Sparse bounds for oscillating multipliers on stratified groups

Abhishek Ghosh[1,2] · Michael Ruzhansky[3]

Abstract

In this article, we address sparse bounds for a class of spectral multipliers that include oscillating multipliers on stratified Lie groups. Our results can be applied to obtain weighted bounds for general Riesz means and for solutions of dispersive equations.

Keywords Oscillating multipliers · Stratified Lie groups · Weights

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1 Introduction

On $\mathbb{R}^n$, operators of the form $\widehat{Tf}(\xi) = m_{\theta,\beta}(\xi)\widehat{f}(\xi)$, where $m_{\theta,\beta} = |\xi|^{-\frac{\theta\beta}{2}} e^{i|\xi|^\theta} \chi_{\{|\xi|>1\}}$ are known as oscillating multipliers. They are extensively studied starting with the pioneering works of Hardy, Hirschman [26] and Wainger [50]. Charles Fefferman proved the crucial weak type $(1, 1)$ estimates in [22], and the sharp range for L^p estimates was obtained by Fefferman and Stein in [23]. They were also studied by Hörmander [27]. We also refer to the articles [42, 43, 48] for results in the context of the wave operators, that is, $\theta = 1$. Weighted estimates for oscillating multipliers

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on $\mathbb{R}^n$ were initiated by Chanillo [13] and extended in [15]. In [16], weighted end-point estimates were obtained by Chanillo, Kurtz, and Sampson. In this article, we shall confine ourselves to weighted L^p estimates for these operators on stratified Lie groups. In order to do that, let us recall the following preliminaries.

Let $\mathfrak{g}$ be a d-dimensional, graded nilpotent Lie algebra so that

$$\mathfrak{g} = \bigoplus_{i=1}^{s} \mathfrak{g}_i$$

as a vector space and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i, j. Suppose that $\mathfrak{g}_1$ generates $\mathfrak{g}$ as a Lie algebra. The associated, connected, simply connected Lie group G is called a stratified Lie group. The homogeneous dimension of G is defined as $Q = \sum_j j \dim(\mathfrak{g}_j)$. Consider the sublaplacian $\mathcal{L} = -\sum_k X_k^2$ on G, where $\{X_k\}$ is a basis for $\mathfrak{g}_1$. For any Borel measurable function m on $\mathbb{R}_+ = [0, \infty)$, we can define the spectral multiplier operator

$$m(\sqrt{\mathcal{L}}) = \int_0^\infty m(\lambda) \, dE_\lambda$$

where $\{E_\lambda\}_{\lambda \geq 0}$ is the spectral resolution of $\sqrt{\mathcal{L}}$. Since the exponential map is a global diffeomorphism, the measure on G can be identified with the d-dimensional Lebesgue measure. In this setting, analogue of the classical Hörmander-Mikhlin multiplier theorem was established in the seminal work by Christ in [18], also fundamental end-point estimates were obtained by Mauceri–Meda in [41], see also [44–46] for other influential works. In recent times, there are many important works in this context, we refer [3, 7–10, 14, 38–40, 49]. We are inspired by the recent work [14] where the authors have introduced a general class of multipliers covering oscillating multipliers and obtained important end-point estimates. On more general graded groups Fourier multiplier operators are studied in [11, 12, 24, 25, 32] and references therein. Throughout this article, for any Borel measurable set R and $1 \leq p < \infty$, $\langle f \rangle_{p,R}$ denotes $(\frac{1}{|R|} \int_R |f|^p)^{1/p}$. Also, a family of sets $\mathcal{S}$ is called $\eta-$sparse if for each $R \in \mathcal{S}$ there exists $E_R \subset R$ such that $|E_R| \geq \eta |R|$ and $\{E_R\}_{\mathcal{S}}$ are pairwise disjoint. Now we state our main result.

Statement of main results

Motivated by [14], we introduce the following class of multipliers. Let $\nu > 1$ be a number which will be specified later. Also, let ϕ be a smooth function on $(0, \infty)$, supported on $\{\nu^{-1} \leq \lambda \leq \nu\}$ and satisfying $\sum_j \phi(\nu^{-j}\lambda) = 1$ for all $\lambda > 0$. Define $m^j(\lambda) := m(\nu^j \lambda)\phi(\lambda)$.

Definition 1.1 Let $\theta \in \mathbb{R} \setminus \{0\}$ and $\beta \geq 0$. We say $m \in \mathcal{M}(\theta, \beta)$ if m is supported in the set $\{\lambda \in \mathbb{R}_+ : \lambda^\theta \geq 1\}$ and

$$\sup_{j\theta>0} \nu^{j\theta\beta/2}\|m^j\|_{L^\infty(\mathbb{R}_+)} < \infty, \tag{1}$$

$$\text{and} \quad \sup_{j\theta>0} \nu^{-j\theta(2s-\beta)/2}\|m^j\|_{L^2_s(\mathbb{R}_+)} < \infty \quad \text{for all} \ s \in \mathbb{N}. \tag{2}$$

Example 1.2 Let $\theta \in \mathbb{R} \setminus \{0\}$. Define $m_{\theta,\beta}(\lambda) := e^{i\lambda^\theta}\lambda^{-\frac{\theta\beta}{2}}\chi_{\{\lambda\in[0,\infty):\lambda^\theta\geq1\}}$. Then it is easy to see that $m_{\theta,\beta} \in \mathscr{M}(\theta,\beta)$.

Now we state our main sparse domination principle for the multiplier class $\mathscr{M}(\theta,\beta)$.

Theorem 1.3 Let $\theta \in \mathbb{R} \setminus \{0\}$ and $\beta \geq 0$, and $m \in \mathscr{M}(\theta,\beta)$. Then there exist sparse families $\mathcal{S}$ and $\mathcal{S}'$ such that for all compactly supported bounded functions f, g we have

$$|\langle m(\sqrt{\mathcal{L}})f, g\rangle| \lesssim_{\theta,\beta,r_1,r_2} \sum_{R\in\mathcal{S}} |R|\langle f\rangle_{r_1,R} \langle g\rangle_{r_2',R}$$

$$\text{and} \ |\langle m(\sqrt{\mathcal{L}})f, g\rangle| \lesssim_{\theta,\beta,r_1,r_2} \sum_{R\in\mathcal{S}'} |R|\langle f\rangle_{r_2',R} \langle g\rangle_{r_1,R},$$

where r_1, r_2 satisfy

$$\left(\frac{1}{r_1} - \frac{1}{2}\right) < \frac{\beta}{2Q}, \quad 1 \leq r_1 \leq r_2 \leq 2, \tag{3}$$

$$\text{or} \ \left(\frac{1}{r_1} - \frac{1}{r_2}\right) < \frac{\beta}{2Q}, \quad 1 \leq r_1 \leq 2 \leq r_2 \leq r_1'. \tag{4}$$

As a special case, we obtain the following corollary.

Corollary 1.4 Let $\theta \in \mathbb{R} \setminus \{0\}$ and $\beta \geq 0$. Then there exist sparse families $\mathcal{S}$ and $\mathcal{S}'$ such that for all compactly supported bounded functions f, g we have

$$|\langle m_{\theta,\beta}(\sqrt{\mathcal{L}})f, g\rangle| \lesssim_{\theta,\beta,r_1,r_2} \sum_{R\in\mathcal{S}} |R|\langle f\rangle_{r_1,R} \langle g\rangle_{r_2',R}$$

$$\text{and} \ |\langle m_{\theta,\beta}(\sqrt{\mathcal{L}})f, g\rangle| \lesssim_{\theta,\beta,r_1,r_2} \sum_{R\in\mathcal{S}'} |R|\langle f\rangle_{r_2',R} \langle g\rangle_{r_1,R},$$

where r_1, r_2 satisfy either condition (3) or (4).

The motivation for proving such an estimate arises from recent works [2, 4, 5, 19, 20, 30, 31, 33–36] where sparse domination is achieved for several classical operators in Harmonic analysis in various settings. The key importance in proving such an estimate lies in the fact that one can obtain a range of quantitative weighted estimates depending on the decay parameter β, we state them here. The Muckenhoupt class of weights (A_p) and reverse Hölder's classes (RH_q) are defined in details in Sect. 4.

