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The difference of weighted composition operators on Fock spaces

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Abstract
In this note, we solve the open problem posted by Tien and Khoi (Monatsh Math 188:183–193, 2019). We prove that when $0 < q < p < \infty$, the difference of two weighted composition operators between Fock spaces $W_{\psi_1,\varphi_1} - W_{\psi_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded if and only if both W_{ψ_1,φ_1} and W_{ψ_2,φ_2} are bounded. Furthermore, we prove that the same conclusion holds for the differences of a weighted composition operator and a weighted composition-differential operator on $\mathcal{F}^p$.

Keywords Fock space · Weighted composition operator · Difference

Mathematics Subject Classification 30H20 · 47B33 · 46E15

1 Introduction

Let $\mathbb{C}$ be the complex plane and $H(\mathbb{C})$ be the space of all entire functions on $\mathbb{C}$. For $0 < p < \infty$, the classical Fock space $\mathcal{F}^p$ is defined as

$$\mathcal{F}^p = \left\{ f \in H(\mathbb{C}) : \|f\|_p^p = \frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) < \infty \right\},$$

where dA is the Lebesgue measure on $\mathbb{C}$. Furthermore, the space $\mathcal{F}^\infty$ consists of all functions $f \in H(\mathbb{C})$ such that

$$\|f\|_\infty = \sup_{z \in \mathbb{C}} |f(z)| e^{-\frac{|z|^2}{2}} < \infty.$$

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It is known that $\mathcal{F}^p$ is a Banach space for $1 \le p \le \infty$. When $0 < p < 1$, $\mathcal{F}^p$ is a complete metric space with distance $d(f, g) = \|f - g\|_p^p$. In particular, $\mathcal{F}^2$ is a Hilbert space with the following inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^2}dA(z).$$

For each $w \in \mathbb{C}$, the linear point evaluation of nth order $f \mapsto f^{(n)}(w)$ is continuous on $\mathcal{F}^2$. It follows from the Riesz representation theorem in Hilbert space theory that there exsits a unique function $K_w^{[n]}$ in $\mathcal{F}^2$ such that

$$f^{(n)}(w) = \langle f, K_w^{[n]}\rangle$$

for all $f \in \mathcal{F}^2$. $K_w^{[n]}$ is called the reproducing kernel function in $\mathcal{F}^2$ at w of order n. It is known that $K_w^{[0]}(z) = e^{\overline{w}z}$ and

$$K_w^{[n]}(z) = \frac{\partial^n K_w^{[0]}}{\partial \overline{w}^n}(z) = z^n e^{\overline{w}z}$$

for $n \ge 1$. Moreover, $\|K_w^{[0]}\|_p = e^{\frac{|w|^2}{2}}$ and $\|K_w^{[n]}\|_p \asymp (1 + |w|)^n e^{\frac{|w|^2}{2}}$ for all $w \in \mathbb{C}$ and $0 < p \le \infty$. Let $k_w(z) = e^{\overline{w}z - \frac{|w|^2}{2}}$, then each k_w is a unit vector in $\mathcal{F}^p$ and converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $|w| \to \infty$. One can refer to the monograph by Zhu [15] for more information about Fock spaces.

If $\varphi, \psi \in H(\mathbb{C})$, the weighted composition operator $W_{\psi,\varphi}$ on $H(\mathbb{C})$ is defined by $W_{\psi,\varphi} f = \psi \cdot (f \circ \varphi)$. When $\psi = 1$, it reduces to the composition operator C_φ. The relationship between the operator-theoretic properties of C_φ and the function-theoretic properties of φ has been studied extensively during the past several decades. We refer the readers to monographs by Cowen and MacCluer [3] and by Shapiro [13] for more details. The boundedness and compactness of $W_{\psi,\varphi}$ between Fock spaces have been completely characterized in [7, 8, 10]. One could also see [12] for the case in several variables and see [1] for large Fock spaces. Let $Df = f'$ be the differentiation operator on $H(\mathbb{C})$ and D^n be the nth iterate of D. Write $W_{\psi,\varphi}^{(n)}$ for the product of D^n and $W_{\psi,\varphi}$, i.e.

$$W_{\psi,\varphi}^{(n)} f = W_{\psi,\varphi} D^n f = \psi \cdot f^{(n)} \circ \varphi.$$

$W_{\psi,\varphi}^{(n)}$ is called a weighted composition-differential operator of order n. It is clear that $W_{\psi,\varphi}$ is the special case $n = 0$. When $n \ge 1$, the boundedness and compactness of $W_{\psi,\varphi}^{(n)}$ between Fock spaces have been studied completely in [4].

In [9], Moorhouse characterized compactness of the difference of two composition operators on classical weighted Bergman spaces over the unit disk. Moorhouse showed that the difference of two composition operators is compact when suitable cancelation occurs and also that there exist two non-compact composition operators whose their difference is compact. However, no cancelation phenomenon exists on Fock spaces.

Precisely, Choe et al. [2] showed that a linear sum of two composition operators is bounded (compact, resp.) on the Hilbert Fock spaces if and only if both composition operators are bounded (compact, resp.) Tien and Khoi [11] studied the differences of weighted composition operators between different Fock spaces and also showed that no cancelation exists. They proved that $W_{\psi_1,\varphi_1} - W_{\psi_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded (compact, resp.) if and only if both W_{ψ_1,φ_1} and $W_{\psi_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are bounded (compact, resp.) for $0 < p \le q < \infty$. But this problem for the case $0 < q < p < \infty$ is left open. In this paper, we completely solve this problem by using Khinchine's inequality. Our first main result reads as follows.

