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Some sharp inequalities for norms in $\mathbb{R}^n$ and $\mathbb{C}^n$

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Abstract

The main result of this paper is that for any norm on a complex or real n-dimensional linear space, every extremal basis satisfies inverted triangle inequality with scaling factor $2^n - 1$. Furthermore, the constant $2^n - 1$ is tight. We also prove that the norms of any two extremal bases are comparable with a factor of $2^n - 1$, which, intuitively, means that any two extremal bases are quantitatively equivalent with the stated tolerance.

Keywords Norm · Convex domain · Maximal/minimal basis

Mathematics Subject Classification 52A40 · 52A21 · 32F17

1 Introduction

Extremal bases, originally introduced in [3, 7], have been established as a useful tool in the study of the properties of the so called $\mathbb{C}$-convex domains, D. On the one hand side they induce a natural orthonormal coordinate system around any given point z in the interior of the domain D. On the other hand, under certain assumptions, see below, every extremal basis B at point z enjoys the following inequalities:

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$$\left(\sum_{b\in B}\frac{|\langle b, v\rangle|}{d_D(z;b)}\right)^{-1} \lesssim d_D(z;v) \lesssim \left(\sum_{b\in B}\frac{|\langle b, v\rangle|}{d_D(z;b)}\right)^{-1}, \tag{1}$$

where $d_D(z; v)$ corresponds to the distance from z to the boundary of D in direction v, i.e.:

$$d_D(z; v) = \sup\{r \in \mathbb{R}^+ | \ z + \lambda v \in D \text{ whenever } |\lambda| < r\}.$$

This means that the extremal bases provide a convenient linear approximation of the structure of the body D in a neighbourhood of any given point z in the interior of the body.

The property (1) of extremal bases has facilitated the construction of plurisubharmonic functions with bounded Hessians and for obtaining of estimates for the Bergman kernels, [7, 8]. In [3, 4], Hefer states and uses the estimates (1) to obtain Hölder and L^p estimates for the solutions of Cauchy-Riemann equations on smooth bounded pseudoconvex domains D of finite type. Estimates (1) have been also applied in the study of the Kobayashi and Bergman metrics, [7, 9–11] and more recently, [12]. For a survey on the geometry of extremal bases and their applications we refer to [2].

The construction of maximal bases can be described as a greedy procedure. One starts with an arbitrary point z in the interior of a $\mathbb{C}$-convex domain D and an empty set of vectors/directions B_0. Next, inductively, the current set B_k is extended to B_{k+1} by adding an *extremal* direction v_{k+1} that is orthogonal to the subspace spanned by B_k. The process terminates once the set $B := B_n$ spans the entire space in which case B is the desired basis.

The notion of *extremity* can be specified as *maximal*, [7, 8], or as *minimal*, [3, 4], and the difference consists in whether one selects:

$$v_{k+1} \in \arg\max\{d_D(z; v) \mid v \perp span(B_k)\} \text{ or}$$
$$v_{k+1} \in \arg\min\{d_D(z; v) \mid v \perp span(B_k)\}.$$

Though important for the applications, the proofs of the estimates (1) departing from geometric or analytical view points, depend on some kind of smoothness conditions for the domain D, [7, 8, 10], and often provide only implicit or rough estimates for the hidden constant. In particular, it is not evident if and how it depends on the domain D.

In this paper, we give answer to the above questions by proving that the estimates (1) are valid with constant $2^n - 1$ where n is the dimensionality of the space in which the (so called weakly linear) convex domain D resides. In this sense this constant is independent of D. Furthermore it turns out that the constant $2^n - 1$ is sharp, that is it cannot be improved.

Our approach is algebraic. It departs from the observation that for a weakly linear convex domain D and any particular point z in the interior of D, the function:

$$f(v) = \frac{1}{d_D(z; v)}$$

is a semi-norm. If further the domain D does not contain complex lines, then f is a norm, but see Remark 1 which suggests that this assumption is not essential and it is only for technical reasons that we restrict our considerations to norms. Thus the estimates (1) can be restated as:

$$\sum_{b\in B} |\langle b, v\rangle| f(b) \gtrsim f(v) \gtrsim \sum_{b\in B} |\langle b, v\rangle| f(b) \tag{2}$$

and whereas the first inequality is satisfied with constant 1, as it easily follows by the triangle inequality, the second inequality seems to be more challenging.

To keep the outline self-contained, in Sect. 2 we prove that every (bounded on the unit sphere) norm can be represented as:

$$f(v) = \sup_{u\in U} |\langle v, u\rangle|$$

for an appropriate set of vectors U. The geometric interpretation of U is that the vectors $u \in U$ define the supporting hyperplanes to the body D centred at z.

In Sect. 3 we state and prove that for any extremal basis B:

$$(2^n - 1) f(v) \geq \sum_{b\in B} |\langle b, v\rangle| f(b).$$

We should stress that the definition of maximal and minimal in our notation, see Definition 2, are reciprocal to the notions used for bodies. The reason is that maximising $d_D(z; .)$ is equivalent to minimising f and vice versa. Thus, Theorem 4 proves the statement for maximal bases w.r.t. $d_D(z; .)$ and for minimal bases w.r.t. f, whereas Theorem 5 proves the statement for minimal bases w.r.t. $d_D(z; .)$ and maximal bases w.r.t. f.

In Sect. 3 we prove that $2^n - 1$ is the best possible bound. Namely, we show that for every $\varepsilon > 0$ there are norms whose maximal, resp. minimal, bases violate the inequality:

$$(2^n - 1 - \varepsilon) f(v) \geq \sum_{b\in B} |\langle b, v\rangle| f(b)$$

for at least one vector v. Since every norm gives rise to a convex body $D' = \{v \mid f(v) \leq 1\}$ such that $d_{D'}(0; v) = \frac{1}{f(v)}$, the results translate immediately for convex bodies. Again, Proposition 6 handles the case of maximal basis w.r.t. $d_D(z; .)$ and minimal basis w.r.t. f, whereas Proposition 7 handles the case of minimal basis w.r.t. $d_D(z; .)$ and maximal basis w.r.t. f.

In Sect. 5 we show that minimal and maximal bases are equivalent in their norms. Whereas similar result has been previously proven in [10], the bounds in Sect. 5 are based on more accurate analysis of the algebraic structure and improve the estimate for the constants from $2^n n!$ to 2^n, thus reducing a factor of $n!$.

We conclude in Sect. 6 with some open problems.

