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Ill-posedness for the gCH-mCH equation in Besov spaces

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Abstract

In this paper, we consider the Cauchy problem to the generalized Fokas–Qiao–Xia–Li/generalized Camassa–Holm-modified Camassa–Holm (gFQXL/gCH-mCH) equation, which includes the Camassa–Holm equation, the generalized Camassa–Holm equation, the Novikov equation, the Fokas–Olver–Rosenau–Qiao/Modified Camassa–Holm equation and the Fokas–Qiao–Xia–Li/Camassa–Holm-modified Camassa–Holm equation. We prove the ill-posedness for the Cauchy problem of the gFQXL/gCH-mCH equation in $B^s_{p,\infty}$ with $s > \max\{2 + 1/p, 5/2\}$ and $1 \le p \le \infty$ in the sense that the solution map to this equation starting from u_0 is discontinuous at $t = 0$ in the metric of $B^s_{p,\infty}$.

Keywords FQXL/gCH-mCH equation · Ill-posedness · Besov space

Mathematics Subject Classification 35Q53 · 37K10

1 Introduction

In this paper, we consider the Cauchy problem of the following generalized Fokas–Qiao–Xia–Li/generalized Camassa–Holm-modified Camassa–Holm (gFQXL/gCH-mCH) equation:

$$\begin{cases} m_t + k_1 \left((u^2 - u_x^2)m\right)_x + k_2 \left(u^k m_x + (k+1)u^{k-1}u_x m\right) = 0, \\ m = u - u_{xx}, \\ m(0,x) = m_0(x), \end{cases} \tag{1.1}$$

where $k_1, k_2 \in \mathbb{R}$ and $k \in \mathbb{Z}^+$. $u = u(t,x)$ is a horizontal velocity.

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When $k_1 = 0$ and $k_2 = 1$, Eq. (1.1) reduces to the following generalized Camassa–Holm (gCH) equation which was proposed in [1, 20, 24]:

$$\begin{cases} m_t + u^k m_x + (k+1)u^{k-1}u_x m = 0, \\ m = u - u_{xx}, \\ m(0,x) = m_0(x). \end{cases} \tag{1.2}$$

It is remarkable that Eq. (1.2) possesses peakon solutions $u(t,x) = c^{1/k}\mathrm{e}^{-|x-ct|}$ with $c > 0$ (see [20, 29]). Himonas–Holliman [24] obtained the local well-posedness in Sobolev spaces by means of the Galerkin approximation method, which was generalized to the Besov spaces by Zhang and Liu [54]. It was further proved that the data-to-solution map is continuous [31] but not uniformly continuous [51].

Equation (1.2) can also be a special case of the g-kbCH equation

$$\begin{cases} m_t + u^k m_x + bu^{k-1}u_x m = 0, \quad b \in \mathbb{R}, \\ m = u - u_{xx}, \\ m(0,x) = m_0(x). \end{cases} \tag{1.3}$$

Zhao–Li–Yan [55] obtained the well-posedness of the Cauchy problem (1.3) in Besov space $B^s_{p,r}(\mathbb{R})$ with $s > \max\{1+1/p, 3/2\}$ and $1 \le p, r \le \infty$. However, for $r = \infty$, they established the continuity of the data-to-solution map in a weaker topology. Chen–Li–Yan [5] solved the critical case for $(s, p, r) = (3/2, 2, 1)$. Li–Yu–Zhu [38] proved the sharp ill-posedness of the Cauchy problem for the g-kbCH equation in $B^s_{p,\infty}$ with $s > \max\{1 + 1/p, 3/2\}$ and $1 \le p \le \infty$ in the sense that the solution map to this equation starting from u_0 is discontinuous at $t = 0$ in the metric of $B^s_{p,\infty}$.

When $k = 1$, Eq. (1.2) becomes the well-known Camassa–Holm (CH) equation

$$\begin{cases} u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \\ u(0,x) = u_0(x). \end{cases} \tag{1.4}$$

The CH equation was originally derived as a bi-Hamiltonian system by Fokas and Fuchssteiner [17] in the context of the KdV model and gained prominence after Camassa–Holm [3] independently re-derived it from the Euler equations of hydrodynamics using asymptotic expansions. The CH equation is completely integrable [3, 8] with a bi-Hamiltonian structure [7, 17] and infinitely many conservation laws [3, 17]. Also, it admits exact peaked soliton solutions (peakons) of the form $ce^{-|x-ct|}$ with $c > 0$, which are orbitally stable [9] and models wave breaking (i.e., the solution remains bounded, while its slope becomes unbounded in finite time [10–12]). In 1998, Misiołek [42] showed that the CH equation re-expresses geodesic motion on the Bott–Virasoro group. Subsequently, the CH equation with periodic boundary conditions had been recast as a geodesic flow on the diffeomorphism group of the circle by Kouranbaeva [35] and Constantin–Kolev [13, 14].

When $k = 2$, Eq. (1.2) is the famous Novikov equation [23, 43, 44, 49, 51]

$$\begin{cases} u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \\ u(0, x) = u_0(x). \end{cases} \tag{1.5}$$

Home–Wang [30] proved that the Novikov equation with cubic nonlinearity shares similar properties with the CH equation, such as a Lax pair in matrix form, a bi-Hamiltonian structure, infinitely many conserved quantities and peakon solutions. The Novikov equation admits multi-peakon traveling wave solutions on both the line and the circle. More precisely, on the line the n-peakon

$$u(x, t) = \sum_{j=1}^{n} p_j(t) e^{-|x - q_j(t)|}$$

is a solution to the Novikov equation if and only if the positions $(q_1, \ldots, q_n)$ and the momenta $(p_1, \ldots, p_n)$ satisfy the following system of $2n$ differential equations

$$\begin{cases} \frac{\mathrm{d}q_j}{\mathrm{d}t} = u^2(q_j), \\ \frac{\mathrm{d}p_j}{\mathrm{d}t} = -u(q_j) u_x(q_j) p_j. \end{cases}$$

Himonas–Holliman–Kenig [26] constructed a 2-peakon solution with an asymmetric antipeakon-peakon initial data and showed the Cauchy problem (1.5) on both the line and the circle is ill-posed in Sobolev spaces H^s with $s < 3/2$. One may note that the CH equation has quadratic rather than cubic nonlinearities, and this plays an important role in the analysis of these two equations.