Theorem 1.5 Let $\theta \in \mathbb{R} \setminus \{0\}$. We have the following results:

i) *Let $m \in \mathscr{M}(\theta, 2Q)$. Then $m(\sqrt{\mathcal{L}})$ maps $L^p(\omega)$ to $L^p(\omega)$ for all $1 < p < \infty$ and $\omega \in A_p$.*

ii) *Let $m \in \mathscr{M}(\theta, \beta)$ with $Q \leq \beta < 2Q$. Then $m(\sqrt{\mathcal{L}})$ maps $L^p(\omega)$ to $L^p(\omega)$ for $p_\beta < p < \infty$ and $\omega \in A_{p/p_\beta}$, where $p_\beta := \frac{2Q}{\beta}$.*

iii) *Let $m \in \mathscr{M}(\theta, \beta)$ with $0 < \beta < Q$. Then $m(\sqrt{\mathcal{L}}) : L^p(\omega) \to L^p(\omega)$ for all $2 < p < s_\beta$, $\omega \in A_{p/2} \cap RH_{(s_\beta/p)'}$, where $\frac{1}{s_\beta} := \frac{1}{2} - \frac{\beta}{2Q}$.*

We believe these results are completely new in the setting of stratified Lie groups. The article is organized as follows. In the next section, we recall some necessary preliminaries and Sect. 3 contains the proof of Theorem 1.3. In Sect. 4, we prove Theorem 1.5 and other applications of Theorem 1.3 to Riesz means and dispersive equations.

2 Preliminaries

Let $\{\delta_r\}_{r>0}$ be the group of dilations associated to G and let $|\cdot|$ be a homogeneous quasi-norm, i.e., $|x| = 0$ if and only if $x = 0$ where 0 denotes the group identity, and $|\delta_r x| = r|x|$ for all $r > 0$ and $x \in G$. Moreover, the right convolution kernel of the operator $m(\sqrt{\mathcal{L}})$ will be denoted by K_m, that is,

$$m(\sqrt{\mathcal{L}})f(x) = \int_G f(x \cdot y^{-1}) \, K_m(y) \, dy.$$

In general, K_m is just a distribution but whenever m is compactly supported, K_m can be identified with an L^2 function on G. See [24] for more details regarding analysis on these groups. The following estimates are well known.

Theorem 2.1 [49] *For any function h and $M > 0$, we denote $h_M(t) := h(tM)$.*

i) *The following Plancherel-type identity holds*

$$\|K_h\|_{L^2(G)}^2 = \int_0^\infty |h(t)|^2 t^{Q-1} \, dt. \tag{5}$$

In particular, if the multiplier is supported on $[0, M]$ then $\|K_h\|_{L^2(G)}^2 \leq M^Q \|h_M\|_2^2$.

ii) *For any compactly supported multiplier h,*

$$\int_G |K_h(x)|^2 (1 + |x|^s)^2 \, dx \lesssim \|h\|_{L^2_s(\mathbb{R}_+)}^2 \tag{6}$$

holds for any $s > 0$. As a consequence, $\|K_h\|_{L^1} \leq \|h\|_{L^2_s(\mathbb{R}_+)}$ for $s > \frac{Q}{2}$.

We also need the following notion of dyadic grids in spaces of homogeneous type. We refer to [17] and [29, 37] for details. Let $0 < c_1 \leq C_1 < \infty$ and $\mu \in (0, 1)$. By a general dyadic grid $\mathscr{D} = \bigcup_{k \in \mathbb{Z}} \mathscr{D}_k$ on G, we mean a countable collection of sets R_k^α

for $k \in \mathbb{Z}$, each associated with a point z_k^α, α coming from a countable index set, with the following properties:

- $G = \bigcup_\alpha R_k^\alpha$ for every $k \in \mathbb{Z}$.
- If $l \geq k$, then either $R_l^\beta \subset R_k^\alpha$ or $R_l^\beta \cap R_k^\alpha = \emptyset$.
- For the constants $c_1, C_1 > 0$ we have $B(R_k^\alpha) := B(z_k^\alpha, c_1\mu^k) \subset R_k^\alpha \subset B(z_k^\alpha, C_1\mu^k) =: C_1 B(R_k^\alpha)$.
- If $l \geq k$ and $R_l^\beta \subset R_k^\alpha$, then $C_1 B(R_l^\beta) \subset C_1 B(R_k^\alpha)$.

For sufficiently small $\mu \in (0, 1)$, Hytönen and Kairema [29, Theorem 4.1] proved the existence of a finite collection of dyadic grids $\mathscr{D}^n$, $n = 1, 2, \ldots, \mathfrak{N}$, such that for every ball $B(z, r) \subset G$ with $\mu^{k+2} \leq r < \mu^{k+1}$, there exists some $n \in \{1, 2, \ldots, \mathfrak{N}\}$ and $R_k^\alpha \in \mathscr{D}^n$ such that $B(z, r) \subset R_k^\alpha$ and $\text{diam}\,(R_k^\alpha) \leq C\,r$, where C depends on μ. For the purposes of this article, the number $\mu \in (0, 1)$ is now considered fixed and ν will denote $\frac{1}{\mu}$.

Remark 2.2 We remark that the sparse families $\mathcal{S}, \mathcal{S}'$ in Theorem 1.3 consist of elements from the dyadic grids $\mathscr{D}^n$, $n = 1, 2, \ldots, \mathfrak{N}$.

3 Proof of Theorems

We shall prove Theorem 1.3 for case $\theta > 0$, other parts can be modified accordingly, see Remark 3.7. Let us fix $\theta > 0$, $\beta \geq 0$, and $m \in \mathscr{M}(\theta, \beta)$. Recall that $m^j(\lambda) := m(\nu^j\lambda)\phi(\lambda)$ for $j \geq 0$, where ϕ is a smooth function on $(0, \infty)$, supported on $\{\nu^{-1} \leq \lambda \leq \nu\}$, satisfying $\sum_j \phi(\nu^{-j}\lambda) = 1$. Then, m^j satisfy the following

$$\|m^j\|_{L^\infty(\mathbb{R}_+)} \lesssim \nu^{-j\theta\beta/2} \quad \text{for } j \geq 0,$$
$$\text{and } \|m^j\|_{L_s^2(\mathbb{R}_+)} \lesssim \nu^{j\theta(2s-\beta)/2} \quad \text{for } j \geq 0, \tag{7}$$

where the implicit constants are independent of j. We also introduce the following notation $m_j(\lambda) := m(\lambda)\phi(\nu^{-j}\lambda)$. Recall that $\sum_j \phi(\nu^{-j}\lambda) = 1$, then $m(\lambda) = \sum_{j\geq 0} m_j(\lambda)$. Moreover, $m_j(\lambda) = m^j(\nu^{-j}\lambda)$. Then we have the following decomposition

$$m(\sqrt{\mathcal{L}}) = \sum_{j\geq 0} T_j, \quad \text{where} \quad T_j f = f * K_{m_j}.$$

It is easy to see by homogeneity that $K_{m_j} = K_m * (\nu^{jQ} K_\phi(\delta_{\nu^j}\cdot))(x)$. Motivated by [4] we make a further decomposition in the space variable, namely

$$T_j^l f(x) = \int f(z) K_{m_j}(z^{-1}x)\phi(\nu^{-l+j(1-\theta)}|z^{-1}x|)\,dz, \quad l \in \mathbb{Z}. \tag{8}$$

Then $T_j f = \sum_{l\in\mathbb{Z}} T_j^l f$ and consequently $m(\sqrt{\mathcal{L}}) = \sum_{j\geq 0} \sum_{l\in\mathbb{Z}} T_j^l$. Now we shall focus on proving certain crucial estimates and for some $\epsilon > 0$ we group the terms

according to their spatial scale, i.e.,

$$T_j f = \sum_{l \leq j\epsilon} T_j^l f + \sum_{l > j\epsilon} T_j^l f.$$

Let us start by proving $L^2 - L^2$ estimates for the pieces T_j^l. Let $l > j\epsilon$ and denote $g := K_{m_j}(\cdot)\phi(v^{-l+j(1-\theta)}|\cdot|)$. By Young's inequality we have

$$
\begin{aligned}
\|T_j^l f\|_2 &\leq \|f\|_2 \|g\|_1 \leq \|f\|_2 \left(\int_{|x|\simeq v^{l-j(1-\theta)}} |K_{m_j}(x)|\, dx \right) \\
&= \|f\|_2 \left(\int_{|x|\simeq v^{l-j(1-\theta)}} v^{jQ}|K_{m^j}(\delta_{v^j}x)|\, dx \right) \\
&\leq \|f\|_2 \left(\int_{|x|\simeq v^{l+j\theta}} |K_{m^j}(x)|\, dx \right) \\
&= \|f\|_2 \left(\int_{|x|\simeq v^{l+j\theta}} (1+|x|^s)(1+|x|^s)^{-1}|K_{m^j}(x)|\, dx \right) \\
&\leq \|f\|_2 \left(\int_{|x|\simeq v^{l+j\theta}} (1+|x|^s)^2 |K_{m^j}(x)|^2\, dx \right)^{1/2} v^{(l+j\theta)(\frac{Q}{2}-s)} \\
&\lesssim v^{(l+j\theta)(\frac{Q}{2}-s)} \|m^j\|_{L_s^2} \|f\|_2 \lesssim v^{(l+j\theta)(\frac{Q}{2}-s)} v^{j\theta(s-\frac{\beta}{2})} \|f\|_2 \\
&\lesssim v^{-\frac{j\theta\beta}{2}} v^{l(\frac{Q}{2}-\frac{s}{2})} v^{\frac{j\theta Q}{2}} v^{-l\frac{s}{2}} \|f\|_2 \lesssim v^{l(\frac{Q}{2}-\frac{s}{4})} v^{-\frac{ls}{4}} v^{\frac{j\theta Q}{2}} v^{-l\frac{s}{2}} \|f\|_2.
\end{aligned}
$$