2 Preliminaries

In what follows we assume that V is a linear space on $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and that V is supplied with a scalar product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$. We denote with $\|.\| : V \to \mathbb{R}^+$ and $\mathbb{S}_1$ the induced norm and the unit sphere in V, respectively, i.e.:

$$\|u\| = \sqrt{\langle u, u \rangle} \text{ for } u \in V$$
$$\mathbb{S}_1 = \{u \in V \mid \langle u, u \rangle = 1\}.$$

Lemma 1 *Let $f : V \to \mathbb{R}^+$ be a norm. Assume that f is bounded on the unit sphere $\mathbb{S}_1 \subset V$.*

(1) There is a set of vectors $U \subseteq V$ such that:

$$f(v) = \sup_{u \in U} | \langle v, u \rangle |.$$

(2) If $v_0 \in V$ is a unit vector such that $f(v_0) = \sup_{v \in \mathbb{S}_1} f(v)$, then for all $u \in V$ it holds that:

$$f(u) \geq | \langle v_0, u \rangle | f(v_0).$$

Proof (1) The first part is an immediate consequence of Hahn-Banach Theorem. Indeed, let $v \in V$. Then denoting by $V_0 = span(v)$, we define $L_0 : V_0 \to \mathbb{F}$ as $L_0(\alpha v) = \alpha f(v)$. Clearly, L_0 is a linear functional on V_0 and $|L_0(\alpha v)| = |\alpha| f(v) = f(\alpha v)$. By Hanh-Banach Theorem, since f is a norm, we can extend L_0 to a linear functional $L_v : V \to \mathbb{F}$ such that $|L_v(u)| \leq f(u)$ for all $u \in V$. Since L_v is linear on V and $\sup_{u \in \mathbb{S}_1} L_v(u)$ is bounded above by $\sup_{u \in \mathbb{S}_1} f(u) < \infty$, it follows that L_v is a bounded linear operator and cosequently there is a vector $v^* \in V$ such that:

$$\left\langle u, v^* \right\rangle = L_v(u) \text{ for all } u \in V.$$

Let $U = \{v^* \mid v \in V\}$. It is straightforward that for any $u \in V$ and $v^* \in U$, $| \langle u, v^* \rangle | = |L_v(u)| \leq f(u)$. On the other hand $f(u) = L_u(u) = | \langle u, u^* \rangle |$. Therefore:

$$f(v) = \sup_{u \in U} | \langle v, u \rangle |.$$

(2) For the second part, consider the linear functional $L = L_{v_0}$ induced by v_0. Since L is linear, it is determined by its values on an orthonormal basis on V. Without loss of generality we may and we do assume that $(e_i)_{i \in I}$ is such a basis with $e_1 = v_0$. Assume that $L(e_i) \neq 0$ for some $i > 1$. Then we consider the vector:

$$u = \overline{L(e_1)} v_0 + \overline{L(e_i)} e_i = f(v_0) v_0 + \overline{L(e_i)} e_i.$$

Then we have that $\|u\| = \sqrt{f^2(v_0) + |L(e_i)|^2}$ and:

$$L(u) = f(v_0)L(v_0) + \overline{L(e_i)}L(e_i) = f(v_0)^2 + |L(e_i)|^2.$$

Hence $u' = \frac{u}{\|u\|}$ is a unit vector and $L(u') = \sqrt{f(v_0)^2 + |L(e_i)|^2}$ implying that $L(u') > f(v_0)$ since $L(e_i) \neq 0$. However, $f(u') \geq L(u') > f(v_0)$ and this contradicts the maximality of v_0. Consequently, $L(e_i) = 0$ for any $i \neq 1$. This proves that $L(u) = \langle u, e_1 \rangle L(e_1) = \langle u, v_0 \rangle f(v_0)$ and therefore:

$$|\langle u, v_0 \rangle| f(v_0) = |L(u)| \leq f(u)$$

as required. □

Remark 1 We can view the first part of the above lemma as a characterisation of (semi)norms. Indeed, if $U \subseteq V$, then $f_U : V \to \mathbb{R}^+$ defined as:

$$f_U(v) = \sup_{u \in U} |\langle v, u \rangle|$$

is a semi-norm. Actually, it is a norm iff $U^{\perp} = \{v \in V \mid \forall u \in U(\langle u, v \rangle = 0)\}$ is the trivial set $\{0\}$. Since, $U^{\perp}$ is a subspace of V, the properties of f_U are uniquely determined by the behaviour of f_U on the orthogonal subspace of $U^{\perp}$. Indeed, if $v = v' + u$ with $u \in U^{\perp}$ and $v' \perp U^{\perp}$ then:

$$f_U(v') \leq f_U(v) + f_U(u) = f(v) \text{ and } f_U(v) \leq f_U(v') + f_U(-u) = f_U(v'),$$

that is $f_U(v') = f_U(v)$.

3 Minimal and maximal bases

In this section we assume that n is a positive integer, and V is an n-dimensional linear space supplied with a scalar product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 2 Let $f : V \to \mathbb{R}^+$ be a norm. We define an *f-minimal (f-maximal, resp.)* (orthonormal) basis for V inductively as follows:

- $n = 1$, then for any $b \in \mathbb{S}_1$, (b) is an f-minimal (f-maximal, resp.) basis.
- $n > 1$, then let:

$$b_1 \in \arg\min\{f(b) \mid b \in \mathbb{S}_1\} (b_1 \in \arg\max\{f(b) \mid b \in \mathbb{S}_1\}, \text{ resp.})$$
$$V_1 = \{u \in V \mid \langle u, b_1 \rangle = 0\}$$
$$f_1 = f \restriction V_1 \text{ be the restriction of } f \text{ to } V_1.$$

If $(b_2, b_3, \ldots, b_n)$ is an f_1-minimal (f_1-maximal, resp.) basis for V_1, then $(b_1, b_2, \ldots, b_n)$ is an f-minimal (f-maximal, resp.) basis for V.

Definition 3 We say that two orthonormal bases $(b_1, \ldots, b_n)$ and $(e_1, \ldots, e_n)$ of V are *equivalent* if for every $i \le n$, b_i and e_i are collinear, i.e. there is an element $\alpha_i \in \mathbb{F}$ with $|\alpha_i| = 1$ such that $e_i = \alpha_i b_i$.

