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Radial symmetry of minimizers to the weighted *p*-Dirichlet energy

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Abstract

Let $\mathbb{A} = \{z : r < |z| < R\}$ and $\mathbb{A}^* = \{z : r^* < |z| < R^*\}$ be annuli in the complex plane. Let $p \in [1, 2]$ and assume that $\mathcal{H}^{1,p}(\mathbb{A}, \mathbb{A}^*)$ is the class of Sobolev homeomorphisms between $\mathbb{A}$ and $\mathbb{A}^*$, $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$. Then we consider the following Dirichlet type energy of h:

$$\mathscr{F}_p[h] = \int_{\mathbb{A}} \frac{\|Dh\|^p}{|h|^p}, 1 \leqslant p \leqslant 2.$$

We prove that this energy integral attains its minimum, and the minimum is a certain radial diffeomorphism $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$, provided a radial diffeomorphic minimizer exists. If $p > 1$ then such diffeomorphism exists always. If $p = 1$, then the conformal modulus of $\mathbb{A}^*$ must not be greater or equal to $\pi/2$. This curious phenomenon is opposite to the Nitsche type phenomenon known for the standard Dirichlet energy.

Keywords Variational integrals · Harmonic mappings · Energy-minimal deformations · Dirichlet-type energy

Mathematics Subject Classification Primary 35J60; Secondary 30C70

1 Introduction

The general law of hyperelasticity tells us that there exists an energy integral $E[h] = \int_{\mathbb{X}} E(x, h, Dh)dx$ where $E : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is a given stored-energy function characterizing mechanical properties of the material. Here $\mathbb{X}$ and $\mathbb{Y}$ are nonempty bounded domains in $\mathbb{R}^n$, $n > 2$. The mathematical models of nonlinear

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elasticity have been first studied by Antman [1], Ball [4, 5], and Ciarlet [8]. One of the interesting and important problems in nonlinear elasticity is whether the radially symmetric minimizers are indeed global minimizers of the given physically reasonable energy. This leads us to study energy minimal homeomorphisms $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ of Sobolev class $\mathscr{W}^{1,2}$ between annuli $\mathbb{A} = \mathbb{A}(r, R) = \{x \in \mathbb{R}^n : r < |x| < R\}$ and $\mathbb{A}^* = \mathbb{A}(r_*, R_*) = \{x \in \mathbb{R}^n : r_* < |x| < R_*\}$. Here $0 \leqslant r < R$ and $0 \leqslant r_* < R_*$ are the inner and outer radii of $\mathbb{A}$ and $\mathbb{A}^*$. The variational approach to Geometric Function Theory [2, 3] makes this problem more important. Indeed, several papers are devoted to understanding the expected radial symmetric properties see [17] and the references therein. Many times experimentally known answers to practical problems have led us to the deeper study of such mathematically challenging problems. We seek to minimize the p-harmonic energy of mappings between two annuli in $\mathbb{R}^2$. We consider the modified Dirichlet energy $\mathscr{F}_p[f] = \int_{\mathbb{A}} \frac{\|Df\|^p}{|f|^p}$, $1 \leqslant p \leqslant 2$ and minimize it.

2 *p*-harmonic equation and statement of the main results

For natural number n, let $A = (a_{i,j})_{n\times n} \in \mathbb{R}^{n\times n}$. We use A^T to denote the transpose of A. The *Hilbert-Schmit norm*, also called the *Frobenius norm*, of A is denoted by $\|A\|$, where

$$\|A\|^2 = \sum_{1\le i,j\le n} \left|a_{i,j}\right|^2 = \mathrm{tr}[A^T A].$$

For $p \geq 1$, we say that a mapping h belongs to the class $\mathcal{W}^{1,p}(\mathbb{A}, \mathbb{A}^*)$, if h belongs to the Sobolev space $\mathcal{W}^{1,p}(\mathbb{A})$ and maps $\mathbb{A}$ onto $\mathbb{A}^*$. Let $h = (h^1, \dots, h^n)$ belong to $\mathcal{W}^{1,p}(\mathbb{A}, \mathbb{A}^*)$. We denote the *Jacobian matrix* of h at the point $x = (x_1, \dots, x_n)$ by $Dh(x)$, where $Dh(x) = \left(\frac{\partial h^i}{\partial x_j}\right)_{n\times n} \in \mathbb{R}^{n\times n}$. Then

$$\|Dh\|^2 = \sum_{1\le i,j\le n} \left|\frac{\partial h^i}{\partial x_j}\right|^2.$$

Here $\frac{\partial h^i}{\partial x_j}$ denotes the weak partial derivatives of h^i with respect to x_j. If h is continuous and belongs to $\mathcal{W}^{1,p}(\mathbb{A}, \mathbb{A}^*)$ $(p \geq 1)$, then the weak and ordinary partial derivatives coincide a.e. in $\mathbb{A}$ (cf. [19, Proposition 1.2]). Let $h = \rho S$, where $S = \frac{h}{|h|}$ and $\rho = |h|$. By [14, Equality (3.2)], we obtain that

$$Dh(x) = \nabla\rho(x) \otimes S(x) + \rho \cdot DS(x)$$

and

$$\|Dh(x)\|^2 = |\nabla\rho(x)|^2 + \rho^2\|DS(x)\|^2, \tag{2.1}$$

where $\nabla\rho$ denotes the gradient of ρ.

