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A Classification of Locally Homogeneous Affine Connections with Skew-Symmetric Ricci Tensor on 2-Dimensional Manifolds

By

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Abstract. We classify, in an explicit form, the locally homogeneous torsionless affine connections as in the title. We also give some motivation for this research coming from the study of Osserman spaces.

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1. Introduction

In the paper [5] the authors introduced so-called affine Osserman connections. This concept originated from the effort to supply new examples of pseudo-Riemannian Osserman spaces (see [2], [3], [4]) via the construction that is called the *Riemann extension*. This construction assigns to every manifold $\mathscr{M}$ with a torsion-free affine connection ∇ a pseudo-Riemannian metric g_∇ of signature (n,n), $n = \dim \mathscr{M}$, on the cotangent bundle $T^*\mathscr{M}$. (See [13], Chapter 7, for more details.)

A pseudo-Riemannian manifold is said to be Osserman if the eigenvalues of the Jacobi operators

$$R_Z: \; X \mapsto R(X,Z)Z, \; Z \in T\mathscr{M}$$

(possibly complex ones!) are constant on the unit tangent sphere bundle $S\mathscr{M}$. A torsion-free connection ∇ on $\mathscr{M}$ is said to be affine Osserman if the Riemann extension $(T^*\mathscr{M}, g_\nabla)$ is an Osserman pseudo-Riemannian manifold.

The authors in [5] pay special attention to dimension $n = 2$. In this case they prove that ∇ is affine Osserman if and only if the Ricci tensor of ∇ is skew-symmetric on $\mathscr{M}$. They also point out the following result by Y.C. Wong (see [12], Th. 4.2), which we present here in a formally modified form:

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Theorem 1. *Let ∇ be a smooth torsion-free connection with skew-symmetric Ricci tensor on a two-dimensional manifold $\mathscr{M}$ and suppose that ∇ does not admit flat points. Then ∇ is Ricci recurrent. Moreover, around each point $p \in \mathscr{M}$ there exists a local coordinate system (u^1, u^2) in which the nonzero components of the connection are*

$$(i) \quad \Gamma_{11}^1 = -\partial_1\theta,\ \Gamma_{22}^2 = \partial_2\theta,$$

where θ is a smooth function such that $\partial_2\partial_1\theta \neq 0$; or

$$(ii) \quad \Gamma_{22}^1 = \varphi,\ \Gamma_{11}^1 = -\partial_1 \log\varphi,\ \Gamma_{22}^2 = \partial_2 \log\varphi,$$

where φ is a smooth function such that $\partial_2\partial_1 \log\varphi \neq 0$; or

$$(iii) \quad \Gamma_{22}^1 = -\psi/(1+u^1u^2),\ \Gamma_{11}^2 = 1/[\psi(1+u^1u^2)],$$
$$\Gamma_{11}^1 = -\partial_1 \log\psi + u^2/(1+u^1u^2),\ \Gamma_{22}^2 = \partial_2 \log\psi + u^1/(1+u^1u^2),$$

where ψ is a smooth function such that $\partial_2\partial_1 \log\psi \neq 0$.

(Compare Theorem 7 in [5] and also related results in [1].)

The authors of [5] used the connections of the simplest type (i) to construct new examples of pseudo-Riemannian Osserman manifolds of signature $(+\ +\ -\ -)$.

The aim of the present paper is to classify all affine connections from Theorem 1 which are *locally homogeneous*. Our classification seems to be not related to that given in Theorem 1 and our method is completely different from the procedure used by Wong. For related topics see [9], [10], [11], and especially [7]. We shall prove

Theorem 2. *Let ∇ be a smooth torsion-free connection with skew-symmetric Ricci tensor on a two-dimensional $\mathscr{M}$. If ∇ is locally homogeneous, then around each point p from a dense open subset of $\mathscr{M}$ there is a local coordinate system (u, v) in which the connection ∇ is expressed by:*

A1) $\nabla_{\partial_u}\partial_u = 0,\ \nabla_{\partial_u}\partial_v = -\frac{1}{3}u^2\partial_u + (1/u)\partial_v,\ \nabla_{\partial_v}\partial_v = -\frac{1}{36}u^5\partial_u - \frac{2}{3}u^2\partial_v,$

or

A2) $\nabla_{\partial_u}\partial_u = 0,\ \nabla_{\partial_u}\partial_v = u\partial_u,\ \nabla_{\partial_v}\partial_v = u\partial_v,$

or

B) $\nabla_{\partial_u}\partial_u = -(2/u)\partial_u - (1/2u)\partial_v,\ \nabla_{\partial_u}\partial_v = (\lambda/u)\partial_u,\ \nabla_{\partial_v}\partial_v = (\lambda/u)\partial_v,$

where λ is an arbitrary real parameter. In the case A1 *the corresponding affine Killing algebra is 3-dimensional and in the cases* A2 *and* B *this algebra is 2-dimensional.*

2. Basic Formulas and Killing Vector Fields

In the following $\mathscr{M}$ denotes a two-dimensional manifold and ∇ a smooth torsion-free connection. The curvature tensor R is uniquely determined by the Ricci tensor due to the formula

$$R(X,Y)Z = \mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y, \tag{2.1}$$

where $X, Y, Z \in T_q\mathscr{M}, q \in \mathscr{M}$.