Theorem 4 *For any norm* $f : V \to \mathbb{R}^+$ *and any* f*-minimal basis* $(b_1, b_2, \ldots, b_n)$ *it holds that:*

$$(2^n - 1) f(v) \ge \sum_{i=1}^{n} |\langle v, b_i \rangle| f(b_i).$$

Proof First note that for any $v \in V$ there are unique $\alpha \in \mathbb{F}$ and vector $u \in span(b_2, \ldots, b_n)$ such that:

$$v = u + \alpha b_1.$$

The statement being obvious for $v = 0$, we assume that $v \neq 0$ and set $v' = \frac{v}{\|v\|}$. Then $v' \in \mathbb{S}_1$ and by the definition of b_1, we get that:

$$f(v') \ge f(b_1) \ge \frac{|\langle v, b_1 \rangle|}{\|v\|} f(b_1) = \frac{|\alpha|}{\|v\|} f(b_1),$$

where we used the Cauchy-Schwartz inequality. So far we have that:

$$f(v) = \|v\| f(v') \ge |\alpha| f(b_1).$$

This settles the case where $n = 1$. Alternatively, i.e. for $n \ge 2$, we use the triangle inequality for $u = v - \alpha b_1$ to conclude that:

$$f(u) = f(v - \alpha b_1) \le f(v) + |\alpha| f(b_1) \le f(v) + f(v) = 2 f(v).$$

With this inequality at hand we can conclude the proof of the theorem by induction on n. As we noticed, the case $n = 1$ is settled. Assume that the conclusion of the theorem holds for any $(n-1)$-dimensional vector space V' and consider the vector space V of dimension n. Let $V' = span(b_2, \ldots, b_n)$ and $f' = f \restriction V'$. It follows, by definition, that $(b_2, \ldots, b_n)$ is an f'-minimal basis for V' and therefore by the induction hypothesis:

$$(2^{n-1} - 1) f'(u) \ge \sum_{i=2}^{n} |\langle u, b_i \rangle| f(b_i) \text{ for any } u \in V'.$$

Finally, for any non-zero $v = u + \alpha b_1$ with $\alpha \in \mathbb{F}$ and $u \in V'$ we have proven that:

$$f(v) \ge 2 f(u) = 2 f(u') \text{ and } f(v) \ge |\alpha| f(b_1).$$

Summing up, and using that $\langle v, b_i \rangle = \langle u, b_i \rangle$ for $i \geq 2$, we conclude that:

$$
\begin{aligned}
(2^n - 1) f(v) &= 2(2^{n-1} - 1) f(v) + f(v) \\
&\geq (2^{n-1} - 1) f(u) + |\alpha| f(b_1) \\
&\geq |\alpha| f(b_1) + \sum_{i=2}^{n} |\langle u, b_i \rangle| f(b_i) \\
&= |\langle v, b_1 \rangle| f(b_1) + \sum_{i=2}^{n} |\langle v, b_i \rangle| f(b_i) \\
&= \sum_{i=1}^{n} |\langle v, b_i \rangle| f(b_i).
\end{aligned}
$$

□

Theorem 5 *For any norm $f : V \to \mathbb{R}^+$ and any f-maximal basis $(b_1, b_2, \ldots, b_n)$ it holds that:*

$$(2^n - 1) f(v) \geq \sum_{i=1}^{n} |\langle v, b_i \rangle| f(b_i).$$

Proof We proceed by induction on $n = \dim V$. For $n = 1$ there is nothing to prove. Assume that the statement of the theorem holds for vector spaces with dimensionality n. Let $f : V \to \mathbb{R}^+$ be a norm. Since $f(b_1) = \max_{v \in \mathbb{S}_1} f(v)$, by Lemma 1 we have that:

$$f(u) \geq |\langle u, b_1 \rangle| f(b_1).$$

Let $V_1 = span(b_2, \ldots, b_n)$ and consider an arbitrary vector $u = \alpha b_1 + v$ with $v \in V_1$. Then, we have:

$$f(u) \geq |\alpha| f(b_1).$$

Furthermore, by the triangle inequality we have:

$$f(u) + |\alpha| f(b_1) = f(\alpha b_1 + v) + f(-\alpha b_1) \geq f(v).$$

Summing up we obtain that $2f(u) \geq f(u) + |\alpha| f(b_1) \geq f(v)$. To complete the inductive step, we use that $\langle u, b_i \rangle = \langle v, b_i \rangle$ for $i \geq 2$ and the inductive hypothesis for V_1 and $f_1 = f \restriction V_1$. Thus, we compute:

$$
\begin{aligned}
(2^n - 1) f(u) &\geq f(u) + 2(2^{n-1} - 1) f(u) \\
&\geq |\alpha| f(b_1) + (2^{n-1} - 1) f(v) \\
&\geq |\alpha| f(b_1) + \sum_{i=2}^{n} |\langle v, b_i \rangle| f(b_i) \\
&= |\alpha| f(b_1) + \sum_{i=2}^{n} |\langle u, b_i \rangle| f(b_i)
\end{aligned}
$$

where the second line follows by the fact $f(u) \geq |\alpha| f(b_1)$ and $2f(u) \geq f(v)$, and the third line follows by the inductive hypothesis. □

4 Lower bounds

In this section we prove that the results from Theorems 4 and 5 are sharp. Our constructions use the characterisation from Lemma 1 and Remark 1 as a basic tool to define norms. In both cases given an $\varepsilon > 0$, we construct a finite set $U \subseteq \mathbb{R}^n$ that defines a norm:

$$
f : \mathbb{C}^n \to \mathbb{R}^+ \text{ s.t. } f(v) = \max_{u \in U} |\langle v, u \rangle|.
$$

The set U is tailored in such a way that it admits:

(1) As f-minimal (f-maximal, resp.) an orthonormal[1] basis

$$
(e_1, \ldots, e_n) \subset \mathbb{R}^n
$$

and

(2) A witness $v_0 \in \mathbb{R}^n$ such that:

$$
(2^n - 1 - \varepsilon) f(v_0) < \sum_{i=1}^{n} |\langle v_0, e_i \rangle| f(e_i).
$$

Since $U \subseteq \mathbb{R}^n$, $(e_1, \ldots, e_n) \subseteq \mathbb{R}^n$ and $v_0 \in \mathbb{R}^n$, the restriction f_R of f to $\mathbb{R}^n$ provides a norm:

$$
f_R : \mathbb{R}^n \to \mathbb{R}^+ \text{ with } f_R(v) = f(v)
$$

that admits as f_R-minimal (f-maximal, resp.) basis again $(e_1, \ldots, e_n)$ and v_0 is a witnesses that:

$$
(2^n - 1 - \varepsilon) f_R(v_0) < \sum_{i=1}^{n} |\langle v_0, e_i \rangle| f_R(e_i).
$$

[1] Actually, it is unique up to equivalence.