Observe that the term $v^{l(\frac{Q}{2}-\frac{s}{4})} \ll 1$ if s is chosen large enough. Moreover, as $l > j\epsilon$, $v^{-\frac{ls}{4}} v^{\frac{j\theta Q}{2}} \leq v^{\frac{j\theta Q}{2}} v^{-\frac{j\epsilon s}{4}} \ll 1$ provided $s \gg \lfloor \frac{Q\theta}{\epsilon} \rfloor$. Finally, choose s large such that $v^{-l\frac{s}{4}} \leq v^{-Q(Q+\frac{\theta\beta}{2})l}$ as well as $v^{-l\frac{s}{4}} \leq v^{-j\epsilon\frac{s}{4}} \leq v^{-Q(Q+\frac{\theta\beta}{2})j}$. Therefore, we obtain

$$\|T_j^l f\|_2 \leq c_\epsilon v^{-Q(Q+\frac{\theta\beta}{2})(j+l)} \|f\|_2 \quad \text{for } l > j\epsilon. \tag{9}$$

At this point we would like to remark that one can in fact improve the bounds for $T_j^l, l > j\epsilon$, to $\|T_j^l f\|_2 \lesssim_\epsilon v^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} \|f\|_2$ for any large constant C. Certainly, $\|T_j f\|_2 \leq v^{-\frac{j\theta\beta}{2}} \|f\|_2$ since $\|m_j\|_{L^\infty} \leq v^{-\frac{j\theta\beta}{2}}$. Combining this with (9) we prove the following lemma:

Lemma 3.1 *We obtain the following estimates:*

i)

$$\|T_j^l f\|_2 \lesssim_\epsilon v^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} \|f\|_2 \quad \text{for } l > j\epsilon.$$

ii)

$$\left\| \sum_{l \le j\epsilon} T_j^l f \right\|_2 \lesssim_\epsilon \nu^{-\frac{j\theta\beta}{2}} \|f\|_2.$$

Now we shall prove $L^1 - L^1$ estimates for the pieces T_j^l. Recall that $g = K_{m_j}(\cdot)\phi(\nu^{-l+j(1-\theta)}|\cdot|)$. The previous argument shows that for any $l > j\epsilon$

$$\|T_j^l f\|_1 \le \|f\|_1 \|g\|_1 \lesssim_\epsilon \nu^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} \|f\|_1. \tag{10}$$

Another observation, together with (6), yields the following for any $s > \frac{Q}{2}$

$$\|T_j f\|_1 \le \|f\|_1 \|K_{m_j}\|_1 \le \|f\|_1 \|K_{m^j}\|_1 \lesssim \nu^{j\theta(s-\frac{\beta}{2})} \|f\|_1. \tag{11}$$

Consequently,

$$\|T_j f\|_1 \lesssim \nu^{j(-\frac{\theta\beta}{2}+\frac{\theta Q}{2}+\varepsilon)} \|f\|_1 \quad \text{for any } \varepsilon > 0.$$

Moreover, summing (10) in l, we obtain

$$\left\| \sum_{l \le j\epsilon} T_j^l f \right\|_1 = \|T_j f - \sum_{l > j\epsilon} T_j^l f\|_1 \lesssim \nu^{j(-\frac{\theta\beta}{2}+\frac{\theta Q}{2}+\varepsilon)} \|f\|_1.$$

Lemma 3.2 *Combining* (10) *and the above discussion we have the following estimates:*

i) For any $\varepsilon > 0$

$$\left\| \sum_{l \le j\epsilon} T_j^l f \right\|_1 \lesssim_\epsilon \nu^{j(-\frac{\theta\beta}{2}+\frac{\theta Q}{2}+\varepsilon)} \|f\|_1.$$

ii) For any $l > j\epsilon$, we have

$$\|T_j^l f\|_1 \lesssim_\epsilon \nu^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} \|f\|_1.$$

Finally, we need $L^1 - L^\infty$ estimates for the operators T_j^l. In order to do that we first need to prove pointwise estimates for the kernel K_h of the operator $h(\sqrt{\mathcal{L}})$ where h is supported on $[M/4, M]$ for some $M > 0$. Let p_t denote the convolution kernel associated with the heat operator $e^{-t\mathcal{L}}$. Recall the following Gaussian estimate (see e.g. [6])

$$|p_t(x)| \le \frac{C}{t^{Q/2}} e^{-\frac{|x|^2}{ct}}. \tag{12}$$

We just sketch the proof, see [9] for details. Denote $H(\lambda) = e^{\frac{\lambda^2}{M^2}} h(\lambda)$, then $h(\lambda) = e^{-\frac{\lambda^2}{M^2}} H(\lambda)$. Also, $\|H_M\|_2 \simeq \|h_M\|_2$ as h is supported on $[M/4, M]$. Therefore, $K_h(y^{-1}x) = \int_G p_{\frac{1}{M^2}}(z^{-1}x) K_H(y^{-1}z)\,dz$. Hölder's inequality and (5) implies

$$|K_h(y^{-1}x)| \leq \|p_{1/M^2}(z^{-1}x)\|_{L^2(dz,G)} \|K_H(y^{-1}z)\|_{L^2(dz,G)} \lesssim M^Q \|H_M\|_2 \lesssim M^Q \|h_M\|_2. \quad (13)$$

Observe that a factor of $M^{Q/2}$ appears from (5) and another from $\|p_{1/M^2}(z^{-1}x)\|_{L^2(dz,G)}$. Using Fourier inversion, we can write

$$K_h(y^{-1}x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{G}(t) p_{(1-it)/M^2}(y^{-1}x)\,dt, \quad (14)$$

where $G(\lambda) := h(M\sqrt{\lambda})e^{\lambda}$. Note that $supp(G)$ is contained $[0, 1]$ due to the support condition on h, so e^{λ} and its derivatives are always bounded. At this point we use the following estimate from [47]

$$|p_{(1-it)/M^2}(y^{-1}x)| \leq CM^Q e^{-\frac{M^2|y^{-1}x|^2}{(1+t^2)}} \leq CM^Q(1 + M|y^{-1}x|)^{-s}(1 + |t|)^s.$$

Therefore, from the above bound with (14), we have for any $s > 0$

$$\begin{aligned}
|K_h(y^{-1}x)| &\leq CM^Q(1 + M|y^{-1}x|)^{-s} \int |\widehat{G}(t)|(1 + |t|)^s\,dt \\
&\lesssim M^Q(1 + M|y^{-1}x|)^{-s} \|G\|_{L^2_{s+\varkappa+\frac{1}{2}}} \\
&\lesssim M^Q(1 + M|y^{-1}x|)^{-s} \|h_M\|_{L^2_{s+\varkappa+\frac{1}{2}}}, \quad (15)
\end{aligned}$$

for any small $\kappa > 0$. Using complex interpolation of (13) and (15), as in [21], we remove the extra exponent $1/2$ in the Sobolev exponent to obtain

$$|K_h(y^{-1}x)| \lesssim M^Q(1 + M|y^{-1}x|)^{-s} \|h_M\|_{L^2_{s+\varkappa}} \quad (16)$$

for any $s > 0$ and any arbitrarily small $\varkappa > 0$. Now we prove the following lemma regarding $L^1 - L^\infty$ estimates for T_j^l.