Choose a system (u,v) of local coordinates in a domain $\mathscr{U} \subset \mathscr{M}$ and denote by U,V the corresponding coordinate vector fields ∂_u, ∂_v. In the domain $\mathscr{U}$, the connection ∇ is uniquely determined by six functions $A,\dots,F$ given by the formulas

$$\nabla_U U = AU + BV,\ \nabla_U V = CU + DV = \nabla_V U,\ \nabla_V V = EU + FV. \quad (2.2)$$

One can easily calculate

$$\begin{aligned} \mathrm{Ric}(U,U) &= B_v - D_u + D(A-D) + B(F-C),\\ \mathrm{Ric}(U,V) &= D_v - F_u + CD - BE,\\ \mathrm{Ric}(V,U) &= C_u - A_v + CD - BE,\\ \mathrm{Ric}(V,V) &= E_u - C_v + E(A-D) + C(F-C). \end{aligned} \quad (2.3)$$

The following assertion is obvious:

Proposition 2.1. *A smooth connection ∇ on $\mathscr{M}$ is locally homogeneous if and only if it admits, in a neighborhood of each point $p \in \mathscr{M}$, at least two linearly independent affine Killing vector fields. $p \in \mathscr{M}$.*

We start with the analysis of the system of partial differential equations for the Killing vector fields.

A Killing vector field X is characterized by the equation

$$[X, \nabla_Y Z] - \nabla_Y [X,Z] - \nabla_{[X,Y]} Z \ = \ 0 \quad (2.4)$$

which has to be satisfied for arbitrary vector fields Y,Z (see [6]). It is sufficient to satisfy (2.4) for the choices $(Y,Z) \in \{(U,U),(U,V),(V,U),(V,V)\}$. Moreover, we easily check from the basic identities for the torsion and the Lie brackets, that the choice $(Y,Z) = (V,U)$ gives the same condition as the choice $(Y,Z) = (U,V)$.

In the sequel, let us express the vector field X in the coordinate form

$$X = a(u,v)U + b(u,v)V. \quad (2.5)$$

If we substitute the corresponding expressions for X, Y and Z in (2.4), we easily see that the condition (2.4) reduces to six linear partial differential equations for the unknown functions a, b:

$$\begin{aligned} &1) \quad a_{uu} + Aa_u - Ba_v + 2Cb_u + A_u a + A_v b = 0,\\ &2) \quad b_{uu} + 2Ba_u + (2D-A)b_u - Bb_v + B_u a + B_v b = 0,\\ &3) \quad a_{uv} + (A-D)a_v + Eb_u + Cb_v + C_u a + C_v b = 0,\\ &4) \quad b_{uv} + Da_u + Ba_v + (F-C)b_u + D_u a + D_v b = 0,\\ &5) \quad a_{vv} - Ea_u + (2C-F)a_v + 2Eb_v + E_u a + E_v b = 0,\\ &6) \quad b_{vv} + 2Da_v - Eb_u + Fb_v + F_u a + F_v b = 0. \end{aligned} \quad (2.6)$$

Next, we shall calculate four integrability conditions (2.7) which must be satisfied by (2.6).

The first condition (2.7-1) is obtained if we differentiate (2.6-1) with respect to v and then subtract (2.6-3) differentiated with respect to u. Moreover, all the second derivatives of a and b are replaced by their values calculated from (2.6). We are left with a partial differential equation of first order. The second integrability condition (2.7-2) follows analogously from the equations (2.6-3) and (2.6-5), the third equation (2.7-3) follows from the equations (2.6-2) and (2.6-4), and the last equation (2.7-4) follows from the equations (2.6-4) and (2.6-6).

After a long but routine calculation we obtain, using also the formulas (2.3), the corresponding integrability conditions in a surprisingly simple form:

$$\begin{aligned}
1)\quad & \mathrm{Ric}(V,U)\ a_u + \mathrm{Ric}(U,U)\ a_v + \mathrm{Ric}(V,V)\ b_u + \mathrm{Ric}(V,U)\ b_v \\
& \quad + \left(\frac{\partial}{\partial u}\mathrm{Ric}(V,U)\right) a + \left(\frac{\partial}{\partial v}\mathrm{Ric}(V,U)\right) b = 0, \\
2)\quad & (\mathrm{Ric}(U,V) + \mathrm{Ric}(V,U))\ a_v + 2\mathrm{Ric}(V,V)\ b_v \\
& \quad + \left(\frac{\partial}{\partial u}\mathrm{Ric}(V,V)\right) a + \left(\frac{\partial}{\partial v}\mathrm{Ric}(V,V)\right) b = 0, \\
3)\quad & 2\,\mathrm{Ric}(U,U)\ a_u + (\mathrm{Ric}(U,V) + \mathrm{Ric}(V,U))\ b_u \\
& \quad + \left(\frac{\partial}{\partial u}\mathrm{Ric}(U,U)\right) a + \left(\frac{\partial}{\partial v}\mathrm{Ric}(U,U)\right) b = 0, \\
4)\quad & \mathrm{Ric}(U,V)\ a_u + \mathrm{Ric}(U,U)\ a_v + \mathrm{Ric}(V,V)\ b_u + \mathrm{Ric}(U,V)\ b_v \\
& \quad + \left(\frac{\partial}{\partial u}\mathrm{Ric}(U,V)\right) a + \left(\frac{\partial}{\partial v}\mathrm{Ric}(U,V)\right) b = 0.
\end{aligned} \tag{2.7}$$

In the next section we shall make the classification of locally homogeneous connections with skew-symmetric Ricci tensor. We shall always assume $\mathrm{Ric} \neq 0$. Because locally symmetric connections in dimension two are known to have symmetric Ricci tensor, we can assume $\nabla\,\mathrm{Ric} \neq 0$ everywhere on $\mathscr{M}$. (Cf. also Theorem 5 in [5].)

