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H-Toeplitz operators on the function spaces

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Abstract
In this paper, we study several classes of H-Toeplitz operators (defined below) on the Hardy space H^2. In particular, we prove that, for $\varphi \in L^\infty$, the adjoint of H-Toeplitz operators is hyponormal. Next, we investigate several properties of H-Toeplitz operators on the weighted Bergman spaces. Finally, we give necessary and sufficient conditions for H-Toeplitz operators to be contractive and expansive on the weighted Bergman spaces.

Keywords Hardy spaces · Weighted Bergman spaces · H-Toeplitz operator

Mathematics Subject Classification Primary 47B35 · 30H10 · 30H20

1 Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, T^* denotes the adjoint of T. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *self-adjoint* if $T = T^*$, *isometric* if $T^*T = I$, *normal* if $[T^*, T] = 0$, *hyponormal* if $[T^*, T] \geq 0$, *quasinormal* if $[T^*T, T] = 0$, and *binormal* if $[T^*T, TT^*] = 0$, respectively, where $[R, S] := RS - SR$.

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H-Toeplitz operators have been studied in various spaces. Recently, the authors in [13] studied the essential conditions for H-Toeplitz operators to become a co-isometry and a partial isometry, explored their invariant subspaces and kernels, and investigated their compactness and Fredholmness. In particular, they showed a nonzero H-Toeplitz operator cannot be a Fredholm operator on the Bergman space. Moreover, they considered the necessary and sufficient conditions for the commutativity of H-Toeplitz operators. In [25], the authors provided a characterization of the commutativity of H-Toeplitz operators with quasihomogeneous symbols on the Bergman space. In [22], the authors explored the characteristics of H-Toeplitz operators on the Bergman space and offered essential criteria for identifying both contractive and expansive operators. Additionally, the authors in [14] studied the slant Toeplitz operators on the Hardy space.

Basic properties of Toeplitz operators on the Hardy space and (weighted) Begman space can be found in [2, 7, 8, 18, 20, 28]. Recently, many authors have characterized the hyponormality of Toeplitz operators on the Bergman spaces and the weighted Bergman spaces (cf. [16, 17, 19, 21, 26, 27, 29]). The theory of Toeplitz operators is a vast and significant field that has made fundamental contributions to several problems in functional analysis and mathematical physics.

Several decades ago, researchers extensively studied contractive and expansive operators (cf. [3, 5, 6]). In particular, in [9], the authors investigated the problem of invariant subspaces for contractive operators. In [22], the authors studied the contractivity and expansivity of H-Toeplitz operators with analytic, co-analytic and harmonic symbols on the Bergman spaces.

In this paper, we study several classes of H-Toeplitz operators on the function spaces. In Sect. 2, we focus on the self-adjointness of H-Toeplitz operators on the Hardy space H^2. Moreover, we consider complex symmetric H-Toeplitz operator on H^2. Furthermore, we investigate hyponormality, quasinormality, and binormality of H-Toeplitz operators. In particular, we show that for $\varphi \in L^\infty$ the adjoint of H-Toeplitz operators is hyponormal. As an application of this, such an operator has a nontrivial invariant subspace. In Sect. 3, we will investigate the the algebraic properties of H-Toeplitz operators on the weighted Bergman spaces $A^2_\alpha(\mathbb{D})$. More concretely, we introduce the notion of H-Toeplitz operators on the weighted Bergman spaces, which combine the properties of both Toeplitz and Hankel operators. The importance of this notion is that it provides a unifying framework for a class of operators on the weighted Bergman spaces, which includes both Toeplitz and Hankel operators. Furthermore, we establish a convenient and explicit criterion for determining the contractivity and expansivity of H-Toeplitz operators.

2 *H*-Toeplitz operators on the Hardy spaces

Let $\mathbb{D}$ be the open unit disk in the complex plane and let $\mathbb{T}(\equiv \partial\mathbb{D})$ be the unit circle. Let $L^\infty(\mathbb{T})$ denote the set of all essentially bounded measurable functions on $\mathbb{T}$. The *Hilbert Hardy space* $H^2(\mathbb{T})$ consists of all analytic functions f with the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ where } \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

For a convenience, we denote $L^\infty(\mathbb{T})$ and $H^2(\mathbb{T})$ by L^∞ and H^2, respectively. For any $\varphi \in L^\infty$, the *multiplication operator* M_φ is defined by $M_\varphi(f) = \varphi f$ for $f \in H^2$, the *Toeplitz operator* $T_\varphi : H^2 \to H^2$ is defined by

$$T_\varphi f = P(\varphi f)$$

for $f \in H^2$ where P denotes the orthogonal projection of L^2 onto H^2, and the *Hankel operator* $H_\varphi : H^2 \to H^2$ is defined by

$$H_\varphi f = P M_\varphi J f$$

where $J : H^2 \to (H^2)^\perp$ denotes the *flip operator* given by $J(e_n) = e_{-n-1}$ for all $n \geq 0$ where $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for L^2. Note that T_φ is bounded if and only if $\varphi \in L^\infty$ and, in which case, $\|T_\varphi\| = \|\varphi\|_\infty$.

Notation 2.1 *Throughout this paper, a dilation operator K from H^2 to L^2 is denoted as $K(e_{2n}) = e_n$ and $K(e_{2n+1}) = e_{-n-1}$ for all $n = 0, 1, 2, \ldots$ where $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for L^2.*

Let $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ be the set of positive integers, non-negative integers, integers, real numbers, and complex numbers, respectively. A dilation operator K is bounded from H^2 to L^2 with $\|K\| = 1$ and its adjoint K^* from L^2 to H^2 is defined as

$$K^*(e_n) = e_{2n} \text{ and } K^*(e_{-n-1}) = e_{2n+1}$$

for all $n = 0, 1, 2, \ldots$. Thus $K^*K = I$ on H^2 and $K^*K = I$ on L^2. Indeed, since $KK^*e_n = Ke_{2n} = e_n$ for each $n \geq 0$, it follows that $KK^* = I$ on H^2. Moreover, since $KK^*e_{-n-1} = Ke_{2n+1} = e_{-n-1}$ for each $n \in \mathbb{N}$, we know that $KK^* = I$ on $(H^2)^\perp$. Thus $KK^* = I$ on L^2. Hence K is unitary from H^2 to L^2.

The authors in [1] have introduced "H-Toeplitz operators" motivated by the Toeplitz, Hankel, and Slant Toeplitz operators.

Definition 2.2 For $\varphi \in L^\infty$, an *H-Toeplitz operator* S_φ with the symbol φ on H^2 is defined by

$$S_\varphi f = P M_\varphi K f$$

for each $f \in H^2$ where P denotes the orthogonal projection of L^2 onto H^2.

In this case, $\|S_\varphi\| = \|P M_\varphi K\| \leq \|M_\varphi\| = \|\varphi\|_\infty$. Note that if $\{e_n\}_{n=0}^{\infty}$ denotes the orthonormal basis for H^2, then

$$S_\varphi e_{2n} = P M_\varphi K e_{2n} = P M_\varphi e_n = T_\varphi e_n$$

and

$$S_\varphi e_{2n+1} = PM_\varphi K e_{2n+1} = PM_\varphi e_{-n-1} = PM_\varphi J e_n = H_\varphi e_n$$

for each $n = 0, 1, 2 \ldots$. Note that for $\varphi \in L^\infty$, the adjoint of S_φ on H^2 is given by

$$S_\varphi^* = K^* M_{\overline{\varphi}}.$$

2.1 Basic properties of an *H*-Toeplitz operator

In this section, we consider the basic properties of an H-Toeplitz operator. We first study the self-adjointness of H-Toeplitz operators on H^2.

