that the unique solution u^n of the problem (46)–(48) in Ω_n satisfies

$$K_f^n(u^n(t)) \leqslant J_h(u_0^n)e^{-2\alpha_1 t}, \qquad t \geqslant 0 \tag{78}$$

and

$$RE^n(u^n|u_{\infty,M^n}^n) \leqslant RE^n(u_0^n|u_{\infty,M^n}^n)e^{-2\alpha_1 t} \qquad t \geqslant 0 \tag{79}$$

for $n \geqslant n_o$.

Step 2: Energy estimates. Each solution u^n is obtained using Theorem 14 as the limit of a regularized problem. Let us take $u^{n,\epsilon}$ the regularized solution for a regularized initial data $u_0^{n,\epsilon}$. Then, by the divergence theorem it is straightforward to show that

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_n} F_\epsilon(u^{n,\epsilon})\,dx = &\int_{\Omega_n} G_\epsilon(u^{n,\epsilon})\Delta V\,dx \\ &- \int_{\Omega_n} |\nabla f_\epsilon(u^{n,\epsilon})|^2\,dx - \int_{\partial\Omega_n} G_\epsilon(u^{n,\epsilon})(\nabla V \cdot n(x))\,dS(x),\end{aligned} \tag{80}$$

where F_ϵ and G_ϵ have the same definitions as F and G which correspond to the regularized f_ϵ. In the regularization procedure for the nonlinearity one can assume without loss of generality that (HE1)–(HE2) are preserved, i.e.,

$$G_\epsilon(u) = uf_\epsilon(u) - F_\epsilon(u) \leqslant AF_\epsilon(u) \text{ for } u > 0 \tag{81}$$

and that

$$\int_{\Omega_n} F(u_0^{n,\epsilon})\,dx \;\to\; \int_{\Omega_n} F(u_0^n)\,dx \text{ as } \epsilon \to 0. \tag{82}$$

Due to (HV8) and the definition of Ω_n we have

$$\frac{\partial V}{\partial n} \geqslant 0 \text{ on } \partial\Omega_n,$$

and therefore using (81)

$$\frac{d}{dt}\int_{\Omega_n} F_\epsilon(u^{n,\epsilon})\,dx \leqslant A\int_{\Omega_n} F_\epsilon(u^{n,\epsilon})\Delta V\,dx.$$

This inequality together with (HE3) implies that

$$\int_{\Omega_n} F_\epsilon(u^{n,\epsilon})\,dx \leqslant \exp\{A\|\Delta V\|_{L^\infty(\mathbb{R}^d)}t\}\int_{\Omega_n} F_\epsilon(u_0^{n,\epsilon})\,dx \text{ for } t \geqslant 0.$$

Now, coming back to the energy evolution (80) we deduce by integrating and using previous estimate that

$$\int_0^T\int_{\Omega_n} |\nabla f_\epsilon(u^{n,\epsilon})|^2\,dx\,dt \leqslant C(T,V)\int_{\Omega_n} F_\epsilon(u_0^{n,\epsilon})\,dx \text{ for any } \; T>0.$$

Now, we take the limit $\epsilon \to 0$ in the above expressions. Using that

1. $u^{n,\epsilon} \to u^n$ uniformly in all compacts $[\tau,T]\times\bar{\Omega}_n$ with $0<\tau\leqslant T$.
2. $f_\epsilon(u^{n,\epsilon}) \to f(u^n)$ strongly in $L^2(Q_T^n)$ and weakly in $L^2(0,T;H^1(\Omega_n))$.

the lower semicontinuity of the norms and (82), we show

$$\int_{\Omega_n} F(u^n)\,dx \leqslant \exp\{A\|\Delta V\|_{L^\infty(\mathbb{R}^d)}t\}\int_{\Omega_n} F(u_0^n)\,dx \text{ for } t\geqslant 0$$

and

$$\int_0^T\int_{\Omega_n} |\nabla f(u^n)|^2\,dx\,dt \leqslant C(T,V)\int_{\Omega_n} F(u_0^n)\,dx \text{ for any } T>0.$$

Step 3: L^∞_{loc} estimate. Let us denote by $[u^n]^{\mathrm{ext}}$ the trivial extension of u^n to $\mathbb{R}^d\times\mathbb{R}_t$. Using (HE3) and (HE4) it is easy to check that $\bar{u}^n$ given by

$$\bar{u}^n(t) := \|u_0\|_{L^\infty(\mathbb{R}^d)} e^{t\|\Delta V\|_{L^\infty(\mathbb{R}^d)}}.$$

are supersolutions for the regularized problem. Therefore, by the comparison principle, one has $u^{n,\epsilon}(t,x)\leqslant \bar{u}^n(t)$ for any $t\geqslant 0$, $x\in\Omega_n$ and $n\geqslant 1$. Using the uniform convergence of $u^{n,\epsilon}$ to u^n, one finally deduces that u^n is bounded in $L^\infty(\mathbb{R}^d\times(0,T))$ for any $T>0$ independently of n.

Step 4: Compactness. Let Ω be any bounded domain in $\mathbb{R}^d$. Using step 3 we have that $[u^n]^{\mathrm{ext}}\nabla V$ is bounded in $L^2_{\mathrm{loc}}(\Omega\times\mathbb{R}_t)$ independently of n. Step 2 assures that $[\nabla f(u^n)]^{\mathrm{ext}}$ is bounded in $L^2_{\mathrm{loc}}(\Omega\times\mathbb{R}_t)$ independently of n for $n\geqslant n_1$. Using the equation for $\frac{\partial u^n}{\partial t}$ we deduce that $[\frac{\partial u^n}{\partial t}]^{\mathrm{ext}}$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R},H^{-1}_{\mathrm{loc}}(\Omega))$ independently of n for $n\geqslant n_1$.

Using the compactness result in [38] in case f convex or in [33] in case f^{-1} is Hölder continuous we have the compactness of the sequence $[u^n]^{\mathrm{ext}}$ for $n\geqslant n_1$ in $L^2_{\mathrm{loc}}(\Omega\times\mathbb{R}_t)$ for any Ω bounded domain in $\mathbb{R}^d$. A standard Cantor diagonal selection argument implies the convergence in $L^2_{\mathrm{loc}}(\mathbb{R}^d\times\mathbb{R}_t)$ and thus, also a.e. in $\mathbb{R}^d$, of a subsequence, that we denote with the same index, to a positive function u.

Let us check that u is a generalized solution of the problem (46)–(47). By step 3 and the a.e. convergence we have $u\in L^\infty(\mathbb{R}^d\times(0,T))$ for any $T>0$. Previous estimates imply that $[f(u^n)]^{\mathrm{ext}}$ converges to $f(u)$ strongly in $L^2_{\mathrm{loc}}(\Omega\times\mathbb{R}_t)$ and weakly in $L^2_{\mathrm{loc}}(\mathbb{R}_t,H^1(\Omega))$ for a subsequence, that we continue denoting with the same index. Therefore, $\nabla f(u)\in L^1_{\mathrm{loc}}(\mathbb{R}^d\times\mathbb{R}_t:\mathbb{R}^d)$ and the equation (46) is satisfied in distributional sense.

Step 5: Proof of a). We have to take the limit $n \to \infty$ in (79). In view of the previous properties of the approximations u_0^n in step 1, the limit as $n \to \infty$ of the right hand sides of (79) is obvious. As a consequence of step 4 we have that for a.e. $t > 0$, $[u^n]^{\text{ext}}(t)$ converges in $L^1_{\text{loc}}(\mathbb{R}^d)$ and a.e. in $\mathbb{R}^d$ to $u(t)$. Let us fix $t > 0$ for which this is true.

Let us focus on the left hand side of (79). If $h(0+) = -\infty$ then by Proposition 5 we have that

$$RE(u(t)|u_{\infty,M}) = E(u(t)|u_{\infty,M})$$

and

$$RE^n(u^n(t)|u^n_{\infty,M^n}) = E^n(u^n(t)|u^n_{\infty,M^n})$$

where

$$E^n(u^n(t)|u^n_{\infty,M^n}) = \int_{\Omega_n} (\Phi(u^n) - \Phi(u^n_{\infty,M^n}) - \Phi'(u^n_{\infty,M^n})\,(u^n - u^n_{\infty,M^n}))(x,t)\,dx,$$

Since $u^n(t)$ converges a.e. to $u(t)$, u^n_{∞,M^n} converges a.e. to $u_{\infty,M}$ and Φ is a convex function we have

$$[(\Phi(u^n) - \Phi(u^n_{\infty,M^n}) - \Phi'(u^n_{\infty,M^n})\,(u^n - u^n_{\infty,M^n}))(x,t)]^{\text{ext}}$$

is a sequence of positive functions converging a.e. in $\mathbb{R}^d$ to

$$(\Phi(u) - \Phi(u_{\infty,M}) - \Phi'(u_{\infty,M})\,(u - u_{\infty,M}))(x,t).$$

Fatou's lemma implies that

$$RE(u(t)|u_{\infty,M}) \leqslant \liminf_{n\to\infty} RE^n(u^n(t)|u^n_{\infty,M^n}).$$

Now, if $h(0+) > -\infty$ we have $\Phi(u) = \tilde{\Phi}(u) + h(0+)u$ with $\tilde{\Phi}(u) \geqslant 0$ (see Remark 14). Then we can write

$$E^n(u^n(t)) = \tilde{E}^n(u^n(t)) + h(0+)M^n$$

where

$$\tilde{E}^n(u^n(t)) = \int_{\Omega_n} (Vu^n + \tilde{\Phi}(u^n))(x,t)\,dx.$$

Since $u^n(t) \geqslant 0$ converges a.e. to $u(t) \geqslant 0$ and $\tilde{\Phi}$ is positive we have $[(Vu^n + \tilde{\Phi}(u^n))(x,t)]^{\text{ext}}$ is a sequence of positive functions converging a.e. in $\mathbb{R}^d$ to $(Vu + \tilde{\Phi}(u))(x,t)$. Fatou's lemma and $M^n \to M$ imply that

$$\tilde{E}(u(t)) + h(0+)M \leqslant \liminf_{n\to\infty} E^n(u^n(t)).$$

Since $M^n \to M$ and $E^n(u^n_{\infty,M_n}) \to E(u_{\infty,M})$ as $n \to \infty$, we have $E^n(u^n_{\infty,M_n}) \to \tilde{E}(u_{\infty,M}) + h(0+)M$ as $n \to \infty$. Therefore,

$$\tilde{RE}(u(t)|u_{\infty,M}) \leqslant \liminf_{n\to\infty} RE^n(u^n(t)|u^n_{\infty,M^n}).$$

Step 6: Proof of b). We have to take the limit $n \to \infty$ in (78). In view of the previous properties of the approximations u_0^n in step 1, the limit as $n \to \infty$ of the right hand sides of (78) is obvious. As a consequence of step 4 we have that for a.e. $t > 0$, $[u^n]^{\text{ext}}(t)$ converges in $L^1_{\text{loc}}(\mathbb{R}^d)$ and a.e. in $\mathbb{R}^d$ to $u(t)$. Let us fixed one of the almost all $t > 0$ for which this is true.

As a consequence of step 4, we know that given a bounded domain Ω in $\mathbb{R}^d$, $f(u^n)$ converges strongly to $f(u)$ in $L^2_{\text{loc}}(\Omega \times \mathbb{R}_t)$ and weakly in $L^2_{\text{loc}}(\mathbb{R}_t, H^1(\Omega))$ for a subsequence which we continue denoting with the same index. Using Lemma 10 we obtain

$$K_f^\Omega(u(t)) \leqslant \liminf_{n\to\infty} K_f^\Omega(u^n(t)) \leqslant \liminf_{n\to\infty} K_f^n(u^n(t))$$

where by K_f^Ω we mean the functional K_f defined on the domain Ω. Using (78) and step 1 we deduce

$$K_f^\Omega(u(t)) \leqslant J_h(u_0)\, e^{-2\,\alpha_1\, t}.$$

for any Ω bounded domain in $\mathbb{R}^d$, which finally proves b).

Step 7: Proof of c). We need first to prove that

$$\int_{\mathbb{R}^d} (V[u^n]^{\text{ext}})(x,t)\, dx \tag{83}$$

is bounded independently of n in $[0,T]$. Using step 5, we know that $E^n(u^n(t))$ is bounded independently of n in $[0,T]$. In the case, $h(0+) > -\infty$ since $E^n(u^n(t)) = \tilde{E}^n(u^n(t)) + h(0+)M^n$, $\tilde{\Phi}$ is positive and $M^n \to M$ as $n \to \infty$, it is easy to deduce the uniform bound on (83).