3. The Preliminary Classification

The skew-symmetry of Ric means that, in any local coordinates,

$$\mathrm{Ric}(U,U) = \mathrm{Ric}(V,V) = 0,\ \mathrm{Ric}(U,V) + \mathrm{Ric}(V,U) = 0\ , \tag{3.1}$$

and the system (2.7) is reduced to only one equation

$$a_u + b_v\ +\ \frac{\rho_u}{\rho}a + \frac{\rho_v}{\rho}b = 0, \tag{3.2}$$

where we put

$$\rho = \mathrm{Ric}(U,V) \neq 0. \tag{3.3}$$

According to (2.3) and (3.1) we have

$$\begin{aligned}
C_u &= A_v + BE - CD - \rho, \\
D_u &= B_v + D(A-D) + B(F-C), \\
E_u &= C_v + E(D-A) + C(C-F), \\
F_u &= D_v + CD - BE - \rho,
\end{aligned} \tag{3.4}$$

For the first covariant derivatives of Ric we have (due to notation (2.2) and (3.3))

$$(\nabla_U \operatorname{Ric})(U,V) = -(\nabla_U \operatorname{Ric})(V,U) = \rho_u - (A+D)\rho,$$
$$(\nabla_V \operatorname{Ric})(U,V) = -(\nabla_V \operatorname{Ric})(V,U) = \rho_v - (C+F)\rho, \tag{3.5}$$

$$(\nabla_X \operatorname{Ric})(U,U) = (\nabla_X \operatorname{Ric})(V,V) = 0 \text{ for } X = U, V \tag{3.6}$$

Put, for the initial point $p \in \mathscr{M}$,

$$r = \rho(p) \neq 0. \tag{3.7}$$

For any $q \in \mathscr{M}$ consider the linear form $\tau_q : Z \mapsto (\nabla_Z \operatorname{Ric})(X,Y)$, where $X, Y \in T_q\mathscr{M}$ are arbitrary but such that $X \wedge Y \neq 0$. Then τ_q is defined up to proportionality by a nonzero factor. Because $\nabla \operatorname{Ric} \neq 0$, τ_q has a nonzero kernel, which is independent of the choice of X and Y. $\operatorname{Ker} \tau$ is a well-defined 1-dimensional distribution on $\mathscr{M}$, which we denote by $\mathscr{D}$. Define a special local coordinate system (u,v) such that $U = \frac{\partial}{\partial u}$ belongs to $\mathscr{D}$ everywhere. We have

$$(\nabla_U \operatorname{Ric})(U,V) = 0, \ (\nabla_V \operatorname{Ric})(U,V) \neq 0 \tag{3.8}$$

in a neighborhood $\mathscr{U}$ of p.

Now, put

$$N = \frac{r}{\rho}(\nabla_V \operatorname{Ric})(U,V). \tag{3.9}$$

From (3.5) we get

$$\frac{\rho_u}{\rho} = A + D, \ \frac{\rho_v}{\rho} = C + F + \frac{N}{r}. \tag{3.10}$$

The obvious integrability condition for (3.10) reads

$$A_v + D_v = C_u + F_u + \frac{1}{r}N_u. \tag{3.11}$$

Further, the first and the last equation (3.4) give

$$A_v + D_v - C_u - F_u = 2\rho. \tag{3.12}$$

Hence we obtain

$$N_u = 2r\rho. \tag{3.13}$$

Next, as in [7], we denote

$$H_{XY} \ = \ (\nabla^2_{XY}\operatorname{Ric})(U,V). \tag{3.14}$$

Then using (3.8) and (3.9) we easily obtain

$$H_{UU} = -\frac{\rho}{r}BN, \ H_{UV} = \frac{\rho}{r}(N_u - DN), \ H_{VU} = -\frac{\rho}{r}DN,$$
$$H_{VV} = -\frac{\rho}{r}\left(N_v + \frac{N^2}{r} - FN\right) \tag{3.15}$$

H_{XY} is a tensor field of type $(0,2)$ which is not symmetric, in general. Let $g(X,Y) = H_{XY} + H_{YX}$ be the symmetrization of H. Then we have the following.

Lemma 3.1. *In a neighborhood $\mathscr{U}$ of any point p from an open dense subset of $\mathscr{M}$ one of the following situations occurs:*

a) $g(U,U)=0$ on $\mathscr{U}$ and there is a system $(\overline{u},\overline{v})$ of local coordinates such that $\overline{U}\in\mathscr{D}$ and either $g(\overline{U},\overline{V})=0$ on $\mathscr{U}$, or $g(\overline{V},\overline{V})=0$ on $\mathscr{U}$.

b) $g(U,U)\neq 0$ on $\mathscr{U}$ and there is a system $(\overline{u},\overline{v})$ of local coordinates such that $\overline{U}\in\mathscr{D}$ and $g(\overline{U},\overline{V})=0$ on $\mathscr{U}$.