Proposition 6 *Let $n \geq 1$ be an integer and $V = \mathbb{C}^n$. Then for any $\varepsilon > 0$, there is a norm $f : V \to \mathbb{R}^+$ and a nonzero vector $v \in \mathbb{R}^n$ such that:*

(1) f admits a unique, up to equivalence, f-minimal basis

$$(e_1, \ldots, e_n) \subseteq \mathbb{R}^n$$

and

(2)

$$(2^n - 1 - \varepsilon) f(v) \leq \sum_{i=1}^{n} |\langle v, e_i \rangle| f(e_i).$$

Proof Let $(e_1, e_2, \ldots, e_n) \subseteq \mathbb{R}^n$ be a fixed orthonormal basis for V and $\varepsilon > 0$. We set $V_i = span(e_i, \ldots, e_n)$. Finally, let $s \in (0, 1)$ whose value will be specified appropriately later. We set out to inductively construct sets $U_i \subseteq V_i \cap \mathbb{R}^n$, norms $f_i : V_i \to \mathbb{R}^+$ and witnesses $v^{(i)} \in V_i \cap \mathbb{R}^n$ with the following properties:

(1) $f_i(v) = \max\{|\langle v, u \rangle| \mid u \in U_i\}$,
(2) $f_i(e_i) = 1 = \min_{v \in \mathbb{S}_1 \cap V_i} f_i(v)$,
(3) $f_i(v) = \frac{1}{s^2(2s+1)} f_{i+1}(v)$ for all $v \in V_{i+1}$,
(4) $\left\langle v^{(i)}, e_j \right\rangle = (2s^2)^{j-i}$ is such that $f_i(v^{(i)}) \in [1, 1 + 2s)$ and:

$$(2^{n-i} - 1) f(v^{(i)}) < \sum_{j=i}^{n} \left|\left\langle v^{(i)}, e_j \right\rangle\right| (1 + 2s)^{2(j-i)+1} f_i(e_j))$$

We start with $U_n = \{e_n\}$ and $f_n(\alpha e_n) = |\alpha|$, $v^{(n)} = e_n$. It is straightforward to see that these objects satisfy the above properties. Assume that for some $i > 1$ the set U_i, the norm f_i and the witness $v^{(i)}$ are defined and have the above properties. We define U_{i-1}, f_{i-1} and $v^{(i-1)}$ as follows. Let

$$U_i^+ = \{u \in U_i \mid \left\langle v^{(i)}, u \right\rangle \geq 0\} \text{ and } U_i^- = \{u \in U_i \mid \left\langle v^{(i)}, u \right\rangle < 0\}.$$

Next we define:

$$\begin{aligned} U_{i-1} = &\left\{e_{i-1} + \tfrac{1}{s(2s+1)} u \mid u \in U_i^+\right\} \cup \left\{e_{i-1} - \tfrac{1}{s^2(2s+1)} u \mid u \in U_i^+\right\} \\ &\cup \left\{e_{i-1} - \tfrac{1}{s(2s+1)} u \mid u \in U_i^-\right\} \cup \left\{e_{i-1} + \tfrac{1}{s^2(2s+1)} u \mid u \in U_i^-\right\}. \end{aligned}$$

Now, we define f_{i-1} and $v^{(i-1)}$ as:

$$\begin{aligned} f_{i-1} &= \max\{|\langle v, u \rangle| \mid u \in U_{i-1}\} \\ v^{(i-1)} &= e_{i-1} + 2s^2 v^{(i)}. \end{aligned}$$

Note that since, by assumption, $U_i \subseteq \mathbb{R}^n$ and $v^{(i)} \in \mathbb{R}^n$, the sets U_i^+ and U_i^- are well-defined. Hence, it should be clear that the $U_{i-1} \subseteq \mathbb{R}^n$ and $v^{(i-1)} \in \mathbb{R}^n$.

Now we verify that U_{i-1}, f_{i-1} and $v^{(i-1)}$ possess the desired properties:

(1) The first property is satisfied by definition.
(2) The third property is also clear for $s \in (0, 1)$.
(3) To see that the second property holds, first note that:

$$\langle e_{i-1}, u \rangle = 1 \text{ for all } u \in U_{i-1}.$$

Next, consider an arbitrary element $v \in U_{i-1} \cap \mathbb{S}_1$. It has a unique representation $v = \alpha e_{i-1} + v'$ with $|\alpha|^2 + \|v'\|^2 = 1$ with $v' \in V_i$. By the induction hypothesis we have that $f_i(v') \geq \|v'\|$. Since U_i is finite, the value $f_i(v') = |\langle v', u' \rangle|$ is attained for some $u' \in U_i$. Let us fix such u' and set $a, b \in \mathbb{R}$ such that:

$$\langle v', u' \rangle = a + ib.$$

We also set $c, d \in \mathbb{R}$ such that $\alpha = c + id$. Thus, for $\sigma \in \{s^{-1}, s^{-2}, -s^{-1}, -s^{-2}\}$ we have:

$$\langle v, e_{i-1} + \sigma u' \rangle = c + id + \sigma(a + ib) = (c + \sigma a) + i(d + \sigma b).$$

Consequently:

$$\begin{aligned} |\langle v, e_{i-1} + \sigma u' \rangle| &= (c + \sigma a)^2 + (d + \sigma b)^2 \\ &= c^2 + d^2 + \sigma^2(a^2 + b^2) + 2\sigma(ac + bd) \\ &= |\alpha|^2 + \sigma^2 |\langle v', u' \rangle|^2 + 2\sigma(ac + bd) \\ &= |\alpha|^2 + \sigma^2 f_i^2(v') + 2\sigma(ac + bd) \\ &\geq |\alpha|^2 + \sigma^2 \|v'\|^2 + 2\sigma(ac + bd). \end{aligned}$$

By construction, we can always choose the sign of σ, i.e. we have either the option $\sigma \in \{s^{-1}, -s^{-2}\}$ or $\sigma \in \{-s^{-1}, s^{-2}\}$. Now, choosing σ such that the sign of σ is the same as the sign of $(ac + bd)$ we conclude that:

$$\begin{aligned} |\langle v, e_{i-1} + \sigma u' \rangle| &\geq |\alpha|^2 + \sigma^2 \|v'\|^2 + 2\sigma(ac + bd) \\ &\geq |\alpha|^2 + \sigma^2 \|v\|^2 \\ &\geq |\alpha|^2 + \|v'\|^2 \\ &= 1, \end{aligned}$$

where the first inequality follows by the choice of σ and the second by the fact that $|\sigma| > 1$. Furthermore the last inequality turns into equality if and only if $v' = 0$. This proves that for any vector $v \in \mathbb{S}_1 \cap V_{i-1}$ which is not collinear with e_{i-1}, it holds that $f_i(v) > 1$. Consequently:

$$f_{i-1}(e_{i-1}) = \min_{v \in V_{i-1} \cap \mathbb{S}_1} f_{i-1}(v)$$

and the minimum is attained only for vectors of the form αe_{i-1} with $|\alpha| = 1$. Hence w.l.o.g. e_{i-1} belongs to the minimal basis.