Theorem A *Let $0 < q < p < \infty$ and $\varphi_1 \neq \varphi_2$. Then the following conditions are equivalent:*

(a) $W_{\psi_1,\varphi_1} - W_{\psi_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded;
(b) $W_{\psi_1,\varphi_1} - W_{\psi_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is compact;
(c) Both W_{ψ_1,φ_1} and $W_{\psi_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are bounded;
(d) Both W_{ψ_1,φ_1} and $W_{\psi_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are compact.

Actually, we can characterize the differences of a weighted composition operator and a weighted composition-differential operator. To the best of our knowledge, no prior results on describing the compactness of two such operators on Fock spaces $\mathcal{F}^p$, and even analytic function spaces on any other domains. Our second main result reads as follows.

Theorem B *Let n be a positive integer and $\varphi_1 \neq \varphi_2$. Then $W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}$ is bounded (compact, resp.) on $\mathcal{F}^p$ if and only if both W_{ψ_1,φ_1} and $W^{(n)}_{\psi_2,\varphi_2}$ are bounded (compact, resp.) on $\mathcal{F}^p$.*

The paper is organized as follows. In Sect. 2, we present some known facts and auxiliary lemmas that will be needed later. Section 3 and Sect. 4 are devoted to the proof of Theorem A and B, respectively.

Throughout the paper, we write $A \lesssim B$ if there exists an absolute constant $C > 0$ such that $A \le CB$. As usual, $A \asymp B$ means $A \lesssim B$ and $B \lesssim A$. We will be more specific if the dependence of such constants on certain parameters becomes critical.

2 Preliminaries

In this section, we collet some preliminary facts and auxiliary lemmas which will be used later. Firstly, according to [5] and [14], we have the following characterizations for Fock spaces via higher order derivatives.

Lemma 2.1 *Let $0 < p \le \infty$ and $f \in H(\mathbb{C})$. Then $f \in \mathcal{F}^p$ if and only if*

$$\frac{f^{(n)}(z)}{(1+|z|)^n} e^{-\frac{|z|^2}{2}} \in L^p(\mathbb{C}, dA)$$

for any non-negative integer n. Moreover,

$$\|f\|_p \asymp \sum_{j=0}^{n-1} |f^{(j)}(0)| + \left(\int_{\mathbb{C}} \frac{|f^{(n)}(z)|^p}{(1+|z|)^{np}} e^{-\frac{p}{2}|z|^2} dA(z) \right)^{1/p}$$

for $0 < p < \infty$. *And*

$$\|f\|_\infty \asymp \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{C}} \frac{|f^{(n)}(z)|}{(1+|z|)^n} e^{-\frac{|z|^2}{2}}.$$

By Lemma 2.1, we can easily get the following estimate for derivatives of functions in Fock space. See also [4] for details.

Lemma 2.2 *Let* $0 < p \leq \infty$ *and* n *be a non-negative integer. Then*

$$|f^{(n)}(z)| \lesssim (1+|z|)^n e^{\frac{|z|^2}{2}} \|f\|_p$$

for all $f \in \mathcal{F}^p$ *and* $z \in \mathbb{C}$.

For $z \in \mathbb{C}$ and $r > 0$, write

$$D(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$$

for the Euclidean disk centered at z with radius r. A sequence $\{a_j\}$ in $\mathbb{C}$ is called an r-lattice if the following conditions are satisfied:

(i) $\cup_{j=1}^{\infty} D(a_j, r) = \mathbb{C}$,
(ii) $\{D(a_j, \frac{r}{2})\}_{j=1}^{\infty}$ are pairwise disjoint.

For example, the set of points on $r\mathbb{Z}^2$ is an r-lattice. With hypotheses (i) and (ii), it is easy to check that

(iii) there exists a positive integer N (depending only on r) such that every point in $\mathbb{C}$ belongs to at most N of the sets $\{D(a_j, r)\}$.

Lemma 2.3 ([6]) *Let* $1 \leq s \leq \infty$, μ *be a positive Borel measure on* $\mathbb{C}$ *and* $\{a_j\}$ *be an* r*-lattice in* $\mathbb{C}$. *Then the following conditions are equivalent:*

(i) $\{\mu(D(a_j, r))\} \in l^s$,
(ii) $\mu(D(\cdot, \delta)) \in L^s(\mathbb{C}, dA)$ *for some (or any)* $\delta > 0$.

The boundedness and compactness of $W_{\psi,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ for $0 < q < p < \infty$ has been charaterized in [10]. We state the results as follows.

Lemma 2.4 ([10]) *Let* $0 < q < p < \infty$, $\psi \in \mathcal{F}^q$ *and* $\varphi(z) = az+b$ *with* $0 < |a| \leq 1$. *Then the following conditions are equivalent:*

(i) $W_{\psi,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ *is bounded;*

(ii) $W_{\psi,\varphi}:\mathcal{F}^p\to\mathcal{F}^q$ *is compact;*

(iii) $|\psi(z)|e^{\frac{|\varphi(z)|^2-|z|^2}{2}}\in L^{\frac{pq}{p-q}}(\mathbb{C},dA)$.

In order to study the difference $W_{\psi_1,\varphi_1}-W^{(n)}_{\psi_2,\varphi_2}$, we need to estimate the norm of the reproducing kernels from below. The following lemmas play important roles in our proof.

Lemma 2.5 *Let n be a positive integer, $\alpha,\beta\in\mathbb{C}\backslash\{0\}$ and $\varepsilon>0$. Then there exists a contant $C=C(\varepsilon,n)>0$, independent of α and β, such that*

$$\|\alpha K_{w_1}+\beta K^{[n]}_{w_2}\|_\infty\ge C\left(|\alpha|\|K_{w_1}\|_\infty+|\beta|\|K^{[n]}_{w_2}\|_\infty\right)$$

for all $w_1,w_2\in\mathbb{C}$ with $|w_1-w_2|\ge\varepsilon$.