Proof. It follows from elementary techniques of linear algebra and partial differential equations. □

We shall prepare some more facts and formulas. First we use (3.10) in (3.2) and we obtain

$$a_u+b_v \;+\; (A+D)a+(C+F+N/r)b=0. \tag{3.16}$$

We can obtain two other equations of 1st order as follows. First we add the equations (2.6-1), (2.6-4) and subtract (3.16) differentiated with respect to u. We easily obtain

$$\frac{N}{r}b_u-\left(A_v+D_v-C_u-F_u-\frac{N_u}{r}\right)b=0 \tag{3.17}$$

and according to (3.11) and the inequality $N\neq 0$ we get $b_u=0$. Hence

$$b=b(v). \tag{3.18}$$

Next, we add the equations (2.6-3), (2.6-6) and subtract (3.16) differentiated with respect to v. We easily obtain

$$(A_v+D_v-C_u-F_u)a+\frac{N}{r}b_v+\frac{N_v}{r}b=0, \tag{3.19}$$

or, according to (3.12),

$$a=-\frac{1}{2\rho r}\left(Nb'(v)+N_v b(v)\right)\,. \tag{3.20}$$

Hence a is uniquely determined if b is fixed. This fact and Proposition 2.1 lead to the following useful

Lemma 3.2. *Suppose that ∇ is locally homogeneous in a neighborhood $\mathscr{U}\subset\mathscr{M}$ of p and let the function $b(v)$ satisfy an ordinary differential equation of the form*

$$P(u,v)b'(v)+Q(u,v)b(v)=0, \tag{3.21}$$

where P and Q are fixed functions. Then P and Q must vanish identically.

Proof. Indeed, in the opposite case there exists *at most one* affine Killing vector field around the point p, which is a contradiction. □

The next step will be that we transform the system (2.6) of partial differential equations for two unknown functions $a(u,v), b(u,v)$ in a system of ordinary differential equations for one unknown function $b(v)$. To this aim, we substitute for b an unknown function $b(v)$ and for a and for its derivatives we substitute the expression (3.20) and its corresponding derivatives. Obviously, the fifth equation obtained in this way will be of 3rd order; the other ones will be of lower order.

Now, a natural idea is to derive as many equations of first order as possible because, according to Lemma 3.2, such equations will give additional information about the functions $A, B, \dots, F$.

First, from (2.6-2) we obtain (using, in addition, formula (3.13))

$$\begin{aligned}&(2B\rho_u N - B_u \rho N - 6rB\rho^2)\ b'(v)\\&\quad + (2B\rho_u N_v + 2r\rho^2 B_v - B_u \rho N_v - 4rB\rho R_v)\ b(v) = 0.\end{aligned} \tag{3.22}$$

According to Lemma 3.2, if ∇ is locally homogeneous, both coefficients must be zero. The second coefficient equation is obviously equivalent to

$$2r\rho\left(\frac{B}{\rho^2}\right)'_v - N_v\left(\frac{B}{\rho^2}\right)'_u = 0. \tag{3.23}$$

The first coefficient equation is equivalent to

$$\left(\frac{B}{\rho^2}\right)'_u = -\frac{6rB}{\rho N}. \tag{3.24}$$

Using (3.24) in (3.23) we obtain

$$\left(\frac{B}{\rho^2}\right)'_v + \frac{3BN_v}{\rho^2 N} = 0. \tag{3.25}$$

We shall now start with our classification. We always assume that ∇ is locally homogeneous and that any base point $p \in \mathscr{M}$ belongs to an *open dense subset* such that Lemma 3.1 can be applied. We distinguish two main cases.

Case A. We suppose that $B = 0$ in the given neighborhood. Then (3.22) is trivially satisfied. Further, from the first formula (3.15) we see that $H_{UU} = 0$. According to Lemma 3.1 we can assume either $H_{UV} + H_{VU} = 0$, or $H_{VV} = 0$, and U still belongs to the distribution $\mathscr{D}$.

Subcase A1. Let first $H_{UV} + H_{VU} = 0$ hold in $\mathscr{U}$ for some coordinate system (u, v) such that $U \in \mathscr{D}$. Because the distribution $\mathscr{D}$ is totally geodesic, a coordinate transformation of the form $\overline{u} = f(u, v)$, $\overline{v} = v$ makes $\overline{U} \in \mathscr{D}$, $\overline{A} = \overline{B} = 0$ and $H_{\overline{U}\,\overline{U}} = 0$, $H_{\overline{U}\,\overline{V}} + H_{\overline{V}\,\overline{U}} = 0$. Because ∇ is curvature homogeneous in each order (see [9], [10]), it is especially curvature homogeneous up to order two and we obtain the example from [7], pp. 129–131. Here

$$A = B = 0,\ C = -\frac{1}{3}u^2,\ D = \frac{1}{u},\ E = -\frac{1}{36}u^5 + e(v)u,\ F = -\frac{2}{3}u^2\ , \tag{3.26}$$

where $e(v)$ is an arbitrary function of v. The corresponding Killing vector fields are all of the form

$$X = q'(v)u\frac{\partial}{\partial u} - 2q(v)\frac{\partial}{\partial v}\ , \tag{3.27}$$

where $q(v)$ is any solution of the ordinary differential equation of 3rd order

$$q'''(v) - 4e(v)q'(v) - 2e'(v)q(v) = 0\ . \tag{3.28}$$

The affine Killing algebra is 3-dimensional.

We shall complete the analysis of subcase A1 in the next section.