Theorem 2.3 *If $\varphi(z) = \sum_{j=-\infty}^{\infty} a_j e_j$ with respect to the orthonormal basis $\mathcal{B} = \{e_n\}_{n=0}^{\infty}$ in L^∞ and S_φ is an H-Toeplitz operator on H^2, then the matrices of S_φ and S_φ^* are represented as*

$$[S_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0 & a_1 & a_{-1} & a_2 & a_{-2} & a_3 & a_{-3} & \cdots \\ a_1 & a_2 & a_0 & a_3 & a_{-1} & a_4 & a_{-2} & \cdots \\ a_2 & a_3 & a_1 & a_4 & a_0 & a_5 & a_{-1} & \cdots \\ a_3 & a_4 & a_2 & a_5 & a_1 & a_6 & a_0 & \cdots \\ a_4 & a_5 & a_3 & a_6 & a_2 & a_7 & a_1 & \cdots \\ a_5 & a_6 & a_4 & a_7 & a_3 & a_8 & a_2 & \cdots \\ a_6 & a_7 & a_5 & a_8 & a_4 & a_9 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

and

$$[S_\varphi^*]_{\mathcal{B}} = \begin{pmatrix} \overline{a_0} & \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} & \overline{a_5} & \overline{a_6} & \cdots \\ \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} & \overline{a_5} & \overline{a_6} & \overline{a_7} & \cdots \\ \overline{a_{-1}} & \overline{a_0} & \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} & \overline{a_5} & \cdots \\ \overline{a_2} & \overline{a_3} & \overline{a_4} & \overline{a_5} & \overline{a_6} & \overline{a_7} & \overline{a_8} & \cdots \\ \overline{a_{-2}} & \overline{a_{-1}} & \overline{a_0} & \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} & \cdots \\ \overline{a_3} & \overline{a_4} & \overline{a_5} & \overline{a_6} & \overline{a_7} & \overline{a_8} & \overline{a_9} & \cdots \\ \overline{a_{-3}} & \overline{a_{-2}} & \overline{a_{-1}} & \overline{a_0} & \overline{a_1} & \overline{a_2} & \overline{a_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Furthermore, $[S_\varphi]_{\mathcal{B}}$ is self-adjoint if and only if $[S_\varphi]_{\mathcal{B}} = 0$.

Proof We know that S_φ is self-adjoint if and only if $a_0, a_1, a_2, a_3, a_5, a_8, \ldots$ are real and $a_2 = \overline{a_{-1}}$, $a_3 = \overline{a_0}$, $a_2 = \overline{a_3}$, $a_{-2} = \overline{a_4}$, $a_3 = \overline{a_5}$, $a_{-3} = \overline{a_6}$, and so on (cf. [1, Page 151]).

On the other hand, the (i, j) entry of the matrix $[S_\varphi]_{\mathcal{B}}$ is given by

$$a_{i,j} = \begin{cases} a_i & \text{if } j = 0 \\ a_{i-n} & \text{if } j = 2n \\ a_{i+n+1} & \text{if } j = 2n+1 \end{cases}$$

(see [1]). And the (i, j) entry of the matrix $[S_\varphi^*]_{\mathcal{B}}$ is given by

$$\overline{a_{j,i}} = \begin{cases} \overline{a_j} & \text{if } i = 0 \\ \overline{a_{j-n}} & \text{if } i = 2n \\ \overline{a_{j+n+1}} & \text{if } i = 2n+1. \end{cases} \tag{2}$$

Thus $[S_\varphi]_{\mathcal{B}} = [S_\varphi^*]_{\mathcal{B}}$ if and only if $a_{i,j} = \overline{a_{j,i}}$ for all i, j. Hence $[S_\varphi]_{\mathcal{B}}$ is self-adjoint if and only if $a_j = a_0 \in \mathbb{R}$ for all $j \in \mathbb{Z}$ if and only if $a_j = 0$ for all j since $[S_\varphi]_{\mathcal{B}}$ is bounded. □

Proposition 2.4 *Let $\varphi \in L^\infty$ and S_φ be an H-Toeplitz operator on H^2. Then S_φ is an isometry on H^2 if and only if $M_{\overline{\varphi}} P M_\varphi = I$ on L^2. In particular, φ is not inner.*

Proof Since $S_\varphi^* = K^* M_{\overline{\varphi}}$, we have $S_\varphi^* S_\varphi = K^* M_{\overline{\varphi}} P M_\varphi K$. Then $K^* M_{\overline{\varphi}} P M_\varphi K = I$ on H^2. Hence $M_{\overline{\varphi}} P M_\varphi = I$ on L^2 since K is unitary from H^2 to L^2. Thus S_φ is an isometry on H^2 if and only if $M_{\overline{\varphi}} P M_\varphi = I$ on L^2.

If φ is inner, then $M_{\overline{\varphi}} P M_\varphi - I = M_{\overline{\varphi}} M_\varphi - I = M_{|\varphi|^2} - I = 0$. Thus S_φ is an isometry on H^2. But, if φ is inner, then S_φ^* is an isometry on H^2 (cf. [1]), and so S_φ^* is normal. Therefore, $\varphi = 0$ from [1], which is a contradiction. □

Next, we study complex symmetric H-Toeplitz operator on H^2. A conjugation on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. If C is a conjugation on $\mathcal{H}$, then there exists an orthonormal basis $\{e_n\}_{n=0}^\infty$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see [10]). An operator $T \in \mathcal{L}(\mathcal{H})$ is complex symmetric if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. Complex symmetric operators have been widely studied by several mathematicians (see [10–12, 23, 24] for more details).

Proposition 2.5 *For $\varphi \in L^\infty$, let S_φ be an H-Toeplitz operator on H^2 and C be a conjugation on L^2 given by $Cf(z) = \overline{f(\overline{z})}$ for $f \in H^2$. Then S_φ is complex symmetric with the conjugation C if and only if $T_{\overline{\varphi(\overline{z})}} e_n = K^* M_{\overline{\varphi(z)}} e_{2n}$ and $H_{\overline{\varphi(\overline{z})}} e_n = K^* M_{\overline{\varphi(z)}} e_{2n+1}$ for $n \in \mathbb{N}_0$.*

Proof Let C be a conjugation on L^2 given by $Cf(z) = \overline{f(\overline{z})}$ for $f \in H^2$. Then $Ce_n = e_n$ for $n \geq 0$ and so $CP = PC$ on L^2 from [24]. Thus for $n \geq 0$,

$$CS_\varphi Ce_{2n} = CS_\varphi e_{2n} = CT_\varphi(e_n) = CP(\varphi e_n) = PC(\varphi e_n) = T_{\overline{\varphi(\overline{z})}} e_n$$

and

$$CS_\varphi Ce_{2n+1} = CH_\varphi(e_n) = CPM_\varphi(Je_n) = CP\varphi(Je_n) = H_{\overline{\varphi(\overline{z})}} e_n$$

hold. Since $S_\varphi^* = K^* M_{\overline{\varphi}}$ for $\varphi \in L^\infty$, we obtain that S_φ is complex symmetric with the conjugation C if and only if $T_{\overline{\varphi(\overline{z})}} e_n = K^* M_{\overline{\varphi(z)}} e_{2n}$ and $H_{\overline{\varphi(\overline{z})}} e_n = K^* M_{\overline{\varphi(z)}} e_{2n+1}$ for $n \in \mathbb{N}_0$. □

Theorem 2.6 *For $\varphi \in L^\infty$, let S_φ be an H-Toeplitz operator on H^2. Assume that C is a conjugation on L^2 given by $Cf(z) = \overline{f(\overline{z})}$ for $f \in H^2$ and $C_{\mu,\lambda}$ is a conjugation on L^2 given by $C_{\mu,\lambda} f(z) = \mu \overline{f(\lambda \overline{z})}$ for $f \in H^2$ with $|\lambda| = |\mu| = 1$. Then the following statements are equivalent:*
(i) *S_φ is complex symmetric with the conjugation C.*
(ii) *S_φ is complex symmetric with the conjugation $C_{\mu,\lambda}$.*
(iii) *$\varphi = 0$.*

