Monatsh. Math. 131, 91–108 (2000)

Monatshefte für
Mathematik

On the Singularities of the Global Small Solutions of the Full Boltzmann Equation

By

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(Received 16 November 1999; in revised form 21 June 2000)

Abstract. We show how the singularities are propagated for the (spatially inhomogeneous) Boltzmann equation (with the usual angular cut-off of Grad) in the context of the small solutions first introduced by Kaniel and Shinbrot.

2000 Mathematics subject classification: 76P
Key words: Boltzmann equation, singularities

1. Introduction

The Boltzmann equation is a standard model of the kinetic theory of gases (cf. [5], [6], [7], [23]). It reads

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \tag{1}$$

where $f(t, x, v)$ is the density of particles of the gas which, at time $t \in \mathbb{R}_+$ and position $x \in \mathbb{R}^3$, have velocity $v \in \mathbb{R}^3$, and Q is a quadratic collision operator which only acts on the variable v and reads

$$Q(f, g) = Q^+(f, g) - f\,Lg, \tag{2}$$

$$Q^+(f, g)(t, x, v) = \int_{v_* \in \mathbb{R}^3} \int_{\omega \in S^2} f(t, x, v')g(t, x, v'_*)B(v - v_*, \omega)\,d\omega\,dv_*, \tag{3}$$

$$Lg(t, x, v) = \int_{v_* \in \mathbb{R}^3} \int_{\omega \in S^2} g(t, x, v_*)B(v - v_*, \omega)\,d\omega\,dv_*. \tag{4}$$

The post-collisional velocities v' and v'_* are here parametrized by

$$\begin{cases} v' = v + ((v_* - v) \cdot \omega)\omega, \\ v'_* = v_* - ((v_* - v) \cdot \omega)\omega. \end{cases}$$

Therefore,

$$Lg = A *_v g,$$

with

$$A(z) = \int_{\omega \in S^2} B(z, \omega)\,d\omega.$$

The cross section B depends on the type of interactions between the particles of the gas. In this paper, we shall always make the so-called "angular cutoff assumption of Grad" (cf. [12]). We shall even limit ourselves to cross sections which satisfy the following assumption:

Assumption 1. The nonnegative cross section B lies in $L^\infty(S^2; W^{1,\infty}(\mathbb{R}^3))$.

Note that the classical cross sections of Maxwellian molecules or regularized soft potentials (with angular cutoff) satisfy this assumption. The case of hard potentials (with angular cutoff), which do not satisfy this assumption, is briefly discussed in a remark at the end of Section 2.

The Cauchy problem for equation (1) in $\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ has been studied by various authors. Global renormalized solutions have been proven to exist for a large class of initial data by DiPerna and P.-L. Lions in [9] (cf. also [17]). Global solutions (in the whole space) close to the equilibrium have been studied by Imai and Nishida in [15] and Ukai and Asano in [24].

Finally, global solutions for small initial data were introduced by Kaniel and Shinbrot (cf. [16]) and studied by Bellomo, Palczewski and Toscani (cf. [1]), Bellomo and Toscani (cf. [2]), Goudon (cf. [11]), Hamdache (cf. [13]), Illner and Shinbrot (cf. [14]), Lu (cf. [18]), Mischler and Perthame (cf. [19]), Polewczak (cf. [20]) and Toscani (cf. [21], [22]).

In this paper, we study how the L^2 singularities of the initial datum are propagated by equation (1). This question seems very difficult to tackle in the general framework of renormalized solutions, because of the lack of L^∞-estimates in this setting.

We shall therefore concentrate on the case of small initial data, where such estimates are available. We think that our work is likely to extend to solutions close to the equilibrium, but we shall not investigate this case.

We recall one of the theorems of existence of such small solutions. We use a formulation adapted to our study, which is inspired from [19].

Theorem 1. *Let B be a cross section satisfying Assumption 1 and f_{in} be an initial datum such that, for all $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$,*

$$0 \leqslant f_{\text{in}}(x, v) \leqslant (81\|A\|_{L^\infty})^{-1} \exp\left(-\frac{1}{2}(|x|^2 + |v|^2)\right). \tag{5}$$

Then there exists a global distributional solution f to (1) with initial datum f_{in}, such that for all $T > 0, t \in [0, T]$ and $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$0 \leqslant f(t, x, v) \leqslant C_T \exp\left(-\frac{1}{2}(|x - vt|^2 + |v|^2)\right) := M_T(t, x, v), \tag{6}$$

where C_T is a constant only depending on T and $\|A\|_{L^\infty}$.

We give in Section 2 the precise form of the singularities of the solution to the Boltzmann equation (in our setting). Our main theorem is

Theorem 2. *Let B be a cross section satisfying Assumption 1 and f_{in} be an initial datum such that (5) holds. Then we can write, for all $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$,*

$$f(t, x, v) = f_{\text{in}}(x - vt, v)\Gamma_1(t, x, v) + \Gamma_2(t, x, v).$$

where $\Gamma_1, \Gamma_2 \in H^{\alpha}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ *for all* $\alpha \in]0, 1/25[$.

Remarks.

- This theorem shows that the singularities of the initial datum (that is, for example, the points around which f_{in} is in L^2 but not in H^s for any $s > 0$) are propagated with the free flow, and decrease exponentially fast (since in fact Γ_1 has an exponential decay).
- Theorem 2 ensures that, if $f(t) \in H^s(\mathbb{R}^3 \times \mathbb{R}^3)$ for some $t > 0$, then $f_{\mathrm{in}} \in H^s(\mathbb{R}^3 \times \mathbb{R}^3)$ (for $s < 1/25$), so that no smoothing can occur. This (less precise) result could however probably be obtained with a simpler method.
- The exponent 1/25 given here is probably not the best one. In order to get the optimal result, one would need to perform many more complicated computations.
- Note that, in the setting of the Boltzmann equation we consider here, it has already been noticed that some memory of the initial datum is preserved by $f(t, \cdot, \cdot)$ for all $t > 0$, as can be seen on the asymptotic behaviour when t goes to $+\infty$, or, more precisely, on the limit when $t \to +\infty$ of $f(t, x + tv, v)$ (cf. [22], [18]).

The proof of Theorem 2 uses the regularizing properties of the kernel Q^+, first studied by P.-L. Lions in [17], and extended by Wennberg in [25] and by Bouchut and Desvillettes in [3]. Note that those properties are exactly what is needed to give the form of the singularities of the solutions to the *spatially homogeneous* Boltzmann equation (with angular cutoff). In order to conclude in our inhomogeneous setting, we also have to use the averaging lemmas of Golse, P.-L. Lions, Perthame and Sentis (cf. [10]).

In Section 3, we give a short proof of a complementary result (that is, the propagation of smoothness instead of the propagation of singularities), under a slightly more stringent assumption. Namely, we show that the smoothness (with respect to t, x, v) of the solution f to equation (1) at time $t > 0$ obtained by theorem 1 is at least as good as that of f_{in} (in x, v). Note that since our solutions satisfy an L^∞ bound, theorem 2 is enough to show this propagation of smoothness as long as Sobolev spaces H^s with $s < 1/25$ are concerned. The theorem that we give here deals with higher derivatives.