Proof Put $L=\|\alpha K_{w_1}+\beta K^{[n]}_{w_2}\|_\infty$ for brevity. It is enough to prove that $L\ge C|\alpha|\|K_{w_1}\|_\infty$ for some $C=C(\varepsilon,n)>0$. After achieving that, it follows immediately from the triangle inequality that

$$\begin{aligned}|\beta|\|K^{[n]}_{w_2}\|_\infty&\le\|\alpha K_{w_1}+\beta K^{[n]}_{w_2}\|_\infty+|\alpha|\|K_{w_1}\|_\infty\\&\le(1+\frac{1}{C})L.\end{aligned}$$

Firstly, we know that

$$L\ge|\alpha K_{w_1}(z)+\beta K^{[n]}_{w_2}(z)|e^{-\frac{|z|^2}{2}}=|\alpha e^{\overline{w_1}z}+\beta z^ne^{\overline{w_2}z}|e^{-\frac{|z|^2}{2}}\tag{1}$$

for all $z\in\mathbb{C}$. By Lemma 2.1, there exists a constant $C_1=C_1(n)>0$ such that

$$\begin{aligned}C_1L&\ge\frac{|\alpha K^{(n)}_{w_1}(z)+\beta(K^{[n]}_{w_2})^{(n)}(z)|}{(n+|z|)^n}e^{-\frac{|z|^2}{2}}\\&=\frac{\left|\alpha\overline{w_1}^ne^{\overline{w_1}z}+\beta h(\overline{w_2}z)e^{\overline{w_2}z}\right|}{(n+|z|)^n}e^{-\frac{|z|^2}{2}}\end{aligned}\tag{2}$$

for all $z\in\mathbb{C}$, where $h(x)=\sum_{k=0}^n\binom{n}{k}\frac{n!}{(n-k)!}x^{n-k}$. Let $R>4$ be a sufficiently large number satisfying

$$\frac{R^n}{(n+R)^n}>\frac{1}{2}(1+e^{-\frac{\varepsilon^2}{2}}).$$

If $|w_1|\le R$, then

$$L\ge|\alpha|\ge e^{-\frac{R^2}{2}}|\alpha|e^{\frac{|w_1|^2}{2}}.$$

If $|w_1| > R$ and $|w_2| \le 1$, then taking $z = w_2 + \zeta$ in (1), where $\zeta = w_2/|w_2|$ if $w_2 \neq 0$ and $\zeta = 1$ if $w_2 = 0$, we get

$$\begin{aligned} L &\ge e^{-\frac{1}{2}}|\beta|(1+|w_2|)^n e^{\frac{|w_2|^2}{2}} - |\alpha| e^{\frac{|w_1|^2}{2}} e^{-\frac{|w_1-w_2-\zeta|^2}{2}} \\ &\ge e^{-\frac{1}{2}}|\beta|(1+|w_2|)^n e^{\frac{|w_2|^2}{2}} - e^{-2}|\alpha| e^{\frac{|w_1|^2}{2}}. \end{aligned} \tag{3}$$

Taking $z = w_1$ in (2), we have

$$\begin{aligned} C_1 L &\ge |\alpha| \frac{|w_1|^n}{(n+|w_1|)^n} e^{\frac{|w_1|^2}{2}} - |\beta| \frac{h(|w_1 w_2|)}{(n+|w_1|)^n} e^{\frac{|w_2|^2}{2}} e^{-\frac{|w_1-w_2|^2}{2}} \\ &\ge \frac{1}{2}|\alpha| e^{\frac{|w_1|^2}{2}} - e^{-\frac{9}{2}}|\beta|(1+|w_2|)^n e^{\frac{|w_2|^2}{2}}. \end{aligned} \tag{4}$$

Adding (3) and (4), we obtain that

$$(1+C_1)L \ge (\frac{1}{2} - \frac{1}{e^2})|\alpha| e^{\frac{|w_1|^2}{2}}.$$

If $|w_1| > R$ and $|w_2| > 1$, then taking $z = w_2$ in (1) and taing $z = w_1$ in (2), we get

$$\begin{aligned} L &\ge |\beta||w_2|^n e^{\frac{|w_2|^2}{2}} - |\alpha| e^{\frac{|w_1|^2}{2}} e^{-\frac{|w_1-w_2|^2}{2}} \\ &\ge |\beta||w_2|^n e^{\frac{|w_2|^2}{2}} - e^{-\frac{\varepsilon^2}{2}}|\alpha| e^{\frac{|w_1|^2}{2}} \end{aligned} \tag{5}$$

and

$$\begin{aligned} C_1 L &\ge |\alpha| \frac{|w_1|^n}{(n+|w_1|)^n} e^{\frac{|w_1|^2}{2}} - |\beta| \frac{h(|w_1 w_2|)}{(n+|w_1|)^n} e^{\frac{|w_2|^2}{2}} e^{-\frac{|w_1-w_2|^2}{2}} \\ &\ge \frac{1}{2}(1+e^{-\frac{\varepsilon^2}{2}})|\alpha| e^{\frac{|w_1|^2}{2}} - |\beta||w_2|^n e^{\frac{|w_2|^2}{2}}. \end{aligned} \tag{6}$$

Adding (5) and (6), we obtain that

$$(1+C_1)L \ge \frac{1}{2}\left(1 - e^{-\frac{\varepsilon^2}{2}}\right)|\alpha| e^{\frac{|w_1|^2}{2}}.$$

Therefore, letting $C = \min\{e^{-\frac{R^2}{2}}, \frac{1}{1+C_1}(\frac{1}{2} - \frac{1}{e^2}), \frac{1}{2(1+C_1)}(1 - e^{-\frac{\varepsilon^2}{2}})\}$, we have

$$L \ge C|\alpha| e^{\frac{|w_1|^2}{2}} = C|\alpha| \|K_{w_1}\|_\infty.$$

The proof is complete. □

Lemma 2.6 *Let n be a positive integer and $\alpha, \beta \in \mathbb{C}\backslash\{0\}$. Then there exists a contant $C > 0$, independent of α and β, such that*

$$\|\alpha K_{w_1} + \beta K_{w_2}^{[n]}\|_\infty^2 \ge C|\alpha||\beta| e^{\frac{|w_1|^2+|w_2|^2}{2}} \frac{|w_1-w_2|^4}{(1+|w_1-w_2|^2)^2}.$$

for all $w_1, w_2 \in \mathbb{C}$.