Lemma 3.3 *i) For $l > j\epsilon$*

$$\|T_j^l f\|_{L^\infty} \lesssim_\epsilon v^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} \|f\|_1.$$

ii) We also have

$$\left\| \sum_{l \leq j\epsilon} T_j^l f \right\|_\infty \leq v^{jQ} v^{-\frac{j\theta Q}{2}} \|f\|_1.$$

Proof Let $l > j\epsilon$. Then we have

$$
\begin{aligned}
|T_j^l f(x)| &\leq \left(\sup_{\{y: |y^{-1}x| \simeq \nu^{l-j(1-\theta)}\}} |K_{m_j}(y^{-1}x)| \right) \int |f(y)| \, dy \\
&\leq \left(\sup_{\{y: |y^{-1}x| \simeq \nu^{l-j(1-\theta)}\}} |\nu^{jQ} K_{m_j}(\delta_{\nu j}(y^{-1}x))| \right) \|f\|_1 \\
&\leq \nu^{jQ} \left(\sup_{\{y: |y^{-1}x| \simeq \nu^{l+j\theta}\}} |K_{m_j}((y^{-1}x))| \right) \|f\|_1 \\
&\leq \nu^{jQ} \left(\sup_{\{y: |y^{-1}x| \simeq \nu^{l+j\theta}\}} (1 + |y^{-1}x|)^{-s} \|m^j\|_{L^2_{s+\varkappa}} \right) \|f\|_1 \quad \text{(using(16))} \\
&\lesssim \nu^{jQ} \nu^{-s(l+j\theta)} \nu^{j\theta(s+\varkappa - \frac{\beta}{2})} \|f\|_1,
\end{aligned}
$$

for any $s > Q$ and a fixed small $\varkappa > 0$ from (16). Since, $l > j\epsilon$, we may choose s sufficiently large, as done in the proof of Lemma 3.1, depending on $Q, \theta, \epsilon, \varkappa$ such that

$$
\|T_j^l f\|_{L^\infty} \lesssim_\epsilon \nu^{-CQ(Q + \frac{\theta\beta}{2})(j+l)} \|f\|_1.
$$

For the second part, observe that

$$
\begin{aligned}
\left| \sum_{l \leq j\epsilon} T_j^l f(x) \right| &= \left| \int f(z) K_{m_j}(z^{-1}x) \sum_{l \leq j\epsilon} \phi(\nu^{-l+j(1-\theta)}|z^{-1}x|) \right| \\
&\leq \int |f(z)| |K_{m_j}(z^{-1}x)| \, dz \\
&\stackrel{\text{using (13)}}{\lesssim} \nu^{jQ} \|m^j\|_{L^\infty} \|f\|_1 \lesssim \nu^{jQ} \nu^{-\frac{j\theta Q}{2}} \|f\|_1,
\end{aligned}
$$

completing the proof. $\qquad\square$

Lemma 3.4 ($L^{r_1} - L^{r_1}$ *estimates*) *Let* $1 \leq r_1 \leq 2$. *Then interpolating Lemmas* 3.2 *and* 3.1 *we obtain the following:*

i) *For any* $\varepsilon > 0$

$$
\left\| \sum_{l \leq j\epsilon} T_j^l f \right\|_{r_1} \lesssim_\epsilon \nu^{-\frac{j\theta\beta}{2}} \nu^{j(\frac{\theta Q}{2} + \varepsilon)(\frac{2}{r_1} - 1)} \|f\|_{r_1}. \tag{17}
$$

ii) *For any* $l > j\epsilon$

$$
\|T_j^l f\|_{r_1} \lesssim_\epsilon \nu^{-CQ(Q + \frac{\theta\beta}{2})(j+l)} \|f\|_{r_1}. \tag{18}
$$

The next lemma concerns the key estimate which is required for our sparse domination estimates.

Lemma 3.5 *Let* $1 \leq r_1 \leq r_2 \leq 2$. *Then we have the following estimates:*

i) For $l > j\epsilon$

$$\|T_j^l f\|_{r_2} \lesssim_\epsilon \nu^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} \nu^{jQ\left(\frac{1}{r_1}-\frac{1}{r_2}\right)} \|f\|_{r_1}. \tag{19}$$

ii) For any $\varepsilon > 0$

$$\left\|\sum_{l \leq j\epsilon} T_j^l f\right\|_{r_2} \lesssim_\epsilon \nu^{-\frac{j\theta\beta}{2}} \nu^{j(\frac{\theta Q}{2}+\varepsilon)(\frac{2}{r_2}-1)} \nu^{jQ\left(\frac{1}{r_1}-\frac{1}{r_2}\right)} \|f\|_{r_1}. \tag{20}$$

Proof Let us introduce a smooth cutoff function ψ such that $\psi = 1$ on the support of ϕ. Then $m_j(\lambda) = m(\lambda)\phi(\nu^{-j}\lambda) = m(\lambda)\phi(\nu^{-j}\lambda)\psi(\nu^{-j}\lambda) = m_j(\lambda)\psi(\nu^{-j}\lambda)$. Let K_ψ be the kernel of $\psi(\sqrt{\mathcal{L}})$. Therefore, for $l > j\epsilon$, using Lemma 3.4 and Young's inequality, we obtain

$$\|T_j^l f\|_{r_2} \lesssim_\epsilon \nu^{-Q(Q+\frac{\theta\beta}{2})(j+l)} \|f * K_{\psi(\nu^{-j}.)}\|_{r_2} \lesssim_\epsilon \nu^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} \|f\|_{r_1} \|K_{\psi(\nu^{-j}.)}\|_t, \tag{21}$$

where $\frac{1}{t} = \frac{1}{r_2} + 1 - \frac{1}{r_1}$. Next recall from (16) and homogeneity that

$$|K_{\psi(\nu^{-j}.)}(x)| = \nu^{jQ}|K_\psi(\delta_{\nu^j}x)| \leq \nu^{jQ} \frac{C}{(1+|\delta_{\nu^j}x|)^N} \leq \nu^{jQ} \frac{C}{(1+\nu^j|x|)^N}$$

for any $N > 0$. From (21) and the above pointwise estimate we obtain the following

$$\|T_j^l f\|_{r_2} \lesssim_\epsilon \nu^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} \|f * K_{\psi(\nu^{-j}.)}\|_{r_2}$$

$$\lesssim_\epsilon \nu^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} \|f\|_{r_1} \|K_{\psi(\nu^{-j}.)}\|_t \lesssim_\epsilon \nu^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} \nu^{jQ\left(\frac{1}{r_1}-\frac{1}{r_2}\right)} \|f\|_{r_1},$$

where we have used that $\|K_{\psi(\nu^{-j}.)}\|_t \lesssim \nu^{jQ(1-\frac{1}{t})}$ by choosing N sufficiently large. Similarly, modifying the above arguments together with (17), we have

$$\left\|\sum_{l \geq j\epsilon} T_j^l f\right\|_{r_2} \lesssim_\epsilon \nu^{-\frac{j\theta\beta}{2}} \nu^{j(\frac{\theta Q}{2}+\varepsilon)(\frac{2}{r_2}-1)} \|f\|_{r_1} \nu^{jQ\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}$$

for any $\varepsilon > 0$. $\qquad\qquad\square$

We also need the following L^p improving estimate.