We say that $h : \mathbb{A} \to \mathbb{A}^*$ is a *radial mapping*, if $h(x) = \rho(|x|)\frac{x}{|x|}$ and if ρ is real and positive function. We use $\mathcal{R}(\mathbb{A}, \mathbb{A}^*)$ to denote the class of radial homeomorphisms in $\mathcal{W}^{1,p}(\mathbb{A}, \mathbb{A}^*)$ and use $\mathcal{P}(\mathbb{A}, \mathbb{A}^*)$ to denote the class of generalized radial homeomorphisms in $\mathcal{W}^{1,p}(\mathbb{A}, \mathbb{A}^*)$. We also use $\mathcal{H}(\mathbb{A}, \mathbb{A}^*)$ to denote the class of homeomorphisms in $\mathcal{W}^{1,p}(\mathbb{A}, \mathbb{A}^*)$.

As it is said before, an important problems in nonlinear elasticity is whether the radially symmetric minimizers are indeed global minimizers. For example, Iwaniec, and Onninen [12] discussed the minimizers of the following two energy integrals:

$$\mathfrak{E}[h] = \int_{\mathbb{A}} \|Dh(x)\|^n dx \quad \text{and} \quad \mathfrak{F}[h] = \int_{\mathbb{A}} \frac{\|Dh(x)\|^n}{|h(x)|^n} dx$$

among all homeomorphisms in $\mathcal{W}^{1,n}(\mathbb{A}, \mathbb{A}^*)$, respectively. The energy integral $\mathfrak{F}$ for $n = 2$, has been considered previously by Astala, Iwaniec, and Martin in [2]. Further such energy has been generalized in planar annuli by Kalaj in [15, 16] and spatial annuli in [13]. On the other hand, Koski and Onninen [17] investigated the minimizers of the p-harmonic energy

$$\mathcal{E}_p[h] = \int_{\mathbb{A}} \|Dh(x)\|^p dx$$

among all homeomorphisms in $\mathcal{W}^{1,p}(\mathbb{A}, \mathbb{A}^*)$, where $\mathbb{A}$ and $\mathbb{A}^*$ are planar annuli and $1 \le p < 2$, provided the homeomorphisms fix the outer boundary. Recently, Kalaj [14] studied the Dirichlet-type energy $\mathscr{F}[h]$ among mappings in $\mathcal{H}(\mathbb{A}, \mathbb{A}^*)$, where

$$\mathscr{F}[h] = \int_{\mathbb{A}} \frac{\|Dh(x)\|^{n-1}}{|h(x)|^{n-1}} dx. \tag{2.2}$$

For $n = 3$, the author proved that the minimizers of $\mathscr{F}[h]$ are certain generalized radial diffeomorphism (cf. [14, Theorem 1.1]). Motivated by the case $n = 3$, in [14] it was posed the following question.

Question 2.1 For $n \neq 3$, does the Dirichlet integral of $h \in \mathcal{H}(\mathbb{A}, \mathbb{A}^*)$, i.e. the integral

$$\mathscr{F}[h] = \int_{\mathbb{A}} \frac{\|Dh(x)\|^{n-1}}{|h(x)|^{n-1}} dx,$$

achieve its minimum for generalized radial diffeomorphisms between annuli?

Then in the subsequent paper by Kalaj and Chen [9] was given the following answer.

Theorem 2.1 *For $n \geq 4$, we have*

$$\inf_{h \in \mathcal{H}(\mathbb{A}, \mathbb{A}^*)} \mathscr{F}[h] = \inf_{h \in \mathcal{P}(\mathbb{A}, \mathbb{A}^*)} \mathscr{F}[h]$$

The last infimum is never attained.

In this paper, we consider the case of the p-energy Sobolev $\mathcal{W}^{1,p}$ homeomorphisms between annuli $\mathbb{A}$ and $\mathbb{A}^*$ in the complex plane. Let

$$\mathscr{F}_p[h] = \int_{\mathbb{A}(1,r)} \frac{\|Dh\|^p}{|h|^p}, \quad 1 \leqslant p < 2.$$

Then we seek the homeomorphisms h of the class $\mathcal{W}^{1,p}$ which are furthermore assumed to preserve the order of the boundary components $|h(z)| \to$r when $|z| \to r^*$ and $|h(z)| \to R^*$ when $|z| \to R$. Such a class of Sobolev homeomorphisms with the above property is denoted by $\mathcal{H}^{1,p}(\mathbb{A}, \mathbb{A}^*)$ and we say that they are *admissible homeomorphisms*. Since we minimize the $\mathscr{F}_p$ energy in the class of homeomorphisms, we can perform the inner variation of the independent variable $z_\epsilon = z + \epsilon\tau(z)$, which leads to the system (see for example [14])

$$\operatorname{div}\left(\frac{1}{|h|^p}\|Dh\|^{p-2}(Dh)^*Dh - \frac{1}{p|h|^p}\|Dh\|^p I\right) = 0, \tag{2.3}$$

where

$$\operatorname{div}\begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix} := \begin{pmatrix} a_x + b_y \\ c_x + d_y \end{pmatrix}.$$

Here $z = (x, y)$. Our argument does not make direct use of the inner variational equation (2.3). Some important facts that follow from (2.3) are as follows.

(1) If we assume that h is radial, then (2.3) reduces to the Euler-Lagrange equation (3.1) below.
(2) Further if f is a solution of (2.3) then so is $\tilde{f} = \frac{1}{f}$.
(3) Let $f_1(z) = \frac{1}{r_*} f(rz)$. Then $f_1 : \mathbb{A}(1, r_1) \overset{\text{onto}}{\longrightarrow} \mathbb{A}(1, R_1)$, provided that $f : \mathbb{A}(r, R) \overset{\text{onto}}{\longrightarrow} \mathbb{A}(r^*, R^*)$, where $R_1 = R_*/r_*$ and $r_1 = R/r$. Moreover, f satisfies (2.3) if and only if f_1 satisfies the same equation.

This is why we reduce the problem to the annuli $\mathbb{A} = \mathbb{A}(1, r)$ and $\mathbb{A}^* = \mathbb{A}(1, R)$. Now we formulate the main results.