Proof Put $L = \|\alpha K_{w_1} + \beta K_{w_2}^{[n]}\|_\infty$ for brevity. Let $R > 4$ be a sufficiently large number satisfying

$$\frac{R^n}{(n+R)^n} > \frac{1}{2}(1 + e^{-\frac{1}{2}}).$$

If $|w_1| \le R$ or $|w_1| > R$ with $|w_1 - w_2| \ge 1$, then by Lemma 2.5, we have

$$L \ge C\left(|\alpha|\|K_{w_1}\|_\infty + |\beta|\|K_{w_2}^{[n]}\|_\infty\right) \ge C(|\alpha|e^{\frac{|w_1|^2}{2}} + |\beta|e^{\frac{|w_2|^2}{2}}),$$

where C is independent of α and β. It follow that $L^2 \ge C|\alpha||\beta|e^{\frac{|w_1|^2+|w_2|^2}{2}}$.

It remains to prove the results for the case where $|w_1| > R$ and $|w_1 - w_2| < 1$. If $|w_2| \ge |w_1|$, then taking $z = w_1$ in (1), we have

$$L \ge |\alpha e^{|w_1|^2} + \beta w_1^n e^{\overline{w_2} w_1}|e^{-\frac{|w_1|^2}{2}} \tag{7}$$

And taking $z = w_2$ in (1), we have

$$L \ge |\alpha e^{\overline{w_1} w_2} + \beta w_2^n e^{|w_2|^2}|e^{-\frac{|w_2|^2}{2}}. \tag{8}$$

Adding (7) and (8), we obtain that

$$\begin{aligned} L &\ge C|\beta||w_2|^n e^{\frac{|w_2|^2}{2}} \left(1 - \left|\frac{w_1}{w_2}\right|^n e^{-\frac{|w_1-w_2|^2}{2}}\right) \\ &\ge C|\beta||w_2|^n e^{\frac{|w_2|^2}{2}} \left(e^{\frac{|w_1-w_2|^2}{2}} - 1\right) \\ &\ge C|\beta||w_2|^n e^{\frac{|w_2|^2}{2}} |w_1 - w_2|^2 \end{aligned}$$

and

$$\begin{aligned} L &\ge C|\alpha|\frac{1}{|w_1|^n} e^{\frac{|w_1|^2}{2}} \left(1 - \left|\frac{w_1}{w_2}\right|^n e^{-\frac{|w_1-w_2|^2}{2}}\right) \\ &\ge C|\alpha|\frac{1}{|w_1|^n} e^{\frac{|w_1|^2}{2}} \left(e^{\frac{|w_1-w_2|^2}{2}} - 1\right) \\ &\ge C|\alpha|\frac{1}{|w_1|^n} e^{\frac{|w_1|^2}{2}} |w_1 - w_2|^2. \end{aligned}$$

Thus, $L^2 \ge C|\alpha||\beta| \left|\frac{w_2}{w_1}\right|^n e^{\frac{|w_1|^2+|w_2|^2}{2}} \ge C|\alpha||\beta|e^{\frac{|w_1|^2+|w_2|^2}{2}}$.

Now suppose $|w_1| > R$, $|w_1 - w_2| < 1$ and $|w_2| < |w_1|$. Let $\zeta = \sqrt{3}n\frac{w_1-w_2}{|w_1-w_2|}i$ or $\zeta = -\sqrt{3}n\frac{w_1-w_2}{|w_1-w_2|}i$ such that the included angle between ζ and w_2 is not more than

$\frac{\pi}{2}$. Taking $z = w_2 + \zeta$ in (1), we get

$$\begin{aligned} L &\geq \left| \alpha e^{\overline{w_1}(w_2+\zeta)} + \beta (w_2+\zeta)^n e^{\overline{w_2}(w_2+\zeta)} \right| e^{\frac{|w_2+\zeta|^2}{2}} \\ &\geq (|\beta||w_2+\zeta|^n e^{\frac{|w_2|^2}{2}} - |\alpha||e^{\overline{w_1}w_2}|e^{-\frac{|w_2|^2}{2}}|e^{\overline{w_1-w_2}\zeta}|)e^{-\frac{|\zeta|^2}{2}} \\ &\geq C\left(|\beta|\left(|w_2| + \frac{n}{|w_2|}\right)^n e^{\frac{|w_2|^2}{2}} - |\alpha| e^{\frac{|w_1|^2}{2}} e^{-\frac{|w_1-w_2|^2}{2}}\right). \end{aligned} \tag{9}$$