Lemma 3.6 *Let* $1 \leq r_1 \leq 2 \leq r_2 \leq r_1'$. *Then the following estimates hold true:*

i) For $l > j\epsilon$

$$\|T_j^l f\|_{r_2} \lesssim_\epsilon v^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} v^{jQ(\frac{1}{r_1}-\frac{1}{r_2})} \|f\|_{r_1}. \tag{22}$$

i) We also have

$$\left\|\sum_{l \leq j\epsilon} T_j^l f\right\|_{r_2} \lesssim_\epsilon v^{-\frac{j\theta\beta}{2}} v^{jQ(\frac{1}{r_1}-\frac{1}{r_2})} \|f\|_{r_1}. \tag{23}$$

Proof Interpolating Lemmas 3.1 and 3.3 we obtain that for any $1 \leq r_1 \leq 2$

$$\|T_j^l f\|_{r_1'} \lesssim_\epsilon v^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} \|f\|_{r_1} \quad \text{for } l > j\epsilon, \tag{24}$$

$$\text{and} \quad \left\|\sum_{l \leq j\epsilon} T_j^l f\right\|_{r_1'} \lesssim_\epsilon v^{-\frac{j\theta Q}{2}} v^{jQ(1-\frac{2}{r_1'})} \|f\|_{r_1}. \tag{25}$$

Recall from the previous lemma that $m_j(\lambda) = m_j(\lambda)\psi(v^{-j}\lambda)$ where ψ is introduced in Lemma 3.5. Also recall

$$|K_{\psi(v^{-j}.)}(x)| \leq v^{jQ} \frac{C}{(1+v^j|x|)^N} \quad \text{for any } N > 0. \tag{26}$$

For $l > j\epsilon$, employing Young's inequality we obtain

$$\|T_j^l f\|_2 \lesssim_\epsilon v^{-CQ(Q+\frac{\theta\beta}{2})(j+l)}\|f * K_{\psi(v^{-j}.)}\|_2 \leq v^{-CQ(Q+\frac{\theta\beta}{2})(j+l)}\|K_{\psi(v^{-j}.)}\|_t \|f\|_{r_1},$$

where $\frac{1}{t} = \frac{1}{2} + \frac{1}{r_1'}$. Therefore, combining the above with (26) yields the following for $l > j\epsilon$

$$\|T_j^l f\|_2 \lesssim_\epsilon v^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} v^{jQ(\frac{1}{r_1}-\frac{1}{2})} \|f\|_{r_1}. \tag{27}$$

Arguing similarly we obtain

$$\left\|\sum_{l \leq j\epsilon} T_j^l f\right\|_2 \lesssim_\epsilon v^{-\frac{j\theta\beta}{2}} v^{jQ(\frac{1}{r_1}-\frac{1}{2})} \|f\|_{r_1}. \tag{28}$$

Interpolating (24) and (27) we obtain that

$$\|T_j^l f\|_{r_2} \lesssim_\epsilon v^{-CQ(Q+\frac{\theta\beta}{2})(j+l)} v^{jQ(\frac{1}{r_1}-\frac{1}{r_2})} \|f\|_{r_1}, \quad \text{holds for all } l > j\epsilon.$$

Similarly, interpolating (25) and (28), we get

$$\left\| \sum_{l \leq j\epsilon} T_j^l f \right\|_{r_2} \lesssim_\epsilon v^{-\frac{j\theta\beta}{2}} v^{jQ(\frac{1}{r_1} - \frac{1}{r_2})} \|f\|_{r_1}.$$

This completes the proof. □

Now we are in a position to prove our main Theorem 1.3 for the case $\theta > 0$. After having the key unweighted estimates, the proof of sparse domination is now quite standard and we provide a brief sketch, for more details we refer to [4].

Proof of Theorem 1.3 Recall the dyadic families $\mathscr{D}^n$ for $n = 1, \cdots, \mathfrak{N}$ and $\mathscr{D}^n = \cup_{k\in\mathbb{Z}} \mathscr{D}_k^n$. Let us define the operators

$$T_{j,n}^l f := \sum_{\substack{R \in \mathscr{D}^n : \\ R \in \mathscr{D}_{\lfloor j(1-\theta)-j\epsilon+10 \rfloor}^n}} T_j^l(f \chi_{c_G B(R)}) \text{ for } l \leq j\epsilon,$$

$$T_{j,n}^l f := \sum_{\substack{R \in \mathscr{D}^n : \\ R \in \mathscr{D}_{\lfloor j(1-\theta)-l+10 \rfloor}^n}} T_j^l(f \chi_{c_G B(R)}) \text{ for } l > j\epsilon,$$

where the universal constant c_G is chosen sufficiently small (by rescaling the metric) to ensure that the support of $T_j^l(f \chi_{c_G B(R)})$ is contained in R. Therefore, it is enough to obtain sparse domination for

$$\mathcal{T}_n := \sum_{\substack{j \geq 0 \\ l \in \mathbb{Z}}} T_{j,n}^l, \text{ for } n = 1, \cdots, \mathfrak{N}.$$

Hence, we only prove sparse domination for one of the $\mathcal{T}_n$ and suppress the index n for simplicity. By localisation and Lemma 3.5, we obtain for $1 \leq r_1 \leq r_2 \leq 2$

$$\left| \left\langle \sum_{l \leq j\epsilon} T_j^l f, g \right\rangle \right|$$

$$\lesssim \sum_{j \geq 0} \sum_{\substack{R \in \mathscr{D} : \\ R \in \mathscr{D}_{\lfloor j(1-\theta)-j\epsilon+10 \rfloor}}} \left\| \left(\sum_{l \leq j\epsilon} T_j^l(f \chi_{c_G B(R)}) \right) \chi_R \right\|_{r_2} \|g \chi_R\|_{r_2'}$$

$$\overset{(20)}{\lesssim}_\epsilon \sum_{j \geq 0} \sum_{\substack{R \in \mathscr{D} : \\ R \in \mathscr{D}_{\lfloor j(1-\theta)-j\epsilon+10 \rfloor}}} v^{-\frac{j\theta\beta}{2}} v^{j(\frac{\theta Q}{2}+\varepsilon)(\frac{2}{r_2}-1)} v^{jQ(\frac{1}{r_1}-\frac{1}{r_2})} \|f \chi_R\|_{r_1} \|g \chi_R\|_{r_2'}$$

$$\lesssim_\epsilon \sum_{j \geq 0} \sum_{\substack{R \in \mathscr{D} : \\ R \in \mathscr{D}_{\lfloor j(1-\theta)-j\epsilon+10 \rfloor}}} |R|^{\frac{1}{r_1}+\frac{1}{r_2'}-1} v^{-\frac{j\theta\beta}{2}} v^{j(\frac{\theta Q}{2}+\varepsilon)(\frac{2}{r_2}-1)} v^{jQ(\frac{1}{r_1}-\frac{1}{r_2})} |R| \langle f \rangle_{r_1,R} \langle g \rangle_{r_2',R}$$

$$\lesssim_\epsilon \sum_{j \geq 0} \sum_{\substack{R \in \mathscr{D} : \\ R \in \mathscr{D}_{\lfloor j(1-\theta)-j\epsilon+10 \rfloor}}} v^{Q(j\epsilon-j(1-\theta))(\frac{1}{r_1}+\frac{1}{r_2'}-1)} v^{-\frac{j\theta\beta}{2}} v^{j(\frac{\theta Q}{2}+\varepsilon)(\frac{2}{r_2}-1)} v^{jQ(\frac{1}{r_1}-\frac{1}{r_2})} |R| \langle f \rangle_{r_1,R} \langle g \rangle_{r_2',R}$$

$$\lesssim_\epsilon \sum_{j \geq 0} v^{j\left(\theta Q\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \frac{\theta\beta}{2} + \frac{\theta Q}{2}\left(\frac{2}{r_2} - 1\right) + \varepsilon\left(\frac{2}{r_2} - 1\right) + \epsilon Q\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right)} \sum_{\substack{R \in \mathscr{D}: \\ R \in \mathscr{D}_{\lfloor j(1-\theta) - j\epsilon + 10 \rfloor}}} |R| \langle f \rangle_{r_1, R} \langle g \rangle_{r_2', R},$$

$$\tag{29}$$

since ϵ and ε are sufficiently small the above gives a geometrically decaying sparse collection if

$$\theta Q\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \frac{\theta\beta}{2} + \frac{\theta Q}{2}\left(\frac{2}{r_2} - 1\right) < 0 \iff \frac{1}{r_1} - \frac{1}{2} < \frac{\beta}{2Q}.$$