Theorem 3. *Let B be a cross section satisfying Assumption 1 and such that $B \in L^1(\mathbb{R}^3 \times S^2)$ and f_{in} be an initial datum such that (5) holds. If, moreover, $f_{\mathrm{in}} \in W^{k,\infty}(\mathbb{R}^3 \times \mathbb{R}^3)$ for some $k \in \mathbb{N} \cup \{\infty\}$, then the solution f to (1) given by Theorem 1 varifies*

$$f \in W^{k,\infty}_{\mathrm{loc}}(\mathbb{R}_{+t} \times \mathbb{R}^3_x \times \mathbb{R}^3_v).$$

Remark. The propagation of smoothness in a very close setting, but only with respect to the variable x, has already been obtained by Polewczak in [20].

2. Propagation of Singularities

This section is devoted to the proof of Theorem 2. The main idea is the following: we write down the Duhamel form of the solution of equation (1). This is also called the mild exponential form. For $(t,x,v)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3$, we have

$$\begin{aligned} f(t,x,v) = &f_{\rm in}(x-vt,v)\exp\left(-\int_0^t Lf(\sigma,x-v(t-\sigma),v)d\sigma\right)\\ &+\int_0^t\Bigg[Q^+(f,f)(s,x-v(t-s),v)\\ &\qquad \exp\left(-\int_s^t Lf(\sigma,x-v(t-\sigma),v)d\sigma\right)\Bigg]ds. \end{aligned} \tag{7}$$

We are going to prove that both Lf and $Q^+(f,f)$ lie in $L^2_{\rm loc}(\mathbb{R}_+;H^\alpha_{\rm loc}(\mathbb{R}^3\times\mathbb{R}^3))$ for any $\alpha\in]0,1/25[$. In order to get this result, we use on the one hand the analysis of regularity of Q^+ in the variable v initiated in [17] and developed in [25] and [3], and on the other hand the averaging lemmas of [10].

2.1. Regularity of Lf. We here prove the following result.

Proposition 4. *If B satisfies Assumption 1 and $f_{\rm in}$ is such that (5) holds, for any $T>0$ and any $R>0$, we have*

$$\|Lf\|_{L^2([0,T];H^{1/2}(\mathscr{B}_R\times\mathscr{B}_R))}\leqslant K_{T,R}\|A\|_{L^\infty(\mathbb{R}^3)},$$

where $K_{T,R}$ is a constant which depends on T (more precisely, on the constant C_T in (6)) and R.

Let us choose $T>0$. Since Lf is a convolution with respect to v, we obviously have that, under Assumption 1, $Lf\in L^2([0,T]_t\times\mathbb{R}^3_x;H^{1/2}_{\rm loc}(\mathbb{R}^3_v))$ (in fact, it lies in $L^2([0,T]_t\times\mathbb{R}^3_x;W^{1,\infty}_{\rm loc}(\mathbb{R}^3_v)))$ and satisfies

$$\|Lf\|_{L^2([0,T]\times\mathscr{B}_r;H^{1/2}(\mathscr{B}_r))}\leqslant K'_{T,r}\|A\|_{L^\infty(\mathbb{R}^3)}.$$

It remains to prove that $Lf\in L^2([0,T]_t\times\mathbb{R}^3_v;H^{1/2}_{\rm loc}(\mathbb{R}^3_x))$.

Let us define the function T_λ, $0<\lambda<1/2$, by $T_\lambda(v_*)=e^{-\lambda v_*}$, and study the following quantity

$$\begin{aligned} &\|Lf\|^2_{L^2([0,T]_t\times\mathbb{R}^3_v;H^{1/2}(\mathbb{R}^3_x))}\\ &\quad=\int_{t,v}\int_{x,h}\left|\int_{v_*}A(v-v_*)(f(t,x+h,v_*)-f(t,x,v_*))dv_*\right|^2dx\frac{dh}{|h|^4}dvdt. \end{aligned} \tag{8}$$

We want to use the averaging lemma of [10], which we here recall in a version very close to that of [4], where the optimal smoothness in the variable t is not given, but where the dependence with respect to the averaging function is kept.

Lemma 1. *Let $f\in C([0,T]_t;L^2_w(\mathbb{R}^3_x\times\mathbb{R}^3_{v_*}))$ solve the equation*

$$\partial_t f+v_*\cdot\nabla_x f=g\quad\text{in }]0,T[\times\mathbb{R}^3\times\mathbb{R}^3,$$

for some $g\in L^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$. Then, for any $\psi\in\mathscr{D}(\mathbb{R}^3)$, the average quantity defined by

$$\rho_\psi(f)(t,x)=\int_{v_*\in\mathbb{R}^3}f(t,x,v_*)\psi(v_*)dv_*$$

belongs to $L^2([0,T];H^{1/2}(\mathbb{R}^3))$ *and satisfies, for any* $s>1$,

$$\|\rho_\psi(f)\|^2_{L^2([0,T];H^{1/2}(\mathbb{R}^3))} \leqslant C_s\left[\int_{x,v}|f(0,x,v_*)|^2\,|\psi(v_*)|^2(1+|v_*|^2)^s dv_* dx + \int_{t,x,v_*}|g(t,x,v_*)|^2|\psi(v_*)|^2(1+|v_*|^2)^s dv_* dx dt\right],$$

where C_s *is a constant only depending on* s.

Using Lemma 1, (8) becomes, for any $s>1$ and any open ball $\mathscr{B}_R$ of $\mathbb{R}^3$,

$$\begin{aligned}
&\|Lf\|^2_{L^2([0,T]_t\times\mathscr{B}_{R_v};H^{1/2}(\mathbb{R}^3_x))}\\
&\leqslant \int_{v\in\mathscr{B}_R}\left\|\rho A(v-\cdot)T_\lambda\left(\frac{f}{T_\lambda}\right)\right\|^2_{L^2([0,T];H^{1/2}(\mathbb{R}^3))} dv\\
&\leqslant C_s\int_{v\in\mathscr{B}_R}\left[\int_{x,v_*}\left|\frac{f_{\text{in}}(x,v_*)}{T_\lambda(v_*)}\right|^2|A(v-v_*)|^2|T_\lambda(v_*)|^2(1+|v_*|^2)^s dv_* dx\right.\\
&\quad+\int_{t,x,v_*}\left|(\partial_t+v_*\cdot\nabla_x)\frac{f}{T_\lambda}\right|^2\\
&\quad\left.\times|A(v-v_*)|^2|T_\lambda(v_*)|^2(1+|v_*|^2)^s dv_* dx dt\right] dv\\
&\leqslant C_{R,s}M^2_{\lambda,s}\|A\|^2_{L^\infty(\mathbb{R}^3)}\\
&\quad\times\left(\left\|\frac{f_{\text{in}}}{T_\lambda}\right\|^2_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}+\left\|(\partial_t+v\cdot\nabla_x)\frac{f}{T_\lambda}\right\|^2_{L^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}\right),
\end{aligned}\tag{9}$$

where $C_{R,s}$ is a constant and

$$M_{\lambda,s}=\sup_{v_*\in\mathbb{R}^3}|T_\lambda(v_*)(1+|v_*|^2)^{s/2}|.\tag{10}$$