(4) Finally, we check the last condition. Let $v^{(i-1)} \in V_{i-1}$ be such that:

$$v^{(i-1)} = e_{i-1} + 2s^2 v^{(i)}.$$

First, let $u \in U_i^+$. In particular, $\left\langle u, v^{(i)} \right\rangle \geq 0$. Therefore:

$$\begin{aligned}\left\langle v^{(i-1)}, e_{i-1} - \frac{1}{s^2(2s+1)} u \right\rangle &= \left\langle e_{i-1} + 2s^2 v^{(i)}, e_{i-1} - \frac{1}{s^2(2s+1)} u \right\rangle \\ &= 1 - \frac{2}{2s+1} \left\langle v^{(i)}, u \right\rangle \\ &= 1 - \frac{2}{2s+1} |\left\langle v^{(i)}, u \right\rangle| \leq 1.\end{aligned}$$

Since $u \in U_i$, we have that $f_i(v^{(i)}) \geq |\left\langle v^{(i)}, u \right\rangle|$ and by the assumption that $f_i(v^{(i)}) \leq 1 + 2s$ we conclude that:

$$\left\langle v^{(i-1)}, e_{i-1} - \frac{1}{s^2(2s+1)} u \right\rangle = 1 - \frac{2}{2s+1} |\left\langle v^{(i)}, u \right\rangle| > 1 - 2 = -1.$$

Hence $|\left\langle v^{(i-1)}, e_{i-1} - \frac{1}{s^2(2s+1)} u \right\rangle| \leq 1$ for all $u \in U_i^+$. On the other hand:

$$\begin{aligned}\left\langle v^{(i-1)}, e_{i-1} + \frac{1}{s(2s+1)} u \right\rangle &= 1 + \frac{2s^2}{s(2s+1)} \left\langle v^{(i)}, u \right\rangle \\ &= 1 + \frac{2s}{1+2s} |\left\langle v^{(i)}, u \right\rangle| \in [1, 1+2s).\end{aligned}$$

Similarly, for $u \in U_i^-$ we have that:

$$\begin{aligned}\left\langle v^{(i-1)}, e_{i-1} + \frac{1}{s^2(2s+1)} u \right\rangle &= 1 - \frac{2}{2s+1} |\left\langle v^{(i)}, u \right\rangle| \in (-1, 1] \\ \left\langle v^{(i-1)}, e_{i-1} - \frac{1}{s(2s+1)} u \right\rangle &= 1 + \frac{2s}{2s+1} |\left\langle v^{(i)}, u \right\rangle| \in [1, 1+2s).\end{aligned}$$

Hence the maximum value of $\left\langle v^{(i-1)}, u'\right\rangle$ when $u' \in U_{i-1}$ is attained for some u' of the form

$$\begin{aligned} u' &= e_{i-1} + \frac{1}{s(2s+1)} u \text{ with } u \in U_i^+ \text{ or} \\ u' &= e_{i-1} - \frac{1}{s(2s+1)} u \text{ with } u \in U_i^- \end{aligned}$$

such that $|\left\langle v^{(i)}, u\right\rangle|$ is maximised. This shows that

$$f_{i-1}(v^{(i-1)}) = 1 + \frac{2s}{2s+1} f_i(v^{(i)}).$$

So far we have that $f_{i-1}(v^{(i-1)}) \in [1, 1+2s)$. We proceed to show that the last inequality holds. To this end, first note:

$$\begin{aligned} \left\langle v^{(i-1)}, e_{i-1}\right\rangle &= 1 \\ \left\langle v^{(i-1)}, e_j\right\rangle &= 2s^2 \left\langle v^{(i)}, e_j\right\rangle \text{ for } j \geq i. \end{aligned}$$

Recalling that $f_{i-1}(e_{i-1}) = 1$ and $f_i(e_j) = s^2(2s+1) f_{i-1}(e_j)$ for $j \geq i$, we conclude that:

$$(1+2s)|\left\langle v^{(i-1)}, e_j\right\rangle| f_{i-1}(e_j) = 2|\left\langle v^{(i)}, e_j\right\rangle| f_i(e_j) \text{ for } j \geq i.$$

Therefore, using that $f_{i-1}(v^{(i-1)}) < 1 + 2s \leq (1+2s) f_i(v^{(i)})$, we compute:

$$\begin{aligned} (2^{n-i+1} - 1) f_{i-1}(v^{(i-1)}) &= f_{i-1}(v^{(i-1)}) + 2(2^{n-i} - 1) f_{i-1}(v^{(i-1)}) \\ &< 1 + 2s + 2(2^{n-i} - 1) f_{i-1}(v^{(i-1)}) \\ &\leq (1+2s)(1 + 2 f_i(v^{(i)})(2^{n-i} - 1)) \\ \text{(inductive hypothesis)} &\leq (1+2s)\left(1 + 2\sum_{j=i}^{n} |\left\langle v^{(i)}, e_j\right\rangle| (1+2s)^{2(j-i)+1} f_i(e_j)\right) \\ &\leq (1+2s)(|\left\langle v^{(i-1)}, e_{i-1}\right\rangle| f(e_{i-1}) \\ &\quad + \sum_{j=i}^{n} (1+2s)^{2(j-i)+3} |\left\langle v^{(i-1)}, e_j\right\rangle| f_{i-1}(e_j)) \\ (\text{ setting } i' = i-1) &= \sum_{j=i'}^{n} (1+2s)^{2(j-i')+1} |\left\langle v^{(i')}, e_j\right\rangle| f_{i'}(e_j) \end{aligned}$$

as required.

Now letting s tend to zero, we see that $f_1(v^{(1)})$ satisfies the conclusion of the lemma.

□

Proposition 7 *Let $n \geq 1$ be an integer and $V = \mathbb{C}^n$. For any real number $\varepsilon > 0$, there is a norm $f : V \to \mathbb{R}^+$ and a non-zero vector $v \in \mathbb{R}^n$ such that:*

(1) f admits a unique up to equivalence f-maximal basis

$$(e_1, e_2, \ldots, e_n) \subseteq \mathbb{R}^n,$$

(2)

$$(2^n - 1 - \varepsilon) f(v) < \sum_{i=1}^{n} |\langle v, e_i\rangle| f(e_i).$$

Proof Let $\varepsilon > 0$ be fixed and $e_1, \ldots, e_n \in \mathbb{R}^n$ be an orthonormal basis of V. We are going to construct a norm $f : V \to \mathbb{R}^+$ whose unique f-maximal basis is $(e_1, \ldots, e_n)$ and satisfies the conclusion of the proposition. To this end consider a real number $c \in (0, 1)$ and an angle $\alpha \in (0, \pi)$ whose precise values will be determined appropriately.