Let us focus now on the case $h(0+) = -\infty$. The main difference is that we have to estimate

$$\int_{\Omega_n} \Phi^-(u^n)(x,t)\, dx.$$

In order to do so, we divide Ω_n into three subsets

$$\Omega_n^1 := \{u^n \leqslant u_A\},\ \Omega_n^2 := \{u_A < u^n < 1\} \text{ and } \Omega_n^3 := \{u^n \geqslant 1\}$$

where $u_A := h^{-1}(C - (V(x)/A))$, $C < 0$ and $A > 0$ to be specified. Taking into account Remark 7 we have $\Phi^-(u^n) \leqslant \Phi^-(u_A)$ in Ω_n^1 and $\Phi^-(u^n) \leqslant \Phi^-(1)u^n$ in Ω_n^3. In Ω_n^2 we use that the function $\Phi^-(s)/s$ is decreasing in s to have

$$\int_{\Omega_n^2} \Phi^-(u^n)(x,t)\, dx \leqslant \int_{\Omega_n^2} \left(u^n \frac{\Phi^-(u_A)}{u_A}\right)(x,t)\, dx.$$

Now, we take C small enough to assure that u_A is well-defined, $u_A \in L^1(\mathbb{R}^d)$ and $\Phi^-(u_A) \in L^1(\mathbb{R}^d)$ by hypotheses (HV4), (HV5), (HV6). Thus, we deduce

$$\int_{\Omega_n} \Phi^-(u^n)(x,t)\, dx \leqslant B_1 + \int_{\Omega_n^2} \left(u^n \frac{\Phi^-(u_A)}{u_A}\right)(x,t)\, dx.$$

Using (HF7) and taking C small enough we prove

$$\frac{\Phi^-(u_A)}{u_A} \leqslant -\, bh(u_A) = \frac{b}{A} V(x) - bC$$

in $\mathbb{R}^d$. Therefore,

$$\int_{\Omega_n} \Phi^-(u^n)(x,t)\,dx \leqslant B_2 + \frac{b}{A}\int_{\Omega_n} (Vu^n)(x,t)\,dx.$$

Since $E^n(u^n(t))$ is decreasing in t, we have that

$$\int_{\mathbb{R}^d} (Vu^n)(x,t)\,dx \leqslant E^n(u^n(0)) + \int_{\Omega_n} \Phi^-(u^n)(x,t)\,dx$$

Take $b < A$ and using step 1, we conclude that

$$\left(1 - \frac{b}{A}\right)\int_{\mathbb{R}^d} (Vu^n)(x,t)\,dx \leqslant B_3$$

and then (83) is proved.

Since u^n converges to u in $L^2_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}_t)$ and the uniform bound of (83), we have that for a.e. $t > 0$ $u^n(t)$ converges weakly in $L^1(\mathbb{R}^d)$. Therefore, $u(t) \in L^1(\mathbb{R}^d)$ for a.e. $t > 0$ and $u^n(t) \to u(t)$ weakly in $L^1(\mathbb{R}^d)$ and strongly in $L^1(\Omega)$ for any Ω bounded domain in $\mathbb{R}^d$.

Now, let us fix $\epsilon > 0$ and $T > 0$, we can find a ball B_R such that for $n \geqslant n_\epsilon$, we have B_R in the support of u^n and

$$\int_{\mathbb{R}^d/B_R} [u^n]^{\text{ext}}(x,t)\,dx \;<\; \epsilon$$

for any $t \in [0,T]$. This is proved just by using that $V(x) \geqslant \alpha_1 |x|^2 \geqslant \alpha_1 R^2$ out of B_R. Let us remark that R is independent of n but depends on ϵ, $R \to \infty$ as $\epsilon \to 0$. Therefore, we have that

$$\int_{B_R} [u^n]^{\text{ext}}(x,t)\,dx \geqslant M^n - \epsilon$$

for $n \geqslant n_\epsilon$. Taking the limit $n \to \infty$ for a.e. $t > 0$ for which $u^n(t)$ converges strongly in B_R we deduce

$$\int_{B_R} u(x,t)\,dx \geqslant M - \epsilon.$$

We find the conservation of mass just taking the limit $\epsilon \to 0$ in the previous expression and using that

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \leqslant M.$$

since $u^n(t) \to u(t)$ weakly in $L^1(\mathbb{R}^d)$. □

In particular, we can take $f(u) = u^m$ and $V(x) = \dfrac{|x|^2}{2}$ to recover the following known result (see [17], [56]), where the generalized solution for the problem (34)–[35] is known to be unique, mass-preserving and $C([0,T], L^1(\mathbb{R}^d))$ for any $T > 0$.

Corollary 4. *Let* $u_0(x) \in L^1_+ \cap L^\infty(\mathbb{R}^d)$ *with mass* M *and* $m \geqslant \dfrac{d-1}{d}$ *and* $m > \dfrac{d}{d+2}$. *Then, the unique generalized solution of problem* (34)–(35) *with* $f(u) = u^m$ *and* $V(x) = \dfrac{|x|^2}{2}$ *satisfies* *a) For a.e.* $t > 0$, $RE(u(t)|u_{\infty,M}) \leqslant RE(u_0|u_{\infty,M})\, e^{-2t}$.

b) For a.e. $t > 0$, $K_f(u(t)) \leqslant J_h(u_0)\, e^{-2t}$.

(HE3) can be dropped if one is able to prove L^∞ estimates for the solution independent of (HE3). The following result is an alternative to the previous theorem.

Theorem 19. *Assume* (HD1)–(HD2), (HV3)–(HV8), (HF1)–(HF7). *In addition, we assume* (U1), (U2), (HE1), (HE2) *and*

(HE3b) *There exists* $0 < \tilde{M} < \bar{\mathbf{M}}$ *such that* $u_0 \leqslant u_{\infty,\tilde{M}}$ *in* $\mathbb{R}^d$ *with* $u_{\infty,\tilde{M}}\Delta V \in L^1(\mathbb{R}^d)$.

Then, there exists a generalized solution of problem (34)–(35) *satisfying*

a) If $h(0+) = -\infty$,

$$RE(u(t)|u_{\infty,M}) \leqslant RE(u_0|u_{\infty,M})\, e^{-2\alpha_1 t}, \quad a.e.\ t > 0.$$

If $h(0+) > -\infty$,

$$\tilde{RE}(u(t)|u_{\infty,M}) \leqslant RE(u_0|u_{\infty,M})\, e^{-2\alpha_1 t}, \quad a.e.\ t > 0.$$

b) For a.e. $t > 0$,

$$K_f(u(t)) \leqslant J_h(u_0)\, e^{-2\alpha_1 t}.$$

c) For a.e. $t > 0$,

$$\|u(t)\|_{L^1(\mathbb{R}^d)} = M.$$

Proof. The proof is exactly the same as in Theorem 18. We have only to prove the energy and the L^∞ estimates using the hypothesis (HE3b) instead of (HE3)–(HE4). We keep the same notations. The L^∞ estimates are an easy consequence of the comparison principle. The restriction of $u_{\infty,\tilde{M}}$ to Ω_n is a stationary solution for the problem in Ω_n with different mass, thus $u_{\infty,\tilde{M}} = u^n_{\infty,\tilde{M}^n}$. By (HE3b) we have $u^n_0 \leqslant u^n_{\infty,\tilde{M}^n}$ in Ω_n, then by comparison principle in the bounded domain problem (see [13]) we have $u^n \leqslant u^n_{\infty,\tilde{M}^n} = u_{\infty,\tilde{M}}$ for any $n \geqslant 1$. This bound gives us uniform L^∞ global estimates in (x,t) of the sequence u^n.

The energy estimates can be obtained following as in step 2 of Theorem 18. We avoid the use of (HE3) by the following argument. Since $F'(0+) = 0$ then, there exists $\lambda > 0$ such that $F(u) \leqslant \lambda u$ for any $u \in [0, \sup(u_{\infty,\tilde{M}})]$. Using (HE2)–(HE3b) we have

$$\int_{\Omega_n} G(u^n)\Delta V\, dx \leqslant A\lambda \int_{\Omega_n} u^n \Delta V\, dx \leqslant A\lambda \int_{\mathbb{R}^d} u_{\infty,\tilde{M}}\Delta V\, dx.$$

The proof continues as in Theorem 18. □

Remark 25. a) Theorem 19 applies to any positive compactly supported bounded initial data when $\bar{\mathbf{M}} = \infty$ provided $u_{\infty,M}\Delta V \in L^1(\mathbb{R}^d)$ for any $M > 0$. The last assertion is true for instance if there exists $D > 0$ such that $\Delta V \leqslant DV$ in $\mathbb{R}^d$ and $u_{\infty,M}V \in L^1(\mathbb{R}^d)$ for any $M > 0$. For example, this is satisfied for $V(x) = \alpha\,|x|^2 + \beta\,|x|^\kappa$ for some $\alpha, \beta, \kappa > 0$, and $f(u) = u^m$, $m > 0$ if

$$\frac{d}{d + \max\{2, \kappa\}} < m.$$

b) Theorems 18 and 19 could be improved if we were able to produce an a priori $L^\infty(\mathbb{R}^d)$ estimate of the solution $u(t)$ for any $t > 0$ in terms of the $L^\infty(\mathbb{R}^d)$ norm of the initial data. Global uniform estimates were obtained for the porous medium and general filtration equation. Their proof is not trivial at all and involves delicate arguments either on nonlinear semigroup theory [5], [68] or on estimates for non integrable initial data [66].

In the linear case, one usually uses propagation of moments in order to study the smoothness of the solution, see for instance [25]. This procedure here is much more complicated because of the nonlinear diffusion. Nevertheless, in the power-law case, uniform bounds of moments can be used to improve previous results in some cases. Here, we shall use the application of this technique in dimension $d = 3$ for the porous media case $f(u) = u^m$ with $m \geqslant 1$, under the assumption of the existence of smooth solutions of (34)–(35).

Corollary 5. *Assume* (HD1)–(HD2), (HV3)–(HV8), (U1), (U2) *in the particular case* $f(u) = u^m$, $m \geqslant 1$, $d = 3$ *and*

(HV9) *There exist* $A_V^1, A_V^2 > 0$ *and* $0 < \gamma_V \leqslant 8$ *such that*

$$A_V^1 |x|^{\gamma_V} \Delta V \leqslant A_V^2 (1 + |x|^{\gamma_V})$$

for all $x \in \mathbb{R}^3$.

Moreover, assume there exists a sequence of approximated smooth fast decaying at ∞ *solutions* u^n *of problem* (34)–(35), *for a regularized initial data sequence* u_0^n, *satisfying the decay properties stated in Theorem 18.*

Then, there exists a generalized solution of problem (34)–(35) *satisfying*

a) For a.e. $t > 0$,

$$\tilde{RE}(u(t)|u_{\infty,M}) \leqslant RE(u_0|u_{\infty,M})\, e^{-2\alpha_1 t}.$$

b) For a.e. $t > 0$,

$$K_f(u(t)) \leqslant J_h(u_0)\, e^{-2\alpha_1 t}.$$

c) For a.e. $t > 0$,

$$\|u(t)\|_{L^1(\mathbb{R}^d)} = M.$$

Proof. The proof follows the lines of the proof of Theorem 18. We only have to prove energy estimates and L^2 estimates in a different way. Basically, we can derive uniform bounds of certain moments of the solution by performing parts of steps 5 and 7 of Theorem 18. By reviewing this proof we note that we can prove apriori the boundedness of

$$\int_{\mathbb{R}^3} (V(x)u^n)(x,t)\, dx \tag{84}$$

independently of n in $[0,T]$. Coming back to the energy estimates in step 2 of Theorem 18, we have in this case

$$\frac{1}{m+1}\frac{d}{dt}\int_{\mathbb{R}^3} (u^n)^{m+1}\, dx \leqslant \frac{m}{m+1}\int_{\mathbb{R}^3} (u^n)^{m+1} \Delta V\, dx - \int_{\mathbb{R}^3} |\nabla (u^n)^m|^2\, dx. \tag{85}$$

Due to (HV9) and Hölder's inequality we have

$$\frac{m}{m+1}\int_{\mathbb{R}^3}(u^n)^{m+1}\Delta V\,dx \leqslant C_1\left(\int_{\mathbb{R}^3}u^n(1+|x|^{\gamma_V})^{6m-1/5m-1}\,dx\right)^{5m-1/6m-1}\left(\int_{\mathbb{R}^3}(u^n)^{6m}\,dx\right)^{m/6m-1}.$$

By using Sobolev's inequality we deduce

$$\int_{\mathbb{R}^3}(u^n)^{m+1}\Delta V\,dx \leqslant C_2\left(\int_{\mathbb{R}^3}u^n(1+|x|^{\gamma_V})^{6m-1/5m-1}\,dx\right)^{5m-1/6m-1}\left(\int_{\mathbb{R}^3}|\nabla(u^n)^m|^2\,dx\right)^{3m/6m-1}$$

and a final application of Young's inequality gives that for any $\delta > 0$ we can find $C_\delta > 0$ such that

$$\int_{\mathbb{R}^3}(u^n)^{m+1}\Delta V\,dx \leqslant C_\delta\left(\int_{\mathbb{R}^3}u^n(1+|x|^{\gamma_V})^{6m-1/5m-1}\,dx\right)^{5m-1/3m-1}+\delta\int_{\mathbb{R}^3}|\nabla(u^n)^m|^2\,dx.$$

Therefore, coming back to the energy equation (85) we finally conclude

$$\frac{1}{m+1}\frac{d}{dt}\int_{\mathbb{R}^3}(u^n)^{m+1}\,dx \leqslant C_\delta\left(\int_{\mathbb{R}^3}u^n(1+|x|^{\gamma_V})^{6m-1/5m-1}\,dx\right)^{5m-1/3m-1}-(1-\delta)\int_{\mathbb{R}^3}|\nabla(u^n)^m|^2\,dx.$$

Moreover, since $\gamma_V \leqslant 8$ due to (HV9) and using conservation of mass we obtain

$$C_\delta\int_{\mathbb{R}^3}u^n(1+|x|^{\gamma_V})^{6m-1/5m-1}\,dx \leqslant C_3 + C_4\int_{\mathbb{R}^3}u^n V(x)\,dx,$$

and thus, it is uniformly bounded in n. Therefore, we find that

$$\int_0^T\int_{\mathbb{R}^3}(u^n)^{m+1}\,dx\,dt$$

and

$$\int_0^T\int_{\mathbb{R}^3}|\nabla(u^n)^m|^2\,dx\,dt$$

are uniformly bounded in n for any $T > 0$. From this, the assertion follow starting at step 4 of the proof of Theorem 18. □

Let us remark that the previous corollary can be generalized to any dimension d by choosing carefully the exponents in the estimates corresponding to the Sobolev inequality. This result will give you the decay result for any potential of the type $|x|^2 + |x|^\kappa$ for κ in a suitable interval. This shows that a priori bounds of moments are key estimates to improve the existence results if we want to allow full generality in the behaviour of $V(x)$ at ∞.