Subcase A2. We have always $H_{UV} + H_{VU} \neq 0$ for U belonging to $\mathscr{D}$. We can again use a coordinate transformation after which $A = B = 0$. The equation (2.6-4) is reduced, due to (3.18), to the form

$$Da_u + D_u a + D_v b = 0. \tag{3.29}$$

Substituting the right-hand side of (3.20) for a and its derivative for a_u into (3.29) we see that the new equation will be a 1st order ODE for $b(v)$. According to Lemma 3.2, the coefficient of $b'(v)$ should be equal to zero. We calculate this coefficient equation explicitly and then use $D_u = -D^2$ and $\rho_u = D\rho$ [(see (3.4-2) and (3.10)]. Finally we obtain,

$$D(N_u - 2DN) = 0. \tag{3.30}$$

According to (3.15) this means $D(H_{UV} + H_{VU}) = 0$ and, according to our assumption $H_{UV} + H_{VU} \neq 0$, we get

$$D = 0. \tag{3.31}$$

(Let us notice that Lemma 3.1 does not imply automatically $H_{VV} = 0$ because we have used already a coordinate transformation to get $A = 0$!) Now, $D = 0$ implies $\rho_u = 0$ and hence

$$\rho = \rho(v). \tag{3.32}$$

Because $A = B = D = 0$, the equation (3.4-1) gives

$$C(u, v) = -\rho(v)(u + c(v)). \tag{3.33}$$

where $c(v)$ is an arbitrary function. Let us make a coordinate transformation

$$\overline{u} = C(u, v),\ \overline{v} = v, \tag{3.34}$$

which does not change the equalities $A = B = D = 0$. We denote $\overline{u}, \overline{v}$ again as u, v. Then, integrating the whole system (3.4) we get

$$C = u,\ F = u + f(v),\ E = -\frac{1}{2}u^2 f(v) + e(v),\ A = B = D = 0, \tag{3.35}$$

where $f(v)$ and $e(v)$ are arbitrary functions and

$$\rho = -1,\ r = -1. \tag{3.36}$$

Moreover, using (3.10), (3.35) and (3.36) we get

$$N = 2u + f(v) \tag{3.37}$$

and (3.20) takes on the form

$$a = -\frac{1}{2}\left[(f(v) + 2u)b' + f'(v)b\right]. \tag{3.38}$$

Now, the equation (2.6-3) can be rewritten in the form

$$2b''(v) + f(v)b'(v) + f'(v)b(v) = 0. \tag{3.39}$$

Let now $b_1(v), b_2(v)$ be independent solutions of (3.39) which are defined by the initial conditions

$$b_1(v_0) = 0,\ b_1'(v_0) = 1,\ b_2(v_0) = 1,\ b_2'(v_0) = 0\ (v_0 = v(p)). \tag{3.40}$$

Consider the Killing vector fields

$$X_1 = a_1(u,v)\frac{\partial}{\partial u} + b_1(v)\frac{\partial}{\partial v},\ X_2 = a_2(u,v)\frac{\partial}{\partial u} + b_2(v)\frac{\partial}{\partial v}, \tag{3.41}$$

where the functions a_1, a_2 are calculated from (3.38). Then the Lie bracket $[X_2, X_1]$ belongs to span (X_1, X_2). Because the corresponding coefficient of $\frac{\partial}{\partial v}$ is $b_2(v)b_1'(v) - b_1(v)b_2'(v)$, which is equal to 1 for $v = v_0$, we get

$$[X_2, X_1] = X_2. \tag{3.42}$$

We shall complete the analysis of subcase A2 in the next section.

Case B. Let us suppose $B \neq 0$ in a whole neighborhood. Then $H_{UU} \neq 0$ holds and, according to Lemma 3.1, we can introduce a system of local coordinates u, v for which $H_{UV} + H_{VU} = 0$, i.e., according to (3.15),

$$N_u - 2DN = 0. \tag{3.43}$$

From (3.13) we get

$$r\rho - DN = 0, \tag{3.44}$$

i.e., $DN \neq 0$ and

$$N = \frac{r\rho}{D}. \tag{3.45}$$

This identity will be still satisfied after any particular changes of local coordinates of the form $\bar{u} = \bar{u}(u),\ \bar{v} = \bar{v}(v)$.

Next, the equation (3.25) can be integrated in the form

$$\frac{B}{\rho^2} = \frac{\varphi(u)}{N^3}, \tag{3.46}$$

where $\varphi(u)$ is an arbitrary function. If we substitute (3.46) in (3.24), we get at once $\varphi'(u) = 0$, i.e., $\varphi(u) = \lambda$ is a nonzero constant. (Using the coordinate transformation $\bar{u} = u,\ \bar{v} = \lambda^{-1/3}v$ one can make $\lambda = 1$. Yet, we shall not use this specialization because it would become inconvenient in the subsequent considerations.)

Thus, we have

$$\frac{B}{\rho^2} = \frac{\lambda}{N^3} \tag{3.47}$$

which makes, in turn, the equation (3.22) satisfied for an arbitrary function $b(v)$. If we substitute (3.45) in (3.47), we obtain an equivalent identity

$$B = \frac{\lambda D^3}{r^3\rho}. \tag{3.48}$$

Now, we come back to the equation (2.6-3) and transform it by use of (3.18), (3.20), (3.10) and (3.13). We obtain an ODE of the second order with respect to $b(v)$. But using also (3.45) we see that the coefficient of $b''(v)$ vanishes and we get an ODE of the *first order*, namely

$$\begin{aligned} &N\left(A_v + D_v - C_u - 2\rho - 2FD - 2CD - \frac{2N}{r}D + \frac{4DN_v}{N} - \frac{2r\rho F}{N}\right)b'(v) \\ &\quad + \left(N_v\left(A_v + D_v - C_u - 2\rho - 2FD - 2CD - \frac{2N}{r}D\right)\right. \\ &\quad \left. + 2DN''_{vv} - 2r\rho F_v\right)b(v) = 0. \end{aligned} \tag{3.49}$$

Now, according to Lemma 3.2, the corresponding coefficient equations must be satisfied. We use (3.45) and differentiate $\log N = \log r + \log\rho - \log D$ with respect to v, which gives

$$\frac{N_v}{N} = \frac{\rho_v}{\rho} - \frac{D_v}{D} = C + F + \frac{\rho}{D} - \frac{D_v}{D} \ . \tag{3.50}$$

