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# Discrepancy Estimates on the Sphere

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**Abstract.** For measures on the unit sphere in $\mathbb{R}^d$, $d \geqslant 3$, we derive discrepancy estimates in terms of the quality of corresponding quadrature formulas and in terms of bounds for potential differences.

## 1. Introduction and Statement of the Results

Distributing points on the unit sphere in $\mathbb{R}^3$ has attracted the interest of many mathematicians (see [6] and [15] for an interesting overview). Although applications go far beyond the construction of quadrature rules on the sphere, this paper is devoted in part to a relationship between the equidistribution of points and the quality of a corresponding quadrature formula. The other part, namely, discrepancy estimates in terms of bounds for Newtonian potentials, has its roots in the complex plane. Starting from quantitative equidistribution results by Erdös and Turán [10] on the unit disk and unit interval for the zeros of polynomials, the second author and Grothmann [7] were able to give a potential-theoretic interpretation, which in the sequel led to various discrepancy estimates for the distribution of zeros of polynomials (see, e.g., [21], [8], [1]).

We place ourselves in $\mathbb{R}^d$, $d \geqslant 3$, and denote by $\mu$ the surface measure on the unit sphere $S = \{x \in \mathbb{R}^d : |x| = 1\}$, normalized to total mass 1. Here, $|\cdot|$ denotes the euclidean norm. We are interested in the question, how well one can approximate $\mu$ by means of certain masses on $S$. More specifically, suppose $\nu$ is a unit measure on $S$. For reasonable classes $\mathscr{B}$ of test sets $B \subset S$ we focus on the discrepancy

$$\sup_{B \in \mathscr{B}} |\mu(B) - \nu(B)|.$$

Our choice of $\mathscr{B}$ is based on the following definition, introduced by Sjögren [20]. A measurable set $B \subset S$ is said to be $K$-regular, if for

$$\partial_\delta^S B := \{y \in S \,|\, \operatorname{dist}(y, B) \leqslant \delta,\ \operatorname{dist}(y, S \setminus B) \leqslant \delta\}$$

it holds that

$$\mu(\partial_\delta^S B) \leqslant K\delta \qquad (\delta > 0).$$

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Here, $\operatorname{dist}(x, A)$ denotes the euclidean distance of a point $x$ from a set $A$, where $\operatorname{dist}(x, \emptyset) := \infty$.

Note that for $\delta$ sufficiently small, $\partial_\delta^S B$ is the $\delta$-neighbourhood of the boundary of $B$ relative to $S$. It is not hard to see that so-called spherical caps, i.e., intersections of balls with the sphere, are $K_0$-regular with some constant $K_0 > 0$ depending only on the dimension $d$. In the case $d = 3$, Sjögren has shown that any rectifiable curve on $S$ having length $l$ is $(K_0 + l\,K_1)$-regular with absolute constants $K_0$, $K_1$. In this paper we will see that the classes $\mathscr{B}_K$ of $K$-regular sets in some sense form the "appropriate classes" of test sets when dealing with quadrature formulae or potential-theoretic estimates.

A relationship between discrepancy and the polynomial quality of the corresponding quadrature formula is given in the following theorem. To formulate the result we choose the integer $s = s(d)$ minimal so that $2s \geqslant d + 2$.

**Theorem 1.** *There exists a constant $C_0 > 0$ depending only on d such that for every unit measure $\nu$ on S and every K-regular set $B \subset S$ it holds that*

$$|\mu(B) - \nu(B)| \leqslant C_0 \inf_{m \in \mathbb{N}} \left( \frac{K}{m} + C(m, \nu, d) \right),$$

*where*

$$C(m, \nu, d) := \sup \left\{ \left| \int p\, d\mu - \int p\, d\nu \right| \ : \ \begin{array}{c} p \textit{ polynomial}, \\ \deg(p) \leqslant ms, |p| \leqslant 1 \text{ on } S \end{array} \right\}.$$

There are immediate consequences for so-called spherical designs: A spherical $(t, n)$-design is a set of different points $y_1, \ldots, y_n \in S$ such that the corresponding quadrature formula of Chebyshev type associating equal weight $1/n$ with each point is exact up to order $t$, i.e.,

$$\int p\, d\mu = \frac{1}{n} \sum_{i=1}^{n} p(y_i)$$

for all multivariate polynomials $p$ of total degree at most $t$.