A similar argument for T_j^l, $l > j\epsilon$, with Lemma 3.5, yields the following

$$\left|\left\langle \sum_{j \geq 0} \sum_{l > j\epsilon} T_j^l(f \chi_{c_G B(R)}), g \right\rangle\right|$$

$$\lesssim \sum_{j \geq 0} \sum_{l > j\epsilon} \sum_{\substack{R \in \mathscr{D}: \\ R \in \mathscr{D}_{\lfloor j(1-\theta) - l + 10 \rfloor}}} \|T_j^l(f \chi_{c_G B(R)}) \chi_R\|_{r_2} \|g \chi_R\|_{r_2'} \tag{30}$$

$$\overset{(19)}{\lesssim} \sum_{j \geq 0} \sum_{l > j\epsilon} \sum_{\substack{R \in \mathscr{D}: \\ R \in \mathscr{D}_{\lfloor j(1-\theta) - l + 10 \rfloor}}} v^{Q(l - j(1-\theta))\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} v^{-CQ\left(Q + \frac{\theta\beta}{2}\right)(j+l)} v^{jQ\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} |R| \langle f \rangle_{r_1, R} \langle g \rangle_{r_2', R}$$

$$\lesssim \sum_{j \geq 0} v^{jQ\left(\theta\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - C\left(Q + \frac{\theta\beta}{2}\right)\right)} \sum_{l > j\epsilon} v^{Ql\left(\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - C\left(Q + \frac{\theta\beta}{2}\right)\right)} \sum_{\substack{R \in \mathscr{D}: \\ R \in \mathscr{D}_{\lfloor j(1-\theta) - l + 10 \rfloor}}} |R| \langle f \rangle_{r_1, R} \langle g \rangle_{r_2', R}.$$

$$\tag{31}$$

As mentioned earlier, the constant C can be chosen sufficiently large, indeed we can always ensure that $\left(\theta\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - C\left(Q + \frac{\theta\beta}{2}\right)\right) < 0$ and $\left(\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - C\left(Q + \frac{\theta\beta}{2}\right)\right) < 0$. Therefore, we again obtain geometrically decaying (r_1, r_2') sparse domination. Hence, combining (29) and (31), we obtain (r_1, r_2') sparse domination for $1 \leq r_1 \leq r_2 \leq 2$ provided $\frac{1}{r_1} - \frac{1}{2} < \frac{\beta}{2Q}$.

Arguing similarly in the case $1 \leq r_1 \leq 2 \leq r_2 \leq r_1'$ with Lemma 3.6 yields

$$\left|\left\langle \sum_{l \leq j\epsilon} T_j^l(f \chi_{c_G B(R)}), g \right\rangle\right|$$

$$\lesssim \sum_{j \geq 0} \sum_{\substack{R \in \mathscr{D}: \\ R \in \mathscr{D}_{\lfloor j(1-\theta) - j\epsilon + 10 \rfloor}}} \left\|\left(\sum_{l \leq j\epsilon} T_j^l(f \chi_{c_G B(R)})\right) \chi_R\right\|_{r_2} \|g \chi_R\|_{r_2'}$$

$$\overset{(23)}{\lesssim}_\epsilon \sum_{j \geq 0} \sum_{\substack{R \in \mathscr{D}: \\ R \in \mathscr{D}_{\lfloor j(1-\theta) - j\epsilon + 10 \rfloor}}} v^{Q(j\epsilon - j(1-\theta))\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} v^{-\frac{j\theta\beta}{2}} v^{jQ\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} |R| \langle f \rangle_{r_1, R} \langle g \rangle_{r_2', R}$$

$$\lesssim_\epsilon \sum_{j \geq 0} v^{j\left(-Q(1-\theta)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \frac{\theta\beta}{2} + Q\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \varepsilon Q\right)} \sum_{\substack{R \in \mathscr{D}: \\ R \in \mathscr{D}_{\lfloor j(1-\theta) - j\epsilon + 10 \rfloor}}} |R| \langle f \rangle_{r_1, R} \langle g \rangle_{r_2', R},$$

$$\tag{32}$$

since $\epsilon > 0$ is sufficiently small we have a geometrically decaying (r_1, r_2') sparse domination for $1 \le r_1 \le 2 \le r_2 \le r_1'$ if

$$-Q(1-\theta)\Big(\frac{1}{r_1} - \frac{1}{r_2}\Big) - \frac{\theta\beta}{2} + Q\Big(\frac{1}{r_1} - \frac{1}{r_2}\Big) < 0 \iff \frac{1}{r_1} - \frac{1}{r_2} < \frac{\beta}{2Q}.$$

A similar argument also produces (r_1, r_2') sparse domination for the pieces $T_j^l, l > j\epsilon$, in the range $1 \le r_1 \le 2 \le r_2 \le r_1'$. Our proof only produces geometrically decaying sparse domination since in the dyadic scale $\mathscr{D}_{\lfloor j(1-\theta)-j\epsilon+10\rfloor}$ cubes are disjoint, however, to obtain a true sparse bound a similar argument can be produced as in [4], see also [35]. Since $m(\sqrt{\mathcal{L}})^* = \overline{m}(\sqrt{\mathcal{L}})$, (r_1, r_2') sparse domination implies (r_2, r_1') sparse domination. This completes the proof of Theorem 1.3 for the case $\theta > 0$. $\square$

Remark 3.7 Let $\theta < 0$ and $m \in \mathscr{M}(\theta, \beta)$. The case $\theta < 0$ represents low frequencies, hence we need to decompose the multiplier as done in [4, 14]. Therefore,

$$m(\lambda) = \sum_{j:j\le 0} m_j(\lambda),$$

where $m_j = m(\lambda)\phi(\nu^{-j}\lambda)$. We can rewrite the above as $m(\lambda) = \sum_{k:k\ge 0} m_{-k}(\lambda)$, where $m_{-k} = m(\lambda)\phi(\nu^k\lambda)$ for $k \ge 0$. Also $m^{-k}(\lambda) := m(\nu^{-k}\lambda)\phi(\lambda)$. Then the facts that $\|m^{-k}\|_{L^\infty} \le C\nu^{k\theta\beta/2}$, and $\|m^{-k}\|_{L_s^2} \le C\nu^{-k\theta(2s-\beta)/2}$ for all $k \ge 0$ and for $s \in \mathbb{N}$, yield the following estimate as in Lemma 3.1 by choosing $s \gg \lfloor -\frac{\theta\beta}{2\epsilon}\rfloor$

$$\|T_k^l f\|_2 \lesssim_\epsilon \nu^{Q(\frac{\theta\beta}{2} - Q)(k+l)}\|f\|_2 \text{ for } l > k\epsilon,$$

also,

$$\|T_k f\|_2 \lesssim \nu^{\frac{k\theta\beta}{2}} \text{ for } k \ge 0.$$

Now one can modify Lemmas 3.2, and 3.3 appropriately to obtain similar results in this case.

4 Applications

4.1 Quantitative estimates

In this subsection we obtain several weighted estimates for oscillating multipliers $m(\sqrt{\mathcal{L}})$. Recall the following notion of Muckenhoupt weights on homogeneous spaces from [2]. Let $1 < p < \infty$, $\omega \in A_p$ if

$$[\omega]_{A_p} := \sup_B \left(\frac{1}{|B|}\int_B \omega\right)\left(\frac{1}{|B|}\int_B \omega^{1-p'}\right)^{p-1} < \infty, \tag{33}$$

where the supremum is over all balls. Also, we say $\omega \in RH_q$ for $1 < q < \infty$, if $[\omega]_{RH_q} := \sup_B \langle \omega \rangle_{1,B}^{-1} \langle \omega \rangle_{q,B} < \infty$. Corresponding to a sparse family $\mathcal{S}$, and $1 \leq r_1, r_2 \leq \infty$, let $\Lambda_{\mathcal{S},r_1,r_2'}$ denote the following bilinear form

$$\Lambda_{\mathcal{S},r_1,r_2'}(f,g) := \sum_{R \in \mathcal{S}} |R| \langle f \rangle_{r_1,R} \langle g \rangle_{r_2',R}.$$

The following quantitative estimate was proved in [2].

Lemma 4.2 [2] *For any $r_1 < p < r_2$, and $\omega \in A_{p/r_1} \cap RH_{(r_2/p)'}$, we have*

$$\Lambda_{\mathcal{S},r_1,r_2'}(f,g) \lesssim_{r_1,r_2,p,\mathcal{S}} \left([\omega]_{A_{p/r_1}}[\omega]_{RH_{(r_2/p)'}}\right)^{\max\{\frac{1}{p-r_1}, \frac{r_2-1}{r_2-p}\}} \|f\|_{L^p(w)} \|g\|_{L^{p'}(w^{1-p'})}.$$

As a consequence of Theorem 1.3, we now prove Theorem 1.5 concerning weighted estimates for $m(\sqrt{\mathcal{L}})$.