Note that, since we have (5), the following estimate holds

$$0\leqslant\frac{f_{\text{in}}(x,v)}{T_\lambda(v)}\leqslant\kappa\,\mathrm{e}^{-|x|^2/2}\mathrm{e}^{(\lambda-1/2)|v|^2},$$

where κ is an absolute constant, so that (recall that $0<\lambda<1/2$) we can find a constant $C_\lambda>0$ such that

$$\left\|\frac{f_{\text{in}}}{T_\lambda}\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}\leqslant C_\lambda.\tag{11}$$

Moreover, we have

$$\left|(\partial_t+v\cdot\nabla_x)\frac{f}{T_\lambda}\right|\leqslant\frac{|Q^+(f,f)|}{T_\lambda}+\frac{|fLf|}{T_\lambda}.\tag{12}$$

It is clear, by (6), that

$$\frac{|f(t,x,v)Lf(t,x,v)|}{T_\lambda(v)} \leqslant \frac{M_T(t,x,v)LM_T(t,x,v)}{T_\lambda(v)}$$
$$\leqslant C_T^2(2\pi)^{3/2}\|A\|_{L^\infty}\mathrm{e}^{-\frac{1}{2}|x-vt|^2}\mathrm{e}^{\left(\lambda-\frac{1}{2}\right)|v|^2}.$$

Hence there exists a constant C_λ such that

$$\left\|\frac{fLf}{T_\lambda}\right\|_{L^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \leqslant C_\lambda. \tag{13}$$

It is also clear that, for $(t,x,v)\in[0,T]\times\mathbb{R}^3\times\mathbb{R}^3$,

$$\frac{|Q^+(f,f)(t,x,v)|}{T_\lambda(v)} = \frac{1}{T_\lambda(v)}\left|\int_{v_*,\omega} f(t,x,v')f(t,x,v'_*)B(v-v_*,\omega)d\omega dv_*\right|$$
$$\leqslant \frac{Q^+(M_T,M_T)(t,x,v)}{T_\lambda(v)}$$
$$= \frac{M_T(t,x,v)LM_T(t,x,v)}{T_\lambda(v)},$$

so that

$$\left\|\frac{Q^+(f,f)}{T_\lambda}\right\|_{L^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \leqslant C_\lambda. \tag{14}$$

Taking (13)–(14) into account, (12) implies that

$$\left\|(\partial_t + v\cdot\nabla_x)\frac{f}{T_\lambda}\right\|_{L^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \leqslant C_\lambda. \tag{15}$$

Then, using (11) and (15) in (9), we get

$$\|Lf\|^2_{L^2([0,T]_t\times\mathbb{R}^3_v;H^{1/2}(\mathbb{R}^3_x))} \leqslant C_s C_\lambda^2 M_{\lambda,s}^2\|A\|^2_{L^\infty}.$$

Recalling that $Lf\in L^2([0,T]_t\times\mathbb{R}^3_x;H^{1/2}_{\mathrm{loc}}(\mathbb{R}^3_v))$, we finally obtain that

$$Lf\in L^2([0,T];H^{1/2}_{\mathrm{loc}}(\mathbb{R}^3_x\times\mathbb{R}^3_v)). \tag{16}$$

2.2. Regularity of $Q^+(f,f)$.

2.2.1. *Study of the average quantities of $Q^+(f,f)$ with respect to the velocity.* This part is devoted to the proof of the

Proposition 5. *Let $\zeta\in\mathscr{D}(\mathbb{R}^3_v)$, B satisfying Assumption 1, and f_{in} such that (5) holds. Then we have, for any $T>0$ and $h\in\mathbb{R}^3$,*

$$\int_{t,x}\left|\int_v [Q^+(f,f)(t,x+h,v)-Q^+(f,f)(t,x,v)]\zeta(v)dv\right|^2 dxdt$$
$$\leqslant K_T\|\zeta\|^2_{W^{1,\infty}(\mathbb{R}^3)}|h|^{2/5}, \tag{17}$$

where K_T is a constant that depends on T (more precisely on the constant C_T in (6)) and on $\|B\|_{L^\infty(S^2;W^{1,\infty}(\mathbb{R}^3))}$.

Proof. Let $\zeta \in \mathscr{D}(\mathbb{R}^3_v)$. We have

$$\int_{\mathbb{R}^3} Q^+(f,f)(v)\zeta(v)dv = \int_{v,v_*,\omega} f(v')f(v'_*)B(v-v_*,\omega)\zeta(v)d\omega dv_* dv. \quad (18)$$

By changing pre/post-collisional variables, (18) becomes

$$\int_{\mathbb{R}^3} Q^+(f,f)(v)\zeta(v)dv$$
$$= \int_{v,v_*} f(v)f(v_*)\left(\int_\omega B(v-v_*,\omega)\zeta(v-((v-v_*)\cdot\omega)\omega)d\omega\right)dv_*dv. \quad (19)$$

Let us set

$$Z(v,v_*) = \int_\omega B(v-v_*,\omega)\zeta(v-((v-v_*)\cdot\omega)\omega)d\omega, \quad (20)$$

which depends neither on t nor on x and belongs to $L^\infty(\mathbb{R}^3\times\mathbb{R}^3)$. As a matter of fact, we have

$$\|Z\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \leqslant 4\pi\|B\|_{L^\infty(\mathbb{R}^3\times S^2)}\|\zeta\|_{L^\infty(\mathbb{R}^3)}.$$

Let us take a mollifying sequence $(\psi_\varepsilon)_{\varepsilon>0}$ of functions of v. Thanks to (19), we get

$$\int_{\mathbb{R}^3} Q^+(f,f)(v)\zeta(v)dv$$
$$= \int_{v,v_*} f(v)f(v_*)\left(\int_{w,w_*} Z(w,w_*)\psi_\varepsilon(v-w)\psi_\varepsilon(v_*-w_*)dw_*dw\right)dv_*dv$$
$$+ \int_{v,v_*} f(v)f(v_*)\left[\int_{w,w_*} (Z(v,v_*)-Z(w,w_*))\right.$$
$$\left.\psi_\varepsilon(v-w)\psi_\varepsilon(v_*-w_*)dw_*dw\right]dv_*dv. \quad (21)$$

We name I_1 (respectively I_2) the first (respectively second) integral in (21). They are functions of $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^3$.

• *Estimate on* I_1. The integral I_1 can be rewritten as

$$I_1 = \int_{w,w_*} Z(w,w_*)\rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x)\rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x)dw_*dw,$$

where $\rho_\psi(f)$ denotes the average quantity of f with respect to ψ.

Let us study the norm $\|\tau_h I_1 - I_1\|_{L^2([0,T]\times\mathbb{R}^3)}$, for $h \in \mathbb{R}^3$, with the notation $\tau_h g(x) = g(x+h)$.