Theorem 2.2 *Let $\mathbb{A}$ and $\mathbb{A}^*$ be planar annuli and $1 < p \leqslant 2$. Then there exists a radially symmetric mapping $h_\circ : \mathbb{A} \to \mathbb{A}^*$ such that*

$$\min_{\mathcal{H}^{1,p}(\mathbb{A},\mathbb{A}^*)} \mathscr{F}_p[h] = \mathscr{F}_p[h_\circ]. \tag{2.4}$$

The map $h_\circ$ is the unique minimizer, up to a rotation, in the class $\mathcal{H}^{1,p}(\mathbb{A}, \mathbb{A}^*)$. Furthermore, the minimizer $h_\circ$ is a homeomorphism.

Theorem 2.3 *Let $\mathbb{A}$ and $\mathbb{A}^*$ be planar annuli. Then there exists a radially symmetric mapping $h_\circ : \mathbb{A} \to \mathbb{A}^*$ which is a homeomorphism such that*

$$\min_{\mathcal{H}^{1,1}(\mathbb{A},\mathbb{A}^*)} \mathscr{F}_1[h] = \mathscr{F}_p[h_\circ], \tag{2.5}$$

if and only if

$$\frac{\pi}{2} - \tan^{-1}\left[\frac{1}{\sqrt{r^2-1}}\right] \geqslant \log R. \tag{2.6}$$

The map $h_\circ$ is the unique minimizer, up to a rotation, in the class $\mathcal{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$.

Remark 2.4 Note that the case $p = 2$ of Theorem 2.2 has been already considered by Astala, Iwaniec, and Martin in [2].

On the other hand side our result can be seen as a variation of minimization property of radial mappings of p-Dirichlet energy throughout Sobolev mappings from the unit ball $\mathbb{B} \subset \mathbb{R}^n$ onto the unit sphere $\mathbb{S}^{n-1}$, fixing the boundary. This is an old problem solved by several authors (see for example [7], [6], [18]).

Furthermore, as was remarked before, Koski and Onninen [17] have considered $\mathcal{E}_p$ energy and proved the minimization property, under a certain constrain. Indeed, if we denote the outer boundary of $\mathbb{A}$ by $\partial_\circ\mathbb{A}$ and consider the subfamily of homomorphisms $\mathcal{H}_\circ = \{f \in \mathcal{H}^{1,p}(\mathbb{A}, \mathbb{A}^*) : f(x) = \frac{R_*}{R}x, \quad \text{for } x \in \partial_\circ\mathbb{A}\}$, then the minimizer of $\mathcal{E}_p$ energy is a radial mapping $h(x) = \rho(x)\frac{x}{|x|}$ provided that R and r satisfies some inequality that depends on p ([17, Theorem 1.5]). In the same paper they proved that this constraint is crucial and there exists annuli, where the minimizer of $\mathcal{E}_p$ is not a radial mapping.

Remark 2.5 By virtue of the density of diffeomorphisms in $\mathcal{H}^{1,p}(\mathbb{A}, \mathbb{A}^*)$, see [10, 11], we can equivalently replace the admissible homeomorphisms by sense preserving diffeomorphims. Indeed, for $p \geqslant 1$, we have

$$\inf_{f\in\mathcal{H}^{1,p}(\mathbb{A},\mathbb{A}^*)} \mathcal{E}_p[h] = \inf_{f\in \mathrm{Diff}(\mathbb{A},\mathbb{A}^*)} \mathcal{E}_p[h]. \tag{2.7}$$

Here by $\mathrm{Diff}(\mathbb{A}, \mathbb{A}^*)$ we denote the class of orientation preserving diffeomorphisms from $\mathbb{A}$ onto $\mathbb{A}^*$ which also preserve the order of the boundary components. A similar result hold for the $\mathscr{F}_p$ energy. Indeed

$$\inf_{f\in\mathcal{H}^{1,p}(\mathbb{A},\mathbb{A}^*)} \mathscr{F}_p[h] = \inf_{f\in \mathrm{Diff}(\mathbb{A},\mathbb{A}^*)} \mathscr{F}_p[h]. \tag{2.8}$$

3 Radial minimizer of the energy $\mathscr{F}_p[h]$, $1 < p < 2$

This section aims is to find the radial minimizer $h_\circ$ of $\mathscr{F}_p$ energy that maps annuli $\mathbb{A}(1, r)$ onto $\mathbb{A}(1, R)$ keeping the boundary order. Moreover, we will use that solution to prove the minimization property of $h_\circ$ in the class of all Sobolev homeomorphisms. Contrary to the case $p = 1$, which will be considered later, we will not have any restriction on r and R. Assume that $h(z) = H(t)e^{i\theta}$, where $z = te^{i\theta}$, where H is a differentiable function and that $t \in [1, r]$, $\theta \in [0, 2\pi]$. Then

$$\|Dh\|^2 = |h_t|^2 + \frac{|h_\theta|^2}{t^2} = \dot{H}(t)^2 + \frac{H(t)^2}{t^2}.$$

Furthermore

$$t\frac{\|Dh\|^p}{|h|^p} = t\left(\frac{1}{t^2} + \frac{\dot{H}(t)^2}{H(t)^2}\right)^{p/2}.$$

Let

$$L(t, H, \dot{H}) = t\left(\frac{1}{t^2} + \frac{\dot{H}(t)^2}{H(t)^2}\right)^{p/2}.$$