Proof of Theorem 1.5 We first prove part (i) and (ii). Assume $m \in \mathcal{M}(\theta, \beta)$ with $\theta \in \mathbb{R} \setminus \{0\}$ with $Q \leq \beta \leq 2Q$. The proof follows from Theorem 1.3 and reverse Hölder's property of A_p weights, see [28]. It is easy to observe from Theorem 1.3 that we have $(r_1, 1)$ sparse domination for all r_1 such that $0 < \frac{1}{r_1} < \frac{1}{p_\beta}$. Let $p_\beta < p < \infty$, and $\omega \in A_{p/p_\beta}$. We can always choose $\varepsilon > 0$ such that $\frac{1}{p} < \frac{1}{p_\beta} - \frac{\varepsilon}{p}$. Denote $\frac{1}{r_1} := \frac{1}{p_\beta} - \frac{\varepsilon}{p}$. Moreover, reverse Hölder's inequality ensures that the quantity ε can be chosen such that $\omega \in A_{\frac{p}{r_1}}$. Theorem 1.3 and Lemma 4.2 imply that for any compactly supported f and g, there exists a sparse family $\mathcal{S}$ such that

$$\left| \left\langle m(\sqrt{\mathcal{L}})f, g \right\rangle \right| \lesssim \Lambda_{\mathcal{S},r_1,1}(f,g) \lesssim C([\omega]_{A_{p/p_\beta}}) \|f\|_{L^p(w)} \|g\|_{L^{p'}(w^{1-p'})}.$$

Now duality concludes the proof.

Let us now prove part (iii). Let $m \in \mathcal{M}(\theta, \beta)$ with $0 < \beta < Q$. Theorem 1.3 implies that we have a $(2, s')$ sparse domination for all s' such that $\frac{1}{2} \leq \frac{1}{s'} < \frac{1}{2} + \frac{\beta}{2Q}$. Let $2 < p < s_\beta$, and $\omega \in A_{p/2} \cap RH_{(s_\beta/p)'}$. By self-improving property of reverse Hölder's classes, $\omega \in RH_{(s_\beta/p)'(1+\delta)}$, for sufficiently small $\delta > 0$. It is easy to choose s such that $2 < p < s < s_\beta$ satisfying $\frac{1}{2} < \frac{1}{s'} < \frac{1}{2} + \frac{\beta}{2Q} = \frac{1}{s_\beta}$ and $\omega \in RH_{(s/p)'}$ simultaneously. Therefore,

$$\left| \left\langle m(\sqrt{\mathcal{L}})f, g \right\rangle \right| \lesssim \Lambda_{\mathcal{S},2,s'}(f,g) \lesssim C([\omega]_{A_{p/2}}, [\omega]_{RH_{(s_\beta/p)'}}) \|f\|_{L^p(w)} \|g\|_{L^{p'}(w^{1-p'})}.$$

Now the proof follows from duality. □

4.3 Riesz means

For $k, \alpha, t > 0$, we define the Riesz means

$$I_{k,\alpha,t}(\mathcal{L}) := kt^{-k} \int_0^t (t-s)^{k-1} e^{is(\sqrt{\mathcal{L}})^\alpha} \, ds. \tag{34}$$

Without loss of generality, let us assume that $t = 1$ and simply denote $I_{k,\alpha,1}(\mathcal{L})$ by $I_{k,\alpha}(\mathcal{L})$. It is well-known that the operator $I_{k,\alpha}(\mathcal{L})$ can be written as $\sigma((\sqrt{\mathcal{L}})^\alpha)$, where the spectral multiplier σ can be decomposed as $\sigma(\lambda) = c_k \psi(\lambda)\lambda^{-k} e^{i\lambda} + \sigma_1(\lambda)$, where σ_1 is a smooth function satisfying the Mikhlin–Hörmander condition, and ψ is a C^∞ function such that $\psi = 0$ if $0 \leq \lambda \leq 1$ and $\psi \equiv 1$ for $\lambda \geq 2$. We refer to [1, 8, 42, 43] and references therein. As $\sigma_1((\sqrt{\mathcal{L}})^\alpha)$ always satisfy $(1, 1)$ sparse domination, the following sparse domination follows from Corollary 1.4

$$|\langle I_{k,\alpha}(\mathcal{L})f, g\rangle| \lesssim_{k,\alpha,r_1,r_2} \sum_{R \in \mathcal{S}} |R| \langle f \rangle_{r_1,R} \langle g \rangle_{r_2',R}$$

$$\text{and } |\langle I_{k,\alpha}(\mathcal{L})f, g\rangle| \lesssim_{k,\alpha,r_1,r_2} \sum_{R \in \mathcal{S}'} |R| \langle f \rangle_{r_2',R} \langle g \rangle_{r_1,R},$$

where r_1, r_2 satisfy

$$\left(\frac{1}{r_1} - \frac{1}{2}\right) < \frac{k}{Q}, \quad 1 \leq r_1 \leq r_2 \leq 2, \quad \text{or} \quad \left(\frac{1}{r_1} - \frac{1}{r_2}\right) < \frac{k}{Q}, \quad 1 \leq r_1 \leq 2 \leq r_2 \leq r_1'.$$

The above sparse domination and Theorem 1.5 yield the following weighted estimates.

i) Let $k \geq Q$. Then $I_{k,\alpha}(\mathcal{L})$ maps $L^p(\omega)$ to $L^p(\omega)$ for all $1 < p < \infty$ and $\omega \in A_p$.

ii) Let $\frac{Q}{2} \leq k < Q$. Then $I_{k,\alpha}(\mathcal{L})$ maps $L^p(\omega)$ to $L^p(\omega)$ for $p_k < p < \infty$ and $\omega \in A_{p/p_k}$, where $p_k := \frac{Q}{k}$.

iii) Let $0 < k < \frac{Q}{2}$. Then $I_{k,\alpha}(\mathcal{L}) : L^p(\omega) \to L^p(\omega)$ for all $2 < p < s_k$, $\omega \in A_{p/2} \cap RH_{(s_k/p)'}$, where $\frac{1}{s_k} := \frac{1}{2} - \frac{k}{Q}$.

4.4 Dispersive equations

Let $f \in C_0^\infty(G)$ and $\alpha \in \mathbb{N}$. Consider the dispersive equation

$$i \, \partial_t u + (\sqrt{\mathcal{L}})^\alpha \, u = 0, \; u(\cdot, 0) = f.$$

Then $u(x, t) = e^{it(\sqrt{\mathcal{L}})^\alpha} f(x, t)$. For a fixed time t, rescaling the operator $\sqrt{\mathcal{L}}$ by $t^{1/\alpha}\sqrt{\mathcal{L}}$, one can prove the following as a consequence of Corollary 1.4

$$|\langle u(\cdot, t), g\rangle| \lesssim_{\beta,\alpha,r_1,r_2,t} \sum_{R \in \mathcal{S}} |R| \langle (I + \sqrt{\mathcal{L}})^\beta f \rangle_{r_1,R} \langle g \rangle_{r_2',R}$$

whenever $\left(\frac{1}{r_1} - \frac{1}{2}\right) < \frac{\beta}{\alpha Q}$, $1 \leq r_1 \leq r_2 \leq 2$ or $\left(\frac{1}{r_1} - \frac{1}{r_2}\right) < \frac{\beta}{\alpha Q}$, $1 \leq r_1 \leq 2 \leq r_2 \leq r_1'$. Let $W_\omega^{s,p}$ denotes the non-homogeneous weighted Sobolev space $W_\omega^{s,p} = \{f : \|f\|_{W_\omega^{s,p}} := \|(I + \sqrt{\mathcal{L}})^s f\|_{L^p(\omega)} < \infty\}$. As an application of Theorem 1.5, we can derive the following weighted estimates:

i) Let $1 < p < \infty$ and $\omega \in A_p$. Then $\|u(\cdot, t)\|_{L^p(\omega)} \lesssim \|f\|_{W_\omega^{\beta,p}}$ provided $\beta \geq \alpha Q$.

ii) Let $\frac{\alpha Q}{2} \leq \beta < \alpha Q$. Then $\|u(\cdot, t)\|_{L^p(\omega)} \lesssim \|f\|_{W_\omega^{\beta,p}}$ holds for all $p_{\alpha,\beta} < p < \infty$ and $\omega \in A_{p/p_{\alpha,\beta}}$, where $p_{\alpha,\beta} := \frac{Q\alpha}{\beta}$.

iii) Finally, let $0 < \beta < \frac{\alpha Q}{2}$. We also have that $\|u(\cdot, t)\|_{L^p(\omega)} \lesssim \|f\|_{W_\omega^{\beta,p}}$ holds for all $2 < p < s_{\alpha,\beta}$, $\omega \in A_{p/2} \cap RH_{(s_{\alpha,\beta}/p)'}$, where $\frac{1}{s_{\alpha,\beta}} := \frac{1}{2} - \frac{\beta}{\alpha Q}$.