3.6. A general result on the exponential decay of the entropy

Theorem 20. *Let $\Omega = \mathbb{R}^d$ and assume that u_0 satisfies*

$$u_0 \in L^1(\mathbb{R}^d), u_0 \geqslant 0 \text{ and } \int_{\mathbb{R}^d} u_0(x)\, dx = M > 0.$$

Furthermore, assume (HF1)–(HF5), (HV3)–(HV8). *Let $u \in L^1_{\text{loc}}(\mathbb{R}^+ : L^1(\mathbb{R}^d))$ be a generalized mass-preserving solution of* (34), (35). *Let $0 \leqslant T_0 < T_1$. Assume for each $t \in [T_0, T_1)$,*

(HG1) $h((u(t)) \in L^1_{\text{loc}}(\mathbb{R}^d)$.

(HG2) *There is a nonvoid, open interval $I_t \subseteq [T_0, T_1)$ such that*

$$(h(u(t)) + V)\, u(s) \in L^1(\mathbb{R}^d), \quad s \in I_t,$$

(HG3) *The function*

$$P_t : I_t \to \mathbb{R}, \quad P_t(s) = \int_{\mathbb{R}^d} (h(u(t)) + V)\, u(s)\, dx$$

is differentiable at t with

$$-P_t'(t) = J_h(u(t)) \left(= \int_{\mathbb{R}^d} u(t)\, |\nabla h(u(t)) + \nabla V|^2\, dx \right) < \infty.$$

(HG4) *$u(t)$ satisfies* (U1), *$E(u(t)) < \infty$ and*

$$\lim_{I_t \ni s \to t} \int_{\mathbb{R}^d} \frac{\Phi(u(s)) - \Phi(u(t)) - h(u(t))\, (u(s) - u(t))}{s - t}\, dx = 0.$$

and assume that the function

$$J : [T_0, T_1) \to \mathbb{R}, \quad J(t) = J_h(u(t))$$

belongs to $L^1(T_0, T_1)$.

Then

$$RE(u(t)|u_{\infty,M}) \leqslant RE(u(T_0)|u_{\infty,M})\, e^{-2\alpha_1(t-T_0)}, \quad T_0 \leqslant t < T_1.$$

Proof. We set for the sake of simplicity for $t > 0$, $E(t) := E(u(t))$. Then we have $E(t) < \infty$ for all $t \in [T_0, T_1)$ and $(h(u(t)) + V)\, u(s) \in L^1(\mathbb{R}^d)$ for any $s \in I_t$. Hence,

$$\Phi(u(s)) + V\, u(s) - \Phi(u(t)) - V\, u(t) - (h(u(t)) + V)\, (u(s) - u(t)) \in L^1(\mathbb{R}^d),$$

for any $s \in I_t$, such that the integrand of (HG4) is well-defined. We calculate for $s \in I_t$ with $s \neq t$,

$$\frac{E(s) - E(t)}{s - t} = \frac{P_t(s) - P_t(t)}{s - t} + \int_{\mathbb{R}^d} \frac{\Phi(u(s)) - \Phi(u(t)) - h(u(t))(u(s) - u(t))}{s - t}\, dx.$$

Hence by (HG3), $E(.)$ is differentiable at t with

$$E'(t) = P_t'(t) = -J(t).$$

By (HG4) we have $J \in L^1(T_0, T_1)$. This sufficies [60] to deduce

$$E(t) - E(T_0) = -\int_{T_0}^{t} J(\sigma)\, d\sigma, \tag{86}$$

with absolutely continuous $E(.) : [T_0, T_1) \to \mathbb{R}$. We obtain from the generalized Sobolev inequality (73),

$$RE(u(t)|u_{\infty,M}) - RE(u(T_0)|u_{\infty,M}) \leqslant\; -2\,\alpha_1 \int_{T_0}^{t} RE(u(\sigma)|u_{\infty,M})\, d\sigma,$$

such that due to the continuity of $RE(u(.)|u_{\infty,M})$,

$$RE(u(t)|u_{\infty,M}) \leqslant RE(u(T_0)|u_{\infty,M})\, e^{-2\alpha_1(t-T_0)}, \quad T_0 \leqslant t < T_1. \qquad \square$$

Remark 26. The intention of Theorem 20 is to formulate "minimal" requirements on a generalized solution of (34), (35) which imply exponential decay of the relative entropy. It is useful to relate the assumptions to the theory of parabolic equations.

a) The requirement $J_h(u(t)) < \infty$ on $[T_0, T_1)$ (see (HG3)) is obligatory for a non-trivial application of the generalized Sobolev inequality which is the main idea behind the proof. Hence assumption (HG1) is a preliminary step to give sense to the integrand of $J_h(u(t))$.

b) The main step in the proof is the verification of (86) with real $E(t)$ and $E(T_0)$. The assumption $E(u(t)) < \infty$ on $[T_0, T_1)$ is therefore obligatory.

c) A more critical assumption is (HG2). Let us, however, note two aspects:

c.1) If one wishes to prove the conservation of mass (we consider here only such solutions) one usually starts with the derivation of estimates on the L^1 norm of $Vu(t)$.

c.2) According to c.1) one has to check whether $h(u(t))\, u(s) \in L^1(\mathbb{R}^d)$ for s in an interval containing t. In the pure power case we have $u(t)\, h(u(t)) = c.\Phi(u(t)) \in L^1(\mathbb{R}^d)$ (since $E(u(t)) < \infty$) such that it is rather natural to believe that $u(t)\, h(u(t)) \in L^1(\mathbb{R}^d)$ can be verified. From that point of view the assumption "$h(u(t))\, u(s) \in L^1(\mathbb{R}^d)$" is a regularity assumption on the time-evolution of u, locally at each $t \in [T_0, T_1)$.

d) Once one has verified (HG2) the proof of local continuity of P_t is probably not out of sight. A much more delicate requirement is the differentiability of P_t at t for all $t \in [T_0, T_1)$. Certainly, one can not go much beyond the requirements "P_t is differentiable on I_t" and "$P_t' \in L^1$ (see the assumption on J)" to establish equation (86), compare well-known counterexamples, to be found, e.g. in [60].

e) Yet another aspect of (HG3) is the identity $P_t'(t) = J_h(u(t))$. This is seemingly a regularity assumption on $u(t)$ for fixed t: The concept of generalized solutions is usually settled on the identification of u_t with an element of an appropriately chosen dual space of a Banach space of Bochner-integrable functions such that

$$\langle u_t, \phi\rangle(s) = \left(\frac{d}{dt}\int_{\mathbb{R}^d} (\nabla f(u(s)) + u(s)\,\nabla V)\,\nabla\phi\, dx\right) \tag{87}$$

for a sufficiently large class of (t-independent) test functions ϕ. Now if we assume continuity of $u(t)$, say for $t > 0$ (which is true for the pure power case), then it is easy to verify that $u(t)\nabla h(u(t)) = \nabla f(u(t))$ for almost all $x \in \mathbb{R}^d$ (with the tacit identification $u(t)\nabla h(u(t)) = 0$ whenever $u = 0$) such that

$$\int_{\mathbb{R}^d} (\nabla f(u(t)) + u(t)\, \nabla V)\, \nabla\phi\, dx = \int_{u(t)>0} u(t)\, (\nabla h(u(t)) + \nabla V)\, \nabla\phi\, dx.$$

Finally, one may try a density argument to replace "ϕ" in (87) by the function $h(u(t)) + V$. This strategy could lead to a verification of $P_t'(t) = J_h(u(t))$ whenever P_t is differentiable at t.

f) It is reasonable to assume $u(t) > 0$ pointwise whenever Φ is not differentiable at 0. Hence it is very reasonable to assume that the integrand of (HG4) tends pointwise to 0 as s approaches t. The differentiability assumption of (HG4) is a regularity assumption on the time-evolution of $u(t)$ then.

g) As already mentioned in d) one cannot go beyond the L^1 property on $J(.)$.

h) Unfortunately, the partial integration in time, i.e. Lemma 2 in Section 2.2, cannot generally be used here. The assumptions of this lemma are that $\partial_t u \in L^{p'}(0, T; X')$ and $h(u) + V \in L^p(0, T; X)$ where $1/p + 1/p' = 1$ and $X = W^{1,p}(\mathbb{R}^d)$. If these conditions are satisfied (and the integrals are defined), we obtain

$$\begin{aligned} RE(u(t)|u_{\infty,M}) - RE(u(s)|u_{\infty,M}) &= \int_s^t \langle u_t, h(u) + V\rangle dt \\ &= \int_s^t \int_{\mathbb{R}^d} u|\nabla(h(u) + V)|^2 dx\, dt, \end{aligned}$$

which is (86). From the proof of [2] it can be seen that the space $W^{1,p}(\mathbb{R}^d)$ can be replaced by a more general Banach space X satisfying some additional assumptions. However, in the porous-medium case with compactly supported initial data the function $h(u) + V$ does not belong to any L^p space.

Yet another strategy to prove exponential decay of the relative entropy in $\mathbb{R}^d$ may be settled on an approximation of $u(t)$ by solutions $u_R(t)$ which solve (34), (35) on an euclidean ball B_R centered at the origin with radius R.

Definition 6. Assume $\Omega = \mathbb{R}^d$ and assume that u_0 satisfies

$$u_0 \in L^1(\mathbb{R}^d), u_0 \geqslant 0 \text{ and } \int_{\mathbb{R}^d} u_0(x)\, dx = M > 0.$$

Furthermore, assume (HF1)–(HF5), (HV3)–(HV8) and let $u \in L^1_{\text{loc}}(\mathbb{R}^+ : L^1(\mathbb{R}^d))$ be a generalized mass-preserving solution of (34), (35). Let $T \in \mathbb{R}^+$. Then u is said to be *consistently approximable at T* iff there is a strictly increasing sequence $(R_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and a sequence $(u_{0,n})_{n\in\mathbb{N}}$ in $\bigcup_{n\in\mathbb{N}} L^1(B_{R_n})$ such that

cat. 1 For all $n \in \mathbb{N}$: $u_{0,n} \in L^\infty(B_{R_n})$, $u_{0,n} \geqslant 0$ and $\int_{B_{R_n}} u_{0,n}(x)\, dx = M_n > 0$.

cat. 2 $\inf_{B_{R_n}} V = 0$.

cat. 3 For all $n \in \mathbb{N}$: $M_n < \bar{\mathbf{M}}(B_{R_n})$.

cat. 4 $E(u_{\infty,M}) = \lim_{n\to\infty} E([u_{\infty,M_n}]^{\text{ext}})$, where $[u_{\infty,M_n}]^{\text{ext}}$ is the trivial extension of $u_{\infty,M_n} \in L^\infty(B_{R_n})$ to $\mathbb{R}^n$.

cat. 5 $E(u_0) = \lim_{n\to\infty} E([u_{0,n}]^{\text{ext}})$.

cat. 6 There is $p \in [1, \infty)$ such that $[u_n(T)]^{\text{ext}} \rightharpoonup u(t)$ weakly in $L^p(\mathbb{R}^d)$ as $n \to \infty$, where $u_n(t)$ is the unique solution of (34), (35) on B_{R_n} with initial datum $u_{0,n}$, see Theorem 12.

Remark 27. The approximating sequence $u_n(t)$ may depend on $T \in \mathbb{R}^+$.

Theorem 21. *Assume* $\Omega = \mathbb{R}^d$ *and assume that* u_0 *satisfies*

$$u_0 \in L^1(\mathbb{R}^d), u_0 \geqslant 0 \text{ and } \int_{\mathbb{R}^d} u_0(x)\,dx = M > 0.$$

Furthermore, assume (HF1)–(HF5), (HV3)–(HV8) *and let* $u \in L^1_{\text{loc}}(\mathbb{R}^+ : L^1(\mathbb{R}^d))$ be a generalized mass-preserving solution of (34), (35). *Let* $T \in \mathbb{R}^+$. *Assume that*

1. $E(u(T)) < \infty$.

2. u is consistently approximable at T.

Then

$$RE(u(T)|u_{\infty,M}) \leqslant RE(u_0|u_{\infty,M})\,e^{-2\alpha_1 T}. \tag{88}$$

Proof. It is left to the reader to verify that the assumptions of Theorem 16 are satisfied for the approximative systems on B_{R_n}, $n \in \mathbb{N}$. Hence we have for any $n \in \mathbb{N}$,

$$RE([u_n(T)]^{\text{ext}}|[u_{\infty,M_n}]^{\text{ext}}) \leqslant RE([u_0]^{\text{ext}}|[u_{\infty,M_n}]^{\text{ext}})\,e^{-2\alpha_1 T}.$$

Due to $E(u(T)) < \infty$ and due to the convexity of Φ we have by lower semi-continuity (apply cat. 6),

$$E(u(T)) \leqslant \liminf_{n\to\infty} E([u_n(T)]^{\text{ext}}).$$

The formula (88) follows from cat. 4 and cat. 5 now. □

4. A Csiszár–Kullback-Type Inequality

This section is devoted to the derivation of a generalized Csiszár–Kullback inequality for the relative entropy $E(.|u_{\infty,M})$ introduced in the previous section, i.e. we are looking for a real function $\mathcal{U}$ such that the estimate

$$\|u - u_{\infty,M}\|_{L^1(\Omega)} \leqslant \mathcal{U}(E(u|u_{\infty,M}))$$

holds for a rather general class of functions u.

For the sake of a broader range of applicability the discussion is performed in a more general measure-theoretic background and a larger class of convex functions Φ than used in the previous section is considered.