Then Euler-Lagrange equation

$$L_H = \partial_t L_{\dot{H}},$$

can be written in the following form

$$\ddot{H} = \frac{\dot{H}\left((p-3)H^3 + tH^2\dot{H} - t^2H\dot{H}^2 + (p-1)t^3\dot{H}^3\right)}{tH^3 + (p-1)t^3H\dot{H}^2}, \tag{3.1}$$

where $H = H(t)$, $\dot{H} = H'(t)$ and $\ddot{H} = H''(t)$. Then by straightforward calculation (3.1) can be reduced to the following differential equation

$$\frac{t\dot{H}(t)}{H(t)} = \frac{\sqrt{g(t)}}{\sqrt{1-g(t)}}, \tag{3.2}$$

where g is a solution to the following differential equation

$$\dot{g}(t) = F[t, g(t)] := \frac{2(2-p)(g(t)-1)g(t)}{t + (p-2)tg(t)}. \tag{3.3}$$

Show that $F < 0$ provided that $t \geqslant 1$ and $g(t) \in (0, 1)$. Namely

$$t + (-2+p)tg(t) \geqslant t + (p-2)t = (p-1)t > 0.$$

Since $2(2-p)(g(t)-1)g(t) < 0$ we infer that g is a decreasing function.

The general solution of (3.3) is given by $g = k^{-1}$, where the function k is defined by

$$k(s) = b\exp\left(\frac{(p-1)\log(1-s) - \log s}{2(2-p)}\right), \tag{3.4}$$

where b is a positive constant and $s \in (0, 1)$.

By (3.2) we infer that H is given by

$$H(t) = C\exp\left[\int_1^t \frac{\sqrt{g(x)}}{\sqrt{1-g(x)}x}\,dx\right]. \tag{3.5}$$

By using the change $t = k(s)$ in (3.5) we obtain

$$H(t) = C \exp\left[\int_{g(t)}^{g(1)} \frac{\left(\frac{p-1}{1-s} + \frac{1}{s}\right)\sqrt{s}}{2(2-p)\sqrt{1-s}} ds\right]. \tag{3.6}$$

Since we seek increasing homeomorphic mappings $H : [1, r] \xrightarrow{\text{onto}} [1, R]$, we have the initial conditions $H(1) = 1$ and $H(r) = R$. Then $C = 1$. Let $0 < \tau < 1$ and chose $b = b(\tau)$ so that

$$b = \exp\left(\frac{(p-1)\log(1-\tau) - \log \tau}{2(p-2)}\right).$$

Denote the corresponding g by g_τ. Then we have $g_\tau(1) = \tau$.

Moreover by (3.4)

$$g_\tau\left[\exp\left(\frac{(p-1)\log(\frac{1-t}{1-\tau}) - \log \frac{t}{\tau}}{2(2-p)}\right)\right] = t.$$

Define the function

$$\mathcal{R}(\tau) = \exp\left[\int_{g_\tau(r)}^{\tau} \frac{\left(\frac{p-1}{1-s} + \frac{1}{s}\right)\sqrt{s}}{2(2-p)\sqrt{1-s}} dx\right].$$

Then we also define

$$H_\tau(t) = \exp\left[\int_{g_\tau(t)}^{\tau} \frac{\left(\frac{p-1}{1-s} + \frac{1}{s}\right)\sqrt{s}}{2(2-p)\sqrt{1-s}} ds\right].$$

Then

$$H_\tau(1) = 1$$

and

$$H_\tau(r) = \mathcal{R}(\tau). \tag{3.7}$$

Let us show that there is a unique $s_\circ = s(r, \tau) \in (0, \tau)$ such that $B(s_\circ) = 0$, where

$$B(s) := \frac{(p-1)\log(\frac{1-s}{1-\tau}) - \log \frac{s}{\tau}}{2(2-p)} - \log r.$$

Note that B is continuous, $B(\tau) = \log \frac{1}{r} < 0$ and $B(0) = +\infty$. Moreover

$$B'(s) = \frac{1 + (-2+p)s}{2(2-p)(-1+s)s} < 0.$$

Thus there is a unique $s_\circ$ so that $B(s_\circ) = 0$. Then $g_\tau(r) = s_\circ$. Since for $0 < s < \tau$ and $p \in (1, 2]$, we have

$$\frac{\log(\frac{1-s}{1-\tau}) - \log\frac{s}{\tau}}{2(2-p)} \geqslant \frac{(p-1)\log(\frac{1-s}{1-\tau}) - \log\frac{s}{\tau}}{2(2-p)},$$

it follows that

$$\frac{\log(\frac{1-s_\circ}{1-\tau}) - \log\frac{s_\circ}{\tau}}{2(2-p)} - \log r \geqslant B(s_\circ) = \frac{(p-1)\log(\frac{1-s_\circ}{1-\tau}) - \log\frac{s_\circ}{\tau}}{2(2-p)} - \log r = 0.$$

Thus

$$0 < s_\circ < \tau_\circ = \frac{1}{1 + r^{4-2p}\left(-1 + \frac{1}{\tau}\right)}. \tag{3.8}$$

Then

$$\mathcal{R}(\tau) = \exp\left[\int_{s_\circ}^{\tau} \frac{\left(\frac{p-1}{1-s} + \frac{1}{s}\right)\sqrt{s}}{2(2-p)\sqrt{1-s}} ds\right].$$

Let us show now that, if $p > 1$, then for every $R \in (1, +\infty)$, there is $\tau \in (0, 1)$ so that $\mathcal{R}(\tau) = R$. It is clear that $\mathcal{R}$ is continuous and also it is clear that $\lim_{\tau\to 0}\mathcal{R}(\tau) = 1$. Let us show that $\lim_{\tau\to 1}\mathcal{R}(\tau) = +\infty$. Observe that $0 \leqslant s \leqslant \sqrt{s} \leqslant 1$. Then from (3.8) we have that

$$\mathcal{R}(\tau) \geqslant K(\tau),$$

where

$$K(\tau) = \exp\left[\int_{\tau_\circ}^{\tau} \frac{\left(\frac{p-1}{s-1} + \frac{1}{s}\right)s}{2(2-p)\sqrt{1-s}} ds\right].$$