The following equality holds

$$\int_{t,x} |\tau_h I_1 - I_1|^2 dxdt$$
$$= \int_{t,x}\left|\int_{w,w_*} Z(w,w_*)[\rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x+h)\rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x+h)\right.$$
$$\left. - \rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x)\rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x)]dw_*dw\right|^2 dxdt.$$

We immediately get

$$\begin{aligned}
&\int_{t,x} |\tau_h I_1 - I_1|^2 dxdt \\
&\quad \leqslant C\|Z\|^2_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \\
&\qquad \times \int_{t,x} dtdx \bigg| \int_{w,w_*} |(\rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x+h) - \rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x))\rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x+h) \\
&\qquad\quad + \rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x)(\rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x+h) - \rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x))|dw_* dw\bigg|^2 \\
&\quad \leqslant C\|Z\|^2_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \\
&\qquad \times \bigg[\int_{t,x} dtdx \bigg| \int_{w,w_*} \\
&\qquad\quad |((\tau_h - Id)\rho_{\psi_\varepsilon(\cdot-w)}(f))(t,x)\tau_h\rho_{\psi_\varepsilon(\cdot-w_*)}(f)(t,x)|dw_* dw\bigg|^2 \\
&\qquad + \int_{t,x} dtdx \bigg| \int_{w,w_*} |((\tau_h - Id)\rho_{\psi_\varepsilon(\cdot-w_*)}(f))(t,x)\rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x)|dw_* dw\bigg|^2 \bigg].
\end{aligned}$$

In the previous inequality, the two terms can be similarly treated. For example, let us study the second one, which we name J.

$$\begin{aligned}
J &= \int_{t,x} \left(\int_w \rho_{\psi_\varepsilon(\cdot-w)}(f)(t,x)dw \right)^2 \left(\int_{w_*} \left|((\tau_h - \mathrm{Id})\rho_{\psi_\varepsilon(\cdot-w_*)}(f))(t,x)\right| dw_* \right)^2 dxdt \\
&\leqslant C_T \int_{t,x} \left(\int_{w_*} \left|((\tau_h - \mathrm{Id})\rho_{\psi_\varepsilon(\cdot-w_*)}(f))(t,x)\right| dw_* \right)^2 dxdt,
\end{aligned}$$

where C_T is the constant in (6). Let us choose $0 < \theta < \lambda < 1/2$. Using the notation T_λ as in Section 2.1, we have

$$\begin{aligned}
J &\leqslant C_T \left(\int_{w_*} e^{-\theta|w_*|^2} dw_* \right) \left(\int_{t,x,w_*} ((\tau_h - \mathrm{Id})\rho_{\psi_\varepsilon(\cdot-w_*)}(f))(t,x)^2 \mathrm{e}^{\theta|w_*|^2} dw_* dxdt \right) \\
&\leqslant C_{T,\theta}|h| \int_{w_*} dw_* \mathrm{e}^{\theta|w_*|^2} \left\| \rho_{\psi_\varepsilon(\cdot-w_*)} T_\lambda \left(\frac{f}{T_\lambda} \right) \right\|^2_{L^2([0,T];H^{1/2}(\mathbb{R}^3))}.
\end{aligned}$$

Then, thanks to the averaging lemma (Lemma 1), we obtain

$$\begin{aligned}
J \leqslant C_{T,\theta,s}|h| &\int_{w_*} dw_* \mathrm{e}^{\theta|w_*|^2} \\
&\times \bigg[\int_{x,v_*} \frac{f_{\mathrm{in}}(x,v_*)^2}{T_\lambda(v_*)^2} \psi_\varepsilon(v_* - w_*)^2 T_\lambda(v_*)^2 (1+|v_*|^2)^s dv_* dx \\
&\quad + \int_{t,x,v_*} \left((\partial_t + v_*\cdot\nabla_x)\frac{f}{T_\lambda} \right)(t,x,v_*)^2 \\
&\quad \times \psi_\varepsilon(v_* - w_*)^2 T_\lambda(v_*)^2 (1+|v_*|^2)^s dv_* dxdt \bigg].
\end{aligned}$$

Let us take care of the term with f_{in} (the other one is treated in the same way thanks to (15)). We notice that, for any $w_* \in \mathscr{B}(v_*, \varepsilon)$,

$$e^{\theta|w_*|^2} \leqslant e^{2\theta|v_*|^2} e^{2\theta\varepsilon^2}.$$

We thus have

$$\begin{aligned}
&\int_{w_*} e^{\theta|w_*|^2} \int_{x,v_*} \frac{f_{\text{in}}(x, v_*)^2}{T_\lambda(v_*)^2} \psi_\varepsilon(v_* - w_*)^2 T_\lambda(v_*)^2 (1 + |v_*|^2)^s dv_* dx dw_* \\
&\leqslant \int_{x,v_*} \frac{f_{\text{in}}(x, v_*)^2}{T_\lambda(v_*)^2} T_\lambda(v_*)^2 (1 + |v_*|^2)^s \left(\int_{w_* \in \mathscr{B}(v_*,\varepsilon)} e^{\theta|w_*|^2} \psi_\varepsilon(v_* - w_*)^2 dw_* \right) dv_* dx \\
&\leqslant \int_{x,v_*} \frac{f_{\text{in}}(x, v_*)^2}{T_\lambda(v_*)^2} T_{\lambda-\theta}(v_*)^2 (1 + |v_*|^2)^s e^{2\theta\varepsilon^2} \|\psi_\varepsilon\|_{L^2}^2 dv_* dx \\
&\leqslant \frac{(e^\theta M_{\lambda-\theta,s})^2}{\varepsilon^3} \left\| \frac{f_{\text{in}}}{T_\lambda} \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2,
\end{aligned}$$

for $0 < \varepsilon < 1$. Note that we have used that $\|\psi_\varepsilon\|_{L^2}^2 \leqslant \varepsilon^{-3}$ and $M_{\lambda-\theta,s}$ is defined by (10). Hence we get, thanks to (11),

$$J \leqslant \frac{C_{\lambda,\theta,s}}{\varepsilon^3},$$

and finally

$$\|\tau_h I_1 - I_1\|_{L^2([0,T]\times\mathbb{R}^3)}^2 \leqslant C_{\lambda,\theta,s} \|Z\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}^2 \varepsilon^{-3} |h|. \tag{22}$$

• *Estimate on* I_2. Let us now study the norm $\|\tau_h I_2 - I_2\|_{L^2([0,T]\times\mathbb{R}^3)}$, with the same notation τ_h as before. We successively have

$$\begin{aligned}
&\|\tau_h I_2 - I_2\|_{L^2([0,T]\times\mathbb{R}^3)}^2 \\
&\quad = \int_{t,x} dt dx \left| \int_{v,v_*} (f(t, x+h, v) f(t, x+h, v_*) - f(t, x, v) f(t, x, v_*)) \right. \\
&\qquad \left. \times \left(\int_{w,w_*} (Z(v, v_*) - Z(w, w_*)) \psi_\varepsilon(v - w) \psi_\varepsilon(v_* - w_*) dw_* dw \right) dv_* dv \right|^2 \\
&\quad \leqslant C \|Z\|_{W^{1,\infty}(\mathbb{R}^3\times\mathbb{R}^3)}^2 \left(\int_w |w| \psi_\varepsilon(w) dw \right)^2 \\
&\qquad \int_{t,x} dt dx \left(\int_{v,v_*} (\tau_h + \text{Id})(|f(t, x, v) f(t, x, v_*)|) dv_* dv \right)^2.
\end{aligned}$$