When $k_1 = 1$ and $k_2 = 0$, Eq. (1.1) reduces to the FORQ/MCH equation

$$\begin{cases} m_t + \big((u^2 - u_x^2)m\big)_x = 0, \\ m = u - u_{xx}, \\ m(0, x) = m_0(x). \end{cases} \tag{1.6}$$

Equation (1.6) written in a slightly different form was first derived by Fokas [16] as an integrable generalisation of the modified KdV equation. Fuchssteiner [18] and Olver–Rosenau [45] independently obtained similar versions of this equation by performing a simple explicit algorithm based on the bi-Hamiltonian representation of the classically integrable system. Later, the concise form written above was recovered by Qiao [46] from the two-dimensional Euler equations by using an approximation procedure. The entire integrable hierarchy related to the FORQ equation was proposed by Qiao [47]. It also has bi-Hamiltonian structure, which was first derived in [45] and then in [46], admits Lax pair [46] and peakon travelling wave solutions that are orbitally stable [21, 39, 48]. It should be mentioned that this equation is also referred as the modified Camassa–Holm equation in [4] (we may call it the FORQ/MCH hierarchy). The local well-posedness and ill-posedness of the Cauchy problem for the FORQ equation (1.6) in Sobolev spaces and Besov spaces were studied in the series of papers [19, 24, 25, 27]. Himonas-Mantzavinos [28] showed that the Cauchy problem (1.6) is well-posed in Sobolev space H^s with $s > 5/2$. Fu et al. [19] established the local well-posedness in Besov space $B^s_{p,r}$ with $s > \max\{2 + 1/p, 5/2\}$ and $1 \leq p, r \leq \infty$. It was further proved the data-to-solution map is continuous [28] but not uniformly continuous [32, 50].

When $k = 1$, Eq. (1.1) reduces to the FQXL/CH-mCH equation [6, 22, 40, 41] as an extension of both the CH and FORQ/MCH equations

$$\begin{cases} m_t + k_1 \left((u^2 - u_x^2)m\right)_x + k_2 (um_x + 2u_x m) = 0, \\ m = u - u_{xx}, \\ m(0, x) = m_0(x), \end{cases} \tag{1.7}$$

which was proposed by Fokas [16] from the two-dimensional hydrodynamical equations for surface waves by using tri-Hamiltonian duality to the bi-Hamiltonian Gardner equation. Xia–Qiao–Li [52] discussed its integrability, bi-Hamilton structure, and conservation laws. Although these models mentioned above have similar properties in several aspects, we would like to point out that these equations are truly different. In fact, only a few of them have a geometrical interpretation as geodesic flow. One of the distinctive features of the CH equation is that it comes up in the description of the geodesic flow on the Bott–Virasoro group with respect to certain (weak) right invariant Riemannian metrics [13–15, 35, 42]. For the CH equation, many interesting results under geometric aspects can be found in Kolev's papers [33, 34].

Equation (1.1) can be viewed as a generalization to the FQXL equation or a combination of both gCH and mCH equtions. Based on this reason, we call Eq. (1.1) the gFQXL/gCH-mCH equation. Very recently, based on the transport equation and Littlewood–Paley theory, Yang–Han–Wang [53] proved that the gFQXL/gCH-mCH equation is locally well-posed in Besov spaces. More precisely, they established

Lemma 1.1 [53] *Let $p, r \in [1, \infty]$ and $s > \max\left\{2 + \frac{1}{p}, \frac{5}{2}\right\}$. Assume that $u_0 \in B^s_{p,r}$, then there exists a time $T > 0$ and a unique solution u to the Cauchy problem (1.7) such that the map $B^s_{p,r} \ni u_0 \mapsto u \in \mathcal{C}([0, T]; B^{s'}_{p,r}) \cap \mathcal{C}^1([0, T]; B^{s'-1}_{p,r})$ is continuous for every $s' < s$ when $r = \infty$ and $s' = s$ when $r < \infty$.*

Naturally, we want to ask a question that whether or not the continuity of the data-to-solution map with values in $L^\infty(0, T; B^s_{p,\infty})$ with $s > \max\{2 + 1/p, 5/2\}$ and $1 \leq p \leq \infty$ holds for the gFQXL/gCH-mCH equation. It should be noticed that well-posedness of the Cauchy problem for the g-kbCH equation holds for $s > \max\{1 + 1/p, 3/2\}$ while for the gFQXL/gCH-mCH equation holds for $s > \max\{2 + 1/p, 5/2\}$. This difference between the well-posedness index of g-kbCH equation and gFQXL/gCH-mCH equation may be explained by the presence of the extra term u_x^3 in (1.7). Recently, Li–Yu–Zhu [38] (see also [37]) proved the solution map to the g-kbCH equation starting from u_0 is discontinuous at $t = 0$ in the metric of $B^s_{p,\infty}(\mathbb{R})$ with $s > \max\{1 + 1/p, 3/2\}$, which implies the ill-posedness of the Cauchy problem for this equation in $B^s_{p,\infty}(\mathbb{R})$. However, this ill-posedness result do not cleanly to the gFQXL/gCH-mCH equation. The main difficulty lies in the presence of the extra term u_x^3. In this paper, motivated by the idea used in [38], we shall bypass this obstacle and prove the ill-posedness of the Cauchy problem for the gFQXL/gCH-mCH equation. In order to present our main result, let us reformulate (1.1). Setting $\Lambda^{-2}=(1-\partial_x^2)^{-1}$, then $\Lambda^{-2}f=G*f$ where $G(x)=\frac{1}{2}e^{-|x|}$ is the kernel of the operator Λ^{-2}. Thus, we can transform (1.1) equivalently into the following transport type equation