Then $K(\tau) = \exp(k(\tau) - k(\tau_0))$, where

$$k(s) = \frac{3 + p(s-2) - 2s}{(2-p)\sqrt{1-s}}.$$

Then

$$\lim_{\tau\to 1^-}\sqrt{1-\tau}\log K(\tau) = \frac{(p-1)\left(r^2 + r^p\right)}{(2-p)r^2}.$$

We notice that here is the moment where $p \in (1, 2)$ is an important assumption. In particular $\lim_{\tau\to 1} R(\tau) = \infty$. So there is $\tau = \tau(r, R)$ so that $\mathcal{R}(\tau) = R$. In view of

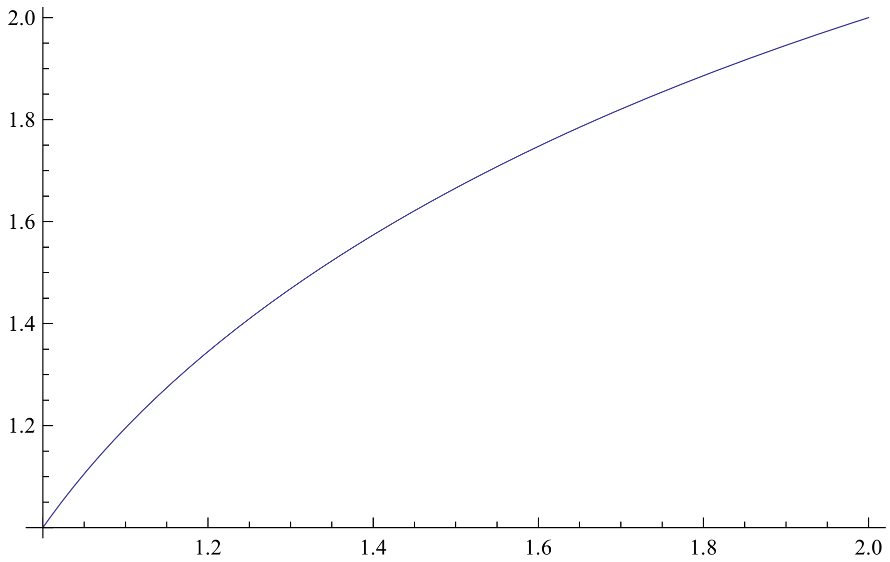

Fig. 1 The graphic of $H_\circ$ satisfying initial conditions $H(1) = 1$, $H(2) = 2$ is far from being identity

(3.7), we have constructed a smooth increasing mapping $H = H_\circ = H_{r,R} : [1, r] \to [1, R]$ so that $H(1) = 1$ and $H(r) = R$, See Fig. 1 below. Let us show that

$$h_\circ(z) = H(t)e^{i\theta}, \quad z = te^{i\theta}, \tag{3.9}$$

is the minimizer in the class of radial homeomorphisms between $\mathbb{A}$ and $\mathbb{A}^*$.

Assume now that $H : [1, r] \to [1, R]$ is any smooth homeomorphism and assume that $h(z) = H(t)e^{i\theta}$. Prove that

$$\mathscr{F}_p[h] \geqslant \mathscr{F}_p[h_\circ]. \tag{3.10}$$

We start from a simple inequality from [17]

$$(a + b)^{q/2} \geqslant s^{1-q/2}a^{q/2} + (1 - s)^{1-q/2}b^{q/2}, \quad q \in [1, 2], \quad s \in [0, 1]. \tag{3.11}$$

By inserting $q = p$, $s = g(t)$,

$$a = t^{\frac{2}{p}-2}, \quad b = t^{2/p}\frac{\dot{H}^2}{H^2}$$

in (3.11) we have

$$t\left(\frac{1}{t^2}+\frac{\dot{H}^2}{H^2}\right)^{p/2} = \left(t^{2/p-2}+t^{2/p}\frac{\dot{H}^2}{H^2}\right)^{p/2} \\ \geqslant (1-g(t))^{1-p/2}t^{1-p}+g(t)^{1-p/2}t\frac{|\dot{H}|^p}{|H|^p}. \tag{3.12}$$

The equality in (3.11) is attained precisely when

$$\frac{b}{a}=\frac{s}{1-s}$$

and thus the equality is attained in (3.12) precisely when

$$\frac{t\dot{H}}{H}=\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}}. \tag{3.13}$$

Then by

$$a^p \geqslant px^{p-1}a-(p-1)x^p, \tag{3.14}$$

where $a=\frac{\dot{H}(t)}{H(t)}$ and $x=\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}t}$ we get

$$t\left(\frac{1}{t^2}+\frac{\dot{H}(t)^2}{H(t)^2}\right)^{p/2} \geqslant t^{1-p}\frac{1-pg(t)}{(1-g(t))^{p/2}}+\frac{s^{2-p}\sqrt{g(t)}}{(1-g(t))^{(p-1)/2}}p\frac{\dot{H}}{H}. \tag{3.15}$$

Notice that, the condition (3.13) is precisely satisfied when we have equality in (3.15).