Proof (i) ⇔ (iii) Let $\varphi(z) = \sum_{j=-\infty}^{\infty} a_j e_j$ be with respect to the basis $\mathcal{B} = \{e_n\}_{n=0}^{\infty}$. Since the matrix of S_φ is of the form (1), it follows that the matrix of $C S_\varphi C$ is the followings:

$$[C S_\varphi C]_{\mathcal{B}} = \begin{pmatrix} \overline{a_0} & \overline{a_1} & \overline{a_{-1}} & \overline{a_2} & \overline{a_{-2}} & \overline{a_3} & \overline{a_{-3}} & \cdots \\ \overline{a_1} & \overline{a_2} & \overline{a_0} & \overline{a_3} & \overline{a_{-1}} & \overline{a_4} & \overline{a_{-2}} & \cdots \\ \overline{a_2} & \overline{a_3} & \overline{a_1} & \overline{a_4} & \overline{a_0} & \overline{a_5} & \overline{a_{-1}} & \cdots \\ \overline{a_3} & \overline{a_4} & \overline{a_2} & \overline{a_5} & \overline{a_1} & \overline{a_6} & \overline{a_0} & \cdots \\ \overline{a_4} & \overline{a_5} & \overline{a_3} & \overline{a_6} & \overline{a_2} & \overline{a_7} & \overline{a_1} & \cdots \\ \overline{a_5} & \overline{a_6} & \overline{a_4} & \overline{a_7} & \overline{a_3} & \overline{a_8} & \overline{a_2} & \cdots \\ \overline{a_6} & \overline{a_7} & \overline{a_5} & \overline{a_8} & \overline{a_4} & \overline{a_9} & \overline{a_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $[S_\varphi]_{\mathcal{B}}$ is complex symmetric with the conjugation C if and only if $a_j = a_0 \in \mathbb{C}$ for all $j \in \mathbb{Z}$. Hence φ is of the form $\varphi = \sum_{j=-\infty}^{\infty} \hat{\varphi}(0) e_j$ and so $\varphi = 0$ since $\varphi \in L^\infty$.

(ii) ⇔ (iii) Let $\varphi(z) = \sum_{j=-\infty}^{\infty} a_j e_j$ be with respect to the basis $\mathcal{B} = \{e_n\}_{n=0}^{\infty}$. It is known from [24] that $C_{\mu,\lambda}$ is unitarily equivalent to $C_{1,\lambda}$. Since the matrix of S_φ is the form of (1), it follows that the matrix of $C_{1,\lambda} S_\varphi C_{1,\lambda}$ is the followings:

$$[C_{1,\lambda} S_\varphi C_{1,\lambda}]_{\mathcal{B}} = \lambda I \begin{pmatrix} \overline{a_0} & \overline{a_1} & \overline{a_{-1}} & \overline{a_2} & \overline{a_{-2}} & \overline{a_3} & \overline{a_{-3}} & \cdots \\ \overline{a_1} & \overline{a_2} & \overline{a_0} & \overline{a_3} & \overline{a_{-1}} & \overline{a_4} & \overline{a_{-2}} & \cdots \\ \overline{a_2} & \overline{a_3} & \overline{a_1} & \overline{a_4} & \overline{a_0} & \overline{a_5} & \overline{a_{-1}} & \cdots \\ \overline{a_3} & \overline{a_4} & \overline{a_2} & \overline{a_5} & \overline{a_1} & \overline{a_6} & \overline{a_0} & \cdots \\ \overline{a_4} & \overline{a_5} & \overline{a_3} & \overline{a_6} & \overline{a_2} & \overline{a_7} & \overline{a_1} & \cdots \\ \overline{a_5} & \overline{a_6} & \overline{a_4} & \overline{a_7} & \overline{a_3} & \overline{a_8} & \overline{a_2} & \cdots \\ \overline{a_6} & \overline{a_7} & \overline{a_5} & \overline{a_8} & \overline{a_4} & \overline{a_9} & \overline{a_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \lambda [C S_\varphi C]_{\mathcal{B}}.$$

Then $[S_\varphi]_{\mathcal{B}}$ is complex symmetric with the conjugation $C_{1,\lambda}$ if and only if $a_j = a_0 \in \mathbb{C}$ for all $j \in \mathbb{Z}$ and $\lambda = 1$. Hence, in this case, φ is of the form $\varphi = \sum_{j=-\infty}^{\infty} \hat{\varphi}(0) e_j$ and so $\varphi = 0$ since $\varphi \in L^\infty$. □

Remark that if $\varphi \in L^2$, then an unbounded H-Toeplitz operator is complex symmetric with the conjugation C if and only if $\hat{\varphi}(j) = \hat{\varphi}(0) \in \mathbb{C}$ for all $j \in \mathbb{Z}$. In previous theorem, if $\hat{\varphi}(0) \neq 0$, then $\varphi = \sum_{j=-\infty}^{\infty} \hat{\varphi}(0) e_j$ does not belong to L^∞.

2.2 Hyponormal, quasinormal, and binormal *H*-Toeplitz operators

In this section, we study hyponormal, quasinormal, and binormal H-Toeplitz operators.

Lemma 2.7 *For $\varphi \in L^\infty$, let S_φ be an H-Toeplitz operator on H^2. Then the following statements hold.*
(i) $S_\varphi S_\varphi^* = T_{|\varphi|^2}$, $S_\varphi^* S_\varphi e_{2n} = K^* M_{\overline{\varphi}} T_\varphi e_n$, *and* $S_\varphi^* S_\varphi e_{2n+1} = K^* M_{\overline{\varphi}} T_\varphi e_n$ *hold for each* $n \in \mathbb{N}_0$.
(ii) S_φ^* *is hyponormal if and only if the following equations hold.*

$$\begin{cases} T_{|\varphi|^2} e_{2n} \geq K^* M_{\overline{\varphi}} T_\varphi e_n \\ T_{|\varphi|^2} e_{2n+1} \geq K^* M_{\overline{\varphi}} H_\varphi e_n \end{cases} \tag{3}$$

for each $n \in \mathbb{N}_0$. In particular, the equalities in (3) hold if and only if S_φ is normal.

Proof (i) For $\varphi \in L^\infty$, let S_φ be an H-Toeplitz operator on H^2. Since $S_\varphi^* = K^* M_{\overline{\varphi}}$, it follows that $S_\varphi^* S_\varphi = K^* M_{\overline{\varphi}} P M_\varphi K$ and

$$S_\varphi S_\varphi^* = (P M_\varphi K)(K^* M_{\overline{\varphi}}) = P M_\varphi M_{\overline{\varphi}} = P M_{|\varphi|^2} = T_{|\varphi|^2}.$$

On the other hand, since $S_\varphi^* S_\varphi = K^* M_{\overline{\varphi}} P M_\varphi K$, it follows that

$$\begin{aligned} S_\varphi^* S_\varphi e_{2n} &= K^* M_{\overline{\varphi}} P M_\varphi K e_{2n} \\ &= K^* M_{\overline{\varphi}} P M_\varphi e_n \\ &= K^* M_{\overline{\varphi}} T_\varphi e_n \end{aligned}$$

and

$$\begin{aligned} S_\varphi^* S_\varphi e_{2n+1} &= K^* M_{\overline{\varphi}} P M_\varphi K e_{2n+1} \\ &= K^* M_{\overline{\varphi}} P M_\varphi e_{-n-1} \\ &= K^* M_{\overline{\varphi}} H_\varphi e_n \end{aligned}$$

for each $n \in \mathbb{N}_0$.

(ii) By (i), we obtain that S_φ^* is hyponormal if and only if for each n, it holds that

$$\begin{cases} T_{|\varphi|^2} e_{2n} \geq K^* M_{\overline{\varphi}} T_\varphi e_n \\ T_{|\varphi|^2} e_{2n+1} \geq K^* M_{\overline{\varphi}} H_\varphi e_n. \end{cases}$$

In particular, we get that S_φ is normal if and only if

$$\begin{cases} T_{|\varphi|^2} e_{2n} = K^* M_{\overline{\varphi}} T_\varphi e_n \\ T_{|\varphi|^2} e_{2n+1} = K^* M_{\overline{\varphi}} H_\varphi e_n \end{cases}$$

for each $n \in \mathbb{N}_0$. □

Using Lemma 2.7, we show that every H-Toeplitz operator on H^2 is cohyponormal.