$$\begin{cases} u_t+\left(k_1u^2-\frac{k_1}{3}u_x^2+k_2u^k\right)u_x+\Lambda^{-2}\left(\frac{k_1}{3}u_x^3+\frac{k_2(k-1)}{2}u^{k-2}u_x^3\right) \\ \quad +\partial_x\Lambda^{-2}\left(\frac{2k_1}{3}u^3+k_1uu_x^2+k_2u^{k+1}+\frac{k_2(2k-1)}{2}u^{k-1}u_x^2\right)=0, \\ u(0,x)=u_0(x). \end{cases} \tag{1.8}$$

Now, we state our main result.

Theorem 1.1 *Let $k_1\in\mathbb{R}\backslash\{0\}$, $k_2\in\mathbb{R}$ and $k\in\mathbb{Z}^+$. Assume that $1\le p\le\infty$ and $s>\max\left\{2+\frac{1}{p},\frac{5}{2}\right\}$. There exists $u_0\in B^s_{p,\infty}(\mathbb{R})$ and a positive constant ε_0 such that the data-to-solution map $u_0\mapsto \mathbf{S}_t(u_0)$ of the Cauchy problem (1.8) satisfies*

$$\limsup_{t\to0^+}\|\mathbf{S}_t(u_0)-u_0\|_{B^s_{p,\infty}}\ge\varepsilon_0.$$

Remark 1.1 Theorem 1.1 demonstrates the ill-posedness of the Cauchy problem (1.8) in $B^s_{p,\infty}$ with $s>\max\{2+1/p,5/2\}$ and $1\le p\le\infty$ in the sense that the solution map to this equation starting from u_0 is discontinuous at $t=0$ in the metric of $B^s_{p,\infty}$.

Organization of our paper In Sect. 2 we present some preliminary results and introduce notation used. In Sect. 3 we rewrite the original system (1.1) by introducing a new unknown quantity. In Sect. 4 we give the proof of main Theorem.

2 Littlewood–Paley analysis

Notation $A\lesssim B$ (resp., $A\gtrsim B$) means that there exists a harmless positive constant c such that $A\le cB$ (resp., $A\ge cB$). Given a Banach space X, we denote its norm by $\|\cdot\|_X$. For $I\subset\mathbb{R}$, we denote by $\mathcal{C}(I;X)$ the set of continuous functions on I with values in X. Sometimes we will denote $L^p(0,T;X)$ by L^p_TX. Next, we will recall some facts about the Littlewood–Paley (L–P) decomposition, the nonhomogeneous Besov spaces and some of their useful properties. Let $\mathcal{B}:=\{\xi\in\mathbb{R}:|\xi|\le\frac{4}{3}\}$ and $\mathcal{C}:=\{\xi\in\mathbb{R}:\frac{3}{4}\le|\xi|\le\frac{8}{3}\}$. There exist two radial functions $\chi\in C_c^\infty(\mathcal{B})$ and $\varphi\in C_c^\infty(\mathcal{C})$ both taking values in [0, 1] such that

$$\chi(\xi)+\sum_{j\ge0}\varphi(2^{-j}\xi)=1\quad\forall\,\xi\in\mathbb{R}.$$

Definition 2.1 [2] For every $u \in \mathcal{S}'(\mathbb{R})$, the L–P dyadic blocks Δ_j are defined as follows

$$\Delta_j u = \begin{cases} 0, & if\ j \leq -2; \\ \chi(D)u = \mathcal{F}^{-1}(\chi \mathcal{F}u), & if\ j = -1; \\ \varphi(2^{-j}D)u = \mathcal{F}^{-1}\big(\varphi(2^{-j}\cdot)\mathcal{F}u\big), & if\ j \geq 0. \end{cases}$$

Definition 2.2 [2] Let $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]^2$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R})$ is defined by

$$B^s_{p,r}(\mathbb{R}) := \Big\{ f \in \mathcal{S}'(\mathbb{R}) :\ \|f\|_{B^s_{p,r}(\mathbb{R})} < \infty \Big\}, \quad \text{where}$$

$$\|f\|_{B^s_{p,r}(\mathbb{R})} = \begin{cases} \Big(\sum_{j\geq -1} 2^{sjr} \|\Delta_j f\|^r_{L^p(\mathbb{R})} \Big)^{1/r}, & if\ 1 \leq r < \infty, \\ \sup_{j\geq -1} 2^{sj} \|\Delta_j f\|_{L^p(\mathbb{R})}, & if\ r = \infty. \end{cases}$$

Remark 2.1 The fact $B^s_{p,\infty}(\mathbb{R}) \hookrightarrow B^t_{p,\infty}(\mathbb{R})$ with $s > t$ will be often used implicity.

We give some important properties which will be also often used throughout the paper.