Given, the constants c and α we define the vector $u'_k \in V$ for $k = 1, 2 \ldots, n$ as follows:

$$u'_1 = e_1$$
$$u'_{k+1} = \sum_{j=1}^{k} e_j \sin^{j-1}\alpha \cos\alpha + e_{k+1}\sin^k \alpha.$$

We set $u_k = c^{k-1} u'_k$ and define the norm $f : V \to \mathbb{R}^+$ as:

$$f(v) = \sup\{|\langle v, u_k\rangle| \mid k \leq n\}.$$

By Remark 1 we know that f is a semi-norm. Since $f(e_k) \geq |\langle e_k, u_k\rangle| > 0$, we see, again by Remark 1, that f is a norm. Further, by Lemma 1, we know that the maximum value $f(v)$ on $\mathbb{S}_1$ is attained at a vector that is collinear to some of the vectors u_k and is equal to $f(v) = \|u_k\|$. Since $\|u_k\| = c^{k-1}$, we conclude that the first vector of the maximal basis is $e_1 = u_1$. Next, the subspace, V_1, of V that is orthogonal to e_1 is spanned by $(e_2, \ldots, e_n)$ and therefore $f_1 = f \restriction V_1$ is actually:

$$f_1(v) = \sup\{|\langle v, u_k - u_1\cos\alpha\rangle| \mid k \leq n\}.$$

Since $\|u_k - u_1 \cos\alpha\| = c^{k-1}\sin\alpha$, applying again Lemma 1, we conclude that the maximal value of f_1 on $\mathbb{S}_1$ is attained at e_2. Proceeding inductively, we may prove that $(e_1, \ldots, e_n)$ is the unique, up to equivalence, f-maximal basis. Note that:

$$f(e_i) = \langle e_i, u_i\rangle = c^{i-1}\sin^{i-1}\alpha.$$

Let $w_1 > 0$ and $w_i = w_1 \mathrm{tg}^{i-1}\frac{\alpha}{2}$. It should be clear that:

$$\begin{aligned} w_i \cos\alpha + w_{i+1}\sin\alpha &= w_i(\cos\alpha + \sin\alpha \mathrm{tg}\frac{\alpha}{2}) \\ &= w_i(2\cos^2\alpha/2 - 1 + 2\sin^2\alpha/2) = w_i. \end{aligned}$$

Therefore, setting $w = \sum_{i=1}^n w_i e_i$, we obtain that

$$\langle w, u_k\rangle = c^{k-1}\left\langle w, u'_k\right\rangle = c^{k-1} w_1$$

and thus $|\langle w, u_k\rangle| = c^{k-1}w_1 \leq w_1$ with equality if and only if $k = 1$. Hence $f(w) = w_1$.

On the other hand:

$$\begin{aligned} w_i f(e_i) &= c^{i-1} w_1 \mathrm{tg}^{i-1}\frac{\alpha}{2}\sin^{i-1}\alpha \\ &= c^{i-1} w_1 \left(2\sin^2\frac{\alpha}{2}\right)^{i-1} \\ &= 2^{i-1}c^{i-1}w_1 \sin^{(2(i-1))}\frac{\alpha}{2}. \end{aligned}$$

It follows that:

$$\sum_{i=1}^n w_i f(e_i) \geq w_1 \sum_{i=1}^n 2^{i-1}[c^{i-1}\sin^{2(i-1)}\frac{\alpha}{2}].$$

Clearly, letting c tend to 1 and α tend to $\pi - 0$, the right hand side tends to $w_1 \sum_{i=1}^n 2^{i-1} = (2^n - 1)w_1 = (2^n - 1)f(w)$. Thus, for any $\varepsilon > 0$, we can find appropriate $c \in (0, 1)$ and $\alpha \in (0, \pi)$ such that $(2^n - 1 - \varepsilon)f(w) < \sum_{i=1}^n w_i f(e_i)$. □

5 Equivalence of bases

Definition 8 Let $f : V \to \mathbb{R}^+$ be a norm. Let $(e_1, e_2, \dots, e_n)$ be an orthonormal basis for V arranged in increasing order w.r.t. f, i.e.:

$$f(e_1) \leq f(e_2) \leq \cdots \leq f(e_n).$$

(1) For a constant $c \in \mathbb{R}^+$, we say that $(e_1, e_2, \dots, e_n)$ satisfies the property $P_f(c)$ if for every $\beta_1, \beta_2, \dots, \beta_n \in \mathbb{F}$ it holds:

$$cf\left(\sum_{i=1}^n \beta_i e_i\right) \geq \sum_{i=1}^n |\beta_i| f(e_i).$$

(2) More generally, for constants $c_1, c_2, \ldots, c_n \in \mathbb{R}^+$, we say that $(e_1, \ldots, e_n)$ satisfies the property $HP_f(c_1, c_2, \ldots, c_n)$ if for every i and every $\beta_i, \beta_{i+1}, \ldots, \beta_n$ it holds that:

$$c_i f\left(\sum_{j=i}^{n} \beta_i e_i\right) \geq \sum_{j=i}^{n} |\beta_i| f(e_i).$$

Remark 2 With these notions, Theorem 5 states that there is an f-maximal basis that satisfies $P_f(2^n - 1)$. Furthermore, since every prefix of an f-maximal basis, $(b_1, b_2, \ldots, b_n)$, is a maximal basis for its linear span, it follows that the f-maximal basis from Theorem 5 actually satisfies $HP_f(2^n - 1, 2^{n-1} - 1, \ldots, 1)$.

On the other hand, Theorem 4 states that every f-minimal basis satisfies $P_f(2^n - 1)$.

It is also obvious that if a basis $(e_1, \ldots, e_n)$ satisfies $P_f(c)$, then it satisfies $HP_f(c, c, \ldots, c)$. Conversely, if an orthonormal basis $(e_1, \ldots, e_n)$ satisfies $HP_f(c_1, \ldots, c_n)$, then it satisfies $P_f(c_1)$.

Lemma 9 *Let $(b_1, b_2, \ldots, b_n)$ and $(e_1, e_2, \ldots, e_n)$ be orthonormal bases on V and $f : V \to \mathbb{R}^+$ be a norm such that:*

$$f(b_1) \leq f(b_2) \leq \cdots \leq f(b_n) \quad and$$
$$f(e_1) \leq f(e_2) \leq \cdots \leq f(e_n).$$

Then:

(1) If $(b_1, \ldots, b_n)$ is f-minimal, then for every $i \leq n$, $f(b_i) \leq \sqrt{i} f(e_i)$,
(2) If $(e_1, e_2, \ldots, e_n)$ satisfies $HP_f(c_1, c_2, \ldots, c_n)$, then for every $i \leq n$ it holds that $f(e_i) \leq c_i \sqrt{i} f(b_i)$.