4.1. Properties of the relative entropy $E(.|.)$. Henceforth we shall make use of the following notations and assumptions.

A.1 $\Phi : I \to \mathbb{R}$, $I = \mathbb{R}_0^+$ or $I = \mathbb{R}^+$, is strictly convex and continuous. If $I = \mathbb{R}^+$, then $\Phi(0+) = \infty$.

A.2 Φ is differentiable on $\mathbb{R}^+$ and we set

$$I' := \{s \in I :\ \Phi \text{ is differentiable at } s\},$$

and $h := \Phi' : I' \to \mathbb{R}$.

A.3 $(S, \mathscr{B}, \mu)$ is a measure space.

We note: $\mathbb{R}^+ \subseteq I' \subseteq I \subseteq \mathbb{R}_0^+$ and if $0 \notin I'$, then $\Phi(0+) = \infty$.

Let us introduce

$$\mathscr{C} := \{v \in L^1(S) : v(z) \in I \text{ for } \mu\text{-}aa\ z \in S\},$$

$$\mathscr{C}' := \{v \in L^1(S) : v(z) \in I' \text{ for } \mu\text{-}aa\ z \in S\}.$$

We certainly have $\mathscr{C}' \subseteq \mathscr{C} \subseteq L^1_+(S)$ (with $\mathscr{C}' = \mathscr{C}$ if $I' = I$ and $\mathscr{C} = L^1_+(S)$ if $I = \mathbb{R}_0^+$). Clearly, $\mathscr{C}', \mathscr{C}, L^1_+(S)$ are closed, convex subsets of $L^1(S)$.

We shall be concerned with relative entropies with respect to a fixed reference function u_∞ and assume

A.4 $u_\infty \in \mathscr{C}'$ with $\int_S u_\infty \, d\mu =: M \in \mathbb{R}^+$.

The derivative h of Φ is strictly increasing. As in the previous section we set

$$h(0+) := \inf h = \lim_{s \to 0} h(s) \in [-\infty, \infty),$$

$$h(\infty) := \sup h = \lim_{s \to \infty} h(s) \in (-\infty, \infty],$$

where we note that $h(0+) = h(0)$ if Φ is differentiable at 0. We recall the definition of the generalized inverse $\bar{h}^{-1}$ of h,

$$\bar{h}^{-1} : \mathbb{R} \to [0, \infty], \quad \bar{h}^{-1}(\sigma) = \begin{cases} 0, & \sigma \leqslant h(0+) \\ h^{-1}(\sigma), & h(0+) < \sigma < h(\infty) \\ \infty, & h(\infty) \leqslant \sigma. \end{cases}$$

Let us consider for $c \in \mathbb{R}$ the function $\bar{h}^{-1}(c + \Phi'(u_\infty))$. We immediately obtain: If $c \leqslant 0$, then $\bar{h}^{-1}(c + \Phi'(u_\infty)) \leqslant u_\infty$, hence $\bar{h}^{-1}(c + \Phi'(u_\infty)) \in L^1_+(S)$. Due to this observation it is easy to see that

$$J := \{c \in \mathbb{R} : \bar{h}^{-1}(c + \Phi'(u_\infty)) \in L^1_+(S)\}$$

is an interval containing the non-positive real axis. For $c < \sup J$ we set

$$\mathbf{M}(c) := \int_S \bar{h}^{-1}(c + \Phi'(u_\infty)) \, d\mu \in \mathbb{R}_0^+.$$

Now we can introduce

$$\bar{\mathbf{M}} := \sup_{c \in J} \mathbf{M}(c),$$

where we immediately obtain $\bar{\mathbf{M}} = \lim_{c \to \sup J} \mathbf{M}(c) \geqslant M$, due to monotonicity. As in the previous section it is easy to see that there is for each $N \in (0, \bar{\mathbf{M}})$ exactly one $c(N) \in J$ with $\int_S \bar{h}^{-1}(c(N) + \Phi'(u_\infty)) \, d\mu = N$. Hence we can define unambigously

$$u_N^* := \bar{h}^{-1}(c(N) + \Phi'(u_\infty)), \quad N \in (0, \bar{\mathbf{M}}).$$

We note: $u_N^* \in \mathscr{C}'$ for any $N \in (0, \bar{\mathbf{M}})$.

We introduce the relative entropy functional

$$E(.|.) : \mathscr{C} \times \mathscr{C}' \to \mathbb{R}_0^+ \cup \{\infty\}, \quad E(v|v^*)$$
$$= \int_S (\Phi(v) - \Phi(v^*) - \Phi'(v^*)(v - v^*)) \, d\mu,$$

where we make use of the fact that – due to the convexity of Φ – the function $\Phi(.|.) : I \times I' \to \mathbb{R}, (s, s^*) \mapsto \Phi(s) - \Phi(s^*) - \Phi'(s^*)(s - s^*)$ is non-negative such that $E(v|v^*)$ has a well-defined value in $[0, \infty]$.

Remark 28. The definition of $E(u|u_\infty)$ neither requires $\Phi(u_\infty) \in L^1(d\mu)$ nor $\Phi'(u_\infty) \in L^\infty(d\mu)$.

We make the following observation.

Proposition 22. *Assume* A.1–A.4. *Let* $v \in \mathscr{C}$ *and let* $v^* \in \mathscr{C}'$ (*such that* $E(v|v^*)$, $E(v^*|u_\infty)$, $E(v|u_\infty)$ *have well-defined values in* $[0, \infty]$). *Assume furthermore*

A. $[(\Phi'(u_\infty) - \Phi'(v^*))(v^* - v)]^- \in L^1(S)$.

Then

$$E(v|v^*) + E(v^*|u_\infty) + \int_S (\Phi'(u_\infty) - \Phi'(v^*))(v^* - v)\, d\mu = E(v|u_\infty). \tag{89}$$

Remark 29. Any of the values $E(v|v^*)$, $E(v^*|u_\infty)$, $\int_S (\Phi'(u_\infty) - \Phi'(v^*))(v^* - v) d\mu$, $E(v|u_\infty)$ may be ∞.

Proof. We have due to non-negativity of the first two integrands and due to A.,

$$\begin{aligned}
&E(v|v^*) + E(v^*|u_\infty) + \int_S (\Phi'(u_\infty) - \Phi'(v^*))(v^* - v)\, d\mu \\
&\quad = \int_S (\Phi(v) - \Phi(v^*) - \Phi'(v^*)(v^* - v))\, d\mu \\
&\qquad + \int_S (\Phi(v^*) - \Phi(u_\infty) - \Phi'(u_\infty)(v^* - u_\infty))\, d\mu \\
&\qquad + \int_S (\Phi'(u_\infty) - \Phi'(v^*))(v^* - v)\, d\mu \\
&\quad = \int_S (\Phi(v) - \Phi(v^*) - \Phi'(v^*)(v - v^*) + \Phi(v^*) - \Phi(u_\infty) - \Phi'(u_\infty)(v^* - u_\infty) \\
&\qquad + \Phi'(u_\infty)(v^* - v) - \Phi'(v^*)(v^* - v))\, d\mu \\
&\quad = \int_S (\Phi(v) - \Phi(u_\infty) - \Phi'(u_\infty)(v - u_\infty))\, d\mu = E(v|u_\infty).
\end{aligned}$$

□

From Proposition 22 one easily deduces

Lemma 23. *Assume* A.1–A.4. *Let* $N \in (0, \bar{\mathbf{M}})$. *Then for all* $v \in \mathscr{C}$ *with* $\int_S v\, d\mu = N$,

$$E(v|u_N^*) + E(u_N^*|u_\infty) \leqslant E(v|u_\infty), \tag{90}$$

in particular

$$E(u_N^*|u_\infty) \leqslant E(v|u_\infty). \tag{91}$$

Remark 30. a) In (90), (91) we have $N = \|u_N^*\|_{L^1(d\mu)} = \|v\|_{L^1(d\mu)}$, which is usually *not equal to* $M = \|u_\infty\|_{L^1(d\mu)}$. This is in contrast to the assumptions on which the (rather similar) derivation of (40) ($\|u\|_{L^1(d\mu)} = M$) is based.

b) As it will become clear from the proof, one has equality in (90) iff $(\Phi'(u_\infty) - \Phi'(u_N^*))(z) = c(N)$ for μ-aa $z \in S$. This is in particular the case for $\Phi'(0+) = -\infty$.

Proof. As already mentioned before, $u_N^* \in \mathscr{C}'$. Hence $E(v|u_N^*)$ is well-defined. We wish to apply (89) of Proposition 22. First we have to verify A. It is easy to deduce from the definition of u_N^* that $\Phi'(u_N^*) = \Phi'(u_\infty) + c(N)$, if $u_N^* > 0$, and $\Phi'(0) \geqslant \Phi'(u_\infty) + c(N)$, if $u_N^* = 0$. Hence due to non-negativity of v

$$\begin{aligned}
&\int_S (\Phi'(u_\infty) - \Phi'(u_N^*))(u_N^* - v)\, d\mu \\
&\quad = -c(N) \int_{u_N^*>0} (u_N^* - v)\, d\mu - \int_{u_N^*=0} [\Phi'(u_\infty) - \Phi'(0)] v \, d\mu \\
&\quad \geqslant\ - c(N) \int_{u_N^*>0} (v - u_N^*)\, d\mu - c(N) \int_{u_N^*=0} v\, d\mu \\
&\quad = -c(N) \int_{u_N^*>0} (v - u_N^*)\, d\mu - c(N) \int_{u_N^*=0} (v - u_n^*)\, d\mu \\
&\quad = -c(N) \int_S (v - u_N^*)\, d\mu = 0,
\end{aligned}$$

such that A has to hold. Furthermore, the integral arising in (89) is non-negative. This proves (90). (91) follows from (90) due to non-negativity. □

For later reference we state

Proposition 24. *Assume* A.1–A.4. *Let* $v_1, v_2 \in \mathscr{C}$.

$$\text{If } v_1 \leqslant v_2 \leqslant u_\infty, \mu\text{-almost everywhere, then } E(v_1|u_\infty) \geqslant E(v_2|u_\infty).$$

Proof. Since Φ is convex, we have for all $t \in \mathbb{R}_0^+$:

$$\Delta_t : [0,t] \to \mathbb{R}, \quad \Delta_t(s) := \begin{cases} \dfrac{\Phi(s) - \Phi(t)}{s-t} - \Phi'(t), & s < t \\ 0, & s = t. \end{cases}$$

is increasing and non-positive. Hence

$$\begin{aligned}
-E(v_2|u_\infty) &= \int_S (\Phi(u_\infty) - \Phi(v_2) - \Phi'(u_\infty)(u_\infty - v_2))\, d\mu \\
&= \int_S \Delta_{u_\infty(z)}(v_2(z))\, (u_\infty(z) - v_2(z))\, d\mu(z) \\
&\geqslant \int_S \Delta_{u_\infty(z)}(v_1(z))\, (u_\infty(z) - v_2(z))\, d\mu(z) \\
&\geqslant \int_S \Delta_{u_\infty(z)}(v_1(z))\, (u_\infty(z) - v_1(z))\, d\mu(z) \\
&= \int_S (\Phi(v_1) - \Phi(u_\infty) - \Phi'(u_\infty)\, (u_\infty - v_1)\, d\mu \\
&= -E(v_1|u_\infty).
\end{aligned}$$

□

We introduce for $\rho \in \mathbb{R}^+$

$$\mathscr{C}_\rho := \left\{ v \in \mathscr{C} : \int_S v \, d\mu = \rho \right\},$$

and

$$R_\infty := \sup \{ E(u|u_\infty) : u \in \mathscr{C}_M \},$$

where we recall $\int_S u_\infty \, d\mu = M$. Furthermore, let

$$\mathscr{C}_M^* := \{ u \in \mathscr{C}_M : E(u|u_\infty) < R_\infty \},$$

and

$$S_\infty := \sup \{ \|u - u_\infty\|_{L^1(d\mu)} : u \in \mathscr{C}_M^* \}.$$

Remark 31. Since $u_\infty \in \mathscr{C}_M$ and since $E(u_\infty|u_\infty) = 0$ we have $R_\infty \in [0, \infty]$.

To avoid trivialities we shall assume henceforth

A.5 $R_\infty \neq 0$ and $\mathscr{C}_M^*$ contains an element different from u_∞.

Remark 32. a) Assumption A.5 is a rather technical requirement on the measure space $(S, \mathscr{B}, \mu)$. A.5 is fullfilled in many cases, e.g. when S is a nonvoid, open subset of $\mathbb{R}^d$, $\mathscr{B}$ is the usual Borel-σ-algebra and μ is the Lebesgue-measure.

b) According to A.5 we have $S_\infty > 0$. On the other hand we trivially have $S_\infty \leqslant 2M$. Hence $S_\infty \in (0, 2M]$.

c) If A.5 did not hold we would have $R_\infty = 0$ (implying $\mathscr{C}_M = \{u_\infty\}$ such that u_∞ is the L^1 function with norm M and well-defined entropy) or $\mathscr{C}_M^* = \{u_\infty\}$ (i.e. u_∞ is the only function in $\mathscr{C}_M$ whose entropy is not maximal.). Both cases are not interesting here.

The reader will have no difficulties to verify that the mapping

$$E(.|u_\infty) : \mathscr{C} \to \mathbb{R} \cup \{\infty\}, \quad u \mapsto E(u|u_\infty)$$

satisfies

$$\forall u \in \mathscr{C} : \quad E(u|u_\infty) \geqslant 0 \quad \text{and} \quad [E(u|u_\infty) = 0 \quad \text{iff} \quad u = u_\infty].$$

This property shows that $E(u|u_\infty)$ can be interpreted as a "distance" between u and u_∞. The question is: Does this notion of "distance" have something in common with the canonical distance $|u - u_\infty|_{L^1(d\mu)}$ of u and u_∞, at least in $\mathscr{C}_M$? The affirmative answer is given in the subsequent section.