Using (3.50), (3.45) and also (3.12) we get the first coefficient equation in the form

$$F_u - 4D_v + 2CD + 2\rho = 0. \tag{3.51}$$

Expressing F_u from (3.4-4) and using this expression in (3.51) we get

$$D_v = CD - \frac{1}{3}BE + \frac{1}{3}\rho \ , \tag{3.52}$$

$$F_u = 2CD - \frac{4}{3}BE - \frac{2}{3}\rho \ . \tag{3.53}$$

Using now the fact that the coefficient of $b'(v)$ in (3.49) vanishes, we can write the second coefficient equation in the form

$$DN\left(\frac{N_v}{N^2}\right)'_v - r\rho\left(\frac{F}{N}\right)'_v = 0. \tag{3.54}$$

Due to (3.45) we get hence

$$\left(\frac{N_v}{N^2} - \frac{F}{N}\right)'_v = 0 \ . \tag{3.55}$$

Using (3.50), (3.45) and (3.52) we obtain

$$\frac{N_v}{N^2} - \frac{F}{N} = \frac{2}{3r} + \frac{1}{3}\frac{BE}{r\rho} \tag{3.56}$$

and (3.55) can be rewritten in the form

$$(BE/\rho)'_v = 0. \tag{3.57}$$

We also obtain

$$N_v = \frac{r\rho}{D^2}\left(\frac{1}{3}BE + DF + \frac{2}{3}\rho\right). \tag{3.58}$$

Differentiating (3.45) in the logarithmic form with respect to u and using (3.10), (3.13) we get

$$D_u = D(A - D). \tag{3.59}$$

From (3.4-2) we then obtain

$$B_v = B(C - F). \tag{3.60}$$

Now, taking the logarithmic derivative of (3.48) with respect to v we get, using also (3.52) and (3.10),

$$\frac{B_v}{B} = 2C - F - \frac{BE}{D}\ . \tag{3.61}$$

Comparing (3.60) and (3.61) we obtain

$$BE - CD = 0 \tag{3.62}$$

and, due to (3.48)

$$E = \frac{CD}{B} = \frac{Cr^3\rho}{\lambda D^2}. \tag{3.63}$$

Now, (3.52) and (3.53) can be written in the form

$$D_v = \frac{2}{3}CD + \frac{1}{3}\rho,\ F_u = \frac{2}{3}CD - \frac{2}{3}\rho\ . \tag{3.64}$$

Next, we differentiate (3.59) with respect to v, then (3.64-1) with respect to u and compare. We also replace ρ_u, D_u and D_v by the corresponding expressions. We obtain first

$$A_v - D_v = \frac{2}{3}C_u + \frac{2}{3}\rho. \tag{3.65}$$

Then (3.4-1) and (3.62) give $A_v = C_u + \rho$ and D_v is given by (3.64-1). Hence we get

$$C_u = \ 2CD. \tag{3.66}$$

Now, let us differentiate the formula (3.63) with respect to u (using logarithms). We easily get

$$E_u = E(5D - A). \tag{3.67}$$

From (3.4-3) we obtain

$$C_v = E_u + E(A - D) + C(F - C) \tag{3.68}$$

and from (3.67) we get, using also (3.63),

$$C_v = \ \frac{4Cr^3\rho}{\lambda D} + C(F - C). \tag{3.69}$$

The integrability condition for (3.66) and (3.69) reads (after we use (3.10), (3.53), (3.59), (3.62) and (3.66) in the final step)

$$\left(6\frac{r^3}{\lambda}-1\right)\rho \;=\; 2CD. \tag{3.70}$$

We have two subcases:

Subcase B1. $C \neq 0$ and, due to (3.63), $E \neq 0$.

Then we get

$$\rho = 2\mu CD,\;\; \mu = \frac{\lambda}{6r^3-\lambda} \tag{3.71}$$

and we obtain (3.48) and (3.63) in the form

$$B = \frac{1}{\kappa}\frac{D^2}{C},\;\; E = \kappa\frac{C^2}{D},\;\; \kappa = \frac{2r^3}{6r^3-\lambda}. \tag{3.72}$$

Here $\kappa \neq 0$ is an arbitrary parameter.

Let us notice that formula (3.57) does not bring new information because it follows from (3.71) and (3.62). Now, (3.64-1) can be rewritten in the form

$$D_v = 2\kappa CD. \tag{3.73}$$

Our next goal is to solve the system of two PDE (3.66) and (3.73). These two equations can be also rewritten in the form

$$(\log C)'_u = 2D,\;\; (\log D)'_v = 2\kappa C\;. \tag{3.74}$$

Hence

$$(\log C)''_{uv} = 2D_v = 4\kappa CD = 2\kappa C_u. \tag{3.75}$$

Integrating this with respect to the variable u, we obtain easily

$$C_v = 2\kappa C^2 + f(v)C, \tag{3.76}$$

where $f(v)$ is an arbitrary function. On the other hand, (3.69) and (3.71) imply

$$C_v = (4\kappa - 1)C^2 + FC. \tag{3.77}$$

Hence it follows

$$F = (1-2\kappa)C + f(v). \tag{3.78}$$

Analogously, we obtain from (3.74) and (3.75)

$$(\log D)''_{uv} = 2D_v.$$

Integrating this with respect to v we get

$$D_u = 2D^2 + g(u)D, \tag{3.79}$$

where $g(u)$ is an arbitrary function.