Proof Let $i \leq n$ and note that $(b_1, b_2, \ldots, b_{i-1})$ spans an $(i-1)$-dimensional subspace of V, whereas $(e_1, e_2, \ldots, e_i)$ spans an i-dimensional subspace of V. Hence there is a non-zero vector $b' \in span(e_1, \ldots, e_i)$ that is orthogonal to all the vectors $b_1, b_2, \ldots, b_{i-1}$. Indeed, it is straightforward that the system:

$$\sum_{j=1}^{i} \alpha_j \left\langle e_j, b_k \right\rangle = 0 \text{ for } k \leq i - 1$$

is overdetermined. Hence it admits a non-zero solution $\alpha = (\alpha_1, \ldots, \alpha_i)$ and since $e_1, \ldots, e_i$ are independent, $b' = \sum_{j=1}^{i} \alpha_j e_j$ is non-zero. Next, without loss of generality, we may and we do assume that $\|b'\| = 1$. Thus $b' \in \mathbb{S}_1$ and b' is orthogonal to all the vectors $b_1, \ldots, b_{i-1}$. By the definition of b_i, it follows that:

$$f(b_i) \leq f(b') = f\left(\sum_{j=1}^{i} \alpha_j e_j\right) \leq \sum_{j=1}^{i} |\alpha_j| f(e_j),$$

where the last inequality follows by the triangle inequality. Finally, by the arrangement of the vectors $e_1, \dots, e_i$ we have $f(e_j) \leq f(e_i)$ for every $j \leq i$ and by the Cauchy-Schwartz inequality we have $\sum_{j=1}^{i} |\alpha_j| \leq \sqrt{i}\sqrt{\sum_{j=1}^{i} |\alpha_j|^2} = \sqrt{i}$. Summing up we obtain:

$$f(b_i) \leq \sum_{j=1}^{i} |\alpha_j| f(e_j) \leq \sum_{j=1}^{i} |\alpha_j| f(e_i) \leq \sqrt{i} f(e_i).$$

For the second part of the statement, we proceed similarly. Let $i \leq n$. Then $(b_1, \dots, b_i)$ span a linear space of dimensionality i whereas $(e_1, \dots, e_{i-1})$ span a linear space of dimensionality $i - 1$. Then, as above, there is a non-zero vector $v \in span(b_1, \dots, b_i)$ that is orthogonal to all the vectors $e_1, \dots, e_{i-1}$. Without loss of generality we may and we do assume that v is a unit vector. Since v is orthogonal to $e_1, \dots, e_{i-1}$, it belongs to the linear space spanned by $(e_i, \dots, e_n)$. Hence v can be written as $v = \sum_{j=i}^{n} \alpha_j e_j$. By the $HP_f(c_1, \dots, c_n)$ of the basis $(e_1, \dots, e_n)$, we conclude that:

$$c_i f(v) = c_i f\left(\sum_{j=i}^{n} \alpha_j e_j\right) \geq \sum_{j=i}^{n} |\alpha_j| f(e_j) \geq \sum_{j=i}^{n} |\alpha_j| f(e_i) \geq f(e_i),$$

where the last but one inequality follows by the ordering of the vectors $(e_1, \dots, e_n)$ and the last inequality follows by the fact that $\sum_{j=i}^{n} |\alpha_j|^2 = 1$, as v is a unit vector.

On the other hand, $v \in span(b_1, \dots, b_i)$ and hence $v = \sum_{j=1}^{i} \beta_j b_j$. Since $\|v\| = 1$, we have that $\sum_{j=1}^{i} |\beta_j|^2 = 1$. Therefore, applying the triangle inequality, we obtain:

$$f(v) = f\left(\sum_{j=1}^{i} \beta_j b_j\right) \leq \sum_{j=1}^{i} |\beta_j| f(b_j) \leq \sum_{j=1}^{i} |\beta_j| f(b_i) \leq \sqrt{i} f(b_i),$$

where the last but one inequality follows by the order of $(b_1, \dots, b_n)$ and the last one is a trivial application of the Cauchy-Schwartz inequality.

Summing up we get:

$$f(e_i) \leq c_i f(v) \leq c_i \sqrt{i} f(b_i)$$

as claimed. □

Corollary 10 *Let $f : V \to \mathbb{R}^+$ be a norm and $(b_1, b_2, \dots, b_n)$ and $(e_1, e_2, \dots, e_n)$ be orthonormal bases such that:*

$$f(b_1) \leq f(b_2) \leq \cdots \leq f(b_n) \text{ and}$$
$$f(e_1) \leq f(e_2) \leq \cdots \leq f(e_n).$$

(1) If $(b_1, \dots, b_n)$ *and* $(e_1, \dots, e_n)$ *are* f*-minimal, then*

$$\frac{1}{\sqrt{i}} \leq \frac{f(b_i)}{f(e_i)} \leq \sqrt{i}.$$

(2) If $(b_1, \dots, b_n)$ *is an* f*-minimal and* $(e_1, \dots, e_n)$ *is an* f*-maximal basis, then* $\frac{1}{\sqrt{i}(2^{n-i+1}-1)} \leq \frac{f(b_i)}{f(e_i)} \leq \sqrt{i}.$

(3) If $(b_1, \dots, b_n)$ *and* $(e_1, \dots, e_n)$ *are* f*-maximal, then*

$$\frac{1}{\sqrt{i}(2^{n-i+1}-1)} \leq \frac{f(b_i)}{f(e_i)} \leq \sqrt{i}(2^{n-i+1}-1).$$

Under certain additional assumptions, Lemma 9 can be inverted as follows:

Proposition 11 *Assume that* $f : V \to \mathbb{R}^+$ *is a norm and* $(b_1, \dots, b_n)$ *and* $(e_1, \dots, e_n)$ *and* $c \geq 1$ *are orthonormal bases for* V *such that:*

$$\begin{aligned} f(b_1) \leq f(b_2) \leq \dots \leq f(b_n) \\ f(e_1) \leq f(e_2) \leq \dots \leq f(e_n) \\ \forall i (f(e_i)/c \leq f(b_i) \leq c f(e_i)). \end{aligned}$$

If $(b_1, \dots, b_n)$ *satisfies* $H_f(c_1)$ *and additionally there is* $\alpha \geq 1$ *such that:*

$$\forall i \neq j \exists \alpha_{i,j} (|\alpha_{i,j}| < \alpha \text{ and } \langle e_i - b_i, e_j - \alpha_{i,j} b_j \rangle = 0),$$

then $(e_1, \dots, e_n)$ *satisfies* $H_f(\alpha c^2 c_1)$.