4.2. A Csiszár–Kullback inequality for $E(.|u_\infty)$. The following Theorem is the main result of this section.

Theorem 25. *Assume* A.1–A.5. *Then there is a function*

$$\mathcal{U} : \mathbb{R}_0^+ \to [0, 2M),$$

such that

1. $\mathcal{U}(0) = 0$.
2. $\mathcal{U}$ *is continuous at* 0, *i.e.* $\lim_{\phi \to 0+} \mathcal{U}(\phi) = 0$.
3. $\mathcal{U}$ *is non-decreasing.*
4. *For all* $u \in \mathscr{C}_M$ *with* $E(u|u_\infty) < \infty$ *one has the Csiszár–Kullback-type inequality*

$$|u - u_\infty|_{L^1(d\mu)} \leqslant \mathcal{U}(E(u|u_\infty)). \tag{92}$$

Proof. The main ingredient of the proof of Theorem 25 is a minimization argument. This can be seen after several preparational steps.

Step 1. Inequality (92) will follow from an estimate involving a function $\mathcal{H}$ whose "generalized inverse" equals $\mathcal{U}$.

Definition 7 [7]. Let $\mathcal{H} : (0, S_\infty) \to \mathbb{R}^+$ be increasing. Then the "generalized inverse $\mathcal{H}^{-*}$ of $\mathcal{H}$" is defined as

$$\mathcal{H}^{-*} : \mathbb{R}_0^+ \to [0, 2M],$$
$$\mathcal{H}^{-*}(\sigma) = \begin{cases} 0, & 0 \leqslant \sigma < \mathcal{H}(0), \\ \sup\{s \in (0, S_\infty) : \mathcal{H}(s) \leqslant \sigma\}, & \mathcal{H}(0) \leqslant \sigma \leqslant \mathcal{H}(S_\infty), \\ S_\infty, & \mathcal{H}(S_\infty) < \sigma, \end{cases}$$

where

$$\mathcal{H}(0) := \lim_{s \to 0} \mathcal{H}(s) = \inf \mathcal{H} \in \mathbb{R}_0^+, \mathcal{H}(S_\infty) := \lim_{s \to S_\infty} \mathcal{H}(s) = \sup \mathcal{H} \in \mathbb{R}^+ \cup \{\infty\}.$$

For the sake of completeness we give a proof of

Lemma 26 [7]. *Let* $\mathcal{H} : (0, S_\infty) \to \mathbb{R}^+$ *be increasing. Then:*
a) $\mathcal{H}^{-*}$ *is increasing and* $\mathcal{H}^{-*}(0) = 0$.
b) For all $\sigma \in \mathbb{R}_0^+$ *and all* $s \in (0, S_\infty)$*: If* $\mathcal{H}(s) \leqslant \sigma$*, then* $s \leqslant \mathcal{H}^{-*}(\sigma)$.
c) $\mathcal{H}^{-*}$ *is continuous at* 0*, i.e.* $\lim_{\sigma \to 0+} \mathcal{H}^{-*}(\sigma) = 0$.

Proof. The proof of a) can be left to the reader.

b) If $0 \leqslant \sigma < \mathcal{H}(0)$, then $\mathcal{H}(s) \leqslant \sigma < \mathcal{H}(0)$, which is not possible. If $\mathcal{H}(0) \leqslant \sigma \leqslant \mathcal{H}(S_\infty)$, then $\mathcal{H}^{-*}(\sigma) = \sup\{t \in (0, S_\infty) : \mathcal{H}(t) \leqslant \sigma\}$. Since $\mathcal{H}(s) \leqslant \sigma$ we obtain $s \leqslant \mathcal{H}^{-*}(\sigma)$. If $\mathcal{H}(S_\infty) < \sigma$, then $\mathcal{H}^{-*}(\sigma) = S_\infty > s$.

c) There is nothing to prove in case of $\mathcal{H}(0) > 0$. Hence assume $\mathcal{H}(0) = 0$ henceforth. We note: $\mathcal{H}(s) \in \mathbb{R}^+$, i.e. $\mathcal{H}(s) > 0$ for all $s \in (0, S_\infty)$. We observe: If $(s_n)_{n \in \mathbb{N}}$ is a sequence in $(0, S_\infty)$ with $\lim_{n \to \infty} \mathcal{H}(s_n) = 0$, then $\lim_{n \to \infty} s_n = 0$. (Indeed, if not, then there is a subsequence $(s_{\mu(n)})_{n \in \mathbb{N}}$ with $s_{\mu(n)} \geqslant s^+ > 0$, hence $\mathcal{H}(s_{\mu(n)}) \geqslant \mathcal{H}(s^+) > 0$ by monotonicity of $\mathcal{H}$ and by the positivity of $\mathcal{H}$.) Furthermore, due to the positivity of $\mathcal{H} - \mathcal{H}(S_\infty) > 0$. Now, let $(\sigma_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}^+$ with $\lim_{n \to \infty} \sigma_n = 0$. We assume without loss of generality: $0 < \sigma_n < \mathcal{H}(S_\infty)$ for all $n \in \mathbb{N}$. By definition of $\mathcal{H}^{-*}$, there is for each $n \in \mathbb{N}$ a number $s_n \in (0, S_\infty)$ with

$$\mathcal{H}(s_n) \leqslant \sigma_n, \quad \mathcal{H}^{-*}(\sigma_n) \leqslant s_n + \frac{1}{n}, \quad n \in \mathbb{N}.$$

As shown above, we obtain $\lim_{n \to \infty} s_n = 0$ which settles $\lim_{n \to \infty} \mathcal{H}^{-*}(\sigma_n) = 0 = \mathcal{H}^{-*}(0)$. □

Step 2: With the aid of the generalized inverse we can re-state the proposition of Theorem 25 as follows.

Lemma 27. *Assume* A.1–A.5. *Let* $\mathcal{H} : (0, S_\infty) \to \mathbb{R}^+$ *such that*
a. $\mathcal{H}$ *is increasing.*

b. For all $u \in \mathscr{C}_M^*$*:*

$$\text{If } |u - u_\infty|_{L^1(d\mu)} \in (0, S_\infty), \textit{then } \mathcal{H}(\|u - u_\infty\|_{L^1(d\mu)}) \leqslant E(u|u_\infty).$$

Then

1. $\mathcal{H}^{-*}(0) = 0$.

2. $\mathcal{H}^{-*}$ *is continuous at* 0, *i.e.* $\lim_{\sigma\to 0+} \mathcal{H}^{-*}(\sigma) = 0$.

3. $\mathcal{H}^{-*}$ *is increasing.*

4. For all $u \in \mathscr{C}_M$ *with* $E(u|u_\infty) < \infty$ *one has the Csiszár-Kullback-type inequality*

$$|u - u_\infty|_{L^1(d\mu)} \leqslant \mathcal{H}^{-*}(E(u|u_\infty)). \tag{93}$$

Proof. 1., 2. and 3. follow from Lemma 26.

4. Let $u \in \mathscr{C}_M$. If $u \in \mathscr{C}_M^*$, then we obtain (93) from assumption b) and Lemma 26. It remains to consider the case $u \in \mathscr{C}_M \setminus \mathscr{C}_M^*$, i.e. $E(u|u_\infty) = R_\infty = \sup\{E(v|u_\infty) : v \in \mathscr{C}_M\}$. Since $E(u|u_\infty) < \infty$ we obtain $R_\infty < \infty$. We furthermore, trivially, have due to b) the estimate $\mathcal{H}(S_\infty) \leqslant R_\infty = E(u|u_\infty)$. If $\mathcal{H}(S_\infty) < R_\infty$, then due to the definition of $\mathcal{H}^{-*}$, $\|u - u_\infty\|_{L^1(d\mu)} < S_\infty = \mathcal{H}^{-*}(R_\infty) = \mathcal{H}^{-*}(E(u|u_\infty))$, and if $\mathcal{H}(S_\infty) = R_\infty$, then $\mathcal{H}(S_\infty) < \infty$ and (93) follows in a straight-forward manner from the definition of $\mathcal{H}^{-*}$. □

Step 3: As shown in Step 2 it is sufficient to find a function which satisfies a. and b. of Lemma 27. Now we shall prove that the existence of such a function can be established by considering a minimization problem.

Lemma 28. *Assume* A.1–A.5. *Let*

$$A_\circ : (0, 1) \to \mathbb{R}^+$$

such that

i. $A_\circ$ *is decreasing.*

ii. For all $\theta \in (0, 1)$*:*

$$A_\circ(\theta) \leqslant \inf\{E(v|u_\infty) : v \in \mathscr{C}_{\theta M}, v(z) \leqslant u_\infty(z) \textit{ for } \mu\textit{-almost all } z \in S\}.$$

Then the function

$$\mathcal{H}_\circ : (0, S_\infty) \to \mathbb{R}^+, \quad \mathcal{H}_\circ(\sigma) := A_0\Big(1 - \frac{\sigma}{2M}\Big)$$

has the following properties:

a. $\mathcal{H}_\circ$ *is increasing.*

b. For all $u \in \mathscr{C}_M^*$*:*

$$\text{If } \|u - u_\infty\|_{L^1(d\mu)} \in (0, S_\infty), \textit{then } \mathcal{H}_\circ(\|u - u_\infty\|_{L^1(d\mu)}) \leqslant E(u|u_\infty).$$

Proof. a. is obvious.

b. Let $u \in \mathscr{C}_M^*$ with $0 < \|u - u_\infty\|_{L^1(d\mu)} < S_\infty$. We put $\theta := \frac{1}{M}\int_S [u - u_\infty]^- \, d\mu$. Since $\int_S u \, d\mu = \int_S u_\infty \, d\mu = M$ we obtain $|u - u_\infty|_{L^1(d\mu)} = 2M\theta$. Hence $\theta \in]0, 1[$. Let $S^+ := \{z \in S : u(z) > u_\infty(z)\}$. We put $\alpha := \int_{S\setminus S^+} u \, d\mu$. Then we obtain

$\int_{S^+} u_\infty \, d\mu = (1-\theta)M - \alpha$ and

$$\begin{aligned}
E(u|u_\infty) &= \int_S \Phi(u|u_\infty)\, d\mu \\
&= \int_{S\setminus S^+} \Phi(u|u_\infty)\, d\mu + \int_{S^+} \Phi(u|u_\infty)\, d\mu \\
&\geqslant \int_{S\setminus S^+} \Phi(u|u_\infty)\, d\mu \\
&\geqslant \inf \left\{ \int_{S\setminus S^+} \Phi(v|u_\infty)\, d\mu : \; v \in \mathscr{C}, v(z) \leqslant u_\infty(z) \text{ for } \mu\text{-aa } z \in S \setminus S^+, \right. \\
&\qquad \left. \text{and } \int_{S\setminus S^+} v \; d\mu = \alpha \right\} \\
&= \inf \left\{ \int_S \Phi(v|u_\infty)\, d\mu : \; v \in \mathscr{C}, v(z) \leqslant u_\infty(z) \text{ for } \mu\text{-aa } z \in S \setminus S^+, \right. \\
&\qquad \left. \text{and } v(z) = u_\infty(z) \text{ for } \mu\text{-aa } z \in S^+, \int_S v \; d\mu = \alpha + \int_{S^+} u_\infty \, d\mu \right\} \\
&= \inf \left\{ E(v|u_\infty) : \; v \in \mathscr{C}, v(z) \leqslant u_\infty(z) \text{ for } \mu\text{-aa } z \in S \setminus S^+, \right. \\
&\qquad \left. \text{and } v(z) = u_\infty(z) \text{ for } \mu\text{-aa } z \in S^+, \int_S v \; d\mu = (1-\theta)M \right\} \\
&\geqslant \inf \left\{ E(v|u_\infty) : v \in \mathscr{C}_{(1-\theta)M}, v(z) \leqslant u_\infty(z) \text{ for } \mu\text{-almost all } z \in S \right\} \\
&= A_\circ(1-\theta) \\
&= A_\circ\left(1 - \frac{\|u - u_\infty\|_{L^1(d\mu)}}{2M}\right) \\
&= \mathcal{H}_\circ(\|u - u_\infty\|_{L^1(d\mu)}).
\end{aligned}$$

□

Step 4: As shown in Step 3 it is sufficient to prove the existence of a function $A_\circ$ which satisfies i. and ii. of Lemma 28. We observe

Lemma 29. *Assume* A.1–A.5. *Then the function*

$$\mathbf{A}_0 : (0,1) \to \mathbb{R}^+ \cup \{\infty\}, \quad \mathbf{A}_0(\theta) = \inf_{\mathscr{C}_{\theta M}} E(.|u_\infty)$$

is decreasing.