**Corollary 1.** *There exists a constant $C_0 > 0$ such that for all spherical $(t, n)$-designs $y_1, \ldots, y_n$ and all K-regular sets $B \subset S$ it holds that*

$$\left| \mu(B) - \frac{\#\{i : y_i \in B\}}{n} \right| \leqslant C_0\, K\, \frac{1}{t}\,. \tag{1}$$

Note that in the case $d = 3$ and when considering only spherical caps, an estimate of the form (1) was derived by GRABNER and TICHY [12].

If $d = 3$, then from any $n$ points $y_1, \ldots, y_n \in S$ more than $\sqrt{n}$ points of these can be connected by a rectifiable arc $l$ on $S$ of length $L_0$ ($L_0$ being an absolute constant). In fact, any $N$ points in the unit square $[0, 1] \times [0, 1]$ can be connected by a path of length $\leqslant c_0 \sqrt{N}$ [17, Pr. 73]. Now, suppose w.l.o.g. that $n$ be the fourth power of an integer. Following an idea of V. Totik, we partition the unit square into $\sqrt{n}$ subsquares of equal side length $1/\sqrt[4]{n}$. If $n$ points are placed on the unit square, then there will be such a subsquare containing at least $N := \sqrt{n}$ points. By the previous reasoning, they can be joined by a path of length $\leqslant c_0 \sqrt{N}/\sqrt[4]{n} = c_0$. The assertion for points on the sphere now follows by means of a parametrization with respect to a finite number of local coordinate systems.

For this arc $l$ joining the points $y_1, \ldots, y_n$ we thus always have the bounds

$$\frac{1}{\sqrt{n}} \leqslant \left| \mu(l) - \frac{\#\{i : y_i \in l\}}{n} \right| \leqslant C_0 \left(K_0 + L_0 K_1\right) \frac{1}{t} \, . \tag{2}$$

This gives the previously known (cf. [13, p.108]) result of

**Corollary 2.** *Let $d = 3$. There exists an absolute constant $C_0$ such that for any $(t, n)$-design, $t \leqslant C_0 \sqrt{n}$.*

It was the work of KOREVAAR and MEYERS [13] that connected polynomial quality of quadrature formulas to the approximate Faraday cage effect, roughly stating that in $d = 3$, $y_1, \ldots, y_n \in S$ are good points for Chebychev type quadrature if and only if the difference

$$\int \frac{1}{|x-y|} \, d\mu(y) - \frac{1}{n} \sum_{j=1}^{n} \frac{1}{|x - y_j|}$$

of the corresponding Newtonian potentials is small inside the sphere. For a precise statement, see their equivalence principle (see also [14, p. 59]).

Some parts of this result can be generalized to arbitrary dimensions $d \geqslant 3$ and arbitrary (unit) measures $\nu$ on $S$: In the following, $||\cdot||_2$ denotes the $L^2(\mu)$-Norm on $S$ and

$$U^{\nu}(x) = \int \frac{1}{|x-y|^{d-2}} \, d\nu(y) \qquad (x \in \mathbb{R}^d)$$

the Newtonian potential of the unit measure $\nu$. Slightly modifying the technique of Korevaar and Meyers one can prove the following result.

**Theorem 2.** *There exists a constant $C_0 > 0$ depending only on d such that for all unit measures $\nu$ on S, all $0 < r < 1$ and all polynomials p of degree $\leqslant n$ it holds that*

$$\left| \int p(z) \, d\mu(z) - \int p(z) \, d\nu(z) \right| \leqslant C_0 \, ||p||_2 \, n \, r^{-n} \, ||U^{\nu}(r\zeta) - 1||_2 . \tag{3}$$

Combining Theorem 1 and 2 gives

**Corollary 3.** *There exists a constant $C_0$ depending only on d such that for every unit measure $\nu$ on S, every $1 > r > 0$ and each K-regular set $B \subset S$ it holds that*

$$|\mu(B) - \nu(B)| \leqslant C_0 \inf_{m \in \mathbb{N}} \left( \frac{K}{m} + m \, r^{-ms} \, \varepsilon(r) \right),$$

*where*

$$\varepsilon(r) := \sup_{|z|=r} |U^{\mu}(z) - U^{\nu}(z)|. \tag{4}$$

Note that $U^\mu(z) = 1$ for $|z| \leqslant 1$, since $\mu$ is the equilibrium measure for the sphere.