By Lemma 2.1, we have

$$\begin{aligned} L &\geq C \sup_{|z|>1} \frac{\left|\alpha \overline{w_1}^n e^{\overline{w_1}z} + \beta h(\overline{w_2}z) e^{\overline{w_z}z}\right|}{|z|^n} e^{-\frac{|z|^2}{2}} \\ &\geq C \frac{|\alpha \overline{w_1}^n e^{|w_1|^2} + \beta h(\overline{w_2}w_1) e^{\overline{w_2}w_1}|}{|w_1|^n} e^{-\frac{|w_1|^2}{2}} \\ &\geq C|\alpha| e^{\frac{|w_1|^2}{2}} - |\beta|\left(|w_2| + \frac{n}{|w_2|}\right)^n e^{\frac{|w|^2}{2}} e^{-\frac{|w_1-w_2|^2}{2}} \end{aligned} \tag{10}$$

Thus, adding (9) and (10), we obtain that

$$\begin{aligned} L &\geq C(|\alpha| e^{\frac{|w_1|^2}{2}} + |\beta||w_2|^n e^{\frac{|w_2|^2}{2}})(1 - e^{-\frac{|w_1-w_2|^2}{2}}) \\ &\geq C(|\alpha| e^{\frac{|w_1|^2}{2}} + |\beta| e^{\frac{|w_2|^2}{2}}) \frac{|w_1-w_2|^2}{1+|w_1-w_2|^2}. \end{aligned}$$

It follows that $L^2 \geq C|\alpha||\beta| e^{\frac{|w_1|^2+|w_2|^2}{2}} \frac{|w_1-w_2|^4}{(1+|w_1-w_2|^2)^2}$. The proof is complete. □

3 The proof of Theorem A

In order to prove Theorem A for the case $0 < q < p < \infty$, we also need to use the classical Khinchine's inequality, which is an important tool in complex and functional analysis. Here we recall the basic facts about this inequality.

Let $\{r_k(t)\}$ denotes the sequence of Rademacher functions defined by

$$r_0(t) = \begin{cases} 1, & \text{if } 0 \leq t - [t] < \frac{1}{2} \\ -1, & \text{if } \frac{1}{2} \leq t - [t] < 1, \end{cases}$$

where $[t]$ denotes the largest integer not greater than t and $r_k(t) = r_0(2^k t)$ for $k = 1, 2, \cdots$. If $0 < p < \infty$, then Khinchine's inequality states that

$$\left(\sum_k |c_k|^2\right)^{p/2} \asymp \int_0^1 \left|\sum_k c_k r_k(t)\right|^p dt$$

for complex sequences $\{c_k\}$.

Now we are ready to prove Theorem A.

Proof of Theorem A It is obvious that (d) implies (b) and (b) implies (a). The equivalence of (c) and (d) follows from Proposition 3.1, Corollary 3.2 in [10] and Lemma 2.4. Thus we only need to prove (a) implies (d).

Suppose $W_{\psi_1,\varphi_1} - W_{\psi_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded. Let

$$M_i = \sup_{z\in\mathbb{C}} |\psi_i(z)| e^{\frac{|\varphi_i(z)|^2 - |z|^2}{2}}, \quad i = 1, 2.$$

By [11, Proposition 2.2], we have $M_i < \infty$ and $\varphi_i(z) = a_i z + b_i$ with $|a_i| \le 1$ for $i = 1, 2$. We consider the following three cases.

Case 1. $a_1 = a_2 = 0$, i.e. $\varphi_1(z) \equiv b_1$ and $\varphi_2(z) \equiv b_2$ with $b_1 \ne b_2$ since $\varphi_1 \ne \varphi_2$. Then

$$\psi_1 - \psi_2 = (W_{\psi_1,\varphi_1} - W_{\psi_2,\varphi_2})1 \in \mathcal{F}^q \tag{11}$$

and $b_1\psi_1 - b_2\psi_2 = (W_{\psi_1,\varphi_1} - W_{\psi_2,\varphi_2})z \in \mathcal{F}^q$. Thus $\psi_1, \psi_2 \in \mathcal{F}^q$. It follows that both W_{ψ_1,φ_1} and $W_{\psi_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are compact by [10, Corollary 3.2].

Case 2. $a_1 = 0$, $a_2 \ne 0$ (it is similar for the case $a_1 \ne 0$, $a_2 = 0$). Then

$$\psi_2(z) \le M_2 e^{\frac{|z|^2 - |a_2 z - b_2|^2}{2}} \lesssim M_2 e^{\frac{(1-|a_2|^2)|z|^2}{2}}, \tag{12}$$

which means that $\psi_2 \in \mathcal{F}^q$. Combining this with (11), we also have $\psi_1 \in \mathcal{F}^q$. By [10, Corollary 3.2] again, we have $W_{\psi_1,\varphi_1} : \mathcal{F}^p \to \mathcal{F}^q$ is compact, then so is W_{ψ_2,φ_2}.

Case 3. $a_1 \ne 0$ and $a_2 \ne 0$. In this case, a similar argument as (12) gives $\psi_1, \psi_2 \in \mathcal{F}^q$. Let $\{a_j\}$ be an $r-$lattice in $\mathbb{C}$. For any $\{\lambda_j\} \in l^p$, we set

$$f(z) = \sum_j \lambda_j k_{a_j}(z).$$

Then $f \in \mathcal{F}^p$ with $\|f\|_p \lesssim \|\{\lambda_j\}\|_{l^p}$. If $W_{\psi_1,\varphi_1} - W_{\psi_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded, then

$$\int_{\mathbb{C}} \left| \sum_j \lambda_j (W_{\psi_1,\varphi_1} - W_{\psi_2,\varphi_2}) k_{a_j}(z) \right|^q e^{-\frac{q}{2}|z|^2} dA(z) \lesssim \|\{\lambda_j\}\|_{l^p}^q. \tag{13}$$