Proof. We observe: $\theta \in (0,1)$ implies $\theta M < M$ and, since $M \leqslant \bar{\mathbf{M}}$, $\theta M < \bar{\mathbf{M}}$. Hence due to Lemma 23 we obtain $E(u^*_{\theta M}|u_\infty) \leqslant E(v|u_\infty)$ for all $v \in \mathscr{C}_{\theta M}$, i.e. $E(u^*_{\theta M}|u_\infty) = \inf_{\mathscr{C}_{\theta M}} E(.|u_\infty)$. We recall:

$$u^*_{\theta M} = \bar{h}^{-1}(c(\theta M) + \Phi'(u_\infty)),$$

such that due to monotonicity of $\bar{h}^{-1}$,

$$\text{If } \theta_1 \leqslant \theta_2, \text{then } c(\theta_1 M) \leqslant c(\theta_2 M) \leqslant 0 \text{ and } u^*_{\theta_1 M} \leqslant u^*_{\theta_2 M} \leqslant u_\infty.$$

Hence by Proposition 24, $E(u^*_{\theta_1 M}|u_\infty) \geqslant E(u^*_{\theta_2 M}|u_\infty)$, i.e. $\mathbf{A}_0$ is decreasing. □

We have for any $\theta \in (0,1)$,

$$\mathbf{A}_0(\theta) \leqslant \inf\{E(v|u_\infty) : v \in \mathscr{C}_{\theta M},\ v(z) \leqslant u_\infty(z) \text{ for } \mu\text{-almost all } z \in S\}.$$

We deduce from Lemma 29: No matter whether $\mathbf{A}_0$ equals ∞ on a subinterval of $(0,1)$ or not, there is always a decreasing, positive function $A_\circ$ which satisfies i. and ii. of Lemma 28. □

4.3. Applications. Hardly any information about $\mathcal{U}$ follows from Theorem 25. This is not surprising, because the function $\mathcal{U}$ will usually depend on local properties of u_∞. However *upper* bounds on $\mathcal{U}$ may be available.

Theorem 30. *Assume* A.1–A.5 *and*

A.6 *There are* $m, \lambda \in \mathbb{R}^+$ *such that for all* $s, t \in I$ *with* $t \leqslant s \leqslant \operatorname{ess\,sup} u_\infty \in \mathbb{R}^+ \cup \{\infty\}$ *the estimate* $\Phi(t) - \Phi(s) - \Phi'(s)(t-s) \geqslant m|t-s|^{1+\lambda}$ *holds. For* $\theta \in (0,1)$ *let* $c_\theta \in \mathbb{R}^+$ *such that*[1] $\int_S [u_\infty - c_\theta]^+ \, d\mu = \theta M$. *Furthermore, let*

$$\mathcal{R} : (0,1) \to \mathbb{R}^+_0, \mathcal{R}(\theta) = \sup_{\eta \in [\theta,1)} \left\{ m\, c_\eta^{1+\lambda} \mu(\{u_\infty > c_\eta\}) + m \int_{\{u_\infty \leqslant c_\eta\}} (u_\infty)^{1+\lambda} \, d\mu \right\}.$$

Then,

1. $\mathcal{R}$ is decreasing and positive.

2. For all $\theta \in (0,1)$, we have

$$\mathcal{R}(\theta) \leqslant \inf\{E(v|u_\infty) : v \in \mathscr{C}_{\theta M},\ v(z) \leqslant u_\infty(z) \text{ for } \mu - \text{almost all } z \in S\}.$$

3. For all decreasing, positive $\mathcal{S} : (0,1) \to \mathbb{R}^+$ with $\mathcal{S} \leqslant \mathcal{R}$, and all $u \in \mathscr{C}^*_M$ with $E(u|u_\infty) < \infty$ one has the Csiszár-Kullback-type inequality

$$|u - u_\infty|_{L^1(d\mu)} \leqslant \mathcal{T}_{\mathcal{S}}^{-*}(E(u|u_\infty)), \tag{94}$$

where $\mathcal{T}_{\mathcal{S}}^{-*}$ is the generalized inverse function of

$$\mathcal{T}_{\mathcal{S}} : (0, 2M) \to \mathbb{R}^+, \quad \mathcal{T}_{\mathcal{S}}(\sigma) = \mathcal{S}\Big(1 - \frac{\sigma}{2M}\Big).$$

Remark 33. a) With respect to 1. one may take $\mathcal{S} = \mathcal{R}$.

b) If $\mu(S) = \infty$ or if – more generally – $\mu(\{u \leqslant c_\theta\}) \neq 0$ for any $\theta \in (0,1)$, then one may take $\mathcal{S} = m \int_{u_\infty \leqslant c_\theta} (u_\infty)^{1+\lambda} \, d\mu$.

c) We note: $\theta \to c_\theta$ is decreasing with $\lim_{\theta \to 1} c_\theta = 0$. Hence the second condition "$\mu(\{u \leqslant c_\theta\}) \neq 0$ for any $\theta \in (0,1)$" of b) is equivalent to "$\mu(\{u \leqslant c_\theta\}) > 0$ for each $\theta \in (0,1)$".

d) If $\mu(\{u \leqslant c_\theta\}) = 0$ for some $\theta \in (0,1)$, then $\mu(S) < \infty$ and there is $\theta^* \in (0,1)$ with $\mu(\{u \leqslant c_\theta\}) > 0$ on $(0, \theta^*)$ and $\mu(\{u \leqslant c_\theta\}) = 0$ on $(\theta^*, 1)$. One may take $\mathcal{S} = m\mu(S) \min\{c_{\theta^*}^{1+\lambda}, c_\theta^{1+\lambda}\}$.

---

[1] It can be left to the reader to verify that $c_\theta \in \mathbb{R}^+$ is uniquely determined.

Proof. The verification of 1. is left to the reader.
2.,3. Let $v \in \mathscr{C}_{\theta M}$ with $\theta \in (0,1)$ and $v(z) \leqslant u_\infty(z)$ for μ-almost all $z \in S$. We calculate

$$\begin{aligned} E(v|u_\infty) &= \int_S (\Phi(v) - \Phi(u_\infty) - \Phi'(u_\infty)(v - u_\infty))\, d\mu \\ &\geqslant m \int_S (u_\infty - v)^{1+\lambda}\, d\mu \\ &\geqslant m \int_S (u_\infty - [u_\infty - c_\theta]^+)^{1+\lambda}\, d\mu \\ &\geqslant m\, c_\theta^{1+\lambda}\, \mu(\{u_\infty > c_\theta\}) + m \int_{\{u \leqslant c_\theta\}} (u_\infty)^{1+\lambda}\, d\mu. \end{aligned}$$

Furthermore, since $\theta \to c_\theta$ is decreasing, we obtain for any $\eta \in [\theta, 1)$ the estimate

$$\begin{aligned} m \int_S (u_\infty - [u_\infty - c_\theta]^+)^{1+\lambda}\, d\mu &\geqslant m \int_S (u_\infty - [u_\infty - c_\eta]^+)^{1+\lambda}\, d\mu \\ &\geqslant m\, c_\eta^{1+\lambda}\, \mu(\{u_\infty > c_\eta\}) + m \int_{\{u \leqslant c_\eta\}} (u_\infty)^{1+\lambda}\, d\mu. \end{aligned}$$

Hence

$$E(v|u_\infty) \geqslant \sup_{\eta \in [\theta,1)} \left\{ m\, c_\eta^{1+\lambda}\, \mu(\{u_\infty > c_\eta\}) + m \int_{\{u \leqslant c_\eta\}} (u_\infty)^{1+\lambda}\, d\mu \right\}.$$

2. and 3. follow from Lemma 28 now. □

As an example, consider $\Phi(t) = t(\log(t) - 1)$, $u_\infty = 1$ and $\mu(S) = 1$. Then for all $0 < t \leqslant s \leqslant 1$ we obtain for a $\xi \in (0,1)$ the estimate

$$\begin{aligned} \Phi(t) - \Phi(s) - \Phi'(s)(t-s) &= \Phi''(\xi t + (1-\xi)s)(t-s)^2 \\ &= \frac{1}{\xi t + (1-\xi)s} |t-s|^2 \geqslant |t-s|^2, \end{aligned}$$

hence $m = \lambda = 1$. Furthermore, $c_\theta = 1 - \theta$ and $\mu(\{u_\infty > c_\theta\}) = 1$. We deduce from Theorem 30 for all $u \in L^1_+(d\mu)$ with $\int_S u\, d\mu = 1$ the estimate

$$|u - 1|_{L^1(d\mu)} \leqslant \sqrt{4 \int_S u \log(u)\, d\mu}.$$

We observe that the constant $\sqrt{4}$ is not optimal (the optimal constant is known to be $\sqrt{2}$), but Theorem 30 provides the right decay rate.

In corollary 30 it is only assumed that $u_\infty \in L^1(d\mu)$ holds. If one additionally assumes $u_\infty^\beta \in L^1(d\mu)$ for some $\beta \in \mathbb{R}$, then one can employ Lemma 28 to extend an argumentation outlined in [56] to get

Theorem 31. *Assume* A.1–A.5 *and additionally*
A.7 *There are* $c \in \mathbb{R}^+$, $a \in (1,\infty)$, $b \in \mathbb{R}$ *such that*

A.7.1 *For all* $t, s \in I$ *with* $t \leqslant s \leqslant \operatorname{ess\,sup} u_\infty \in]0,\infty]$ *the estimate*

$$\Phi(t) - \Phi(s) - \Phi'(s)(t-s) \geqslant c\, s^{b-a}\,(s-t)^a$$

holds and

A.7.2 $u_\infty^{\frac{a-b}{a-1}} \in L^1(d\mu)$.

Then for all $u \in \mathscr{C}_M^*$ *with* $|u-u_\infty|_{L^1(d\mu)} \in (0, S_\infty)$:

$$|u-u_\infty|_{L^1(d\mu)} \leqslant \frac{2}{c^{1/a}} \left(\int_S (u_\infty)^{\frac{a-b}{a-1}}\, d\mu\right)^{\frac{a-1}{a}} (E(u|u_\infty))^{1/a}.$$

Proof. We wish to apply Lemma 28. Let $v \in \mathscr{C}_{\theta M}$ with $\theta \in (0,1)$ and $v(z) \leqslant u_\infty(z)$ for μ-almost all $z \in S$. Due to assumption A.7 we have the estimate

$$E(v|u_\infty) \geqslant \int_S c\, u_\infty^{b-a}\,(u_\infty - v)^a\, d\mu.$$

On the other hand we can employ $u_\infty \in L^{\frac{a-b}{a-1}}(d\mu)$ to obtain from Hölder's inequality the estimate

$$\begin{aligned}
(1-\theta)M &= \int_S (u_\infty - v)\, d\mu \\
&= \int_S u_\infty \left(1 - \frac{v}{u_\infty}\right) d\mu \\
&= \int_S (u_\infty)^{\frac{a-b}{a}} \left[(u_\infty)^{\frac{b}{a}} \left(1 - \frac{v}{u_\infty}\right)\right] d\mu \\
&\leqslant \left(\int_S (u_\infty)^{\frac{a-b}{a-1}}\, d\mu\right)^{\frac{a-1}{a}} \left(\int_S u_\infty^b \left(1 - \frac{v}{u_\infty}\right)^a d\mu\right)^{\frac{1}{a}} \\
&= \left(\int_S (u_\infty)^{\frac{a-b}{a-1}}\, d\mu\right)^{\frac{a-1}{a}} \left(\int_S u_\infty^{b-a}\,(u_\infty - v)^a\, d\mu\right)^{\frac{1}{a}} \\
&\leqslant c^{-1/a} \left(\int_S (u_\infty)^{\frac{a-b}{a-1}}\, d\mu\right)^{\frac{a-1}{a}} (E(v|u_\infty))^{1/a}.
\end{aligned}$$

We set

$$A_\circ(\theta) := (1-\theta)^a M^a\, c \left(\int_S (u_\infty)^{\frac{a-b}{a-1}}\, d\mu\right)^{1-a}$$

and apply Lemma 28. □

We wish to apply Theorem 31 in cases where A.1–A.5 holds with

$$\Phi(t) = t\, h(t) - f(t), \quad t\, h'(t) = f'(t),$$

where f is of one of the following types:

a) $f(t) = t$. Then $\Phi(t) = t\,(\log(t) - 1)$, $I = \mathbb{R}_0^+$ and $\Phi'' = 1/t$ is decreasing. Hence we have for all $t, s \in I$ with $t \leqslant s \leqslant \operatorname{ess\,sup} u_\infty \in \mathbb{R}^+ \cup \{\infty\}$ the estimate

$$\begin{aligned}
\Phi(t) - \Phi(s) - \Phi'(s)(t-s) &= \frac{1}{2}\, \Phi''(\eta\, s + (1-\eta)\, t)(s-t)^2 \\
&\geqslant \frac{1}{2}\, \Phi''(s)\,(s-t)^2 = \frac{1}{2}\, s^{-1}\,(s-t)^2,
\end{aligned}$$

because $\eta \in \,]0,1[$. We can apply Theorem 31 with $c = 1/2$, $b = 1$ and $a = 2$ to obtain

$$\|u - u_\infty\|_{L^1(d\mu)} \leqslant \sqrt{8 \int_S E(u|u_\infty)}.$$

b) $f(t) = t^m$ with $m \in (0,1)$. Then $\Phi(t) = -\frac{1}{1-m}\, t^m$, $I = \mathbb{R}_0^+$ and $\Phi'' = m\, t^{m-2}$ is decreasing. As above we have for all $t, s \in I$ with $t \leqslant s \leqslant \operatorname{ess\,sup}\ u_\infty \in \mathbb{R}^+ \cup \{\infty\}$

$$\Phi(t) - \Phi(s) - \Phi'(s)(t-s) = \frac{1}{2}\, \Phi''(\eta s + (1-\eta)\, t)(s-t)^2 \geqslant \frac{m}{2}\, s^{m-2}\, (s-t)^2,$$

such that we obtain with $c = m/2$, $a = 2$ and $b = m$ the estimate [56]

$$|u - u_\infty|_{L^1(d\mu)} \leqslant \left[\frac{8}{m} \int_S (u_\infty)^{2-m}\, d\mu\right]^{1/2} \sqrt{E(u|u_\infty)}.$$

Clearly, this estimate is non-trivial only in cases where $u_\infty \in L^{2-m}(d\mu)$.