Thanks to (6), the second integral term is bounded by a constant $K_T \geqslant 0$. Hence there exists a constant $C_T \geqslant 0$ such that

$$\|\tau_h I_2 - I_2\|_{L^2([0,T]\times\mathbb{R}^3)}^2 \leqslant C_T \|Z\|_{W^{1,\infty}(\mathbb{R}^3\times\mathbb{R}^3)}^2 \varepsilon^2. \tag{23}$$

• *Estimate on the average quantity.* Under Assumption 1, the following inequalities clearly hold

$$\|Z\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \leqslant C\|\zeta\|_{L^\infty(\mathbb{R}^3)}, \tag{24}$$

$$\|Z\|_{W^{1,\infty}(\mathbb{R}^3\times\mathbb{R}^3)} \leqslant C\|\zeta\|_{W^{1,\infty}(\mathbb{R}^3)}, \tag{25}$$

where C is a constant depending on T and $\|B\|_{L^\infty(S^2;W^{1,\infty}(\mathbb{R}^3))}$. Consequently, using (21)–(25), we get, for $h \in \mathbb{R}^3$,

$$\int_{t,x}\left|\int_v [Q^+(f,f)(t,x+h,v) - Q^+(f,f)(t,x,v)]\zeta(v)dv\right|^2 dxdt \leqslant K_T\|\zeta\|^2_{W^{1,\infty}(\mathbb{R}^3)}(\varepsilon^2+\varepsilon^{-3}|h|),$$

that gives (17), if we choose $\varepsilon \simeq |h|^{1/5}$. □

2.2.2. *Study of* $Q^+(f,f)$. Let us once again choose a mollifying sequence $(\psi_\delta)_{\delta>0}$ of functions of v. We obviously have, for all $\delta > 0$,

$$Q^+(f,f) = (Q^+(f,f) - \psi_\delta *_v Q^+(f,f)) + \psi_\delta *_v Q^+(f,f).$$

Note that, thanks to (17), for any $h \in \mathbb{R}^3$ and $\delta > 0$,

$$\begin{aligned}\int_{t,x}\left|\int_w [Q^+(f,f)(t,x+h,w) - Q^+(f,f)(t,x,w)]\psi_\delta(v-w)dw\right|^2 dxdt &\leqslant C\|\psi_\delta(v-\cdot)\|^2_{W^{1,\infty}(\mathbb{R}^3)}|h|^{2/5} \\ &\leqslant C\delta^{-8}|h|^{2/5}.\end{aligned} \tag{26}$$

On the other hand, we know that thanks to the regularizing properties of Q^+ (cf. [3]), for all $R > 0$,

$$\|Q^+(f,f) - \psi_\delta *_v Q^+(f,f)\|_{L^2([0,T]\times\mathscr{B}_R\times\mathscr{B}_R)} \leqslant C\delta. \tag{27}$$

Using again the translations τ_h ($h \in \mathbb{R}^3$) in the variable x, and assuming that $|h| \leqslant 1$, we successively have

$$\begin{aligned}&\int_{(t,x,v)\in[0,T]\times\mathscr{B}_R\times\mathscr{B}_R} |[\tau_h Q^+(f,f) - Q^+(f,f)]|^2 dvdxdt \\ &\leqslant C\Bigg[\int_{t,x,v} |(Q^+(f,f) - \psi_\delta *_v Q^+(f,f))(t,x,v)|^2 dvdxdt \\ &\quad + \int_{t,x,v} |(\tau_h(\psi_\delta *_v Q^+(f,f)) - \psi_\delta *_v Q^+(f,f))(t,x,v)|^2 dvdxdt\Bigg] \\ &\leqslant C_R(\delta^2 + |h|^{2/5}\delta^{-8}),\end{aligned} \tag{28}$$

thanks to (26)–(27). Then for a good choice of $\delta(\simeq |h|^{1/25})$ in (28), we find the following estimate

$$\left(\int_0^T \int_{(\mathscr{B}_R)_x}\int_{(\mathscr{B}_R)_v} |\tau_h Q^+(f,f) - Q^+(f,f)|^2 dvdxdt\right)^{1/2} \leqslant C|h|^{1/25},$$

that ensures that $Q^+(f,f) \in L^2([0,T]\times(\mathscr{B}_R)_v; H^\alpha((\mathscr{B}_R)_x))$, for any $0<\alpha<1/25$.

Besides, we already know that $Q^+(f,f) \in L^2([0,T]\times((\mathscr{B}_R)_x); H^1((\mathscr{B}_R)_v))$.

Then, by a standard interpolation result, we can state that for all $\alpha \in]0, 1/25[$,

$$Q^+(f, f) \in L^2([0, T]; H^\alpha_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)). \tag{29}$$

Let us now justify (7). Note that, at least formally, (7) is easily rewritten as

$$\begin{aligned} f^\#(t, x, v) = \exp\Big(-\int_0^t Lf^\#(\sigma, x, v)d\sigma\Big) \\ \times \Big(f_{\mathrm{in}}(x, v) + \int_0^t \Big[Q^+(f, f)^\#(s, x, v) \\ \exp\Big(\int_0^s Lf^\#(\sigma, x, v)d\sigma\Big)\Big]ds\Big), \end{aligned} \tag{30}$$

where we set

$$h^\#(t, x, v) = h(t, x + vt, v).$$

In (30), we name E_1 the first exponential term in the previous product, and E_2 the whole integral term with Q^+.

We first notice that since Lf has the same $H^{1/2}$ smoothness in both variables x and v, it is clear that $Lf^\# \in L^2([0, T]; H^{1/2}_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^3))$. In the same way, $Q^+(f, f)^\#$ lies in $L^2([0, T]; H^\alpha_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^3))$ for all $\alpha \in]0, 1/25[$.

Besides, we have, for any $h \in L^2([0, T] : H^\alpha(\mathscr{B}_R \times \mathscr{B}_R)), R > 0, \alpha \in]0, 1/25[$,

$$\int_0^T \Big\|\int_0^t h(\sigma)d\sigma\Big\|^2_{L^2([0,T];H^\alpha(\mathscr{B}_R\times\mathscr{B}_R))} dt \leqslant T^2 \|h\|^2_{L^2([0,T];H^\alpha(\mathscr{B}_R\times\mathscr{B}_R))}. \tag{31}$$

Using (31) with $h = Lf^\#$, we immediately obtain that for any $t \in [0, T]$,

$$\int_0^t Lf^\#(\sigma)d\sigma \in L^2([0, T]; H^{1/2}_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)).$$

Its time derivative is exactly $Lf^\#$ which also lies in $L^2([0, T]; H^{1/2}_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^3))$. Consequently, we have proven that

$$\int_0^t Lf^\#(\sigma)d\sigma \in H^1_{\mathrm{loc}}(\mathbb{R}_+; H^{1/2}_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)) \subset H^{1/2}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3).$$

Since $x \mapsto \mathrm{e}^x$ is a bounded C^∞ function on $[-T \max Lf, T \max Lf]$, we can conclude that E_1 belongs to $H^{1/2}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$.