With the choice

$$m := \max\left(1\,,\,\left[\frac{\log\dfrac{1}{\min\,(\varepsilon(r),1/2)}}{2s\,\log\dfrac{1}{r}}\right]\right)$$

Corollary 3 implies

**Corollary 4.** *There exists a constant $C_0$ depending only on $d$ such that for every unit measure $\nu$ on $S$, every $0 < r < 1$ and every $K$-regular set $B \subset S$ it holds that*

$$|\mu(B) - \nu(B)| \leqslant C_0\,K \log\frac{2}{r}\left(\log\frac{1}{\min\,(\varepsilon(r),1/2)}\right)^{-1},$$

*where $\varepsilon(r)$ is defined in* (4).

Corollary 4 has its two-dimensional counterpart in a theorem of ERDÖS and TURÁN [9], where a similar estimate of discrepancy for the unit disk can be used to give a quantitative equidistribution result for zeros of polynomials. This was quite recently generalized by the first two authors to discrepancy estimates on analytic curves and Dini-smooth arcs [2], [3].

We now want to formulate a special case of Corollary 4, namely, when $\nu$ consists of point masses.

**Corollary 5.** *Let* $0 < r < 1$, $C$, $c$, $\alpha > 0$. *There exists a constant $C_0$ depending only on $d$, $r$, $C$, $c$ and $\alpha$ such that for all $n \in \mathbb{N}$, all $y_1^{(n)}, \ldots, y_n^{(n)} \in S$ with*

$$\sup_{|x|\leqslant r}\left|1 - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{|x - y_i^{(n)}|^{d-2}}\right| \leqslant C\exp\left(-c\,n^{\alpha}\right) \tag{5}$$

*and for all $K$-regular sets $B \subset S$ it holds that*

$$\left|\mu(B) - \frac{\#\{i : y_i^{(n)} \in B\}}{n}\right| \leqslant C_0\,K\,\frac{1}{n^{\alpha}}. \tag{6}$$

The formulation of the last statement is related to the following. In [13], KOREVAAR and MEYERS consider the case $d = 3$ and the problem of finding points with property (5). Roughly speaking, they are seeking to approximate the Faraday cage effect, i.e., constant potential on the conductor, exponentially well in the interior of the unit ball. For $n$ along some subsequence they construct points satisfying (5) with parameter $\alpha = 1/3$ and show, that point sequences for $\alpha > 1/2$ do not exist (this also follows from Corollary 5 and (2)). In addition, they conjecture that such a sequence of $n$-tupels of points can be found for $\alpha = 1/2$. In this case the discrepancy estimate (6) would at least up to a log-term be of the order $O(\log n/\sqrt{n})$ of the discrepancy estimate for Fekete-points as derived by the third author in his Ph.D.-thesis [11].

It is interesting to note that preliminary numerical results of Kuijlaars and Voogd (see [14, p. 50]) give some evidence that for Fekete points there is no very small potential difference in (5). Based on this observation we do not expect an immediate converse to Theorem 1.

We have also seen that when $d = 3$, the discrepancy, based on the concept of $K$-regular sets, between $\mu$ and a measure $\nu_n$ giving mass to at most $n$ points is bounded from below by $C_0/\sqrt{n}$, which then gives lower bounds for potentials and for quadrature formula estimates. However, in taking as test sets the class $\mathscr{B}_{\text{caps}}$ of spherical caps, one can do much better w.r.t. discrepancy. This was shown by Beck [5], who proved that a previously known lower bound due to Schmidt [19] is optimal up to a logarithmic term. Thus, no sharp estimates for $\mathscr{B}_{\text{caps}}$ by means of potential bounds can be expected. For an extensive list of references on the spherical cap discrepancy and for a discussion of other concepts in the Weyl-Hlawka discrepancy theory, see [6].

## 2. Proofs of the Results

Our proof of Theorem 1 is based on the following construction using a convolution technique of Nikol'skiĭ and Lizorkin [18]. In the sequel, by $C_0$ we denote positive constants, depending at most on $d$ and possibly different at different occurences, even if they appear in the same formula or estimate.