c) $f(t) = \log(t)$. Then $\Phi(t) = -1 - \log(t)$, $I = \mathbb{R}^+$ and $\Phi'' = t^{-2}$ is decreasing. As above we have for all $t, s \in I$ with $t \leqslant s \leqslant \operatorname{ess\,sup} u_\infty \in \mathbb{R}^+ \cup \{\infty\}$

$$\Phi(t) - \Phi(s) - \Phi'(s)(t-s) = \frac{1}{2}\, \Phi''(\eta s + (1-\eta)\, t)(s-t)^2 \geqslant \frac{1}{2}\, s^{-2}\, (s-t)^2,$$

such that we obtain with $c = 1/2$, $a = 2$ and $b = 0$ the estimate

$$|u - u_\infty|_{L^1(d\mu)} \leqslant \left[8 \int_S (u_\infty)^2\, d\mu\right]^{1/2} \sqrt{E(u|u_\infty)}.$$

Clearly, this estimate is non-trivial only in cases where $u_\infty \in L^2(d\mu)$.

d) $f(t) = t^m$ with $m \in (1,2)$. Then $\Phi(t) = \frac{1}{m-1}\, t^m$, $I = \mathbb{R}_0^+$ and $\Phi'' = m\, t^{m-2}$ is decreasing. As above we have for all $t, s \in I$ with $t \leqslant s \leqslant \operatorname{ess\,sup} u_\infty \in \mathbb{R}^+ \cup \{\infty\}$

$$\Phi(t) - \Phi(s) - \Phi'(s)(t-s) = \frac{1}{2}\, \Phi''(\eta s + (1-\eta)\, t)(s-t)^2 \geqslant \frac{m}{2}\, s^{m-2}\, (s-t)^2,$$

such that we obtain with $c = m/2$, $a = 2$ and $b = m$ the estimate

$$\|u - u_\infty\|_{L^1(d\mu)} \leqslant \left[\frac{8}{m} \int_S (u_\infty)^{2-m}\, d\mu\right]^{1/2} \sqrt{E(u|u_\infty)}.$$

This estimate is non-trivial only in cases where $u_\infty^{2-m} \in L^1(d\mu)$, where we note that $0 < 2 - m < 1$.

e) $f(t) = t^2$. We can proceed as in d) to obtain

$$|u - u_\infty|_{L^1(d\mu)} \leqslant 2 \left[\int_S (u_\infty)^0\, d\mu\right]^{1/2} \sqrt{E(u|u_\infty)} = \sqrt{4\, \mu(S)\, E(u|u_\infty)}.$$

This is a non-trivial estimate if and only if $\mu(S) < \infty$. What happens in case of $\mu(S) = \infty$? Will there be an estimate involving square roots available? The answer to the second question is according to f) below: usually not.

f) $f(t) = t^2$, $S = \,]0,\infty[$, $\mathscr{B} =$ set of all Borel-measurable subsets of $]0,\infty[$, $\mu =$ Lebesgue measure, $u_\infty(z) = \mathrm{e}^{-z}$. We have for all $u \in L^1(d\mu)$ with $u(z) \geqslant 0$ μ-aa and $\int_0^\infty u(z)\,dz = 1$,

$$E(u|u_\infty) = \int_0^\infty (u(z) - \mathrm{e}^{-z})^2\,dz.$$

For $\theta \in \,]0, 2/\mathrm{e}[$ let $c = c(\theta) \in \,]0, 1/\mathrm{e}[$ and $B = B(\theta) \in \,]0,\infty[$ such that

$$\theta = -2\,c(\theta)\log(c(\theta)), \quad 2\,c(\theta)\,B(\theta) = \theta.$$

We note $\lim_{\theta\to 0} c(\theta) = 0$. For $\theta \in \,]0, 1/\mathrm{e}[$ let

$$u_\theta : \mathbb{R}^+ \to \mathbb{R}_0^+$$

$$u_\theta(z) = \begin{cases} e^{-z} - c(\theta), & 0 < z \leqslant -\log(c(\theta)), \\ e^{-z} + c(\theta), & -\log(c(\theta)) < z \leqslant -\log(c(\theta)) + B(\theta), \\ e^{-z}, & -\log(c(\theta)) + B(\theta) < z. \end{cases}$$

We certainly have for all $\theta \in \,]0, 2/\mathrm{e}[$: $u_\theta(z) \geqslant 0$ for μ-aa $z \in \,]0,\infty[$ and

$$\begin{aligned} \|u_\theta\|_{L^1(d\mu)} &= \int_0^\infty u_\theta(z)\,dz \\ &= \int_0^{-\log(c(\theta))} (e^{-z} - c(\theta))\,dz \\ &\quad + \int_{-\log(c(\theta))}^{-\log(c(\theta))+B(\theta)} (e^{-z} + c(\theta))\,dz + \int_{-\log(c(\theta))+B(\theta)}^\infty e^{-z}\,dz \\ &= 1 + c\log(c) + c\,B = 1, \end{aligned}$$

and

$$\|u_\theta - u_\infty\|_{L^1(d\mu)} = -c\log(c) + c\,B = \theta, \quad E(u_\theta|u_\infty) = -c^2\log(c) + c^2\,B = c\,\theta.$$

Hence

$$\lim_{\theta\to 0} \frac{\|u_\theta - u_\infty\|^2_{L^1(d\mu)}}{E(u_\theta|u_\infty)} = \lim_{\theta\to 0} \frac{\theta}{c(\theta)} = \lim_{\theta\to 0} -2\log(c(\theta)) = \infty.$$

Hence there is no $K \in \,]0,\infty[$ such that for all $u \in L^1(d\mu)$, $u(z) \geqslant 0$ for μ-aa $z \in \,]0,\infty[$, $\int_0^\infty u\,d\mu = 1$ an estimate of the form $\|u - u_\infty\|_{L^1(d\mu)} \leqslant K\sqrt{E(u|u_\infty)}$ would be available.

f) (continued) We keep the notations and assumptions of the previous example. The above discussion suggests an estimate of the form

$$\|u - u_\infty\|_{L^1(d\mu)} \leqslant K_1\,\mathscr{F}^{-1}(K_2\,E(u|u_\infty)), \tag{95}$$

where $K_1, K_2 \in \,]0,\infty[$, at least for "small" values of $\|u - u_\infty\|_{L^1(d\mu)}$, where $\mathscr{F}$ is the strictly increasing function $\theta \mapsto c(\theta)\,\theta$, $\theta \in \,]0, 2/\mathrm{e}[$. Estimate (95) actually follows from Lemma 28. This can be seen as follows. For $\theta \in \,]0,1[$ it is easy to see that we have for all $u \in L^1_+(d\mu)$ with $u_\infty(z) = \mathrm{e}^{-z} \geqslant u(z)$, $\int_0^\infty u\,d\mu = \theta$ the estimate

$$E(u|u_\infty) \geqslant E([u-c]^+|u_\infty) = -c^2\log(c) + c^2/2 \geqslant -c^2\log(c) = \frac{c\,(1-\theta)}{2},$$

where $c = c(1-\theta)$. (95) follows with $K_1 = K_2 = 2$ from Lemma 28 now.

g) $f(t) = t^m$ with $m \in [2,\infty[$ (hence $\Phi(t) = \frac{1}{m-1}\,t^m$) and $\mu(\{u_\infty > 0\}) < \infty$. According to the discussions of e) and f) the norm $\|u - u_\infty\|_{L^1(d\mu)}$ can usually not be estimated by $K\sqrt{E(u|u_\infty)}$. This is shown at hand of the counter example of f). In this example the support of u_∞ has *infinite* measure. On the other hand the stationary solutions u_∞ of the evolution systems investigated in previous sections are compactly supported, i.e. $\mu(\{u_\infty > 0\}) < \infty$. The question arises whether this additional property of u_∞ can be exploited to achieve a more transparent description of $\mathcal{U}$. The affirmative answer is given in

Theorem 32. *Assume A.3, A.4. Let $m \in [2,\infty[$ and let $\Phi : \mathbb{R}_0 \to \mathbb{R}_0$, $\Phi(t) = c\,t^m$ with $c \in \mathbb{R}^+$. Assume A.5 and additionally*

$$\mu(\{u_\infty > 0\}) < \infty.$$

Then there is $K \in \mathbb{R}^+$ (only depending on m, c, $\mu(\{u_\infty > 0\})$), such that for all $u \in \mathscr{C}_1^*$ with $\|u - u_\infty\|_{L^1(d\mu)} \in (0, S_\infty)$:

$$\|u - u_\infty\|_{L^1(d\mu)} \leqslant K\,(E(u|u_\infty))^{1/m}.$$

Proof. We wish to apply Lemma 28. Hence we have to estimate

$$\inf\{E(v|u_\infty) : v \in L^1(d\mu), 0 \leqslant v \leqslant u_\infty, \textstyle\int v\,d\mu = \theta\},$$

where $\theta \in \,]0,1[$. Let $v \in L^1(d\mu)$ with $0 \leqslant v \leqslant u_\infty, \int v\,d\mu = \theta$. We note: $\theta = \int_{\{u_\infty > 0\}} v\,d\mu$ and

$$\begin{aligned}
E(v|u_\infty) &= \int [\Phi(v) - \Phi(u_\infty) - \Phi'(u_\infty)\,(v - u_\infty)]\,d\mu \\
&\geqslant \int_{\{u_\infty > 0\}} [\Phi(v) - \Phi(u_\infty) - \Phi'(u_\infty)\,(v - u_\infty)]\,d\mu \\
&= \int_{\{u_\infty > 0, v \neq u_\infty\}} [\Phi(v) - \Phi(u_\infty) - \Phi'(u_\infty)\,(v - u_\infty)]\,d\mu.
\end{aligned}$$

Now comes an important estimate: There is $K_0(m) \in \mathbb{R}^+$ such that for all $\sigma \in [0,1[$

$$\frac{\sigma^m - m\,\sigma + (m-1)}{(1-\sigma)^m} \geqslant K_0.$$

Hence

$$\begin{aligned}
&\int_{\{u_\infty>0\},v\neq u_\infty} [\Phi(v)-\Phi(u_\infty)-\Phi'(u_\infty)\,(v-u_\infty)]\,d\mu\\
&\quad = c\int_{\{u_\infty>0\},v\neq u_\infty} u_\infty^m[(v/u_\infty)^m - m(v/u_\infty)+m-1]\,d\mu\\
&\quad \geqslant c\,K_0\int_{\{u_\infty>0\},v\neq u_\infty} u_\infty^m[1-(v/u_\infty)]^m\,d\mu\\
&\quad = c\,K_0\int_{\{u_\infty>0\},v\neq u_\infty} (u_\infty - v)^m\,d\mu\\
&\quad = c\,K_0\int_{\{u_\infty>0\}} (u_\infty - v)^m\,d\mu\\
&\quad \geqslant c\,K_0\,(\mu\{u_\infty>0\})^{1-m}\left(\int_{\{u_\infty>0\}}(u_\infty - v)\,d\mu\right)^m\\
&\quad =: K_1\,(1-\theta)^m.
\end{aligned}$$

Hence, due to Lemma 28: If $u \in \mathscr{C}_1^*$ and if $\|u-u_\infty\|_{L^1(d\mu)} \in (0, S_\infty)$, then

$$K_1\left(\frac{\|u-u_\infty\|_{L^1(d\mu)}}{2}\right)^m \leqslant E(u|u_\infty),$$

which finishes the proof. □

Remark 34. In the previous sections exponential decay has been established for superentropies $E(t)$ of $E(u(t)|u_\infty)$ (i.e. $E(t) \geqslant E(u(t)|u_\infty)$). Due to Lemma 32 we have under the assumptions mentioned there for $f(t) = \log(t)$ or $f(t) = t^m$, $m \in \mathbb{R}^+$, the estimate

$$\|u(t)-u_\infty\|_{L^1(d\mu)} \leqslant K\,(E(t))^{1/\kappa},$$

where $K \in \mathbb{R}^+$ and $\kappa = 2$ for $\lim_{t\to\infty} f(t)\,t^{-2} = 0$ and $\kappa = m$ for $f(t) = t^m$ with $m \in [2,\infty)$.

5. Entropy Dissipation for Higher-Order Parabolic Equations

In this section we present some extensions of the previous results. For most of these extensions the proofs are straightforward adaptations of the proof methods already used. In particular, we shall discuss further applications of the entropy method to higher order (possibly) degenerate scalar parabolic equations. The following discussion will be largely formal, due to the fact that the existence theory for the class of equations we shall introduce is at present not well developed, except in particular cases. One of the main reasons of this fact is that maximum principles are in general not available for fourth order equations such that positivity or nonnegativity properties has to be proven by *ad hoc* techniques. In many cases, like for the so-called thin film equations, positivity has been derived considering entropy dissipation ([9], [11], [12], [6]).