Theorem 2.8 *For $\varphi \in L^\infty$, let S_φ be an H-Toeplitz operator on H^2. Then S_φ^* is hyponormal.*

Proof. Set $\varphi = \sum_{j=-\infty}^{\infty} \hat{\varphi}(j)e_j \in L^\infty$. Let S_φ be an H-Toeplitz operator on H^2. Then S_φ^* is hyponormal if and only if

$$\|S_\varphi f\|^2 \leq \|S_\varphi^* f\|^2$$

for each $f \in H^2$. Taking $f = e_{2n}$ for each n, Lemma 2.7 implies that

$$\begin{aligned}
\|S_\varphi^* e_{2n}\|^2 - \|S_\varphi e_{2n}\|^2 &= \|K^* M_\varphi^* e_{2n}\|^2 - \|PT_\varphi e_n\|^2 \\
&= \|M_\varphi^* e_{2n}\|^2 - \|PT_\varphi e_n\|^2 \\
&= \left\| \sum_{j=-\infty}^{\infty} \overline{\hat{\varphi}(j)} e_{2n-j} \right\|^2 - \left\| \sum_{j=-n}^{\infty} \hat{\varphi}(j) e_{n+j} \right\|^2 \\
&= \sum_{j=-\infty}^{\infty} |\hat{\varphi}(j)|^2 - \sum_{j=-n}^{\infty} |\hat{\varphi}(j)|^2 \\
&= \sum_{j=n-1}^{\infty} |\hat{\varphi}(-j)|^2 \geq 0
\end{aligned}$$

since K^* is unitary. Put $f(z) = e_{2n+1}$ for each n. Then Lemma 2.7 ensures that

$$\begin{aligned}
\|S_\varphi^* e_{2n+1}\|^2 - \|S_\varphi e_{2n+1}\|^2 &= \|S_\varphi^* e_{2n+1}\|^2 - \|H_\varphi e_n\|^2 \\
&= \|K^* M_\varphi^* e_{2n+1}\|^2 - \|PM_\varphi J e_n\|^2 \\
&= \|M_\varphi^* e_{2n+1}\|^2 - \|PM_\varphi J e_n\|^2 \\
&= \left\| \sum_{j=-\infty}^{\infty} \overline{\hat{\varphi}(j)} e_{2n+1-j} \right\|^2 - \left\| P(\sum_{j=-\infty}^{\infty} \hat{\varphi}(j) e_{j-n-1}) \right\|^2 \\
&= \sum_{j=-\infty}^{\infty} |\hat{\varphi}(j)|^2 - \sum_{j=n+1}^{\infty} |\hat{\varphi}(j)|^2 \\
&= \sum_{j=-\infty}^{n} |\hat{\varphi}(j)|^2 \geq 0.
\end{aligned}$$

Hence we conclude that S_φ^* is hyponormal. □

Theorem 2.9 *Let $\varphi \in L^\infty$ and S_φ be an H-Toeplitz operator on H^2. Then the following statement hold.*
(i) *If φ is a nonzero constant function, then S_φ is not quasinormal, but its adjoint S_φ^* is quasinormal.*
(ii) *If $\varphi = \lambda u$ for an inner function u and $\lambda \in \mathbb{C}$, then S_φ^* is quasinormal.*

Proof (i) Let $\varphi = \varphi_1 + \overline{\varphi_2} \in L^\infty$ where $\varphi_1, \varphi_2 \in H^\infty$. Then S_φ is quasinormal if and only if, for each $n \in \mathbb{N}_0$,

$$\begin{aligned}
0 &= (S_\varphi^* S_\varphi S_\varphi - S_\varphi S_\varphi^* S_\varphi) e_{2n} \\
&= K^* M_{\overline{\varphi}} P M_\varphi K P M_\varphi K e_{2n} - P M_\varphi K K^* M_{\overline{\varphi}} P M_\varphi K e_{2n} \\
&= K^* M_{\overline{\varphi}} P M_\varphi K P M_\varphi e_n - P M_\varphi M_{\overline{\varphi}} P M_\varphi e_n \\
&= K^* M_{\overline{\varphi}} P M_\varphi K P M_\varphi e_n - P M_{|\varphi|^2} P M_\varphi e_n
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
0 &= (S_\varphi^* S_\varphi S_\varphi - S_\varphi S_\varphi^* S_\varphi) e_{2n+1} \\
&= K^* M_{\overline{\varphi}} P M_\varphi K P M_\varphi K e_{2n+1} - P M_\varphi K K^* M_{\overline{\varphi}} P M_\varphi K e_{2n+1} \\
&= K^* M_{\overline{\varphi}} P M_\varphi K P M_\varphi e_{-n-1} - P M_\varphi M_{\overline{\varphi}} P M_\varphi e_{-n-1} \\
&= K^* M_{\overline{\varphi}} P M_\varphi K P M_\varphi e_{-n-1} - P M_{|\varphi|^2} P M_\varphi e_{-n-1}.
\end{aligned}$$

If $\varphi = c$ is nonzero constant and n is odd, then (4) becomes

$$\begin{aligned}
K^* M_{\overline{\varphi}} P M_\varphi K P M_\varphi e_n - P M_{|\varphi|^2} P M_\varphi e_n &= K^* M_{\overline{\varphi}} P M_\varphi K c e_n - P M_{|c|^2} c e_n \\
&= K^* M_{\overline{\varphi}} P (c^2 e_{\frac{-n-1}{2}}) - |c|^2 c e_n = -|c|^2 c e_n \\
&\neq 0.
\end{aligned}$$

Hence S_φ is not quasinormal.

On the other hand, S_φ^* is quasinormal if and only if $S_\varphi S_\varphi^* S_\varphi^* - S_\varphi^* S_\varphi S_\varphi^* = 0$. Since $S_\varphi S_\varphi^* = T_{|\varphi|^2}$, it follows that S_φ^* is quasinormal if and only if

$$T_{|\varphi|^2} S_\varphi^* = S_\varphi^* T_{|\varphi|^2}. \tag{5}$$

If φ is a constant function, i.e. $\varphi = c$, then

$$\begin{aligned}
(T_{|\varphi|^2} S_\varphi^* - S_\varphi^* T_{|\varphi|^2}) e_{2n} &= (T_{|c|^2} S_c^* - S_c^* T_{|c|^2}) e_{2n} \\
&= T_{|c|^2} K^* M_{\overline{c}} e_{2n} - K^* M_{\overline{c}} T_{|c|^2} e_{2n} \\
&= P(\overline{c}|c|^2 K^* e_{2n}) - \overline{c}|c|^2 K^* e_{2n} = 0
\end{aligned}$$

and

$$\begin{aligned}
(T_{|\varphi|^2} S_\varphi^* - S_\varphi^* T_{|\varphi|^2}) e_{2n+1} &= (T_{|c|^2} S_c^* - S_c^* T_{|c|^2}) e_{2n+1} \\
&= T_{|c|^2} K^* M_{\overline{c}} e_{2n+1} - K^* M_{\overline{c}} T_{|c|^2} e_{2n+1} \\
&= P(\overline{c}|c|^2 K^* e_{2n+1}) - \overline{c}|c|^2 K^* e_{2n+1} = 0
\end{aligned}$$

for each $n \in \mathbb{N}_0$. Therefore, S_φ^* is quasinormal.

(ii) Since $\varphi = \lambda u$ for an inner function u and $\lambda \in \mathbb{C}$, it follows that

$$S_\varphi S_\varphi^* = (P M_\varphi K)(K^* M_{\overline{\varphi}}) = P M_{\lambda u} M_{\overline{\lambda u}} = P M_{|\lambda|^2 |u|^2} = T_{|\lambda|^2} = |\lambda|^2 I.$$

Thus (5) holds. Hence S_φ^* is quasinormal. □

We next consider the hyponormality and the binormality of S_φ.