Lemma 2.1 [2] *(1) Let $(p, r) \in [1, \infty]^2$ and $s > \max\left\{1 + \frac{1}{p}, \frac{3}{2}\right\}$. Then we have*

$$\|uv\|_{B^{s-2}_{p,r}(\mathbb{R})} \leq C\|u\|_{B^{s-2}_{p,r}(\mathbb{R})} \|v\|_{B^{s-1}_{p,r}(\mathbb{R})}.$$

(2) For $(p, r) \in [1, \infty]^2$, $B^{s-1}_{p,r}(\mathbb{R})$ with $s > 1 + \frac{1}{p}$ is an algebra. Moreover, for any $u, v \in B^{s-1}_{p,r}(\mathbb{R})$ with $s > 1 + \frac{1}{p}$, we have

$$\|uv\|_{B^{s-1}_{p,r}(\mathbb{R})} \leq C\|u\|_{B^{s-1}_{p,r}(\mathbb{R})} \|v\|_{B^{s-1}_{p,r}(\mathbb{R})}.$$

(3) Let $m \in \mathbb{R}$ and f be an S^m-multiplier (i.e., $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies that $\forall \alpha \in \mathbb{N}$, there exists a constant $\mathcal{C}_\alpha$ such that $|\partial^\alpha f(\xi)| \leq \mathcal{C}_\alpha(1 + |\xi|)^{m-|\alpha|}$ for all $\xi \in \mathbb{R}$). Then the operator $f(D)$ is continuous from $B^s_{p,r}(\mathbb{R})$ to $B^{s-m}_{p,r}(\mathbb{R})$.

(4) For any $s \in \mathbb{R}$, $(1-\partial_x)^{-1}$ is an isomorphic mapping from $B^{s-1}_{p,r}(\mathbb{R})$ into $B^s_{p,r}(\mathbb{R})$.

(5) For $1 \leq p \leq \infty$ and $s > 0$, there exists a positive constant C such that

$$\Big\| 2^{js} \, \big\| [\Delta_j, v]\partial_x f \big\|_{L^p} \Big\|_{\ell^\infty} \leq C\big(\|\partial_x v\|_{L^\infty} \|f\|_{B^s_{p,\infty}} + \|\partial_x f\|_{L^\infty} \|\partial_x v\|_{B^{s-1}_{p,\infty}} \big),$$

where we denote the standard commutator $[\Delta_j, v]\partial_x f = \Delta_j(v\partial_x f) - v\Delta_j \partial_x f$.

3 Reformulation of system

Due to the presence of the extra term u_x^3 in (1.8), it seems difficult to deal with Eq. (1.8) directly. From Eq. (1.1), we infer that

$$\begin{aligned}&\left(1-\partial_x^2\right)\left(u_t+\left(k_1\left(u^2-u_x^2\right)+k_2u^k\right)u_x\right)\\&\quad=-2k_1\left(u_x^2m\right)_x-2k_1uu_xm+k_2(2k-1)u^{k-1}u_xm\\&\quad-k_2k(k-1)u^{k-2}u_x^3-3k_2ku^ku_x,\end{aligned}$$

which implies

$$\begin{aligned}&u_t+\left(k_1\left(u^2-u_x^2\right)+k_2u^k\right)u_x\\&\quad=k_2\Lambda^{-2}\left((2k-1)u^{k-1}u_xm-k(k-1)u^{k-2}u_x^3-3ku^ku_x\right)\\&\quad-2k_1\partial_x\Lambda^{-2}\left(u_x^2m\right)-2k_1\Lambda^{-2}(uu_xm).\end{aligned}\tag{3.9}$$

Differentiating (3.9) with respect to x yields

$$\begin{aligned}&u_{xt}+\left(k_1\left(u^2-u_x^2\right)+k_2u^k\right)u_{xx}\\&\quad=k_2\partial_x\Lambda^{-2}\left((2k-1)u^{k-1}u_xm-k(k-1)u^{k-2}u_x^3-3ku^ku_x\right)\\&\quad-2k_1\Lambda^{-2}\left(u_x^2m\right)-2k_1\partial_x\Lambda^{-2}(uu_xm)-k_2ku^{k-1}u_x^2.\end{aligned}\tag{3.10}$$

Introducing $v:=(1-\partial_x)u$, which implies $u^2-u_x^2=(2u-v)v$, then we have from (3.9) and (3.10)

$$\begin{cases}\partial_tv+\left(k_1(2u-v)v+k_2u^k\right)\partial_xv=\Phi_1(u)+\Phi_2(v)+\Phi_3(v),\\u=(1-\partial_x)^{-1}v,\\v_0=(1-\partial_x)u_0,\end{cases}\tag{3.11}$$

where the terms $\Phi_1(u)$, $\Phi_2(v)$ and $\Phi_3(v)$ are defined by

$$\begin{aligned}&\Phi_1(u)=-k_2ku^{k-1}u_x^2,\\&\Phi_2(v)=2k_1\partial_x\Lambda^{-2}\left(v_xu_xm\right)-2k_1\Lambda^{-2}(vu_xm),\\&\Phi_3(v)=k_2(1-\partial_x)\Lambda^{-2}\left((2k-1)u^{k-1}u_xm-k(k-1)u^{k-2}u_x^3-3ku^ku_x\right).\end{aligned}$$

Since $(1-\partial_x)^{-1}$ is an isomorphic mapping from $B^{s-1}_{p,r}(\mathbb{R})$ into $B^s_{p,r}(\mathbb{R})$, the ill-posedness of u in $B^{s+1}_{p,r}$ can be transformed into that of v in $B^s_{p,r}$. Based on this observation, we shall consider Eq. (3.11) satisfied by v in the rest of this paper. Now we restate our main result as follows.