Comparing (3.79) and (3.59) we obtain

$$A = 3D + g(u). \tag{3.80}$$

Next, (3.74-1) and (3.79) form a system of PDEs

$$(\log C)'_u = 2D,\ \ (\log D)'_u = 2D + g(u). \tag{3.81}$$

Hence

$$(\log(D/C))'_u = g(u) \tag{3.82}$$

and thus

$$\log(D/C) = G(u) + H(v), \tag{3.83}$$

where $G(u)$ and $H(v)$ are arbitrary functions, $G'(u) = g(u)$.

Analogously, using (3.74-2) and (3.76) we obtain at once

$$(\log(D/C))'_v = -f(v) \tag{3.84}$$

and hence $H'(v) = -f(v)$. We obtain

$$D = C\mathbf{e}^Q,\ \ Q = G(u) + H(v)\ , \tag{3.85}$$

where

$$Q_u = g(u),\ \ Q_v = -f(v). \tag{3.86}$$

Expressing D in (3.74) through formula (3.85) we get the system

$$C_u = 2C^2\mathbf{e}^Q,\ \ C_v = 2\kappa C^2 + f(v)C, \tag{3.87}$$

which satisfies the integrability condition.

An easy integration gives

$$C = -\frac{L'(v)}{2\kappa(K(u) + L(v))},\ \ D = -\frac{K'(u)}{2(K(u) + L(v))}, \tag{3.88}$$

where

$$K'(u) = \mathbf{e}^{G(u)},\ \ L'(v) = \kappa\mathbf{e}^{-H(v)}, \tag{3.89}$$

$$g(u) = K''(u)/K'(u),\ f(v) = L''(v)/L'(v). \tag{3.90}$$

Because $K'(u) \neq 0,\ L'(v) \neq 0$, we can make the coordinate transformation

$$\overline{u} = K(u),\ \ \overline{v} = L(v).$$

If we denote the coordinate system $(\overline{u}, \overline{v})$ as (u, v) again, then (3.88) takes on the form

$$C = \frac{-1}{2\kappa(u + v)},\ \ D = \frac{-1}{2(u + v)}. \tag{3.91}$$

Further, (3.72) implies

$$B = \frac{-1}{2(u + v)},\ \ E = \frac{-1}{2\kappa(u + v)} \tag{3.92}$$

and (3.78), (3.80) and (3.90) imply

$$A = \frac{-3}{2(u + v)},\ \ F = \left(1 - \frac{1}{2\kappa}\right)\frac{1}{u + v}. \tag{3.93}$$

It remains to calculate the Killing vector fields. Using, in addition, (3.20), (3.45) and (3.71), we get

$$b = b(v),\ a = (u+v)b'(v) - b(v). \tag{3.94}$$

Then the equation (2.6-4) implies $b''(v) = 0$ and hence

$$b(v) = C_1 v + C_2,\ a = a(u) = C_1 u - C_2. \tag{3.95}$$

We easily see that *all* equations (2.6) are satisfied and hence the generating Killing vector fields are

$$X_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\ X_2 = \frac{\partial}{\partial v} - \frac{\partial}{\partial u}. \tag{3.96}$$

We shall complete the analysis of subcase B1 in the next section.

Subcase B2. Here we have $C = 0$ and $E = 0$. From (3.70) we obtain

$$6r^3 - \lambda = 0. \tag{3.97}$$

Further, we summarize the conditions (3.59), (3.64) and (3.65). We get

$$A_v = \rho,\ D_v = \frac{1}{3}\rho,\ F_u = -\frac{2}{3}\rho,\ D_u = D(A-D). \tag{3.98}$$

Hence $(A-3D)'_v = 0$ and we can substitute for A the expression

$$A = 3D + g(u) \tag{3.99}$$

in the last equation (3.98). We get

$$D_u = 2D^2 + Dg(u), \tag{3.100}$$

where $g(u)$ is an arbitrary function. If we divide (3.100) by D^2, we obtain a linear equation of 1st order for the unknown function $1/D$. This equation can be integrated in an explicit form by standard methods and we obtain the general solution as follows:

$$D = \frac{K'(u)}{-2K(u)+L(v)} \neq 0, \tag{3.101}$$

where $K(u), L(v)$ are arbitrary functions such that

$$K''(u)/K'(u) = g(u). \tag{3.102}$$

Now, according to (3.98), $D_v = \rho/3 \neq 0$ and hence $L'(v) \neq 0$. Hence we can choose a new system of local coordinates $(\overline{u}, \overline{v})$ putting $\overline{u} = K(u),\ \overline{v} = L(v)$. If we denote the new coordinates again as u, v, we obtain easily

$$D = \frac{1}{-2u+v},\ A = 3D,\ \rho = -3D^2,\ N = -3rD. \tag{3.103}$$

The last equality follows from (3.45).

Finally, we can calculate B from (3.48) and F from (3.10-2). We get, using also (3.97),

$$B = -2D,\ F = D. \tag{3.104}$$

The equation (2.6-4) now easily yields the equation

$$b''(v) = 0. \tag{3.105}$$

Hence and from (3.20), (3.103) we obtain

$$b(v) = C_1 v + C_2,\ a(u,v) = C_1 u + \frac{1}{2}C_2. \tag{3.106}$$

The generating Killing vector fields (as it is easy to check) are

$$X_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\ X_2 = \frac{1}{2}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \tag{3.107}$$

Finally, we check at once that all equations (2.6) and (3.4) are satisfied as consequences of (3.103) and (3.104).

We shall complete the analysis of subcase B2 in the next section.