In (13), we replace λ_j by $r_j(t)\lambda_j$, so that the right-hand side does not change. Then we integrate both sides with respact to t from 0 to 1 to obtain

$$\int_0^1 \int_{\mathbb{C}} \left| \sum_j r_j(t) \lambda_j (W_{\psi_1,\varphi_1} - W_{\psi_2,\varphi_2}) k_{a_j}(z) \right|^q e^{-\frac{q}{2}|z|^2} dA(z) \lesssim \|\{\lambda_j\}\|_{l^p}^q.$$

By Fubini's Theorem and Khinchine's inequality,

$$\begin{aligned}
&\int_{\mathbb{C}}\left(\sum_j |\lambda_j|^2 |(W_{\psi_1,\varphi_1}-W_{\psi_2,\varphi_2})k_{a_j}(z)|^2 e^{-|z|^2}\right)^{\frac{q}{2}} dA(z)\\
&\lesssim \int_{\mathbb{C}}\int_0^1 \left|\sum_j r_j(t)\lambda_j (W_{\psi_1,\varphi_1}-W_{\psi_2,\varphi_2})k_{a_j}(z)e^{-\frac{1}{2}|z|^2}\right|^q dt dA(z)\\
&\lesssim \int_0^1 \int_{\mathbb{C}} \left|\sum_j r_j(t)\lambda_j (W_{\psi_1,\varphi_1}-W_{\psi_2,\varphi_2})k_{a_j}(z)e^{-\frac{1}{2}|z|^2}\right|^q dA(z) dt\\
&\lesssim \|\{\lambda_j\}\|_{l^p}^q.
\end{aligned}$$

Recall that there is a positive N such that each point $z \in \mathbb{C}$ belongs to at most N of the disks $\{D(a_j, r)\}$. Let

$$E = \{z \in \mathbb{C} : |\varphi_1(z) - \varphi_2(z)| > 3r\}.$$

Applying Minkowski's inequality if $\frac{2}{q} \leq 1$ and Hölder's inequality if $\frac{2}{q} > 1$, we obtain

$$\begin{aligned}
&\int_{\varphi_1^{-1}(D(a_j,r))} \sum_j |\lambda_j|^q |(W_{\psi_1,\varphi_1}-W_{\psi_2,\varphi_2})k_{a_j}(z)|^q e^{-\frac{q}{2}|z|^2}\chi_E(z)dA(z)\\
&\leq \max\{1, N^{1-\frac{q}{2}}\}\int_{\varphi_1^{-1}(D(a_j,r))\cap E}\left(\sum_j |\lambda_j|^2 |(W_{\psi_1,\varphi_1}-W_{\psi_2,\varphi_2})k_{a_j}(z)|^2 e^{-|z|^2}\right)^{\frac{q}{2}} dA(z)\\
&\lesssim \int_{\mathbb{C}}\left(\sum_j |\lambda_j|^2 |(W_{\psi_1,\varphi_1}-W_{\psi_2,\varphi_2})k_{a_j}(z)|^2 e^{-|z|^2}\right)^{\frac{q}{2}} dA(z)\\
&\lesssim \|\{\lambda_j\}\|_{l^p}^q.
\end{aligned}$$

It follows from duality argument that

$$\int_{\varphi_1^{-1}(D(a_j,r))} |(W_{\psi_1,\varphi_1}-W_{\psi_2,\varphi_2})k_{a_j}(z)|^q e^{-\frac{q}{2}|z|^2}\chi_E(z)dA(z) \in l^{\frac{p}{p-q}}.$$

If $z \in \varphi_1^{-1}(D(a_j, r)) \cap E$, then

$$\begin{aligned}
&|(W_{\psi_1,\varphi_1}-W_{\psi_2,\varphi_2})k_{a_j}(z)|e^{-\frac{1}{2}|z|^2}\\
&\geq |\psi_1(z)|e^{\frac{|\varphi_1(z)|^2-|z|^2}{2}}e^{-\frac{|a_j-\varphi_1(z)|^2}{2}} - |\psi_2(z)|e^{\frac{|\varphi_1(z)|^2-|z|^2}{2}}e^{-\frac{|a_j-\varphi_2(z)|^2}{2}}\\
&\geq e^{-\frac{1}{2}r^2}\left(|\psi_1(z)|e^{\frac{|\varphi_1(z)|^2-|z|^2}{2}} - e^{-\frac{3}{2}r^2}|\psi_2(z)|e^{\frac{|\varphi_1(z)|^2-|z|^2}{2}}\right)
\end{aligned}$$

Thus

$$\int_{\varphi_1^{-1}(D(a_j,r))\cap E} \left(|\psi_1(z)|e^{\frac{|\varphi_1(z)|^2-|z|^2}{2}} - e^{-\frac{3}{2}r^2}|\psi_2(z)|e^{\frac{|\varphi_2(z)|^2-|z|^2}{2}}\right)^q dA(z) \in l^{\frac{p}{p-q}}.$$

It follows from Lemma 2.3 that

$$\int_{\varphi_1^{-1}(D(w,r))\cap E} \left(|\psi_1(z)|e^{\frac{|\varphi_1(z)|^2-|z|^2}{2}} - e^{-\frac{3}{2}r^2}|\psi_2(z)|e^{\frac{|\varphi_2(z)|^2-|z|^2}{2}}\right)^q dA(z) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA).$$

This implies that

$$\int_{D(w,\frac{r}{|a_1|})} \left(|\psi_1(z)|e^{\frac{|\varphi_1(z)|^2-|z|^2}{2}} - e^{-\frac{3}{2}r^2}|\psi_2(z)|e^{\frac{|\varphi_2(z)|^2-|z|^2}{2}}\right)^q \chi_E(z)dA(z) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA). \tag{14}$$