Theorem 3.1 *Let $k_1 \in \mathbb{R}\backslash\{0\}$, $k_2 \in \mathbb{R}$ and $k \in \mathbb{Z}^+$. Assume that $1 \le p \le \infty$ and $s > \max\left\{1+\frac{1}{p}, \frac{3}{2}\right\}$. There exists $v_0 \in B^s_{p,\infty}(\mathbb{R})$ and a positive constant ε_0 such that the data-to-solution map $v_0 \mapsto \mathbf{S}_t(v_0)$ of the Cauchy problem (3.11) satisfies*

$$\limsup_{t\to 0^+} \|\mathbf{S}_t(v_0) - v_0\|_{B^s_{p,\infty}} \ge \varepsilon_0.$$

4 Proof of Theorem 3.1

In this section, we will give the proof of Theorem 3.1.

4.1 Construction of initial data

We need to introduce smooth, radial cut-off functions to localize the frequency region. Precisely, let $\widehat{\phi} \in \mathcal{C}_0^\infty(\mathbb{R})$ be an even, real-valued and non-negative function on $\mathbb{R}$ and satisfy

$$\widehat{\phi}(\xi) = \begin{cases} 1, & if\ |\xi| \le \frac{1}{4}, \\ 0, & if\ |\xi| \ge \frac{1}{2}. \end{cases}$$

Remark 4.1 By the Fourier–Plancherel formula, we have $\phi(x) = \mathcal{F}^{-1}(\widehat{\phi}(\xi))$. Obviously

$$\phi(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\phi}(\xi)\mathrm{d}\xi > 0.$$

Lemma 4.1 [36] *Let $n \gg 1$. Define the function $f_n(x)$ by*

$$f_n(x) = \phi(x)\sin\left(\frac{17}{12}2^n x\right)$$

or

$$f_n(x) = \phi(x)\cos\left(\frac{17}{12}2^n x\right).$$

Then we have

$$\Delta_j(f_n) = \begin{cases} f_n, & if\ j = n, \\ 0, & if\ j \neq n. \end{cases}$$

Lemma 4.2 *Define the initial data $u_0(x)$ and $v_0(x)$ as*

$$u_0(x) := \sum_{n=0}^{\infty} 2^{-n(s+1)}\phi(x)\sin\left(\frac{17}{12}2^n x\right),$$

$$v_0(x) := (1-\partial_x)u_0(x).$$

Then for any $s > \max\left\{\frac{3}{2}, 1+\frac{1}{p}\right\}$ *and* $k \in \mathbb{Z}^+$, *we have for some n large enough*

$$\|v_0\|_{B^s_{p,\infty}} \approx \|u_0\|_{B^{s+1}_{p,\infty}} \leq C, \tag{4.1}$$

$$\left\|\left(k_1(2u_0-v_0)v_0 + k_2u_0^k\right)\partial_x\Delta_n v_0\right\|_{L^p} \geq \frac{c_0c_1}{2}2^{n(1-s)}, \tag{4.2}$$

where C and c_0, c_1 *are some positive constants. In particular,*

$$c_0 := \begin{cases} \left(\frac{\delta}{\pi}\int_0^\pi |\sin x|^p \mathrm{d}x\right)^{1/p}, & \text{if} \quad p\in[1,\infty), \\ 1, & \text{if} \quad p=\infty, \end{cases} \quad \text{and} \quad c_1 := \frac{|k_1|}{(1-2^{-s})^2}\phi^3(0).$$

Proof Using Lemma 4.1 yields

$$\Delta_n u_0(x) = 2^{-n(s+1)}\phi(x)\sin\left(\frac{17}{12}2^n x\right). \tag{4.3}$$

By the definition of Besov space, we have

$$\|u_0\|_{B^{s+1}_{p,\infty}} = \sup_{j\geq -1} 2^{(s+1)j}\|\Delta_j u_0\|_{L^p} = \sup_{j\geq 0}\left\|\phi(x)\sin\left(\frac{17}{12}2^j x\right)\right\|_{L^p} \leq C.$$

Due to the relation $u_0 = (1-\partial_x)^{-1}v_0$, we have $\|u_0\|_{B^{s+1}_{p,\infty}} \lesssim \|v_0\|_{B^s_{p,\infty}}$, which implies (4.1).

From (4.3), we have

$$\partial_x\Delta_n u_0(x) = 2^{-n(s+1)}\phi'(x)\sin\left(\frac{17}{12}2^n x\right) + \frac{17}{12}2^{-ns}\phi(x)\cos\left(\frac{17}{12}2^n x\right). \tag{4.4}$$

Then

$$\begin{aligned}\partial_x\partial_x\Delta_n u_0(x) = 2^{-n(s+1)}\phi''(x)\sin\left(\frac{17}{12}2^n x\right) + \frac{17}{6}2^{-ns}\phi'(x)\cos\left(\frac{17}{12}2^n x\right) \\ -\left(\frac{17}{12}\right)^2 2^n 2^{-ns}\phi(x)\sin\left(\frac{17}{12}2^n x\right).\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) directly gives that

$$\begin{aligned}\partial_x\Delta_n v_0 &= \partial_x\Delta_n u_0 - \partial_x\partial_x\Delta_n u_0 \\ &= \left(\frac{17}{12}\right)^2 2^n 2^{-ns}\phi(x)\sin\left(\frac{17}{12}2^n x\right)\end{aligned}$$

$$
+ 2^{-n(s+1)}(\phi'(x) - \phi''(x)) \sin\left(\frac{17}{12}2^n x\right)
+ \frac{17}{12}2^{-ns}\left(\phi(x) - 2\phi'(x)\right)\cos\left(\frac{17}{12}2^n x\right).
$$

Since $u_0(x)$, $v_0(x)$ and $\phi(x)$ are real-valued and continuous functions on $\mathbb{R}$, then there exists some $\delta > 0$ such that for any $x \in B_\delta(0)$

$$
\begin{aligned}
&|[\left(k_1(2u_0 - v_0)v_0 + k_2u_0^k\right)\phi](x)| \\
&\geq \frac{1}{2}|[\left(k_1(2u_0 - v_0)v_0 + k_2u_0^k\right)\phi](0)| = \frac{|k_1|}{2}|v_0^2(0)|\phi(0) \\
&= \frac{|k_1|}{2}\left(\frac{17}{12}\phi(0)\sum_{n=0}^{\infty}2^{-ns}\right)^2\phi(0) \geq c_1.
\end{aligned}
\tag{4.6}
$$