It is evident that the generalized Sobolev inequalities we obtained in the previous sections allow to obtain, at least in a formal way, the asymptotic behaviour of nonnegative solutions of fourth-order parabolic equations of the form

$$\frac{\partial u}{\partial t} = -\gamma\Delta(\Psi(u)\Delta(V(x) + h(u))) + \operatorname{div}(u\nabla(V(x) + h(u))), \qquad (x \in \mathbb{R}^d, t > 0), \tag{96}$$

where $\gamma > 0$, $\Psi \geqslant 0$ and V and h verify the hypotheses used in Section 4 for the general nonlinear diffusion equation (1). In fact, if we assume that the considered solution of equation (96) is positive and regular enough to perform the necessary integrations by parts, the entropy functional is nonincreasing in time, and

$$\begin{aligned}\frac{d[E(u(t)) - E(u_\infty)]}{dt} &= -\int_{\mathbb{R}^d} u\,|\Delta(V + h(u))|^2\,dx \\ &\quad - \int_{\mathbb{R}^d} \Psi(u)|\Delta[V + h(u)]|^2\,dx \\ &\leqslant -\lambda[E(u(t)) - E(u_\infty)].\end{aligned} \tag{97}$$

In other words, the fourth order term "improves" the exponential decay of the entropy towards its minimum. Particular choices of both the constant γ, and the functions Ψ, V in (96) permit to obtain well-known equations. To this aim, let us set $V(x) = |x|^2/2$, and $\gamma = 1/d$. Since $\Delta|x|^2 = 2d$, equation (96) become

$$\frac{\partial u}{\partial t} = -\frac{1}{d}\Delta[\Psi(u)\Delta h(u)] - \Delta(\Psi(u) - f(u)) + \operatorname{div}(xu) \qquad (x \in \mathbb{R}^d, t > 0), \tag{98}$$

Thus, choosing $\Psi(u) = f(u)$, equation (98) reduces to

$$\frac{\partial u}{\partial t} = -\frac{1}{d}\Delta[f(u)\Delta h(u)] + \operatorname{div}(xu) \quad (x \in \mathbb{R}^d, t > 0), \tag{99}$$

with $f(u)$ and $h(u)$ related as in (HF3), namely

$$h(u) := \int_1^u \frac{f'(s)}{s}\,ds, \quad u \in (0, \infty). \tag{100}$$

This implies that, with the choice $V(x) = |x|^2/2$, for any given Fokker–Planck type equation (1), we can construct a fourth-order parabolic equation, given by (99), which has the same equilibrium solution as equation (1), and such that, at least formally, has nonnegative solutions which converge exponentially in relative entropy towards its steady state (with the same mass). From now on, we shall call equation (99) the fourth order parabolic equation conjugate to the Fokker–Planck type equation (1). In the next two subsections we will discuss in some detail two cases of particular interest. The first one is related to the linear Fokker–Planck equation, while the second refers to the porous medium equation with exponent $m = 2$.

5.1. A parabolic equation describing interface fluctuations. In this paragraph, we consider the fourth-order equation conjugate to the linear Fokker–Planck equation. In this simple case, $f(u) = u$, so that $h(u) = \log u$, and equation (99) reads

$$\frac{\partial u}{\partial t} = -\frac{1}{d}\Delta(u\Delta \log u) + \operatorname{div}(xu) \quad (x \in \mathbb{R}^d, t > 0), \tag{101}$$

By a standard time dependent rescaling (see Section 3.2), one shows that the function

$$v(x,t) = \frac{1}{\alpha^d(t)} u\left(\frac{x}{\alpha(t)}, \tau(t)\right), \tag{102}$$

where

$$\alpha(t) = (1+4t)^{1/4} \ ; \qquad \tau(t) = \log \alpha(t), \tag{103}$$

satisfies the fourth-order diffusion equation

$$\frac{\partial v}{\partial t} = -\frac{1}{d}\Delta(v\Delta \log v) \quad (x \in \mathbb{R}^d, t > 0). \tag{104}$$

Equation (104) was considered in [14] among possible generalizations to higher dimensions of the one–dimensional partial differential equation

$$v_t = -(v(\log v)_{xx})_{xx}, \quad (x \in \mathbb{R}, t > 0). \tag{105}$$

Equation (105) arose originally as a scaling limit in the study of interface fluctuations in a certain spin system. The same equation also arises in the modeling of quantum semiconductor devices [57]. The initial boundary value problem for equation (105) has been first considered in [14] with periodic boundary conditions. Assuming (strictly) positive initial H^1-data, they showed that there exists a unique positive classical solution, locally in time. For suitably small initial data, the solution is even global in time. However, the problem whether non-negative solutions for general (non-negative) initial data exist globally in time, even in one dimension, remained open. A first step in this direction was done in [40], for equation (105) in a bounded domain $\Omega = (0,1)$, subject to the boundary conditions

$$v(0) = v(1) = 1, \quad v_x(0) = v_x(1) = 0.$$

However, these results are not strong enough to give a rigorous treament of the asymptotic behaviour of equation (105) posed on the whole real line. From a formal point of view, the polynomial decay of the nonnegative solution of (105) towards the (Gaussian) similarity solution with the same mass M

$$w(x,t) = \frac{M}{\sqrt{2\pi(1+t)^{1/2}}} \exp\left[-x^2/4(1+t)^{1/2}\right] \tag{106}$$

is a consequence of the structure of the fourth order diffusion equation (101), conjugate to the Fokker–Planck equation, which in one dimension can be written as

$$u_t = -\left(u\left(\log u + \frac{x^2}{2}\right)_{xx}\right)_{xx} + \left(u\left(\log u + \frac{x^2}{2}\right)_x\right)_x. \tag{107}$$

Exponential $L^1(\mathbb{R})$-convergence of the nonnegative solution of equation (107) to the Gaussian u_∞ follows by the convergence in relative entropy, which in this case, as for the linear Fokker–Planck equation, is the standard Boltzmann relative entropy $H(u|u_\infty) = H(u) - H(u_\infty)$, with

$$H(f) = \int_{\mathbb{R}} \left(\frac{x^2}{2} f + f \log f \right) dx. \tag{108}$$

As discussed in the previous sections, the relative entropy $H(u|u_\infty)$ satisfies the differential inequality

$$\frac{d}{dt} H(u|u_\infty) \leqslant -\int_{\mathbb{R}} u \left(\frac{x^2}{2} + u \log u \right)_x^2 dx = -D(u) \leqslant 0. \tag{109}$$

We remark that $D(u)$ is the entropy dissipation associated to the Fokker–Planck equation. The classical logarithmic Sobolev inequality then permits to bound the entropy production from below in terms of the relative entropy.

A study of positivity and global existence of solutions of the equation (105) on the real line, and the rigorous derivation of the asymptotic behaviour of its solutions are presently being carried out [19].

The second example we present here is the fourth-order diffusion equation conjugate to the porous medium equation with exponent $m = 3/2$.

5.2. Droplet breakup in a Hele–Shaw cell. Let us consider the one–dimensional fourth-order nonlinear degenerate diffusion equation

$$\frac{\partial v}{\partial t} = -(v v_{xxx})_x, \qquad (x \in \mathbb{R}, t > 0). \tag{110}$$

This equation, derived from a lubrication approximation, models the surface-tension-dominated-motion of thin viscous films and spreading droplets [52]

$$\frac{\partial v}{\partial t} = \operatorname{div}\,(f(v) \nabla_x \Delta_x v). \tag{111}$$

Equation (111) is a particular case of the thin film equation

$$\frac{\partial v}{\partial t} = -(|v|^n v_{xxx})_x, \qquad (x \in \mathbb{R}, t > 0), \tag{112}$$

where $n > 0$. These equations attracted a lot of attention in the mathematical literature in the last ten years (see [9], [11], [8], [12], [6], and the references therein). The majority of these papers deal with the problem of existence of solutions in a bounded domain, subject to appropriate boundary conditions. The asymptotic behaviour of (112) has been studied in [12] for the initial-boundary value problem with periodic boundary conditions. On the real line $\mathbb{R}$, the problem of the asymptotic behaviour of equation (110) has been recently considered by Carrillo and Toscani in [18]. They remarked that (110) can be written as

$$v_t = -2 \left[v^{\frac{3}{2}} \left(v^{\frac{1}{2}} \right)_{xx} \right]_{xx}. \tag{113}$$

Let us set

$$\alpha(t) = e^t \text{ and } \beta(t) = \left(e^{5t} - 1\right)/5.$$

Due to the standard time dependent change of variables

$$u(x,t) = \alpha(t)v(\alpha(t)x, \beta(t)), \tag{114}$$

Equation 113 becomes

$$u_t = -2\left[u^{\frac{3}{2}}\left(u^{\frac{1}{2}}\right)_{xx}\right]_{xx} + (xu)_x. \tag{115}$$

Equation (115) has a unique $C^1(\mathbb{R})$ compactly supported steady state of given mass M,

$$u_\infty(x) = \frac{1}{6}\left(C^2 - \frac{x^2}{2}\right)_+^2 \tag{116}$$

with $C = C(M)$, and, as usual, g_+ indicates the positive part of g. This solution has been found first by Smyth and Hill [61]. The steady state (116) is nothing but the Barenblatt–Pattle steady state of a rescaled porous medium equation with exponent $m = 3/2$.

It is seen by a simple calculation that equation (115) corresponds to the choice $f(u) = \sqrt{\frac{2}{3}}u^{3/2}$ and $h(u) = \sqrt{6}(u^{1/2} - 1)$ in (99). In other words, equation (115) is the fourth-order equation conjugate to the Fokker–Planck type equation

$$w_t = \sqrt{\frac{2}{3}}\left(w^{3/2}\right)_{xx} + (xw)_x, \tag{117}$$

Thus, by studying entropies of the second order nonlinear degenerate diffusion equation (117) we obtain entropies for the fourth-order nonlinear diffusion equation (115). The exact form of the entropy associated to the steady state u_∞ given in (116) is given by:

$$H(f) = \int_{\mathbb{R}} \left(\frac{x^2}{2}f + \sqrt{\frac{8}{3}}f^{3/2}\right) dx. \tag{118}$$

Let $u_\infty(x)$ be the stationary solution defined by (116). As discussed in the previous sections, the relative entropy $H(u|u_\infty)$ satisfies the differential inequality

$$\frac{d}{dt}H(u|u_\infty) \leqslant -\int_{\mathbb{R}} u\left(\frac{x^2}{2} + \sqrt{6}u^{1/2}\right)_x^2 dx = -D_p(u) \leqslant 0. \tag{119}$$

We remark that $D_p(u)$ is the entropy production associated to the porous medium type equation. Lower bounds for the entropy production in terms of the relative entropy have been obtained in Section 2. These bounds assure that

$$H(u|u_\infty) \leqslant \frac{1}{2}D_p(u). \tag{120}$$

Applying (120) to $u(t)$ we finally deduce

$$\frac{d}{dt}H(u(t)|u_\infty) \leqslant -2H(u(t)|u_\infty),$$

which implies exponential convergence to equilibrium in relative entropy with the explicit rate 2. In [18] these computations were rigorously justified. It is important to remark that, while the results about existence and regularity of nonnegative solutions, as well as the results on the asymptotic behaviour of the solution in a bounded domain hold for any exponent $n > 0$ in equation (112), the entropy method by conjugation to the second-order equation on $\mathbb{R}$ seems to be limited to the exponent $n = 1$, since only in this case the fourth-order diffusion equation is conjugate to a second-order diffusion equation. In the next subsection we shall discuss briefly possible extensions to the case $0 < n < 3$.

5.3. About the thin-film equation in $\mathbb{R}^d$. We shall now briefly discuss possible extensions of the entropy method to the thin-film equation in $\mathbb{R}^d$, with $d \geqslant 1$,

$$\frac{\partial v}{\partial t} = \operatorname{div}(v^n \nabla_x \Delta_x v), \quad 0 < n < 3. \tag{121}$$

For this range of the exponent n, it has been shown, first in dimension $d = 1$ [10], and subsequently in dimension $d \geqslant 2$ [29], that a unique C^1 source-type radial self-similar nonnegative solution exists. This solution, from now on called $w_\infty^{(n)}(r,t)$, has bounded support $[0,a]$, and is positive and decreasing for the radius $0 \leqslant r < a$. As usual, it can be found by looking for steady states of the equation

$$\frac{\partial u}{\partial t} = -\operatorname{div}(u^n \nabla_x \Delta_x u) + \operatorname{div}(xu), \quad 0 < n < 3. \tag{122}$$

Unless $n = 1$, where $w_\infty^{(1)}$ is given by the steady solution is not known explicitly. On the other hand, its properties permit to conclude the existence of a nondecreasing function $f^{(n)}(w)$ satisfying hypotheses (HF1)–(HF3) such that

$$\nabla f^{(n)}(w) + xw = 0, \qquad \int_\Omega w\,dx = M. \tag{123}$$

for $w = w_\infty^{(n)}$. This is an easy consequence of the discussion we gave in Section 3.1. Hence, for any $0 < n < 3$ we can write a nonlinear Fokker–Planck equation

$$\frac{\partial w}{\partial t} = \operatorname{div}(\nabla f^{(n)}(w) + xw), \quad 0 < n < 3. \tag{124}$$

which has $w_\infty^{(n)}(x)$ as the equilibrium solution. By the results of Section 3, the generalized Sobolev inequality implies the exponential convergence to zero of the relative entropy $E^{(n)}(w|w_\infty^{(n)}) = E^{(n)}(w) - E^{(n)}(w_\infty^{(n)})$, where

$$E^{(n)}(w) = \int_{\mathbb{R}^d} \left(\frac{x^2}{2} w + \Phi^{(n)}(w)\right) dx. \tag{125}$$

The equation (122) is however not conjugate to the second-order equation (124). Thus we can not conclude directly the exponential convergence of the solution of equation (121) towards the equilibrium $w_\infty^{(n)}(x)$.

In fact, for all $0 < n < 3$ elementary computations show that we can write (122) as

$$\frac{\partial u}{\partial t} = -\mathrm{div}\left[u^n \nabla_x \left(\Delta_x u + G^{(n)}(u)\right)\right] + \mathrm{div}\left[u\,\nabla_x\left(h^{(n)}(u) + \frac{x^2}{2}\right)\right], \tag{126}$$

where $h^{(n)}(u)$ is related to $f^{(n)}(u)$ by (100), and

$$G^{(n)}(u) = \int_1^u \frac{[f^{(n)}]'(\rho)}{\rho^{n-1}}\,d\rho. \tag{127}$$

We believe that (125) is the natural entropy to study the asymptotic behaviour of the thin-film equation. So far, however it remains an open problem to find lower bounds for the entropy production of equation (126), which would imply convergence of the nonnegative solution in relative entropy towards the equilibrium solution $w_\infty^{(n)}(x)$ with an explicit rate.

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