Define

$$P(t)=t^{2-p}\,(1-g(t))^{\frac{1}{2}(1-p)}\,\sqrt{g(t)},$$

and show that it is a constant. *This fact is crucial for our approach.*

By (3.3) we obtain that

$$\frac{P'(t)}{P(t)}=\frac{2-p}{t}+\frac{(1+(p-2)g(t))\dot{g}(t)}{2(1-g(t))g(t)}=0.$$

Thus

$$P(t)\equiv c=P(r)=r^{2-p}\,(1-g(r))^{\frac{1}{2}(1-p)}\,\sqrt{g(r)}. \tag{3.16}$$

Observe that

$$g(r)=g_\tau(r)=c_\circ(r,\tau)=c_\circ(r,\tau(r,R)).$$

Thus $c = c(r, R)$. Now we have

$$\begin{aligned}\mathscr{F}_p[h] &= 2\pi \int_1^r t\left(\frac{1}{t^2} + \frac{\dot{H}(t)^2}{H^2(t)}\right)^{p/2} dt \\ &\geqslant 2\pi \int_1^r \left(t^{1-p}\frac{1-pg(t)}{(1-g(t))^{p/2}} + c(r,R)\frac{\dot{H}}{H}\right) dt \\ &= 2\pi \int_1^r \left(t^{1-p}\frac{1-pg(t)}{(1-g(t))^{p/2}}\right) dt + 2\pi \int_1^r c(r,R)\frac{\dot{H}(t)}{H(t)} dt \\ &= 2\pi \int_1^r \left(t^{1-p}\frac{1-pg(t)}{(1-g(t))^{p/2}}\right) dt + 2\pi c(r,R)\log R \\ &= \mathscr{F}_p[h^\circ].\end{aligned}$$

4 Radial minimizers for the case $p = 1$

The corresponding subintegral expression for the functional $\mathscr{F}_1[h] = \int_{\mathbb{A}(1,r)} \frac{|Df(z)|}{|f(z)|}$, for radial function $h(z) = H(t)e^{i\theta}$, $z = te^{i\theta}$ is given by

$$L(t, H, \dot{H}) = \left(1 + \frac{t^2\dot{H}(t)^2}{H(t)^2}\right)^{1/2}.$$

The corresponding differential equation (3.1) for $p = 1$ reduces to

$$\left(-tH(t)\dot{H}(t)^2 + t^2\dot{H}(t)^3 + H(t)^2\left(2\dot{H}(t) + t\ddot{H}(t)\right)\right) = 0 \tag{4.1}$$

which can be written in the following form

$$\frac{t\dot{H}(t)}{H(t)} = \frac{\sqrt{g(t)}}{\sqrt{1-g(t)}}$$

where g is a solution of the differential equation (see (3.3) for $p = 1$):

$$2g(t) + t\dot{g}(t) = 0. \tag{4.2}$$

Then the general solution of (4.2) is given by $g(t) = bt^{-2}$. Then the solution of (4.1) is the solution of the equation

$$\frac{t\dot{H}(t)}{H(t)} = \frac{1}{\sqrt{b^2t^2 - 1}}$$

and it is given by

$$H(t) = c\exp\left(-\cot^{-1}\left[\sqrt{b^2t^2-1}\right]\right).$$

If we let that $H(1) = 1$ then

$$H(t) = \exp\left(\cot^{-1}\left[\sqrt{b^2-1}\right] - \cot^{-1}\left[\sqrt{b^2t^2-1}\right]\right). \tag{4.3}$$

Here $b \geqslant 1$. Moreover, if we assume that $H(r) = R$, then after straightforward computations we get

$$b = \frac{\sqrt{(1+r^2-2r\cos\log R)}\csc[\log R]}{r}.$$

The corresponding minimizer is denoted by $h_\circ(z) = H(r)e^{i\theta}$, $z = re^{i\theta}$. Hence

$$\mathscr{F}[h] = 2\pi \int_1^r \left(1 + \frac{t^2(\dot{H}(t))^2}{H^2(t)}\right)^{1/2} dt \geqslant 2\pi \int_1^r \sqrt{1 - \frac{1}{b^2t^2}} + \frac{\dot{H}(t)}{bH(t)} dt$$

Thus

$$\mathscr{F}[h] \geqslant \mathscr{F}[h_\circ]$$

where

$$\mathscr{F}[h_\circ] = 2\pi \frac{-\sqrt{b^2-1} + \sqrt{b^2r^2-1} - \csc^{-1}[b] + \csc^{-1}[br]}{b} + \frac{2\pi \log R}{b}.$$

Lemma 4.1 *It exists a radial homeomorphism* $h : \mathbb{A}(1,r) \to \mathbb{A}(1,R)$ *if and only if*

$$\frac{\pi}{2} - \tan^{-1}\left[\frac{1}{\sqrt{r^2-1}}\right] > \log R.$$

Proof By differentiating (4.3) w.r.t. b we get

$$\partial_b H(t) = \frac{\exp\left(\cot^{-1}\left[\sqrt{-1+b^2}\right] - \cot^{-1}\left[\sqrt{-1+b^2t^2}\right]\right)\left(-\frac{1}{\sqrt{-1+b^2}} + \frac{1}{\sqrt{-1+b^2t^2}}\right)}{b}.$$

Hence H is decreasing in b. The largest value is for $b = 1$ and it is equal to

$$R_\circ(r) := \exp\left(\frac{\pi}{2} - \tan^{-1}\left[\frac{1}{\sqrt{r^2-1}}\right]\right)$$

for $t = r$. In other words, there is a increasing diffeomorphism of $[1, r]$ onto $[1, R]$ if and only if $R \leqslant R_\circ(r)$. □

Remark 4.2 Observe that $\lim_{r\to\infty}\mathcal{R}(r) = e^{\pi/2}$, so there is not any homeomorphic minimizer of the $\mathscr{F}$ energy between annuli $\mathbb{A}(1, r)$ and $\mathbb{A}(1, e^{\pi/2})$. Note that the conformal modulus of mod $\mathbb{A}(1, e^{\pi/2})$ is $\log e^{\pi/2} = \pi/2$. So the case $p = 1$ differs from the case $p > 1$. Moreover, this case is also opposite to the Nitsche type phenomenon for Dirichlet energy $\mathcal{E}$. Namely Nitsche type phenomenon asserts that the modulus of image domain could be arbitrarily large, but not small enough.