Obviously,

$$
\|\left(k_1(2u_0 - v_0)v_0 + k_2u_0^k\right)(x)\|_{L^\infty(\mathbb{R})} \leq C. \tag{4.7}
$$

Thus, for some n large enough, we have from (4.6) and (4.7)

$$
\begin{aligned}
&\|\left(k_1(2u_0 - v_0)v_0 + k_2u_0^k\right)\partial_x\Delta_n u_0\|_{L^p} \\
&\geq c_1 2^n 2^{-ns}\left\|\sin\left(\frac{17}{12}2^n x\right)\right\|_{L^p(B_\delta(0))} \\
&\quad - C2^{-ns}\left\|(\phi'(x) - \phi''(x))\sin\left(\frac{17}{12}2^n x\right)\right\|_{L^p} \\
&\quad - C2^{-ns}\left\|\left(\phi(x) - 2\phi'(x)\right)\cos\left(\frac{17}{12}2^n x\right)\right\|_{L^p} \\
&\geq (c_0c_1 2^n - C)2^{-ns},
\end{aligned}
$$

where we have used

$$
\left\|\sin\left(\frac{17}{12}2^n x\right)\right\|_{L^p(B_\delta(0))} \geq c_0 > 0.
$$

In fact, for $p \in [1, \infty)$, we have

$$
\left\|\sin\left(\frac{17}{12}2^n x\right)\right\|_{L^p(B_\delta(0))}^p = \frac{2\delta}{\lambda_n}\int_0^{\lambda_n}|\sin x|^p \mathrm{d}x \quad \text{with} \quad \lambda_n := \frac{17}{12}\delta 2^n.
$$

Due to the fact

$$
\lim_{n\to\infty}\frac{1}{\lambda_n}\int_0^{\lambda_n}|\sin x|^p \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}|\sin x|^p \mathrm{d}x,
$$

then there exists a positive integer number N such that for $n > N$

$$\frac{1}{\lambda_n}\int_0^{\lambda_n} |\sin x|^p \mathrm{d}x \geq \frac{1}{2\pi}\int_0^{\pi} |\sin x|^p \mathrm{d}x,$$

For $p = \infty$, we have for some n large enough

$$\left\|\sin\left(\frac{17}{12}2^n x\right)\right\|_{L^\infty(B_\delta(0))} = \|\sin x\|_{L^\infty(B_{\lambda_n}(0))}.$$

We choose n large enough such that $C \leq \frac{c_0c_1}{2}2^n$ and then finish the proof of Lemma 4.2. □

4.2 Error estimates

From now on, we denote $v(t) = \mathbf{S}_t(v_0)$ for the sake of convenience.

Lemma 4.3 *Assume that* $\|v_0\|_{B^s_{p,\infty}} \lesssim 1$. *Under the assumptions of Theorem 1.1, we have*

$$\|(v^2 - 2uv)\partial_x v\|_{B^{s-1}_{p,\infty}} + \|u^k\partial_x v\|_{B^{s-1}_{p,\infty}} \leq 1,$$
$$\|\Phi_1(u)\|_{B^s_{p,\infty}} + \|\Phi_2(v)\|_{B^s_{p,\infty}} + \|\Phi_3(v)\|_{B^s_{p,\infty}} \leq 1.$$

Proof By the local well-posedness result [53], there exists a short time $T = T(\|u_0\|_{B^{s+1}_{p,\infty}})$ such that Eq. (1.1) has a unique solution $u(t) \in \mathcal{C}([0,T]; B^{s+1}_{p,r})$. Moreover, for all $t \in [0,T]$, there holds

$$\|u(t)\|_{B^{s+1}_{p,\infty}} \leq C\|u_0\|_{B^{s+1}_{p,\infty}} \quad \text{or} \quad \|v(t)\|_{B^s_{p,\infty}} \leq C\|v_0\|_{B^s_{p,\infty}}. \tag{4.8}$$

It can be inferred from (4.8) that which will be frequently used later

$$\|v\|_{B^{s-1}_{p,\infty}} \lesssim \|v_0\|_{B^{s-1}_{p,\infty}} \lesssim \|v_0\|_{B^s_{p,\infty}} \lesssim 1.$$

Using the fact that $B^{s-1}_{p,r}$ is a Banach algebra with $s-1 > \max\{\frac{1}{p},\frac{1}{2}\}$ and Lemma 2.1, one has

$$\begin{aligned}
&\|(v^2 - 2uv)\partial_x v\|_{B^{s-1}_{p,\infty}} + \|u^k\partial_x v\|_{B^{s-1}_{p,\infty}} \\
&\quad \leq \|(v^2 - 2uv)\|_{B^{s-1}_{p,\infty}}\|\partial_x v\|_{B^{s-1}_{p,\infty}} + \|u\|^k_{B^{s-1}_{p,\infty}}\|v\|_{B^s_{p,\infty}} \\
&\quad \leq \|v\|^2_{B^{s-2}_{p,\infty}}\|v\|_{B^s_{p,\infty}} + \|v\|^3_{B^{s-1}_{p,\infty}} \leq 1.
\end{aligned}$$