Proposition 2.10 *For $\varphi \in L^\infty$, let S_φ be an H-Toeplitz operator on H^2. Then the following statements are equivalent.*
(i) *S_φ is normal.*
(ii) *S_φ is hyponormal.*
(iii) $\varphi = 0$.

Proof If $\varphi = 0$, then S_φ is normal, and hence hyponormal. If S_φ is hyponormal, the proof follows from [1]. □

Theorem 2.11 *Let $\varphi \in L^\infty$and S_φ be an H-Toeplitz operator on H^2. Assume that one of the following statements hold.*
(i) *φ is a constant function.*
(ii) *$\varphi = \lambda u$ for an inner function u and $\lambda \in \mathbb{C}$.*
(iii) *$\varphi = \lambda \overline{u}$ for an inner function u and $\lambda \in \mathbb{C}$. Then S_φ is binormal.*

Proof Let $\varphi \in L^\infty$. Then S_φ is binormal if and only if $S_\varphi^* S_\varphi$ and $S_\varphi S_\varphi^*$ commute. This is equivalent to $S_\varphi^* S_\varphi$ and $T_{|\varphi|^2}$ commute. Thus S_φ is binormal if and only if

$$[S_\varphi^* S_\varphi, S_\varphi S_\varphi^*] = [S_\varphi^* S_\varphi, T_{|\varphi|^2}] = [K^* M_{\overline{\varphi}} P M_\varphi K, T_{|\varphi|^2}] = 0. \tag{6}$$

(i) If φ is a constant function, then (6) clearly holds.
(ii) If $\varphi = \lambda u$ for an inner function u and $\lambda \in \mathbb{C}$, then S_φ^* is quasinormal and so S_φ^* is binormal. Hence S_φ is binormal.
(iii) If $\varphi = \lambda \overline{u}$ for an inner function u and $\lambda \in \mathbb{C}$, then

$$S_\varphi S_\varphi^* = (P M_\varphi K)(K^* M_{\overline{\varphi}}) = P M_{\lambda \overline{u}} M_{\overline{\lambda} u} = P M_{|\lambda|^2 |u|^2} = T_{|\lambda|^2} = |\lambda|^2 I.$$

Thus (6) clearly holds. Hence S_φ is binormal. □

Example 2.12 If $\varphi(z) = z^m$ for some m, then by Theorem 2.11, S_{z^m} is binormal and by Theorem 2.9, S_{z^m} is not quasinormal and $S_{z^m}^*$ is quasinormal.

Example 2.13 Let $\varphi(z) = \lambda\left(\frac{z-\mu}{1-\overline{\mu}z}\right)$ for $\mu \in \mathbb{D}$ and $\lambda \in \mathbb{C}$. Then S_φ is binormal from Theorem 2.11.

Corollary 2.14 *For $\varphi \in L^\infty$, let S_φ be an H-Toeplitz operator on H^2. Assume that one of the following statements hold.*
(i) *φ is a constant function.*
(ii) *$\varphi = \lambda u$ for an inner function u and $\lambda \in \mathbb{C}$.*
(iii) *$\varphi = \lambda \overline{u}$ for an inner function u and $\lambda \in \mathbb{C}$.*
Then S_φ^* has a nontrivial invariant subspace.

Proof By Theorem 2.11, S_φ is binormal. Hence S_φ^* is binormal. Since S_φ^* is hyponormal by Theorem 2.8, we conclude that S_φ^* has a nontrivial invariant subspace from [4]. □

3 *H*-Toeplitz operators on the weighted Bergman spaces

3.1 Preliminaries and auxiliary lemmas

For $-1 < \alpha < \infty$, the *weighted Bergman spaces* $A^2_\alpha(\mathbb{D})$ is the space of analytic functions in $L^2(\mathbb{D}) \equiv L^2(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).$$

The inner product on $L^2(\mathbb{D})$ is given by

$$\langle f, g\rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}dA_\alpha(z) \qquad (f, g \in L^2(\mathbb{D}, dA_\alpha)).$$

If $\alpha = 0$, then, $A^2_0(\mathbb{D})$ is the Bergman spaces. For $n \in \mathbb{N}_0$, let

$$e_n(z) = \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}}\, z^n \quad (z \in \mathbb{D}).$$

Here, $\Gamma(s)$ stands for the usual Gamma functions. It is easy to check that $\{e_n\}_{n=0}^\infty$ be an orthonormal set in $A^2_\alpha(\mathbb{D})$ ([15]). Because the set of polynomials is dense in $A^2_\alpha(\mathbb{D})$, we conclude that e_n forms an orthonormal basis for $A^2_\alpha(\mathbb{D})$. If $f, g \in A^2_\alpha(\mathbb{D})$ are functions of the form

$$f(z) = \sum_{n=0}^\infty a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^\infty b_n z^n,$$

then

$$\langle f, g\rangle_\alpha = \sum_{n=0}^\infty \frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)} a_n \overline{b}_n.$$

The *weighted harmonic Bergman spaces* $L^2_\alpha(\mathbb{D})$ denote the space of all harmonic functions f on $\mathbb{D}$ such that

$$\|f\| := \left(\int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z)\right)^{1/2} < \infty.$$

The space $L^2_\alpha(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D})$ and therefore inherits the structure of a Hilbert space from $L^2(\mathbb{D})$. Let P_{harm} denote the orthogonal projection from $L^2(\mathbb{D})$ onto $L^2_\alpha(\mathbb{D})$.

For $\varphi \in L^\infty(\mathbb{D})$, the multiplication operators M_φ on $A^2_\alpha(\mathbb{D})$ is defined by $M_\varphi(f) = \varphi f$, and the *Toeplitz operators* T_φ on $A^2_\alpha(\mathbb{D})$ is defined by

$$T_\varphi(f) = P_\alpha(\varphi f),$$

where P_α denotes the orthogonal projection of $L^2(\mathbb{D})$ onto $A^2_\alpha(\mathbb{D})$ and $f \in A^2_\alpha(\mathbb{D})$. It is evident that those operators are bounded when $\varphi \in L^\infty(\mathbb{D})$. The *Hankel operators* H_φ on the $A^2_\alpha(\mathbb{D})$ is defined by

$$H_\varphi(f) = P_\alpha M_\varphi J(f),$$

where the operators $J : A^2_\alpha(\mathbb{D}) \to \overline{A^2_\alpha(\mathbb{D})}$ is given by $J(e_n(z)) = \overline{e_{n+1}(z)}$ for all $n \in \mathbb{N}_0$.

Now, we introduce the notion of H-Toeplitz operators on the weighted Bergman spaces and discuss their various familiar properties. First of all, we recall the well-known facts.

Lemma 3.1 [19] *For any $s, t \in \mathbb{N}_0$,*

$$P_\alpha(\overline{z}^t z^s) = \begin{cases} \frac{\Gamma(s+1)\Gamma(s-t+\alpha+2)}{\Gamma(s+\alpha+2)\Gamma(s-t+1)} z^{s-t} & \text{if } s \geq t \\ 0 & \text{if } s < t. \end{cases}$$

In [13], the orthogonal projection from the space $L^2(\mathbb{D})$ onto the harmonic Bergman space is given. Using a similar method, the following results can be induced.