References

1. Alexopoulos, G.: Oscillating multipliers on Lie groups and Riemannian manifolds. Tohoku Math. J. (2) **46**(4), 457–468 (1994)
2. Bernicot, F., Frey, D., Petermichl, S.: Sharp weighted norm estimates beyond Calderón–Zygmund theory. Anal. PDE **9**(5), 1079–1113 (2016)
3. Bramati, R., Ciatti, P., Green, J., Wright, J.: Oscillating spectral multipliers on groups of Heisenberg type. Rev. Mat. Iberoam. **38**(5), 1529–1551 (2022)
4. Beltran, D., Cladek, L.: Sparse bounds for pseudodifferential operators. J. Anal. Math. **140**(1), 89–116 (2020)
5. Beltran, D., Roos, J., Seeger, A.: Multi-scale sparse domination, to appear in Mem. Am. Math. Soc. arXiv:2009.00227
6. Bonfiglioli, A., Lanconelli, E., Uguzzoni, F.: Uniform Gaussian estimates for the fundamental solutions for heat operators on Carnot groups. Adv. Differ. Equ. **7**(10), 1153–1192 (2002)
7. Bui, T.A., Ly, F.K.: Sharp estimates for Schrödinger groups on Hardy spaces for $0 < p \leq 1$,. J. Fourier Anal. Appl. **28**(4), 23 (2022)
8. Bui, T.A., Hong, Q., Hu, G.: On boundedness of oscillating multipliers on stratified Lie groups. J. Geom. Anal. **32**(8), 20 (2022)
9. Bui, T.A., Duong, X.T.: Spectral multipliers of self-adjoint operators on Besov and Triebel-Lizorkin spaces associated to operators. Int. Math. Res. Not. (23), 18181–18224 (2021)
10. Bui, T.A., D'Ancona, P., Nicola, F.: Sharp L^p estimates for Schrödinger groups on spaces of homogeneous type. Rev. Mat. Iberoam. **36**(2), 455–484 (2020)
11. Cardona, D., Delgado, J., Ruzhansky, M.: L^p-bounds for pseudo-differential operators on graded Lie groups. J. Geom. Anal. **31**(12), 11603–11647 (2021)
12. Cardona, D., Ruzhansky, M.: Björk–Sjölin condition for strongly singular convolution operators on graded Lie groups. Math. Z. **302**(4), 1957–1981 (2022)
13. Chanillo, S.: Weighted norm inequalities for strongly singular convolution operators. Trans. Am. Math. Soc. **281**(1), 77–107 (1984)
14. Ciatti, P., Wright, J.: Strongly Singular Integrals on Stratified Groups, Geometric Aspects of Harmonic Analysis, 299-320, Springer INdAM Ser., 45, Springer, Cham (2021)
15. Chanillo, S., Kurtz, D., Sampson, G.: Weighted L^p estimates for oscillating kernels. Ark. Mat. **21**, 233–257 (1983)
16. Chanillo, S., Kurtz, D., Sampson, G.: Weighted weak (1, 1) and weighted L^p estimates for oscillating kernels. Trans. Am. Math. Soc. **295**(1), 127–145 (1986)

17. Christ, M.: Lectures on singular integral operators. In: CBMS Regional Conference Series in Mathematics, vol. 77, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (1990)

18. Christ, M.: L^p bounds for spectral multipliers on nilpotent groups. Trans. Am. Math. Soc. **328**(1), 73–81 (1991)

19. Conde-Alonso, J.M., Culiuc, A., Di Plinio, F., Ou, Y.: A sparse domination principle for rough singular integrals. Anal. PDE **10**(5), 1255–1284 (2017)

20. Conde-Alonso, J. M., Di Plinio, F., Parissis, I., Vempati, M.N.: A metric approach to sparse domination. Ann. Mat. Pura Appl. **4** 201(4), 1639–1675 (2022)

21. Duong, X.T., Ouhabaz, E.M., Sikora, A.: Plancherel-type estimates and sharp spectral multipliers. J. Funct. Anal. **196**(2), 443–485 (2002)

22. Fefferman, C.: Inequalities for strongly singular integral operators. Acta Math. **24**, 9–36 (1970)

23. Fefferman, C., Stein, E.M.: H^p spaces of several variables. Acta Math. **129**, 137–193 (1972)

24. Fischer, V., Ruzhansky, M.: Quantization on Nilpotent Lie groups, Progress in Mathematics, 314. Birkhäuser/Springer, [Cham], 2016. xiii+557 pp

25. Fischer, V., Ruzhansky, M.: Fourier multipliers on graded Lie groups. Colloq. Math. **165**(1), 1–30 (2021)

26. Hirschman, I.: On multiplier transformations. Duke Math. J. **26**, 222–242 (1959)

27. Hörmander, L.: Pseudo-differential operators and hypoelliptic equations. Proc. Sympos. Pure Math., **10**, Am. Math. Soc. 138–183 (1967)

28. Hytönen, T., Pérez, C., Rela, E.: Sharp reverse Hölder property for A^∞ weights on spaces of homogeneous type. J. Funct. Anal. **263**(12), 3883–3899 (2012)

29. Hytönen, T., Kairema, A.: Systems of dyadic cubes in a doubling metric space. Colloq. Math. **126**(1), 1–33 (2012)

30. Hytönen, T.P.: The sharp weighted bound for general Calderón–Zygmund operators. Ann. Math. (2) **175**(3), 1473–1506 (2012)

31. Hytönen, T.P., Roncal, L., Tapiola, O.: Quantitative weighted estimates for rough homogeneous singular integrals. Israel J. Math. **218**(1), 133–164 (2017)

32. Hong Q.Q., Hu, G., Ruzhansky, M.: Fourier multipliers for Hardy spaces on graded Lie groups. Proc. R. Soc. Edinb. Sect. A153 , (5), 1729–1750 (2023)

33. Lacey, M.T.: An elementary proof of the A_2 bound. Israel J. Math. **217**(1), 181–195 (2017)

34. Lacey, M.T.: Sparse bounds for spherical maximal functions. J. Anal. Math. **139**(2), 613–635 (2019)

35. Lacey, M.T., Mena, D.: The sparse T1 theorem. Houston J. Math. **43**(1), 111–127 (2017)

36. Lerner, A.K.: On pointwise estimates involving sparse operators. N. Y. J. Math. **22**, 341–349 (2016)

37. Lorist, E.: On pointwise ℓ^r-sparse domination in a space of homogeneous type. J. Geom. Anal. **31**(9), 9366–9405 (2021)

38. Martini, A.: Joint functional calculi and a sharp multiplier theorem for the Kohn Laplacian on the sphere. Math. Z. **286**, 1539–1574 (2017)

39. Martini, A., Müller, D.: Spectral multipliers of Euclidean type on new classes of two-step stratified groups. Proc. Lond. Math. Soc. **109**, 1229–1263 (2014)

40. Martini, A., Müller, D.: Spectral multipliers on 2-step groups: topological versus homogeneous dimension. GAFA **26**, 680–702 (2016)

41. Mauceri, G., Meda, S.: Vector-valued multipliers on stratified groups. Revista Ibero. Mat. **6**(3), 141–154 (1990)

42. Miyachi, A.: On the weakly strongly singular integrals. Japan. J. Math. **4**, 221–262 (1978)

43. Miyachi, A.: On some estimates for the wave equation in L^p and H^p. J. Fac. Sci. Tokyo Sci. IA **27**, 331–354 (1980)

44. Müller, D., Seeger, A.: Sharp L^p-bounds for the wave equation on groups of Heisenberg type. Anal. PDE **8**(5), 1051–1100 (2015)

45. Müller, D., Stein, E.M.: On spectral multipliers for Heisenberg and related groups. J. Math. Pures Appl **73**, 413–440 (1994)

46. Müller, D., Stein, E.M.: L^p-estimates for the wave equation on the Heisenberg group. Rev. Mat. Ibero. **15**, 297–334 (1990)

47. Ouhabaz, E.M.: Analysis of Heat Equations on Domains. London Mathematical Society Monographs Series, 31. Princeton University Press, Princeton, NJ . xiv+284 pp (2005)

48. Peral, J.: L^p estimates for the wave equation. J. Funct. Anal. **36**, 114–145 (1980)

49. Sikora, A.: Multiplier theorem for sub-Laplacians on homogeneous groups. C. R. Acad. Sci. Paris Sér. I Math. **315**(4), 417–419 (1992)
50. Wainger, S.: Special trigonometric series in k dimesnions. Mem. Am. Math. Soc. (59) (1965)