Similarly, we have

$$\int_{D(w,\frac{r}{|a_2|})} \left(|\psi_2(z)|e^{\frac{|\varphi_2(z)|^2-|z|^2}{2}} - e^{-\frac{3}{2}r^2}|\psi_1(z)|e^{\frac{|\varphi_1(z)|^2-|z|^2}{2}}\right)^q \chi_E(z)dA(z) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA). \tag{15}$$

Letting $\delta = \min\{\frac{r}{|a_1|}, \frac{r}{|a_2|}\}$ and adding (14) and (15), we obtain

$$\int_{D(w,\delta)} |\psi_i(z)|^q e^{\frac{q}{2}(|\varphi_i(z)|^2-|z|^2)} \chi_E(z)dA(z) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA), \quad i = 1, 2.$$

It follows that

$$\int_{D(w,\delta)} |\psi_i(z)|^q e^{\frac{q}{2}(|\varphi_i(z)|^2-|z|^2)} dA(z) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA), \quad i = 1, 2$$

since $M_i < \infty$ and E^c is bounded in $\mathbb{C}$. Notice that

$$|\psi_i(z)|e^{\frac{|\varphi_i(z)|^2-|z|^2}{2}} = |\psi_i(z)e^{a_i\overline{b_i}z}|e^{\frac{(|a_i|^2-1)|z|^2+|b_i|^2}{2}}.$$

By [15, Lemma 2.32], there exists a constant $C = C(q, r)$ such that

$$\int_{D(w,\delta)} |\psi_i(z)|^q e^{\frac{q}{2}(|\varphi_i(z)|^2-|z|^2)} dA(z) \geq C|\psi_i(w)|^q e^{\frac{q}{2}(|\varphi_i(w)|^2-|w|^2)}$$

for $i = 1, 2$. Therefore, $|\psi_i(w)|e^{\frac{|\varphi_i(w)|^2-|w|^2}{2}} \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$. Then the compactness of W_{ψ_i,φ_i} for $i = 1, 2$ is established by Lemma 2.4. The proof is complete. □

4 The proof of Theorem B

In this section, we give the proof of Theorem B.

Lemma 4.1 *Let $0 < p \leq \infty$, n be a positive integer and $\varphi_1 \neq \varphi_2$. If $W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}$ is bounded on $\mathcal{F}^p$, then $\varphi_1(z) - \varphi_2(z) = az + b$ for some $a, b \in \mathbb{C}$.*

Proof Suppose $W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}$ is bounded on $\mathcal{F}^p$. By Lemma 2.2, we have

$$\begin{aligned} \|W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}\| &\geq \|(W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2})k_w\|_p \\ &\geq |(W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2})k_w(z)|e^{-\frac{|z|^2}{2}} \\ &= |\psi_1(z)K_w(\varphi_1(z)) - \psi_2(z)K^{(n)}_w(\varphi_2(z))|e^{-\frac{|w|^2}{2}}e^{-\frac{|z|^2}{2}} \\ &= |\overline{\psi_1(z)}K_{\varphi_1(z)}(w) - \overline{\psi_2(z)}K^{[n]}_{\varphi_2(z)}(w)|e^{-\frac{|w|^2}{2}}e^{-\frac{|z|^2}{2}} \end{aligned} \tag{16}$$

for all $z, w \in \mathbb{C}$. This means that

$$\|\overline{\psi_1(z)}K_{\varphi_1(z)} - \overline{\psi_2(z)}K^{[n]}_{\varphi_2(z)}\|_\infty e^{-\frac{|z|^2}{2}} \leq \|W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}\| \tag{17}$$

for all $z \in \mathbb{C}$. Combining this with Lemma 2.6, we get

$$|\psi_1(z)\psi_2(z)|e^{\frac{|\varphi_1(z)|^2+|\varphi_2(z)|^2}{2}-|z|^2}\frac{|\varphi_1(z)-\varphi_2(z)|^4}{1+|\varphi_1(z)-\varphi_2(z)|^4} \lesssim \|W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}\|$$

for all $z \in \mathbb{C}$. Therefore

$$\sup_{z\in\mathbb{C}} |\psi_1(z)\psi_2(z)|\frac{|\varphi_1(z)-\varphi_2(z)|^4}{1+|\varphi_1(z)-\varphi_2(z)|^4}e^{\left|\frac{\varphi_1(z)-\varphi_2(z)}{2}\right|^2-|z|^2} < \infty.$$

Modifying the proof of [8, Proposition 2.1], we obtain $\varphi_1(z) - \varphi_2(z) = az + b$ for some $a, b \in \mathbb{C}$. The proof is complete. □

We are now ready to prove Theorem B. For any $t > 0$, we denote

$$\Omega_t = \{z \in \mathbb{C} : |\varphi_1(z) - \varphi_2(z)| < t\}.$$

Proof of Theorem B The "if part" is trivial, we only need to prove the "only if part".

First assume $W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}$ is bounded on $\mathcal{F}^p$. Lemma 4.1 tells us that $\varphi_1(z) - \varphi_2(z) = az + b$. Let $t = \frac{b}{2}$ if $a = 0$ and $t = 1$ if $a \neq 0$. Then there exists $R > 0$ such that $\{|z| > R\} \subset \Omega_t^c$. By (17) and Lemma 2.5, we can find a constant $C > 0$, independent of ψ_1 and ψ_2, such that

$$\|W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}\| \geq C|\psi_1(z)|e^{\frac{|\varphi_1(z)|^2-|z|^2}{2}}$$

for all $z \in \Omega_t^c$. It follows that $\sup_{z\in\mathbb{C}} |\psi_1(z)|e^{\frac{|\varphi_1(z)|^2-|z|^2}{2}} < \infty$ since Ω_t is bounded in $\mathbb{C}$. Obviously, $\psi_1 = (W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2})1 \in \mathcal{F}^q$. Thus according to [10, Theorem 3.4], W_{ψ_1,φ_1} is bounded, then so is $W^{(n)}_{\psi_2,\varphi_2}$.