Proof Since $(b_1, \dots, b_n)$ satisfies $H_f(c_1)$, it follows that:

$$c_1 f(e_i) \geq \sum_{j=1}^{n} |\langle e_i, b_j \rangle| f(b_j).$$

Hence:

$$\begin{aligned} f(b_i) &\geq f(e_i)/c \\ &\geq c_1 f(e_i)/(c c_1) \\ &\geq \sum_{j=1}^{n} |\langle e_i, b_j \rangle| f(b_j)/c \\ &\geq \frac{1}{c^2} \sum_{j=1}^{n} |\langle e_i, b_j \rangle| f(e_j). \end{aligned}$$

Note that the condition $\langle e_i - b_i, e_j - \alpha_{i,j} b_j \rangle = 0$ can be rewritten as:

$$\langle e_i, e_j \rangle + \overline{\alpha_{i,j}} \langle b_i, b_j \rangle - \langle b_i, e_j \rangle - \overline{\alpha_{i,j}} \langle e_i, b_j \rangle = 0.$$

For $i \neq j$ it holds that $\langle b_i, b_j \rangle = \langle e_i, e_j \rangle = 0$ and therefore for $i \neq j$ we have:

$$|\langle b_i, e_j \rangle| = |\alpha_{i,j}||\langle e_i, b_j \rangle| \leq \alpha |\langle e_i, b_j \rangle|.$$

Hence the above inequality implies that:

$$f(b_i) \geq \frac{1}{c^2} \sum_{j=1}^{n} |\langle e_i, b_j \rangle| f(e_j) \geq \frac{1}{\alpha c^2} \sum_{j=1}^{n} |\langle b_i, e_j \rangle| f(e_j).$$

Therefore:

$$\alpha c^2 f(b_i) \geq \sum_{j=1}^{n} |\langle b_i, e_j \rangle| f(e_j).$$

Finally, for arbitrary v the above inequality and the validity of $H_f(c_1)$ for the basis $(b_1, \ldots, b_n)$ imply:

$$\begin{aligned}
\alpha c^2 c_1 f(v) &\geq \alpha c^2 \sum_{i=1}^{n} |\langle v, b_i \rangle| f(b_i) \\
&\geq \sum_{i=1}^{n} \sum_{j=1}^{n} |\langle v, b_i \rangle| |\langle b_i, e_j \rangle| f(e_j) \\
&= \sum_{j=1}^{n} f(e_j) \sum_{i=1}^{n} |\langle v, b_i \rangle \langle b_i, e_j \rangle| \\
&\geq \sum_{j=1}^{n} f(e_j) \left| \sum_{i=1}^{n} \langle v, b_i \rangle \langle b_i, e_j \rangle \right| \\
&= \sum_{j=1}^{n} |\langle v, e_j \rangle| f(e_j).
\end{aligned}$$

This proves that $(e_1, \ldots, e_n)$ satisfies $H_f(\alpha c^2 c_1)$. □

6 Open problems

The definitions of f-minimal and f-maximal bases of a norm suggest a simple greedy strategy to find an orthonormal basis $(e_1, \ldots, e_n)$ which satisfies the inequality:

$$f\left(\sum_{i=1}^{n} \alpha_i e_i\right) \geq \frac{1}{2^n - 1} \sum_{i=1}^{n} |\alpha_i| f(e_i).$$

Of course, the feasibility of this approach depends on the possibility to efficiently solve the optimisation problem:

$$\max_{v \in \mathbb{S}_1} f(v) \text{ or } \min_{v \in \mathbb{S}_1} f(v).$$

Yet, since every (semi)norm f is convex and $\mathbb{S}_1$ is compact, especially the second problem is well studied and efficient methods for its solution stay at hand.

However, the problem with the greedy approach is that the constant $2^n - 1$ grows exponentially with n and it may be inconvenient to prove precise bounds in general. As we have proven in Proposition 6 and Proposition 7, the constant $2^n - 1$ cannot be improved under the suggested greedy strategy. Thus, the natural question that arises is how this constant can be improved while preserving the clear structure of the bases that it implies. In this respect, we consider the following theoretical problems.

First, for a natural number $n \geq 1$, and a linear vector space V with inner product, where $V = \mathbb{C}^n$ or $V = \mathbb{R}^n$ we define $c_n^{\perp} := c_n^{\perp}(V)$ to be the least real number such that for every norm $f : V \to \mathbb{R}^+$ there is an orthonormal basis $(b_1, \ldots, b_n)$ such that:

$$f\left(\sum_{i=1}^{n} \alpha_i b_i\right) \geq \frac{1}{c_n^{\perp}} \sum_{i=1}^{n} |\alpha_i| f(b_i) \text{ for all } \alpha_1, \ldots, \alpha_n \in \mathbb{F}.$$

It is known that for $V = \mathbb{R}^2$, $c_2^{\perp} = 2$, [6]. The construction in [6] relies upon defining appropriate areas and the continuity principle to show existence. Is there a more explicit way to define such a basis? To the best knowledge of the authors, the techniques from $n = 2$ do not extend to higher dimensions.

Secondly, for a natural number $n \geq 1$, and a linear vector space V with inner product, where $V = \mathbb{C}^n$ or $V = \mathbb{R}^n$ we define $c_n^{\angle} = c_n^{\angle}(V)$ to be the least real number such that for every norm $f : V \to \mathbb{R}^+$ there is a basis $(b_1, \ldots, b_n)$ of unit vectors such that:

$$f\left(\sum_{i=1}^{n} \alpha_i b_i\right) \geq \frac{1}{c_n^{\angle}} \sum_{i=1}^{n} |\alpha_i| f(b_i) \text{ for all } \alpha_1, \ldots, \alpha_n \in \mathbb{F}.$$

It is known that for $V = \mathbb{R}^2$, $c_2^{\angle} = \frac{3}{2}$, [1]. This question is tightly related with John's Theorem [5] which relies on volumes' optimisation.

Both questions can be uniformly stated as follows. Let $n \geq 1$, and $V = \mathbb{C}^n$ or $V = \mathbb{R}^n$ and $\alpha \in [0, 1]$. Define $c_n^{\alpha} := c_n^{\alpha}(V)$ to be the least real number such that for every norm $f : V \to \mathbb{R}^+$ there is a basis $(b_1, \ldots, b_n)$ of unit vectors such that:

$$f\left(\sum_{i=1}^{n} \alpha_i b_i\right) \geq \frac{1}{c_n^{\alpha}} \sum_{i=1}^{n} |\alpha_i| f(b_i) \text{ for all } \alpha_1, \ldots, \alpha_n \in \mathbb{F}$$
$$\text{subject to} : |\langle b_i, b_j \rangle| \leq \alpha \text{ for all } i \neq j.$$

In this framework, $c_n^{\perp}(V) = c_n^0(V)$ and $c_n^{\angle}(V) = c_n^1(V)$. We consider that the freedom to vary α may be useful in applications where this kind of inequalities are to be combined with other classical inequalities where the scalar products of the basis' vectors has to be controlled.

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