Similarly, one has

$$\|\Phi_1(u)\|_{B^s_{p,\infty}} \leq \|u^{k-1}u_x^2\|_{B^s_{p,\infty}} \leq \|u\|^{k-1}_{B^s_{p,\infty}}\|u\|^2_{B^{s+1}_{p,\infty}} \leq 1,$$

$$\|\Phi_2(v)\|_{B^s_{p,\infty}} \leq \|v_x u_x m\|_{B^{s-1}_{p,\infty}} + \|v u_x m\|_{B^{s-1}_{p,\infty}} \leq \|v\|_{B^s_{p,\infty}} \|u\|_{B^s_{p,\infty}} \|u - u_{xx}\|_{B^{s-1}_{p,\infty}} \leq 1,$$
$$\|\Phi_3(v)\|_{B^s_{p,\infty}} \leq \|u^{k-1} u_x m\|_{B^{s-1}_{p,\infty}} + \|u^{k-2} u_x^3\|_{B^{s-1}_{p,\infty}} + \|u^k u_x\|_{B^{s-1}_{p,\infty}} \leq 1.$$

Thus, we finish the proof of Lemma 4.3. □

Proposition 4.1 *Assume that* $\|v_0\|_{B^s_{p,\infty}} \lesssim 1$. *Under the assumptions of Theorem 1.1, we have*

$$\|\mathbf{S}_t(v_0) - v_0\|_{B^{s-1}_{p,\infty}} \lesssim t.$$

Proof By the mean value theorem and the Minkowski inequality, we obtain

$$\begin{aligned}
\|v(t) - v_0\|_{B^{s-1}_{p,\infty}} &\leq \int_0^t \|\partial_\tau v\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau \\
&\leq k_1 \int_0^t \|(v^2 - 2uv)\partial_x v\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau + k_2 \int_0^t \|u^k \partial_x v\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau \\
&\quad + \int_0^t \|\Phi_1(u)\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau + \int_0^t \|\Phi_1(v)\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau + \int_0^t \|\Phi_2(v)\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau.
\end{aligned}$$

Thus, using Lemma 4.3 enable us to finish the proof of Proposition 4.1. □

Next, we shall establish the key estimate which plays an important role in the proof of main Theorem.

Proposition 4.2 *Assume that* $\|v_0\|_{B^{s-1}_{p,\infty}} \lesssim 1$. *Under the assumptions of Theorem 1.1, there holds*

$$\|\mathbf{w}\|_{B^{s-2}_{p,\infty}} \lesssim t^2,$$

where we denote $\mathbf{w} := \mathbf{S}_t(v_0) - v_0 - t\mathbf{U}_0$ *and*

$$\mathbf{U}_0 := \left(k_1(2u_0 - v_0)v_0 + k_2 u_0^k\right) \partial_x v_0 + \Phi_1(u_0) + \Phi_2(v_0) + \Phi_3(v_0). \tag{4.9}$$

Proof Using differential mean value theorem and (3.11), we obtain

$$\begin{aligned}
\|\mathbf{w}\|_{B^{s-2}_{p,\infty}} &\leq \int_0^t \|\partial_\tau v - \mathbf{U}_0\|_{B^{s-2}_{p,\infty}} \mathrm{d}\tau \\
&\leq \int_0^t \|v^2 \partial_x v - v_0^2 \partial_x v_0\|_{B^{s-2}_{p,\infty}} \mathrm{d}\tau + \int_0^t \|uv\partial_x v - u_0 v_0 \partial_x v_0\|_{B^{s-2}_{p,\infty}} \mathrm{d}\tau \\
&\quad + \int_0^t \|\Phi_1(u) - \Phi_1(u_0)\|_{B^{s-2}_{p,\infty}} \mathrm{d}\tau + \int_0^t \|\Phi_2(v) - \Phi_2(v_0)\|_{B^{s-2}_{p,\infty}} \mathrm{d}\tau \\
&\quad + \int_0^t \|\Phi_3(v) - \Phi_3(v_0)\|_{B^{s-2}_{p,\infty}} \mathrm{d}\tau.
\end{aligned} \tag{4.10}$$

Now we need to estimate each term on the right hand side of (4.10). Notice that $B^{s-1}_{p,r}$ is a Banach algebra with $s-1>\max\{\frac{1}{p},\frac{1}{2}\}$, combining with Lemma 2.1 yields

$$
\begin{aligned}
\|v^3-v_0^3\|_{B^{s-2}_{p,\infty}} &\le \|(v-v_0)(v^2+vv_0+v_0^2)\|_{B^{s-1}_{p,\infty}} \lesssim \|v-v_0\|_{B^{s-1}_{p,\infty}},\\
\|uv\partial_x v-u_0v_0\partial_x v_0\|_{B^{s-2}_{p,\infty}} &= \|(u-u_0)v\partial_x v+u_0(v\partial_x v-v_0\partial_x v_0)\|_{B^{s-2}_{p,\infty}}\\
&\lesssim \|u-u_0\|_{B^{s-2}_{p,\infty}}\|v\partial_x v\|_{B^{s-1}_{p,\infty}}+\|u_0\|_{B^{s-1}_{p,\infty}}\|v^2-v_0^2\|_{B^{s-1}_{p,\infty}}\\
&\lesssim \|v-v_0\|_{B^{s-1}_{p,\infty}}.
\end{aligned}
$$