4. The Canonical Forms à la Sophus Lie

Subcase A1. Consider the space of all vector fields $Y = q(v)\frac{\partial}{\partial v}$, where $q(v)$ satisfies the equation (3.28). This is a 3-dimensional Lie algebra of "infinitesimal transformations" of a line interval. We shall now use the following

Lemma. (S. Lie) *Each 3-dimensional Lie algebra $\mathfrak{g}$ of infinitesimal transformations acting on a one-dimensional manifold can be expressed locally, with respect to a convenient local coordinate v, in the form*

$$\mathfrak{g} = \left(\frac{\partial}{\partial v},\ v\frac{\partial}{\partial v},\ v^2\frac{\partial}{\partial v} \right). \tag{4.1}$$

The corresponding local transformation group is locally equivalent to the group of all projective transformations of the real line.

Proof. See [8], p. 6, Theorem 1.

From the above lemma and (3.27) we see that, with respect to a new system of local coordinates (u, v), the algebra of all affine Killing vector fields has the form

$$\mathfrak{k} = \operatorname{span}\left(\frac{\partial}{\partial v},\ u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v},\ vu\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v} \right). \tag{4.2}$$

Further, from (4.1) and (3.28) we obtain $e(v) = 0$. We obtain a unique locally homogeneous affine connection with the Christoffel symbols

$$A = B = 0,\ C = -\frac{1}{3}u^2,\ D = \frac{1}{u},\ E = -\frac{1}{36}u^5,\ F = -\frac{2}{3}u^2. \tag{4.3}$$

Subcase A2. Introduce a new variable $\overline{v}$ by the formula

$$\overline{v} = \int_{v_0}^{v} \frac{dv}{b_2(v)}. \tag{4.4}$$

Denoting $\overline{v}$ again as v we see that we can assume $b_2(v) = 1$ (and $v_0 = 0$). From (3.41) and (3.42) we obtain $b_1'(v) = 1$ and from the corresponding initial conditions $b_1(v) = v$. Because (3.39) has two independent solutions $b_1(v) = v,\ b_2(v) = 1$, we obtain $f(v) = 0$ and the basic Killing vector fields (3.41) are given by

$$X_1 = -u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\ X_2 = \frac{\partial}{\partial v}. \tag{4.5}$$

Moreover, the equalities $A = B = D = 0$ are still preserved.

Integrating the equations (3.4) in the last coordinate system, we obtain

$$\begin{aligned} &C = u + \overline{c}(v),\ F = u + \overline{f}(v), \\ &E = -\frac{1}{2}u^2\big(\overline{f}(v) - \overline{c}(v)\big) + \big[\overline{c}'(v) + (c(v) - f(v))c(v)\big]u + \overline{e}(v), \end{aligned} \tag{4.6}$$

where $\overline{c}(v)$,$\overline{f}(v)$ and $\overline{e}(v)$ are arbitrary functions.

If we now substitute in the system (2.6) for the pair of functions (a, b) the pairs $(-u, v)$ and $(0, 1)$ respectively, and for C, F and E the expressions given by (4.6), we see easily that $\overline{c}(v) = \overline{f}(v) = \overline{e}(v) = 0$. We obtain a unique locally homogeneous affine connection

$$A = B = D = E = 0,\ C = F = u. \tag{4.7}$$

Subcase B1. Here we make the coordinate transformation $\overline{u} = u + v,\ \overline{v} = v$. It is easy to calculate the new Christoffel symbols $\overline{A}, \overline{B}, \ldots, \overline{F}$ from (2.2). Turning back to the original notation of our coordinates, we get finally the basic affine Killing vector fields (3.96) in the form

$$X_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\ X_2 = \frac{\partial}{\partial v}. \tag{4.8}$$

Further, one gets a one-parameter family of locally homogeneous affine connections with the Christoffel symbols

$$A = -\frac{2}{u},\ B = -\frac{1}{2u},\ C = F = \frac{\lambda}{u}\ \left(\lambda \neq \frac{3}{2}\right),\ D = E = 0\ . \tag{4.9}$$

Subcase B2. Here we use the coordinate transformation $\overline{u} = 2u - v,\ \overline{v} = -v$. By a similar computations we obtain, rewriting again $\overline{u}, \overline{v}$ as u, v, the basic affine Killing vector fields in the form

$$X_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\ X_2 = \frac{\partial}{\partial v} \tag{4.10}$$

and the corresponding Christoffel symbols in the form

$$A = -\frac{2}{u},\ B = -\frac{1}{2u},\ C = F = \frac{3}{2u},\ D = E = 0. \tag{4.11}$$

We obtain a unique locally homogeneous affine connection.

Hence the subcases B1, B2 can be unified if we assume λ to be an arbitrary real parameter.

This concludes the proof of Theorem 2.

Remark 1. The simplest locally homogeneous connections, that is those with constant Christoffel symbols, are not Ricci skew-symmetric.

Remark 2. One can see easily that ∇ is Ricci recurrent (Theorem 1) as follows: first define, for any system of local coordinates (u, v),

$$\omega = \left[\frac{\rho_u}{\rho} - (A + D)\right] du \; + \; \left[\frac{\rho_v}{\rho} - (C + F)\right] dv.$$

A direct computation shows that ω is independent of the choice of the local coordinate system and hence ω is a global Pfaffian form on $\mathscr{M}$. Then we see at once from (3.1), (3.3) and (3.5) that

$$\nabla_X \mathrm{Ric} \; = \; \omega(X).\mathrm{Ric}$$

for each vector field X on $\mathscr{M}$.

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