Then we notice that E_2 is the integral of the product of two terms which are both in $A = L^2([0, T]; H^\alpha_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ for all $\alpha \in]0, 1/25[$. The previous vector space A is in fact an algebra, so E_2 is the integral of a term that lies in A. Using once again (31), we find that E_2 belongs to $H^\alpha_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ for all $\alpha \in]0, 1/25[$.

Since E_1 and E_2 are obviously in A, $\tilde{\Gamma}_1 = E_1$ and $\tilde{\Gamma}_2 = E_1 \times E_2$ lie in A too, so that both quantities belong to $H^\alpha_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ for all $\alpha \in]0, 1/25[$.

And then, from (30) back to the standard formulation, we obtain (7) with the required smoothness on both Γ_1 and Γ_2, because $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ have the same smoothness in the three variables t, x and v.

Remark. In this proof, we have only considered cross sections B lying in $L^\infty(S^2; W^{1,\infty}(\mathbb{R}^3))$, which covers the case of Maxwellian molecules and regularized soft potentials (with angular cutoff). We briefly explain here how to transform the proof to get a result in the case of hard potentials (with angular cutoff) or hard spheres.

Note first that the solutions of [19], which have an exponential decay in both x and v, are replaced by solutions with an algebraic decay in at least one of the variables, like those of [2] or [20]. Then, throughout the proof, if the algebraic decay concerns the variable v, the function T_λ is replaced by $S_\lambda(v_*) = (1+|v_*|^2)^{-\frac{\lambda}{2}}$. The estimate on $\frac{Q^+(f,f)}{S_\lambda}$ becomes then more intricate (but is still valid).

Then, one has to replace the estimates in $W^{1,\infty}$ by estimates in $C^{0,\beta}$ (except for hard spheres) because the cross sections of hard potentials are only Hölder continuous, not Lipschitz continuous.

Finally, the L^∞ estimates must be replaced by weighted L^∞ estimates because the cross sections of hard potentials (and hard spheres) tend to infinity when $|v-v_*|$ tends to infinity. At the end, the exponent in the Sobolev space is less than 1/25 (and may be very small for hard potentials close to Maxwellian molecules).

3. Smoothness Estimates

We give in this section the proof of theorem 3. Thanks to our assumption on B and to the L^∞-estimate (6) of theorem 1, we can directly estimate the derivatives of f using a Gronwall type lemma, namely

Lemma 2. *We suppose that, for some $T>0$, $(U_t)_{t\in[0,T]}$ is a family of uniformly bounded linear operators from $L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v)^P$ to $L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v)^P$, for $P\in\mathbb{N}$, and $S\in L^\infty([0,T]_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v)^P$. We also assume that $g\in L^\infty([0,T]_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v)^P\cap C([0,T]_t;L^\infty_{w^*}(\mathbb{R}^3_x\times\mathbb{R}^3_v)^P)$ satisfies the equation*

$$\partial_t g + v\cdot\nabla_x g = U_t g + S, \tag{32}$$

in the sense of distributions. Then there exists a constant C_T only depending on T, $\sup_{t\in[0,T]}|||U_t|||_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)^P}$ and $\|S\|_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)^P}$, such that

$$\|g\|_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)^P} \leqslant C_T(1+\|g(0)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)^P}).$$

Proof. We use, for any $h\in L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)^P$, the standard notation

$$h^\#(t,x,v) = h(t,x+vt,v). \tag{33}$$

Equation (32) can be written under the Duhamel's form

$$g^\#(t,x,v) = g(0,x,v) + \int_0^t S^\#(s,x,v)ds + \int_0^t (U_s(g))^\#(s,x,v)ds.$$

Taking L^∞ norms, we get

$$\|g(t)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)^P} \leqslant \|g(0)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)^P} + T\|S\|_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)^P}$$
$$+ \sup_{\sigma\in[0,T]} |||U_\sigma|||_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)^P}\int_0^t \|g(s)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)^P}ds.$$

Then lemma 2 is an immediate consequence of Gronwall's lemma. □

3.1. Derivatives with respect to x. We first study the derivatives of f with respect to x. Using the fact that Q only acts on the variable v, we can give an intermediate result, in which the smoothness of f_{in} with respect to v is not required. We recall that a very similar result, in a slightly different context, is given in [20].

Proposition 6. *Let B be a cross section satisfying Assumption 1 and such that $B \in L^1(\mathbb{R}^3 \times S^2)$ and f_{in} be an initial datum such that (5) holds. If, moreover, $\nabla_x^p f_{\text{in}} \in L^\infty(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ for $p = 1, \ldots, k$, then the solution f to (1) given by Theorem 1 is such that $\nabla_x^p f \in L^\infty([0, T]_t \times \mathbb{R}_x^3 \times \mathbb{R}_v^3)$* for *$p = 1, \ldots, k$ and $T > 0$.*

Proof. We introduce the quantities

$$\tau_h^1 f(t, x, v) = f(t; x_1 + h, x_2, x_3; v), \tag{34}$$

$$R_h^1 f = \frac{\tau_h^1 f - f}{h}, \quad h \neq 0, \tag{35}$$

and, in the same way, $\tau_h^2, \tau_h^3, R_h^2$ and R_h^3.

Applying R_h^i to (1), we get

$$\begin{aligned}\partial_t(R_h^i f) + v \cdot \nabla_x(R_h^i f) + (R_h^i f)(\tau_h^i Lf) + f(LR_h^i f) \\ = Q^+(R_h^i f, \tau_h^i f) + Q^+(f, R_h^i f).\end{aligned} \tag{36}$$

We now use Lemma 2 with $S = 0, g = R_h^i f$ and

$$U_t = Q^+(\cdot, \tau_h^i f(t)) + Q^+(f(t), \cdot) - \cdot(\tau_h^i Lf(t)) - f(t)(L\cdot). \tag{37}$$

Since $f \in L^\infty([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$, it is quite easy to see that each term of U_t is a bounded operator of $L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ the norm of which is smaller than $\|A\|_{L^1(\mathbb{R}^3)} \|f\|_{L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)}$. We just show the computation for the first term.

$$\begin{aligned}&\|Q^+(g, \tau_h^i f)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \\ &\quad = \sup_{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3} \left| \int_{v_* \in \mathbb{R}^3} \int_{\omega \in S^2} g(x, v') \tau_h^i f(x, v'_*) B(v - v_*, \omega) d\omega dv_* \right|,\end{aligned}$$

and then it is clear that

$$\|Q^+(g, \tau_h^i f)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \leqslant \|g\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \|f\|_{L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} \|A\|_{L^1(\mathbb{R}^3)}.$$

Thanks to Lemma 2, we obtain for any $T > 0$ a constant C_T independent on h such that

$$\|R_h^i f\|_{L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} \leqslant C_T(1 + \|R_h^i f_{\text{in}}\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}).$$

Using now the fact that $\nabla_x f_{\text{in}} \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, we see that, for all $i \in \{1, 2, 3\}, R_h^i$ is uniformly bounded with respect to h in $L^\infty([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$, so that

$$\nabla_x f \in L^\infty([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$$

for any $T > 0$.