Lemma 3.2 *In the weighted harmonic Bergman spaces $L^2_\alpha(\mathbb{D})$, for $s, t \in \mathbb{N}_0$,*

$$P_{harm}(\overline{z}^t z^s) = \begin{cases} \frac{\Gamma(s-t+\alpha+2)\Gamma(s+1)}{\Gamma(s+\alpha+2)\Gamma(s-t+1)} z^{s-t} & \text{if } s \geq t \\ \frac{\Gamma(t-s+\alpha+2)\Gamma(t+1)}{\Gamma(t+\alpha+2)\Gamma(t-s+1)} \overline{z}^{t-s} & \text{if } s < t. \end{cases}$$

Proof If $s \geq t$, then

$$\begin{aligned} \langle P_{harm}(\overline{z}^t z^s), z^k\rangle &= \langle \overline{z}^t z^s, z^k\rangle \\ &= \begin{cases} \frac{\Gamma(\alpha+2)\Gamma(s+1)}{\Gamma(s+\alpha+2)} & \text{if } k = s-t \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{\Gamma(k+\alpha+2)\Gamma(s+1)}{\Gamma(s+\alpha+2)\Gamma(k+1)} \langle z^k, z^k\rangle & \text{if } k = s-t \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{\Gamma(s-t+\alpha+2)\Gamma(s+1)}{\Gamma(s+\alpha+2)\Gamma(s-t+1)} \langle z^{s-t}, z^k\rangle. \end{aligned}$$

On the other hands, if $s < t$, then

$$\begin{aligned} \langle P_{harm}(\overline{z}^t z^s), \overline{z}^k\rangle &= \langle \overline{z}^t z^s, \overline{z}^k\rangle \\ &= \begin{cases} \frac{\Gamma(\alpha+2)\Gamma(t+1)}{\Gamma(t+\alpha+2)} & \text{if } k = t-s \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{\Gamma(k+\alpha+2)\Gamma(t+1)}{\Gamma(t+\alpha+2)\Gamma(k+1)} \langle \overline{z}^k, \overline{z}^k\rangle & \text{if } k = t-s \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$= \frac{\Gamma(t-s+\alpha+2)\Gamma(t+1)}{\Gamma(t+\alpha+2)\Gamma(t-s+1)} \langle \bar{z}^{t-s}, \bar{z}^k \rangle.$$

□

Next, we find the matrix representations of Toeplitz operators T_φ and of Hankel operators H_φ with harmonic symbols φ on the weighted Bergman spaces. For the harmonic symbol $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \bar{z}^j \in L^\infty(\mathbb{D})$, the $(m,n)^{th}$ entry of the matrix of T_φ with respect to orthonormal basis $\mathcal{B} = \{e_n\}_{n=0}^{\infty}$ of $A_\alpha^2(\mathbb{D})$ is given by

$$\begin{aligned}
\langle T_\varphi e_n, e_m \rangle &= \langle P_\alpha(\varphi e_n), e_m \rangle \\
&= \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}} \sqrt{\frac{\Gamma(m+\alpha+2)}{\Gamma(m+1)\Gamma(\alpha+2)}} \left\langle \left(\sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \bar{z}^j \right) z^n, z^m \right\rangle \\
&= \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}} \sqrt{\frac{\Gamma(m+\alpha+2)}{\Gamma(m+1)\Gamma(\alpha+2)}} \left(\sum_{i=0}^{\infty} a_i \langle z^{i+n}, z^m \rangle + \sum_{j=1}^{\infty} b_j \langle z^n, z^{m+j} \rangle \right).
\end{aligned}$$

There are two cases to consider. If $m \geq n$, then we have

$$\begin{aligned}
\langle T_\varphi e_n, e_m \rangle &= \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}} \sqrt{\frac{\Gamma(m+\alpha+2)}{\Gamma(m+1)\Gamma(\alpha+2)}} \sum_{i=0}^{\infty} a_i \langle z^{i+n}, z^m \rangle \\
&= \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}} \sqrt{\frac{\Gamma(m+\alpha+2)}{\Gamma(m+1)\Gamma(\alpha+2)}} \frac{\Gamma(m+1)\Gamma(\alpha+2)}{\Gamma(m+\alpha+2)} a_{m-n} \\
&= \sqrt{\frac{\Gamma(n+\alpha+2)\Gamma(m+1)}{\Gamma(n+1)\Gamma(m+\alpha+2)}} a_{m-n}.
\end{aligned}$$

If $m < n$, then we have

$$\begin{aligned}
\langle T_\varphi e_n, e_m \rangle &= \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}} \sqrt{\frac{\Gamma(m+\alpha+2)}{\Gamma(m+1)\Gamma(\alpha+2)}} \sum_{j=1}^{\infty} b_j \langle z^n, z^{m+j} \rangle \\
&= \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}} \sqrt{\frac{\Gamma(m+\alpha+2)}{\Gamma(m+1)\Gamma(\alpha+2)}} \frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)} b_{n-m} \\
&= \sqrt{\frac{\Gamma(m+\alpha+2)\Gamma(n+1)}{\Gamma(m+1)\Gamma(n+\alpha+2)}} b_{n-m}.
\end{aligned}$$

Thus, we have

$$\langle T_\varphi e_n, e_m \rangle = \begin{cases} \sqrt{\frac{\Gamma(n+\alpha+2)\Gamma(m+1)}{\Gamma(n+1)\Gamma(m+\alpha+2)}} a_{m-n} & \text{for } m \geq n \\ \sqrt{\frac{\Gamma(m+\alpha+2)\Gamma(n+1)}{\Gamma(m+1)\Gamma(n+\alpha+2)}} b_{n-m} & \text{for } m < n, \end{cases}$$

where $m, n \in \mathbb{N}_0$. Therefore, the matrix representation of T_φ is given by

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0 & \sqrt{\frac{1}{\alpha+2}}b_1 & \sqrt{\frac{1\cdot 2}{(\alpha+2)(\alpha+3)}}b_2 & \sqrt{\frac{1\cdot 2\cdot 3}{(\alpha+2)(\alpha+3)(\alpha+4)}}b_3 & \cdots \\ \sqrt{\frac{1}{\alpha+2}}a_1 & a_0 & \sqrt{\frac{1\cdot 2}{\alpha+3}}b_1 & \sqrt{\frac{1\cdot 2\cdot 3}{(\alpha+3)(\alpha+4)}}b_2 & \cdots \\ \sqrt{\frac{1\cdot 2}{(\alpha+2)(\alpha+3)}}a_2 & \sqrt{\frac{1\cdot 2}{\alpha+3}}a_1 & a_0 & \sqrt{\frac{3}{\alpha+4}}b_1 & \cdots \\ \sqrt{\frac{1\cdot 2\cdot 3}{(\alpha+2)(\alpha+3)(\alpha+4)}}a_3 & \sqrt{\frac{1\cdot 2\cdot 3}{(\alpha+3)(\alpha+4)}}a_2 & \sqrt{\frac{3}{\alpha+4}}a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the adjoint of the matrix representation of T_φ is given by

$$[T_\varphi^*]_{\mathcal{B}} = \begin{pmatrix} \overline{a_0} & \sqrt{\frac{1}{\alpha+2}}\overline{a_1} & \sqrt{\frac{1\cdot 2}{(\alpha+2)(\alpha+3)}}\overline{a_2} & \sqrt{\frac{1\cdot 2\cdot 3}{(\alpha+2)(\alpha+3)(\alpha+4)}}\overline{a_3} & \cdots \\ \sqrt{\frac{1}{\alpha+2}}\overline{b_1} & \overline{a_0} & \sqrt{\frac{1\cdot 2}{\alpha+3}}\overline{a_1} & \sqrt{\frac{1\cdot 2\cdot 3}{(\alpha+3)(\alpha+4)}}\overline{a_2} & \cdots \\ \sqrt{\frac{1\cdot 2}{(\alpha+2)(\alpha+3)}}\overline{b_2} & \sqrt{\frac{1\cdot 2}{\alpha+3}}\overline{b_1} & \overline{a_0} & \sqrt{\frac{3}{\alpha+4}}\overline{a_1} & \cdots \\ \sqrt{\frac{1\cdot 2\cdot 3}{(\alpha+2)(\alpha+3)(\alpha+4)}}\overline{b_3} & \sqrt{\frac{1\cdot 2\cdot 3}{(\alpha+3)(\alpha+4)}}\overline{b_2} & \sqrt{\frac{3}{\alpha+4}}\overline{b_1} & \overline{a_0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and hence, we check that $T_\varphi^* = T_{\overline{\varphi}}$.