Here, we need only to estimate $\|v_xu_xm-v_{0x}u_{0x}m_0\|_{B^{s-2}_{p,\infty}}$ since the other terms can be processed in a similar more relaxed way. Using Lemma 2.1 yields

$$
\begin{aligned}
&\|v_xu_xm-v_{0x}u_{0x}m_0\|_{B^{s-2}_{p,\infty}}\\
&\quad\le \|(v-v_0)_xu_xm\|_{B^{s-2}_{p,\infty}}+\|v_{0x}(u-u_0)_xm\|_{B^{s-2}_{p,\infty}}\\
&\qquad+\|v_{0x}u_{0x}(m-m_0)\|_{B^{s-2}_{p,\infty}}\\
&\quad\le \|v-v_0\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}\|m\|_{B^{s-1}_{p,\infty}}+\|v_0\|_{B^{s}_{p,\infty}}\|u-u_0\|_{B^{s}_{p,\infty}}\|m\|_{B^{s-1}_{p,\infty}}\\
&\qquad+\|v_0\|_{B^{s}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|m-m_0\|_{B^{s-2}_{p,\infty}}\\
&\quad\le \|v-v_0\|_{B^{s-1}_{p,\infty}}.
\end{aligned}
$$

Putting the above estimates into (4.10) and using Proposition 4.1 yield

$$
\|\mathbf{w}\|_{B^{s-1}_{p,\infty}}\lesssim \int_0^t\|v(\tau)-v_0\|_{B^{s-1}_{p,\infty}}\mathrm{d}\tau\lesssim t^2.
$$

Thus, we complete the proof of Proposition 4.2. □

Now we present the proof of Theorem 3.1.

Proof of Theorem 3.1 Notice that $\mathbf{S}_t(v_0)-v_0=t\mathbf{U}_0+\mathbf{w}$ where $\mathbf{U}_0$ is given by (4.9), and

$$
\begin{aligned}
\Delta_n\Big(\big(k_1(2u_0-v_0)v_0+k_2u_0^k\big)\partial_x v_0\Big)&=\Big(k_1(2u_0-v_0)v_0+k_2u_0^k\Big)\partial_x\Delta_n v_0\\
&\quad+[\Delta_n,\big(k_1(2u_0-v_0)v_0+k_2u_0^k\big)]\partial_x v_0,
\end{aligned}
$$

by the triangle inequality and Proposition 4.2, we deduce that

$$
\begin{aligned}
\|\mathbf{S}_t(v_0)-v_0\|_{B^{s}_{p,\infty}} &\ge 2^{ns}\,\|\Delta_n\,(\mathbf{S}_t(v_0)-v_0)\|_{L^p}\\
&= 2^{ns}\,\|\Delta_n\,(t\mathbf{U}_0+\mathbf{w})\|_{L^p}\\
&\ge t2^{ns}\|\Delta_n\mathbf{U}_0\|_{L^p}-2^{2n}2^{n(s-2)}\|\Delta_n\mathbf{w}\|_{L^p}\\
&\ge t2^{ns}\left\|\Delta_n\left((k_1(2u_0-v_0)v_0+k_2u_0^k\right)\partial_x v_0)\right\|_{L^p}
\end{aligned}
$$

$$
\begin{aligned}
&- t2^{ns}\left\|\Delta_n\left(\Phi_1(u_0)+\Phi_2(v_0)+\Phi_3(v_0)\right)\right\|_{L^p} - C2^{2n}\|\mathbf{w}\|_{B^{s-2}_{p,\infty}} \\
&\geq t2^{ns}\left\|\left(k_1(2u_0-v_0)v_0+k_2u_0^k\right)\partial_x\Delta_n v_0\right\|_{L^p} \\
&\quad - t2^{ns}\left\|[\Delta_n,\left(k_1(2u_0-v_0)v_0+k_2u_0^k\right)]\partial_x u_0\right\|_{L^p} \\
&\quad - t\left\|\Phi_1(u_0)+\Phi_2(v_0)+\Phi_3(v_0)\right\|_{B^s_{p,\infty}} - C2^{2n}t^2 \\
&\geq t2^{ns}\left\|\left(k_1(2u_0-v_0)v_0+k_2u_0^k\right)\partial_x\Delta_n v_0\right\|_{L^p} \\
&\quad - Ct\left\|2^{ns}[\Delta_n,\left(k_1(2u_0-v_0)v_0+k_2u_0^k\right)]\partial_x u_0\|_{L^p}\right\|_{\ell^\infty} \\
&\quad - Ct - C2^{2n}t^2, \qquad (4.11)
\end{aligned}
$$

where we have used Lemma 4.2 and the commutator estimate

$$
\begin{aligned}
&\left\|2^{ns}\|[\Delta_n,\left(k_1(2u_0-v_0)v_0+k_2u_0^k\right)]\partial_x v_0\|_{L^p}\right\|_{\ell^\infty} \\
&\leq \|\partial_x\left(k_1(2u_0-v_0)v_0+k_2u_0^k\right)\|_{L^\infty}\|v_0\|_{B^s_{p,\infty}} \\
&\quad + \|\partial_x v_0\|_{L^\infty}\|\partial_x\left(k_1(2u_0-v_0)v_0+k_2u_0^k\right)\|_{B^{s-1}_{p,\infty}} \leq 1.
\end{aligned}
$$

Gathering all the above estimates and Lemma 4.2 together with (4.11), we obtain

$$
\|\mathbf{S}_t(v_0)-v_0\|_{B^s_{p,\infty}} \geq \frac{c_0c_1}{2}t2^n - Ct - C2^{2n}t^2.
$$

Taking large n such that $\frac{c_0c_1}{2}2^n \geq 2C$, we have

$$
\|\mathbf{S}_t(v_0)-v_0\|_{B^s_{p,\infty}} \geq \frac{c_0c_1}{4}t2^n - C2^{2n}t^2.
$$

Thus, picking $t2^n \approx \varepsilon$ with small ε, we have

$$
\|\mathbf{S}_t(v_0)-v_0\|_{B^s_{p,\infty}} \gtrsim \frac{c_0c_1}{4}\varepsilon - \varepsilon^2 \gtrsim \frac{c_0c_1}{8}\varepsilon.
$$

This completes the proof of Theorem 3.1. □

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Conflict of interest The authors declare that they have no conflict of interest.

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