The equation satisfied by $\partial_{x_i} f$ is the same as (36), but with $h = 0$, namely

$$\begin{aligned}\partial_t(\partial_{x_i} f) + v \cdot \nabla_x(\partial_{x_i} f) = Q^+(\partial_{x_i} f, f) + Q^+(f, \partial_{x_i} f) \\ - (\partial_{x_i} f)Lf - f(L\partial_{x_i} f).\end{aligned} \tag{38}$$

Applying R_h^j to this equation, we get

$$\begin{aligned}
&\partial_t(R_h^j\partial_{x_i}f) + v\cdot\nabla_x(R_h^j\partial_{x_i}f)\\
&\quad = Q^+(R_h^j\partial_{x_i}f, \tau_h^j f) + Q^+(\partial_{x_i}f, R_h^j f)\\
&\qquad + Q^+(R_h^j f, \tau_h^j\partial_{x_i}f) + Q^+(f, R_h^j\partial_{x_i}f)\\
&\qquad - (R_h^j\partial_{x_i}f)(\tau_h^j Lf) - (\partial_{x_i}f)(LR_h^j f) - (R_h^j f)(\tau_h^j L\partial_{x_i}f) - f(LR_h^j\partial_{x_i}f).
\end{aligned}$$

At this level, we use Lemma 2 with

$$\begin{aligned}
S = {} & Q^+(\partial_{x_i}f, R_h^j f) + Q^+(R_h^j f, \tau_h^j\partial_{x_i}f)\\
& - (\partial_{x_i}f)(LR_h^j f) - (R_h^j f)(\tau_h^j L\partial_{x_i}f)
\end{aligned}$$

and U_t is still given by (37).

Using the fact that $f, \nabla_x f \in L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$, it is easy to see that S is bounded in $L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$ uniformly with respect to h.

Thanks to lemma 2 and the assumption that $\nabla_x\nabla_x f_{\text{in}} \in L^\infty(\mathbb{R}^3\times\mathbb{R}^3)$, we see that $R_h^j\partial_{x_i}f$ is bounded (uniformly in h) in $L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$ for all $T>0$, $i,j\in\{1,2,3\}$, so that finally $\nabla_x\nabla_x f \in L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$.

The derivatives of higher order of f w. r. t. x are then obtained by a simple induction, in which only the source term is changed. □

3.2. Derivatives with respect to v. We now turn to the derivatives with respect to v. Since the proof gets quite intricate, we shall directly use derivatives in the sense of distributions, instead of precisely writing down quantities like $R_h f$. Note however that a complete justification of our computations would require the use of such quantities.

We first write down the equation satisfied by $\partial_{v_i}f$ for $i\in\{1,2,3\}$:

$$\partial_t(\partial_{v_i}f) + v\cdot\nabla_x(\partial_{v_i}f) = -\partial_{x_i}f - (\partial_{v_i}f)(Lf) - f(L\partial_{v_i}f) + \partial_{v_i}Q^+(f,f). \quad (39)$$

Using [3], note that we could immediately deduce from (39) that $\nabla_v f$ lies in $L^\infty_{\text{loc}}([0,T]_t\times\mathbb{R}^3_x; L^2_{\text{loc}}(\mathbb{R}^3_v))$, under a slightly more stringent assumption on B.

However, we rather use a more elementary method, which directly gives estimates in the L^∞ setting.

Thanks to our study of the derivatives of f with respect to x, we shall be able to put the term $\nabla_x f$ in the source, and conclude with Lemma 2. Note that it was important to first treat the derivatives with respect to x.

We now study $\partial_{v_i}Q^+(f,f)$. Let us denote, for a given $(t,x)\in\mathbb{R}_+\times\mathbb{R}^3$, the functions

$$\begin{aligned}
F : {} & \mathbb{R}^3\times\mathbb{R}^3\times S^2 \to \mathbb{R}\\
& (Z,z,\omega) \mapsto f(Z + ((z-Z)\cdot\omega)\omega)\\
& \qquad\qquad f(z - ((z-Z)\cdot\omega)\omega),
\end{aligned}$$

and

$$\begin{aligned}
G : {} & \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}\\
& (Z,z) \mapsto \int_{z_*\in\mathbb{R}^3}\int_{\omega\in S^2} F(Z,z_*,\omega)B(z-z_*,\omega)\,d\omega\,dz_*.
\end{aligned}$$

Note that

$$G(Z,z) = \int_{\omega \in S^2} (F(Z,\cdot,\omega) *_z B(\cdot,\omega))(z)d\omega,$$

and that

$$G(v,v) = Q^+(f,f)(v), \quad v \in \mathbb{R}^3.$$

We have, for $i \in \{1,2,3\}$,

$$\begin{aligned} \partial_{v_i} Q^+(f,f)(v) &= \frac{\partial G}{\partial Z_i}(v,v) + \frac{\partial G}{\partial z_i}(v,v) \\ &= \int_{\omega \in S^2} \left[\left(\frac{\partial F}{\partial Z_i}(v,\cdot,\omega) + \frac{\partial F}{\partial z_i}(v,\cdot,\omega) \right) * B(\cdot,\omega) \right](v)d\omega. \end{aligned} \tag{40}$$

With obvious notations, it is easy to compute

$$\frac{\partial F}{\partial Z_i}(Z,z,\omega) = [(e_i - \omega_i\omega)\cdot \nabla_v f(Z')]f(z') + [\omega_i\omega \cdot \nabla_v f(z')]f(Z') \tag{41}$$

and, in the same way,

$$\frac{\partial F}{\partial z_i}(Z,z,\omega) = [\omega_i\omega \cdot \nabla_v f(Z')]f(z') + [(e_i - \omega_i\omega)\cdot \nabla_v f(z')]f(Z'). \tag{42}$$

Taking (40)–(42) into account, it is clear that $\partial_{v_i} Q^+(f,f) \equiv H_t(\nabla_v f)$ is linearly depending on $\nabla_v f$, and that

$$\left\| \frac{\partial F}{\partial z_i} + \frac{\partial F}{\partial Z_i} \right\|_{L^\infty} \leqslant 2\|f\|_{L^\infty} \|\nabla_v f\|_{L^\infty}. \tag{43}$$

Using the L^∞-estimate (6) on f, we get, for some constant $C_T > 0$,

$$\|\nabla_v Q^+(f,f)(t)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \leqslant C_T \|\nabla_v f(t)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}, \quad 0 \leqslant t < T. \tag{44}$$

We are now in a position to apply Lemma 2, with

$$S = -\nabla_x f$$

and

$$U_t = H_t(\cdot) - \cdot(Lf) - fL(\cdot).$$

We get at the end that, for any $T > 0$, there exists $C_T > 0$ such that

$$\|\nabla_v f\|_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \leqslant C_T\big(1 + \|\nabla_v f_{\text{in}}\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\big).$$

Note that the constant C_T in (44) depends in fact on $\|\nabla_x f_{\text{in}}\|_{L^\infty}$, so that we really need that $f_{\text{in}} \in W^{1,\infty}(\mathbb{R}^3 \times \mathbb{R}^3)$ to conclude that

$$\nabla_v f \in L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3).$$

In order to study the second derivatives of f with respect to v, we are led to consider the derivatives $\partial^2_{x_i v_j} f$ and $\partial^2_{v_i v_j}(Q^+(f,f))$. More precisely, we first prove that $\nabla_x \nabla_v f \in L^\infty$, and then we can conclude that $\nabla_v \nabla_v f \in L^\infty$.