Next, for the harmonic symbol $\varphi(z) = \sum_{i=0}^\infty a_i z^i + \sum_{j=1}^\infty b_j \overline{z}^j \in L^\infty(\mathbb{D})$, the $(m, n)^{th}$ entry of the matrix of H_φ with respect to orthonormal basis $\mathcal{B} = \{e_n\}_{n=0}^\infty$ of $A_\alpha^2(\mathbb{D})$ is given by

$$\begin{aligned} \langle H_\varphi e_n, e_m\rangle &= \langle P_\alpha M_\varphi J e_n, e_m\rangle \\ &= \sqrt{\frac{\Gamma(n+\alpha+3)}{\Gamma(n+2)\Gamma(\alpha+2)}}\sqrt{\frac{\Gamma(m+\alpha+2)}{\Gamma(m+1)\Gamma(\alpha+2)}}\left\langle\left(\sum_{i=0}^\infty a_i z^i + \sum_{j=1}^\infty b_j \overline{z}^j\right)\overline{z}^{n+1}, z^m\right\rangle \\ &= \sqrt{\frac{\Gamma(n+\alpha+3)}{\Gamma(n+2)\Gamma(\alpha+2)}}\sqrt{\frac{\Gamma(m+\alpha+2)}{\Gamma(m+1)\Gamma(\alpha+2)}} \\ &\quad \left(\sum_{i=0}^\infty a_i \langle z^i, z^{m+n+1}\rangle + \sum_{j=1}^\infty b_j \langle \overline{z}^j, z^{m+n+1}\rangle\right) \\ &= \sqrt{\frac{\Gamma(n+\alpha+3)\Gamma(m+\alpha+2)}{\Gamma(n+2)\Gamma(m+1)}}\frac{\Gamma(m+n+2)}{\Gamma(m+n+\alpha+3)}a_{m+n+1} \end{aligned}$$

for $m, n \in \mathbb{N}_0$. Therefore, the matrix representation of H_φ is given by

$$[H_\varphi]_{\mathcal{B}} = \begin{pmatrix} \sqrt{\frac{1}{\alpha+2}}a_1 & \sqrt{\frac{1\cdot 2}{(\alpha+2)(\alpha+3)}}a_2 & \sqrt{\frac{1\cdot 2\cdot 3}{(\alpha+2)(\alpha+3)(\alpha+4)}}a_3 & \cdots \\ \frac{1\cdot 2}{\alpha+3}a_2 & \frac{3\sqrt{2}}{(\alpha+4)\sqrt{\alpha+3}}a_3 & \frac{4\sqrt{6}}{(\alpha+5)\sqrt{(\alpha+3)(\alpha+4)}}a_4 & \cdots \\ \frac{3\sqrt{2}}{(\alpha+4)\sqrt{\alpha+3}}a_3 & \frac{12}{(\alpha+4)(\alpha+5)}a_4 & \frac{20\sqrt{3}}{(\alpha+5)(\alpha+6)\sqrt{\alpha+4}}a_5 & \cdots \\ \frac{4\sqrt{6}}{(\alpha+5)\sqrt{(\alpha+3)(\alpha+4)}}a_4 & \frac{20\sqrt{3}}{(\alpha+5)(\alpha+6)\sqrt{\alpha+4}}a_5 & \frac{4\cdot 5\cdot 6}{(\alpha+5)(\alpha+6)(\alpha+7)}a_6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Notation 3.3 *For our convenience, we introduce the following notations:*

$$\Lambda_\alpha(s) = \frac{\Gamma(s+1)\Gamma(\alpha+2)}{\Gamma(s+\alpha+2)} \quad \textit{and} \quad \Lambda_\alpha(s,t) = \frac{\Gamma(s+1)^2\Gamma(s-t+\alpha+2)\Gamma(\alpha+2)}{\Gamma(s+\alpha+2)^2\Gamma(s-t+1)}.$$

Lemma 3.4 [19] *For $m \geq 0$, we have that*

(i) $\left\|\bar{z}^m \sum_{j=0}^{\infty} c_j z^j\right\|^2 = \sum_{j=0}^{\infty} \Lambda_\alpha(j+m)|c_j|^2$, *and*

(ii) $\left\|P_\alpha\left(\bar{z}^m \sum_{j=0}^{\infty} c_j z^j\right)\right\|^2 = \begin{cases} \sum_{j=0}^{\infty} \Lambda_\alpha(j,m)|c_j|^2 & \textit{if} \;\; m \leq j \\ \sum_{j=1}^{\infty} \Lambda_\alpha(j,m)|c_j|^2 & \textit{if} \;\; m > j. \end{cases}$

Applying Lemmas 3.2 and 3.4, we obtain the following Remarks.

Remark 3.5 For $m \geq 0$, we have

$$\|P_{harm}\left(\bar{z}^m \sum_{j=0}^{\infty} c_j z^j\right)\|^2 = \sum_{j=0}^{m} \Lambda_\alpha(m,j)|c_j|^2 + \sum_{j=m+1}^{\infty} \Lambda_\alpha(j,m)|c_j|^2.$$

To define the notion of H-Toeplitz operators on $A^2_\alpha(\mathbb{D})$, we start by considering the operators $K : A^2_\alpha(\mathbb{D}) \to L^2_\alpha(\mathbb{D})$ defined by

$$K(e_{2n}(z)) = e_n(z) \text{ and } K(e_{2n+1}(z)) = \overline{e_{n+1}(z)} \tag{7}$$

for all $n \geq 0$ and $z \in \mathbb{D}$. The operator K can be shown to be a bounded linear operator on $A^2_\alpha(\mathbb{D})$ with $\|K\| = 1$. Furthermore, the adjoint operator K^* is given by

$$K^*(e_n(z)) = e_{2n}(z) \text{ and } K^*(\overline{e_{n+1}(z)}) = e_{2n+1}(z)$$

for all $n \geq 0$. From the definitions of the operators K and K^*, we have that $KK^* = I_{L^2_\alpha(\mathbb{D})}$ and $K^*K = I_{A^2_\alpha(\mathbb{D})}$.

Remark 3.6 It follows from the definition of operator K, we have

$$K(z^{2n}) = \frac{\sqrt{\Gamma(2n+1)\Gamma(n+\alpha+2)}}{\sqrt{\Gamma(n+1)\Gamma(2n+\alpha+2)}} z^n, \; K(z^{2n+1}) = \frac{\sqrt{\Gamma(2n+2)\Gamma(n+\alpha+3)}}{\sqrt{\Gamma(n+2)\Gamma(2n+\alpha+3)}} \bar{z}^{n+1},$$

$K^*(z^n) = \frac{\sqrt{\Gamma(n+1)\Gamma(2n+\alpha+2)}}{\sqrt{\Gamma(2n+1)\Gamma(n+\alpha+2)}} z^{2n}$, and $K^*(\bar{z}^{n+1}) = \frac{\sqrt{\Gamma(n+2)\Gamma(2n+\alpha+3)}}{\sqrt{\Gamma(2n+2)\Gamma(n+\alpha+3)}} z^{2n+1}$.

We next define H-Toeplitz operators on the weighted Bergman spaces $A^2_\alpha(\mathbb{D})$.

Definition 3.7 For $\varphi \in L^\infty(\mathbb{D})$, the H-Toeplitz operator B_φ on the weighted Bergman space is defined by $B_\varphi(f) = P_\alpha M_\varphi K(f)$ for all $f \in A^2_\alpha(\mathbb{D})$ where K is defined as in (7).