Now suppose $W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}$ is compact on $\mathcal{F}^p$, then both W_{ψ_1,φ_1} and $W^{(n)}_{\psi_2,\varphi_2}$ are bounded. It follows from [10, Propsition 3.1] and [4, Lemma 4.3] that

$$M_1 = \sup_{z\in\mathbb{C}} |\psi_1(z)|e^{\frac{|\varphi_1(z)|^2-|z|^2}{2}} < \infty$$

and

$$M_2' = \sup_{z\in\mathbb{C}} |\psi_2(z)||\varphi_2(z)|^n e^{\frac{|\varphi_2(z)|^2-|z|^2}{2}} < \infty.$$

Moreover, $\varphi_i(z) = a_i z + b_i$ for $i = 1, 2$. If $a_1 = 0$, then [10, Corollary 3.2] tells us that W_{ψ_1,φ_1} is bounded on $\mathcal{F}^p$. So is $W^{(n)}_{\psi_2,\varphi_2}$. Now we consider the case $a_1 \neq 0$.

Taking $w = \varphi_1(z)$ in (16), we obtain

$$\begin{aligned} &|\psi_1(z)|e^{\frac{|\varphi_1(z)|-|z|^2}{2}} - M_2' \left|\frac{\varphi_1(z)}{\varphi_2(z)}\right|^n e^{-\frac{|\varphi_1(z)-\varphi_2(z)|^2}{2}} \\ \le &\|(W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2})k_{\varphi_1(z)}\|_p. \end{aligned} \tag{18}$$

The compactness of $W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}$ yields that $\|(W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2})k_{\varphi_1(z)}\|_p \to 0$ as $|z| \to \infty$. If $a_1 \neq a_2$, then $\lim_{|z|\to\infty} |\varphi_1(z) - \varphi_2(z)| = \infty$. Then by (18), we have

$$\lim_{|z|\to\infty} |\psi_1(z)|e^{\frac{|\varphi_1(z)|-|z|^2}{2}} = 0.$$

If $0 \neq a_1 = a_2$, then $b_1 \neq b_2$ and $|\varphi_1(z)|, |\varphi_2(z)| \to \infty$ as $|z| \to \infty$. By the proof of Lemma 2.5, we obtain

$$\begin{aligned} &|\psi_1(z)|e^{\frac{|\varphi_1(z)|-|z|^2}{2}} \\ \lesssim &\|(W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}k_{\varphi_1(z)}\|_p + \|(W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}k_{\varphi_2(z)}\|_p \end{aligned}$$

for $|z|$ large enough. Then the compactness of $W_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2}$ yields that

$$\lim_{|z|\to\infty} |\psi_1(z)|e^{\frac{|\varphi_1(z)|-|z|^2}{2}} = 0.$$

Therefore, by [10, Theorem 3.4], W_{ψ_1,φ_1} is compact. Then so is $W^{(n)}_{\psi_2,\varphi_2}$. The proof is complete. □

Remark The methods in this paper can be applied to study the difference $W^{(m)}_{\psi_1,\varphi_1} - W^{(n)}_{\psi_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ with $p \neq q$, $\varphi_1 \neq \varphi_2$ and $m \neq n$ through more elaborate computations.

Availability of data and materials No data was generalited by this project.

Declarations

Conflict of interest The author declares that there are no Conflict of interest regarding the publication of this paper.

References

1. Arroussi, H., Tong, C.: Weighted composition operators between large Fock spaces in several complex variables. J. Funct. Anal. **277**, 3436–3466 (2019)
2. Choe, H., Izuchi, K., Koo, H.: Linear sums of two composition operators on the Fock space. J. Math. Anal. Appl. **369**, 112–119 (2010)
3. Cowen, C., MacCluer, B.: Composition Operators on Spaces of Analytic Functions. CRC Press, New York (1995)
4. Hu, J., Li, S., Qu, D.: Embedding derivatives of Fock spaces and generalized weighted composition operators. J. Nolinear. Var. Anal. **5**, 589–613 (2021)
5. Hu, Z.: Equivalent norms on Fock spaces with some applications to extended Cesàro operators. Proc. Amer. Math. Soc. **141**, 2829–2840 (2013)
6. Hu, Z., Lv, X.: Toeplitz operators from one Fock space to another. Integral Equ. Oper. Theory **70**, 541–559 (2011)
7. Khoi, L., Thom, L., Tien, P.: Weighted composition operators between Fock spaces $\mathcal{F}^\infty(\mathbb{C})$ and $\mathcal{F}^p(\mathbb{C})$. Internat J. Math. **30**, 1950015 (2019)
8. Le, T.: Normal and isometric weighted composition operators on Fock spaces. Bull. London Math. Soc. **46**, 847–856 (2014)
9. Moorhouse, J.: Compact differences of composition operators. J. Funct. Anal. **219**, 70–92 (2005)
10. Tien, P., Khoi, L.: Weighted composition operators between Fock spaces. Potential Anal. **50**, 171–195 (2019)
11. Tien, P., Khoi, L.: Differences of weighted composition operators between the Fock space. Monatsh. Math. **188**, 183–193 (2019)
12. Tien, P., Khoi, L.: Weighted composition operators between Fock spaces in several variables. Math. Nachr. **293**, 1200–1220 (2020)
13. Shapiro, J.: Composition Operators and Classical Function Theory. Springer-Verlag, New York (1993)
14. Ueki, S.: Characterization for Fock-type space via higher order derivatives and its application. Complex Anal. Oper. Theory. **8**, 1457–1486 (2014)
15. Zhu, K.: Analysis on Fock spaces. Springer, New York (2012)