We recall that, for a given $i \in \{1,2,3\}, \partial_{x_i} f$ satisfies equation (38). Consequently, the derivative $\partial^2_{x_i v_j} f$, $j \in \{1,2,3\}$, verifies

$$\begin{aligned} \partial_t(\partial^2_{x_i v_j} f) + v \cdot \nabla_x(\partial^2_{x_i v_j} f) = & - (\partial^2_{x_i v_j} f)(Lf) - f(L\partial^2_{x_i v_j} f) \\ & + \partial_{v_j}(Q^+(\partial_{x_i} f, f) + Q^+(f, \partial_{x_i} f)) \\ & - (\partial_{x_i} f)(L\partial_{v_j} f) - (\partial_{v_j} f)(L\partial_{x_i} f) - \partial^2_{x_i x_j} f. \end{aligned} \tag{45}$$

We want to apply Lemma 2. It is clear that the last three terms in (45) lie in $L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$. In the same way as in the study of $\nabla_v Q^+(f,f)(t)$, we can prove that both $\partial_{v_j} Q^+(\partial_{x_i} f, f)(t)$ and $\partial_{v_j} Q^+(f, \partial_{x_i} f)(t)$ are linearly depending on $\nabla_v(\partial_{x_i} f)$ and that

$$\left.\begin{array}{l} \|\nabla_v Q^+(\partial_{x_i} f, f)(t)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \\ \|\nabla_v Q^+(f, \partial_{x_i} f)(t)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \end{array}\right\} \leqslant K_T \|\nabla_v(\partial_{x_i} f)(t)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)},$$

for some constant $K_T > 0$.

Then, using Lemma 2, we get

$$\nabla_v(\partial_{x_i} f) \in L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$$

for any $T > 0$.

Let us study the second derivatives of $Q^+(f,f)$ with respect to v. From (40), we immediately compute that

$$\begin{aligned} & \partial^2_{v_i v_j} Q^+(f,f)(v) \\ & \quad = \int_{\omega \in S^2} [(\partial^2_{Z_i Z_j} F + \partial^2_{Z_i z_j} F + \partial^2_{z_i Z_j} F + \partial^2_{z_i z_j} F)(v, \cdot, \omega) * B(\cdot, \omega)](v) d\omega. \end{aligned}$$

It is then clear that $\nabla_v \nabla_v Q^+(f,f)(t) \equiv I_t(\nabla_v \nabla_v f)$ linearly depends on $\nabla_v \nabla_v f$ and that, for any $T > 0$, there exists a constant K_T depending on $T, \|f\|_{L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)}$ and $\|\nabla_v f\|_{L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)}$, such that

$$\|\nabla_v \nabla_v Q^+(f,f)(t)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \leqslant K_T \|\nabla_v \nabla_v f(t)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}.$$

We are now able to prove that the derivative $\nabla_v \nabla_v f$ lies in $L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ for any $T > 0$. Let us write down the equation satisfied by $\partial^2_{v_i v_j} f$. For $i, j \in \{1,2,3\}$, we have

$$\begin{aligned} & \partial_t(\partial^2_{v_i v_j}) + v \cdot \nabla_x(\partial^2_{v_i v_j}) \\ & \quad = -[\partial^2_{x_i v_j} f + \partial^2_{v_i x_j} f + (\partial_{v_j} f)(L\partial_{v_i} f) + (\partial_{v_i} f)(L\partial_{v_j} f)] \\ & \qquad - (\partial^2_{v_i v_j})(Lf) - f(L\partial^2_{v_i v_j}) + \partial^2_{v_i v_j} Q^+(f,f). \end{aligned} \tag{46}$$

We apply Lemma 2 with $P = 6$, where S is the vector whose coordinates are like the term in brackets in (46), and $U_t = - \cdot (Lf) - fL(\cdot) + I_t(\cdot)$. Then, for any $T > 0$, we find a constant $C_T > 0$ such that

$$\|\nabla_v \nabla_v f\|_{L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} \leqslant C_T (1 + \|\nabla_v \nabla_v f_{\text{in}}\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}).$$

That ends the study of the second derivatives of f with respect to x, v.

In order to study the smoothness of the derivatives of p-th order, $p \geqslant 3$, of f with respect to x, v, we use an induction on p. Once we know that all derivatives of f with respect to x, v of order $\leqslant p-1$ is bounded, we study, for any $i_1, \cdots, i_p \in \{1,2,3\}$, $\dfrac{\partial^p f}{\partial x_{i_1} \cdots \partial x_{i_{p-1}} \partial v_{i_p}}$, then $\dfrac{\partial^p f}{\partial x_{i_1} \cdots \partial x_{i_{p-2}} \partial v_{i_{p-1}} \partial v_{i_p}}, \ldots$, up to $\dfrac{\partial^p f}{\partial v_{i_1} \cdots \partial v_{i_p}}$ (in this order).

Note that we do need that $f_{\text{in}} \in W^{p,\infty}$ of both x and v variables to conclude that the derivatives of p-th order with respect to v only lie in $L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ for all $T > 0$.

3.3. Derivatives with respect to t. As we did in Subsection 3.2, we use derivatives in the sense of distributions.

From (1)–(2), we immediately obtain that

$$\partial_t f = -v \cdot \nabla_x f + Q^+(f,f) - f\, Lf. \tag{47}$$

Using (6), it is clear that

$$\partial_t f \in L^\infty_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3). \tag{48}$$

We next study, for a given $i \in \{1,2,3\}$, the term $\partial^2_{tx_i} f$. In fact, we know that $\partial_{x_i} f$ satisfies equation (38), which similarly implies that

$$\partial^2_{tx_i} f \in L^\infty_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3). \tag{49}$$

Then we differentiate (47) with respect to t and get

$$\partial^2_{tt} f = -v \cdot \nabla_x(\partial_t f) + Q^+(f, \partial_t f) + Q^+(\partial_t f, f) - (\partial_f f)(Lf) - f(L\partial_t f).$$

Using (6), (48) and (49), we obtain that

$$\partial^2_{tt} f \in L^\infty_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3).$$

Besides, from (39) and the estimates on $f, \nabla_x f, \nabla_v f, \nabla_x \nabla_v f, \nabla_v Q^+(f,f)$, it is clear that

$$\partial^2_{tv_i} f \in L^\infty_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3).$$

Then we conclude by induction in the same way as in Subsection 3.2, by first studying the mixed derivatives with respect to x and t, and next finding the smoothness of the mixed derivatives with respect to t, x and v.

Acknowledgement: This work falls within the framework of the European TMR contract "Asymptotic Methods in Kinetic Theory", # ERB FMRX CT97 0157.

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