5 Proof of Theorem 2.2 and Theorem 2.3

We begin with the following proposition

Proposition 5.1 *Assume that $h = \rho(z)e^{i\Theta(z)}$ is a diffeomorphism between annuli $\mathbb{A}(1, r)$ and $\mathbb{A}(1, R)$. Then for every $t \in [1, r]$ and $\theta \in [0, 2\pi]$ we have*

$$\int_{t\mathbb{T}} |\nabla\Theta(z)||dz| \geqslant 2\pi. \tag{5.1}$$

If the equality hold in (5.1) for every $\theta \in [0, 2\pi]$, then $\Theta(z) = e^{i\varphi(\theta)}$, $z = te^{i\theta}$, for a diffeomorphism $\varphi : [0, 2\pi] \xrightarrow{\text{onto}} [\alpha, 2\pi + \alpha]$. Further, we have

$$\int_1^R \frac{|\nabla\rho(te^{i\theta})|}{\rho(te^{i\theta})}dt \geqslant \log R. \tag{5.2}$$

If the equality hold in (5.2) for every $t \in [1, R]$, then $\rho(te^{i\theta}) = \rho(t)$.

Proof of Proposition 5.1 First of all, for fixed t, $\gamma(\theta) = e^{i\Theta(te^{i\theta})}$ is a surjection of $[0, 2\pi]$ onto $\mathbb{T} = \{z : |z| = 1\}$. Further

$$|\nabla\Theta(te^{i\theta})|^2 = |\Theta_t|^2 + \frac{|\Theta_\theta|^2}{t^2}.$$

So

$$|\gamma'(\theta)| = |\Theta_\theta| \leqslant t|\nabla\Theta(te^{i\theta})|. \tag{5.3}$$

The equality is attained in (5.3) if and only if $\Theta_t \equiv 0$. In this case $\gamma(\theta) = e^{i\varphi(\theta)}$, for a smooth function of $\varphi : [0, 2\pi] \xrightarrow{\text{onto}} [\alpha, 2\pi + \alpha]$.

We obtain that

$$|\mathbb{T}| = 2\pi \leqslant \int_0^{2\pi} |\gamma'(\theta)|d\theta \leqslant \int_{t\mathbb{T}} |\nabla\Theta(z)||dz|,$$

with an equality if and only if $\Theta(se^{i\theta})$ does not depend on t. Thus the first statement of the proposition is proved.

Similarly the function $\alpha(t) = \log\rho(te^{i\theta})$ is a surjection of $[1, r]$ onto $[0, \log R]$ and hence

$$\log R = \int_1^r \alpha'(t)dt \leqslant \int_1^r \frac{|\nabla\rho(te^{i\theta})|}{\rho(te^{i\theta})}dt.$$

The equality statement can be proved in the same way as the former part. We only need to use the formula

$$|\nabla\rho(te^{i\theta})|^2 = |\rho_t|^2 + \frac{|\rho_\theta|^2}{t^2} \geqslant |\rho_t|^2.$$

□

Proof of Theorem 2.2 Assume as before that $h(z) = \rho(z)e^{i\Theta(z)}$ is a mapping from the annulus $\mathbb{A}$ onto the annulus $\mathbb{A}^*$. We start from the following inequality which follows from Hölder inequality

$$\mathscr{F}_p[h] = \int_{\mathbb{A}(1,r)} \frac{\|Dh\|^p}{|h|^p} \geqslant \frac{\left(\int_{\mathbb{A}(1,r)} \frac{\|Dh\|}{|h|} \cdot \frac{\|Dh_\circ\|^{p-1}}{|h_\circ|^{p-1}}\right)^p}{\left(\int_{\mathbb{A}(1,r)} \frac{\|Dh_\circ\|^p}{|h_\circ|^p}\right)^{p-1}}.$$

In view of (2.1)

$$\|Dh\|^2 = |\nabla\rho|^2 + \rho^2|\nabla\Theta|^2,$$

where $\rho(z) = |h(z)|$. And thus

$$\frac{\|Dh\|}{|h|} = \left(|\nabla\Theta|^2 + \frac{|\nabla\rho|^2}{\rho^2}\right)^{1/2}.$$

Then by (3.10), for $q = 1$ we have

$$\frac{\|Dh\|}{|h|} \geqslant \left(\sqrt{1-g(t)}|\nabla\Theta| + \sqrt{g(t)}\frac{|\nabla\rho|}{\rho}\right). \tag{5.4}$$

From (5.4) we get

$$\begin{aligned}
&\int_{\mathbb{A}(1,r)} \frac{\|Dh\|}{|h|} \cdot \frac{\|Dh_\circ\|^{p-1}}{|h_\circ|^{p-1}} \\
&= \int_{\mathbb{A}(1,r)} \left(|\nabla\Theta|^2 + \frac{|\nabla\rho|^2}{\rho^2}\right)^{1/2} \cdot \frac{\|Dh_\circ\|^{p-1}}{|h_\circ|^{p-1}} \\
&\geqslant \int_0^{2\pi}\int_1^r t\frac{\|Dh_\circ\|^{p-1}}{|h_\circ|^{p-1}}\left(\sqrt{1-g(t)}|\nabla\Theta| + \sqrt{g(t)}\frac{|\nabla\rho|}{\rho}\right) dt d\theta.
\end{aligned}$$

Let

$$K(t) = t\sqrt{g(t)}\frac{\|Dh_\circ\|^{p-1}}{|h_\circ|^{p-1}}.$$

Then

$$K(t)=t\sqrt{g(t)}\left(\frac{1}{t^2}+\frac{g(t)}{t^2(1-g(t))}\right)^{\frac{1}{2}(p-1)}=P(t).$$