For the integer $s \geqslant d/2 + 1$ introduced earlier we consider the generalized Jackson kernel

$$D_m(t) := \left( \frac{\sin \dfrac{m+1}{2}\, t}{\sin \dfrac{t}{2}} \right)^{2s} \qquad (t \in \mathbb{R}),$$

which has a representation of the form

$$D_m(t) = P_m(\cos t) \qquad (t \in \mathscr{R}) \tag{7}$$

with a real, algebraic polynomial $P_m$ of degree $m\,s$. If $B \subset S$ is a measurable set, we define

$$T_m(x, B) := \frac{1}{\kappa_m} \int_B P_m(\langle x, y \rangle)\, d\mu(y) \qquad (x \in S), \tag{8}$$

where the constant $\kappa_m$ is chosen so that $T_m(\cdot, S) = 1$. $T_m(\cdot, B)$ is a polynomial and due to the representation (7), $\kappa_m\, T_m(\cdot, B)$ is the spherical convolution of the indicator function of $B$ with the Jackson kernel $D_m$,

$$T_m(x, B) = \frac{1}{\kappa_m} \int 1_B(y)\, D_m(\breve{xy})\, d\mu(y) \qquad (x \in S),$$

where $\breve{xy}$ denotes the angle between $x$ and $y$. In addition we note that for a certain constant $c > 0$ independent of $m$ the norming constant $\kappa_m$ satisfies the inequalities

$$\frac{1}{c}\, m^{2s-d+1} \leqslant \kappa_m \leqslant c\, m^{2s-d+1} \tag{9}$$

(see [18 p. 217]). Because of (8) we have that $T_m(\cdot\,,B)$ is the restriction to $S$ of a polynomial in $d$ variables of degree $\leqslant m\,s$ having the property

$$\int T_m(z,B)\,d\mu(z) = \mu(B).$$

We derive other specific properties of $T_m(\cdot\,,B)$.

**Lemma 1.** *There exists a constant $C_0 > 0$ depending only on $d$ such that for every measurable set $B \subset S$ and all $x \in S$ with* $\operatorname{dist}(x, S\backslash B) \geqslant 1/m$ *we have:*

$$T_m(x,B) \geqslant C_0.$$

*Proof.* Let $x \in S$. For $2 \geqslant r > 0$ we consider the spherical cap

$$S \cap B(x,r) = \{\, y \in S :\, |x-y| < r \,\} = \{\, y \in S :\, \langle x,y\rangle > 1 - \frac{r^2}{2} \}.$$

We have

$$\begin{aligned} T_m(x, S\cap B(x,r)) &= \frac{\beta_{d-1}}{\kappa_m}\int_{1-r^2/2}^{1} P_m(u)\,(1-u^2)^{(d-3)/2}\,du \\ &= \frac{\beta_{d-1}}{\kappa_m}\int_0^{\arccos(1-r^2/2)} D_m(t)\,(\sin t)^{d-2}\,dt. \end{aligned}$$

By definition of the Jackson kernel, for $\pi/(m+1) \geqslant v > 0$ the following estimate holds:

$$\begin{aligned} \int_0^v D_m(t)\,(\sin t)^{d-2}\,dt \geqslant \int_0^v \frac{\left(\sin\dfrac{m+1}{2}t\right)^{2s}}{(t/2)^{2s-d+2}}\,dt \\ = 2\,(m+1)^{2s-d+1}\int_0^{(m+1)v/2}\left(\frac{\sin y}{y}\right)^{2s} y^{d-2}\,dy \\ \geqslant C_0\,(m+1)^{2s-d+1}\left(\frac{m+1}{2}\,v\right)^{d-1}. \end{aligned}$$

If $\operatorname{dist}(x, S\backslash B) \geqslant 1/m$, then chosing $r = 1/m$ in the previous inequality gives

$$T_m(x,B) \geqslant T_m\left(x, S\cap B\left(x,\frac{1}{m}\right)\right) \geqslant \frac{C_0}{\kappa_m}\,(m+1)^{2s-d+1}$$

From this taking into account (9) the assertion follows. □

**Lemma 2.** *There exists a constant $C_0 > 0$ depending only on $d$ such that for all measurable sets $B \subset S$, all $k \in \mathbb{N}$ and all $x \in S$ with*

$$\frac{k}{m} \leqslant \operatorname{dist}(x,B),$$

*it holds that:*

$$T_m(x,B) \leqslant C_0\,\frac{1}{k^3}.$$

*Proof.* We again look at the spherical cap $S \cap B(x,r)$ considered above, where this time we choose $r := \operatorname{dist}(x,B)$. We have

$$T_m(x,B) \leqslant T_m(x, S\setminus B(x,r)) = \frac{\beta_{d-1}}{\kappa_m}\int_{\arccos(1-r^2/2)}^{\pi} D_m(t)\,(\sin t)^{d-2} dt.$$