Thus we again use (3.16) to conclude that $K(t)=c(r,R)$. Furthermore

$$\begin{aligned} t\frac{\|Dh\|}{|h|}\cdot\frac{\|Dh_\circ\|^{p-1}}{|h_\circ|^{p-1}} &\geqslant t\left(t^2(1-g(t))\right)^{\frac{1}{2}(1-p)}\left[\sqrt{1-g(t)}|\nabla\theta|+\sqrt{g(t)}\frac{|\nabla\rho|}{\rho}\right] \\ &= t^{2-p}\left(1-g(t)\right)^{1-p/2}|\nabla\Theta|+c(r,R)\frac{|\nabla\rho|}{\rho}. \end{aligned}$$

Now by Proposition 5.1 we have

$$\int_{\mathbb{A}}\frac{|\nabla\rho|}{|\rho|}\geqslant 2\pi\log R$$

and

$$t\int_0^{2\pi}|\nabla\Theta(te^{i\theta})|d\theta\geqslant 2\pi.$$

So we have

$$\int_{\mathbb{A}(1,r)}\frac{\|Dh\|}{|h|}\cdot\frac{\|Dh_\circ\|^{p-1}}{|h_\circ|^{p-1}}\geqslant 2\pi\left(c\log R+\int_1^r t^{2-p}(1-g(t))^{1-p/2}dt\right)=\mathscr{F}_p[h_\circ].$$

Thus

$$\mathscr{F}_p[h]\geqslant\frac{\mathscr{F}_p^p[h_\circ]}{\mathscr{F}_p^{p-1}[h_\circ]}=\mathscr{F}_p[h_\circ].$$

The uniqueness part of this theorem follows from Proposition 5.1. The equation in (5.4) is satisfied if and only if

$$\frac{\rho(te^{i\theta})|\nabla\Theta(te^{i\theta})|}{|\nabla\rho(te^{i\theta})|}$$

is a function that depends only on t. Since $\Theta(\theta)=e^{i\varphi(\theta)}$, we get $|\nabla\Theta(\theta)|=\varphi'(\theta)=$ const. Because $\varphi:[0,2\pi]\xrightarrow{\text{onto}}[\alpha,2\pi+\alpha]$, it follows that $\varphi(\theta)=\theta+\alpha$. In other words $h(z)$ is a minimizer if and only if $h(z)=H_\circ(t)e^{i(\theta+\alpha)}=e^{i\alpha}h_\circ(z)$. This finishes the proof. □

Proof of Theorem 2.3 The proof of Theorem 2.3 is the same as the proof of Theorem 2.2 up to the part concerning the existence of the radial solutions given in Sect. 4 (See Lemma 4.1). □

Data availibility Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author declares that he has not Conflict of interest.

References

1. Antman, S.S.: Nonlinear Problems of Elasticity. Applied Mathematical Sciences, vol. 107. Springer, New York (1995)
2. Astala, K., Iwaniec, T., Martin, G.: Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. Princeton University Press (2009)
3. Astala, K., Iwaniec, T., Martin, G.: Deformations of annuli with smallest mean distortion. Arch. Ration. Mech. Anal. **195**(3), 899–921 (2010)
4. Ball, J.M.: Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. Philos. Trans. R. Soc. Lond. A **306**, 557–611 (1982)
5. Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Ration. Mech. Anal. **63**(4), 337–403 (1976, 77)
6. Bourgoin, J.-C.: The minimality of the map $x/|x|$ for weighted energy. Calc. Var. Partial. Differ. Equ. **25**(4), 469–489 (2006)
7. Brezis, H., Coron, J.-M., Lieb, E.H.: Harmonic Maps with Defects. Commun. Math. Phys. **107**, 649–705 (1986)
8. Ciarlet, P.G.: Mathematical elasticity Vol. I. Three-dimensional elasticity. Studies in Mathematics and its Applications 20, North-Holland Publishing Co., Amsterdam (1988)
9. Chen, J., Kalaj, D.: Dirichlet-type energy of mappings between two concentric annuli. Calc. Var. Partial Differ. Equ. **60**(6), 205 (2021)
10. Hencl, S., Pratelli, A.: Diffeomorphic approximation of $\mathcal{W}^{1,1}$ planar Sobolev homeomorphisms. J. Eur. Math. Soc. **20**(3), 597–656 (2018)
11. Iwaniec, T., Kovalev, L.V., Onninen, J.: Diffeomorphic approximation of Sobolev homeomorphisms. Arch. Rat. Mech. Anal. **201**(3), 1047–1067 (2011)
12. Iwaniec, T., Onninen, J.: n-harmonic mappings between annuli: the art of integrating free Lagrangians. Mem. Am. Math. Soc. **218**, 105 (2012)
13. Kalaj, D.: (n, ρ)-harmonic mappings and energy minimal deformations between annuli. Calc. Var. **58**, 19 (2019)
14. Kalaj, D.: Harmonic maps between two concentric annuli in $\mathbb{R}^3$. Adv. Calc. Var. **14**(3), 303–312 (2021)
15. Kalaj, D.: Hyperelastic deformations and total combined energy of mappings between annuli. J. Differ. Equ. **268**, 6103–6136 (2020)
16. Kalaj, D.: Deformations of annuli on Riemann surfaces and the generalization of Nitsche conjecture. J. Lond. Math. Soc. **93**(3), 683–702 (2016)
17. Koski, A., Onninen, J.: Radial symmetry of p- harmonic minimizers. Arch. Ration. Mech. Anal. **230**, 321–342 (2018)
18. Min-Chun, H.: On the minimality of the p-harmonic map $x/|x| : \mathbb{B}^n \to \mathbb{S}^{n-1}$. Calc. Var. Partial. Differ. Equ. **13**(4), 459–468 (2001)
19. Rickman, S.: Quasiregular Mappings. Springer, Berlin (1993)