Since for $0 < u < \pi$,

$$\int_u^{\pi} D_m(t)\,(\sin t)^{d-2}\,dt \leqslant \int_u^{\pi} \pi^{2s}\,\frac{1}{t^{2s}}\,t^{d-2} \leqslant C_0\,\pi^{2s}\,\frac{1}{u^{2s-d+1}},$$

it follows that

$$T_m(x,B) \leqslant C_0\,\frac{1}{\kappa_m}\,(\operatorname{dist}(x,B))^{-2s+d-1}.$$

Taking into account (9) and the choice of $s$, the asserted estimate follows. □

*Proof of Theorem 1.* For every measurable set $B \subset S$ and every $m \in \mathbb{N}$ it holds that $0 \leqslant T_m(\cdot,B) \leqslant T_m(\cdot,S) = 1$ and, consequently,

$$\left|\int T_m(z,B)\,d\mu(z) - \int T_m(z,B)\,d\nu(z)\right| \leqslant C(m,\nu,d). \tag{10}$$

Now, suppose $B \subset S$ is $K$-regular. For $k = 0,1,2,\ldots$ set

$$A_k := \left\{ y \in S : \frac{k}{m} < \operatorname{dist}(y,B) \leqslant \frac{k+1}{m} \right\}$$

and

$$\tilde{A}_k := \left\{ y \in S\,:\,\operatorname{dist}(y,A_k) \leqslant \frac{1}{m} \right\}.$$

We have $\tilde{A}_k \subset \partial^S_{(k+2)/m} B$ and thus $\mu(\tilde{A}_k) \leqslant C_0\,K(k+1)/m$. Because of Lemma 1 and (10) we have that for $k = 0,1,\ldots,$

$$\begin{aligned} C_0\,\nu(A_k) &\leqslant \int_{A_k} T_m(y,\tilde{A}_k)\,d\nu(y) \\ &\leqslant C(m,\nu,d) + \int T_m(y,\tilde{A}_k)\,d\mu(y) \\ &\leqslant C(m,\nu,d) + C_0\,K\,\frac{k+1}{m}. \end{aligned}$$

We have established that

$$\nu(A_k) \leqslant C_0\left(\frac{(k+1)K}{m} + C(m,\nu,d)\right) \quad (k = 0,1,\ldots).$$

Taking into account (10) and Lemma 2, this estimate implies that

$$\begin{aligned}\nu(B) &\geqslant \int_B T_m(x,B)d\nu \\ &= \int T_m(x,B)\,d\nu - \sum_{k=0}^{\infty}\int_{A_k} T_m(x,B)\,d\nu \\ &\geqslant \int T_m(x,B)\,d\mu(x) - C(m,\nu,d) - C_0\left(\nu(A_0) + \sum_{k=1}^{\infty}\frac{1}{k^3}\,\nu(A_k)\right) \\ &\geqslant \mu(B) - C_0\left(\frac{K}{m} + C(m,\nu,d)\right).\end{aligned}$$

From this, the desired upper estimate for $\mu(B) - \nu(B)$ follows. The corresponding lower estimate follows from the upper one when replacing $B$ by $S\backslash B$, which is also a $K$-regular set. □

For completeness, we present also a proof of Theorem 2, which in major parts follows the ideas of Korevaar and Meyers.

*Proof of Theorem 2.* Let $\zeta, z \in S$, let $r \in ]0,1[$. Because of the representation [16, p. 3] of the Gegenbauer polynomials $C_k^{(\lambda)}$ by a generating function, the Newtonian kernel has a development

$$\frac{1}{|r\,\zeta - z|^{d-2}} = (1 - 2\,r\,\langle\zeta,z\rangle + r^2)^{-(d-2)/2} = \sum_{k=0}^{\infty} r^k\,C_k^{(\lambda)}(\langle\zeta,z\rangle),$$

where $\lambda := (d-2)/2$. Consequently,

$$U^{\nu}(r\zeta) - 1 = \int\sum_{k=1}^{\infty} r^k\,C_k^{(\lambda)}(\langle\zeta,z\rangle)\,d\nu(z) = \sum_{k=1}^{\infty} r^k\int C_k^{(\lambda)}(\langle\zeta,z\rangle)\,d\nu(z). \qquad (11)$$

If one sets

$$Q_k(\zeta) := \int C_k^{(\lambda)}(\langle\zeta,z\rangle)\,d\nu(z) \quad (\zeta \in S),$$

then the functions $Q_k$ are spherical harmonics of degree $k$ and in particular form an orthogonal system for integration with respect to the surface measure on $S$. Squaring and integrating (11) yields

$$||U^{\nu}(r\zeta) - 1||_2^2 = \int\left|\sum_{k=1}^{\infty} r^k Q_k(\zeta)\right|^2 d\mu(\zeta) = \sum_{k=1}^{\infty} r^{2k}\,||Q_k||_2^2. \qquad (12)$$

Now, let $p$ be an arbitrary polynomial of degree $\leqslant n$. Any such polynomial can be developed into a series

$$p(z) = \sum_{k=0}^{n} Y_k(z) \qquad (z \in S) \qquad (13)$$

with respect to orthogonal spherical harmonics $Y_k = Y_k(\cdot, p)$ of degree $k$. Because of orthogonality, we have

$$||p||_2^2 = \sum_{k=0}^{n} ||Y_k||_2^2.$$

The $Y_k$'s can be continued into $\mathbb{R}^d$ admitting a representation

$$Y_k(z) = Y_k(z,p) = \int p(\zeta)\, Z_k(z,\zeta)\, d\mu(\zeta) \quad (z \in \mathbb{R}^d), \tag{14}$$

where $Z_k(\cdot,\zeta)$ denotes the zonal spherical harmonic of degree $k$ with pole in $\zeta$.

Following [4, Th. 5.24], for $x \in \mathbb{R}^d$ and $\zeta \in S$, the zonal function $Z_k(x,\zeta)$ is equal to the expression

$$(d+2k-2)\sum_{j=0}^{[k/2]}(-1)^j \frac{d\,(d+2)\,\ldots\,(d+2k-2j-4)}{2^j\, j\,!\,(k-2j)!}\,(\langle x,\zeta\rangle)^{k-2j}\,|x|^{2j},$$

and applying formula [16, p. 50] yields the identity

$$Z_k(z,\zeta) = \frac{d+2k-2}{d-2}\, C_k^{(\lambda)}(\langle z,\zeta\rangle) \qquad (z,\, \zeta \in S).$$

In particular, when inserting in (13) and (14) the function $p = Y_k$, it follows that

$$Y_k(z) = \frac{d+2k-2}{d-2}\int Y_k(\zeta)\, C_k^{(\lambda)}(\langle z,\zeta\rangle)\, d\mu(\zeta) \qquad (z \in S).$$

Because of

$$\int p\, d\mu = Y_0 = \int Y_0\, d\nu$$

this implies that

$$\begin{aligned}\int p\, d\nu - \int p\, d\mu &= \sum_{k=1}^{n}\int Y_k\, d\nu = \sum_{k=1}^{n}\frac{d+2k-2}{d-2}\int Y_k(\zeta)\, Q_k(\zeta)\, d\mu(\zeta)\\ &= \int\left(\sum_{k=1}^{n}\frac{d+2k-2}{d-2}\, r^{-k}\, Y_k(\zeta)\right)\left(\sum_{j=1}^{n} r^j\, Q_j(\zeta)\right) d\mu(\zeta).\end{aligned}$$

Squaring this identity and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}&\left|\int p\, d\mu - \int p\, d\nu\right|^2\\ &\quad\leqslant \left(\sum_{k=1}^{n}\left(\frac{d+2k-2}{d-2}\right)^2 r^{-2k}\, ||Y_k(z)||_2^2\right)\left(\sum_{j=1}^{n} r^{2j}\, ||Q_j(\zeta)||_2^2\right)\\ &\quad\leqslant \left(\frac{d+2n-2}{d-2}\right)^2 r^{-2n}\, ||p||_2^2\, \sum_{j=1}^{n} r^{2j}\, ||Q_j||_2^2.\end{aligned}$$

The assertion of the theorem now follows by inserting (12) into this inequality. □

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