We find the matrix representation of H-Toeplitz operators B_φ with harmonic symbol φ on the weighted Bergman spaces. If the harmonic symbol of the form $\varphi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \overline{z}^j \in L^\infty(\mathbb{D})$, then

$$B_\varphi(e_{2n}) = P_\alpha M_\varphi K(e_{2n}) = P_\alpha M_\varphi(e_n) = T_\varphi(e_n)$$

and

$$B_\varphi(e_{2n+1}) = P_\alpha M_\varphi K(e_{2n+1}) = P_\alpha M_\varphi(\overline{e_{n+1}}) = P_\alpha M_\varphi J(e_n) = H_\varphi(e_n)$$

where $\{e_n\}_{n=0}^{\infty}$ is an orthonormal set in $A_\alpha^2(\mathbb{D})$. Thus

$$\begin{aligned}\langle B_\varphi e_{2n}, e_m\rangle &= \langle T_\varphi e_n, e_m\rangle \\ &= \begin{cases} \sqrt{\frac{\Gamma(n+\alpha+2)\Gamma(m+1)}{\Gamma(n+1)\Gamma(m+\alpha+2)}}\, a_{m-n} & \text{for } m \geq n \\ \sqrt{\frac{\Gamma(m+\alpha+2)\Gamma(n+1)}{\Gamma(m+1)\Gamma(n+\alpha+2)}}\, b_{n-m} & \text{for } m < n, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\langle B_\varphi e_{2n+1}, e_m\rangle &= \langle H_\varphi e_n, e_m\rangle \\ &= \sqrt{\frac{\Gamma(n+\alpha+3)\Gamma(m+\alpha+2)}{\Gamma(n+2)\Gamma(m+1)}}\,\frac{\Gamma(m+n+2)}{\Gamma(m+n+\alpha+3)}\, a_{m+n+1}\end{aligned}$$

where $m, n \in \mathbb{N}_0$. Thus $(m, n)^{th}$ entry of the matrix representation of B_φ with respect to orthonormal basis $\mathcal{B} = \{e_n\}_{n=0}^{\infty}$ of $A_\alpha^2(\mathbb{D})$ is given by

$$[B_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0 & \sqrt{\frac{1}{\alpha+2}}\,a_1 & \sqrt{\frac{1}{\alpha+2}}\,b_1 & \sqrt{\frac{1\cdot 2}{(\alpha+2)(\alpha+3)}}\,a_2 & \cdots \\ \sqrt{\frac{1}{\alpha+2}}\,a_1 & \frac{1\cdot 2}{\alpha+3}\,a_2 & a_0 & \frac{3\sqrt{2}}{(\alpha+4)\sqrt{\alpha+3}}\,a_3 & \cdots \\ \sqrt{\frac{1\cdot 2}{(\alpha+2)(\alpha+3)}}\,a_2 & \frac{3\sqrt{2}}{(\alpha+4)\sqrt{\alpha+3}}\,a_3 & \sqrt{\frac{1\cdot 2}{\alpha+3}}\,a_1 & \frac{12}{(\alpha+4)(\alpha+5)}\,a_4 & \cdots \\ \sqrt{\frac{1\cdot 2\cdot 3}{(\alpha+2)(\alpha+3)(\alpha+4)}}\,a_3 & \frac{4\sqrt{6}}{(\alpha+5)\sqrt{(\alpha+3)(\alpha+4)}}\,a_4 & \sqrt{\frac{1\cdot 2\cdot 3}{(\alpha+3)(\alpha+4)}}\,a_2 & \frac{20\sqrt{3}}{(\alpha+5)(\alpha+6)\sqrt{\alpha+4}}\,a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The following proposition presents some basic properties of H-Toeplitz operators on the weighted Bergman spaces (cf. [13]).

Proposition 3.8 *For $\varphi, \psi \in L^\infty(\mathbb{D})$, the operator B_φ satisfies the following:*
(i) *B_φ is a bounded linear operators on $A_\alpha^2(\mathbb{D})$ with $\|B_\varphi\| \leq \|\varphi\|_\infty$.*
(ii) *For any scalars α and β, it holds $B_{\alpha\varphi+\beta\psi} = \alpha B_\varphi + \beta B_\psi$.*
(iii) *The adjoint of the H-Toeplitz operators B_φ is given by $B_\varphi^* = K^* P_{harm} M_{\overline{\varphi}}$.*

The following remark provides an important information regarding adjoint operators, showing the difference between the adjoint of Toeplitz operators and the adjoint of H-Toeplitz operators.

Remark 3.9 If f, g are in $L^{\infty}(\mathbb{D})$, then, by the definition of Toeplitz operators, we have that

$$T_f^* = T_{\overline{f}} \quad \text{and} \quad T_{\overline{f}} T_g = T_{\overline{f}g} \text{ if } f \text{ or } g \text{ is analytic.}$$

However, for the case of the H-Toeplitz operators,

$$B_z^*(az) = K^* P_{harm} M_{\overline{z}}(az) = K^* P_{harm}(a\overline{z}z) = K^* \left(\frac{\Gamma(2)\Gamma(\alpha+2)}{\Gamma(\alpha+3)} a \right) = \frac{a}{\alpha+2}$$

and

$$B_{\overline{z}}(az) = P_\alpha M_{\overline{z}} K(az) = P_\alpha M_{\overline{z}} a\overline{z} = P_\alpha(a\overline{z}^2) = 0.$$

Therefore, $B_z^*(az) \neq B_{\overline{z}}(az)$. It can be easily verified by computation that $B_z B_z \neq B_{z^2}$.

Recall that a bounded linear operator T on a Hilbert space is called *expansive* if $T^*T \geq I$, *contractive* if $T^*T \leq I$, and *isometric* if $T^*T = I$, respectively. For $k \in A_\alpha^2(\mathbb{D})$, let $k(z) = k_e(z) + k_o(z)$, where

$$k_e(z) := \sum_{n=0}^{\infty} c_{2n} z^{2n} \quad \text{and} \quad k_o(z) := \sum_{n=0}^{\infty} c_{2n+1} z^{2n+1}.$$

3.2 *H*-Toeplitz operators with analytic symbols

In this subsection, we examine the characteristics of H-Toeplitz operators B_φ with analytic symbol functions φ. First, we study the necessary condition for contractivity and expansivity of B_φ where $\varphi(z) = \sum_{j=0}^{\infty} a_j z^j$ with $a_j \in \mathbb{C}$ under a certain additional assumptions concerning the symbol φ.

Theorem 3.10 *Let* $\varphi(z) = \sum_{j=0}^{\infty} a_j z^j$ *and* $a_j \in \mathbb{C}$.
(i) *If* B_φ *is contractive, then*

$$\sum_{j=0}^{\infty} |a_j|^2 \leq 1 \quad \text{and} \quad \sum_{j=s+1}^{\infty} \frac{\Lambda_\alpha(j, s+1)}{\Lambda_\alpha(s+1)} |a_j|^2 \leq 1$$

for any $s \in \mathbb{N}_0$.
(ii) *If* B_φ *is expansive, then*

$$\sum_{j=0}^{\infty} \Lambda_\alpha(j) |a_j|^2 \geq 1 \quad \text{and} \quad \sum_{j=s+1}^{\infty} \frac{\Lambda_\alpha(j, s+1)}{\Lambda_\alpha(s+1)} |a_j|^2 \geq 1$$

for any $s \in \mathbb{N}_0$.

Proof For any $k \in A^2_\alpha(\mathbb{D})$, we have

$$\begin{aligned} B_\varphi k(z) &= P_\alpha M_\varphi K(k_e(z) + k_o(z)) \\ &= P_\alpha M_\varphi \sum_{n=0}^{\infty} \left(\frac{\sqrt{\Gamma(2n+1)\Gamma(n+\alpha+2)}}{\sqrt{\Gamma(n+1)\Gamma(2n+\alpha+2)}} c_{2n} z^n + \frac{\sqrt{\Gamma(2n+2)\Gamma(n+\alpha+3)}}{\sqrt{\Gamma(n+2)\Gamma(2n+\alpha+3)}} c_{2n+1} \overline{z}^{n+1} \right) \\ &= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sqrt{\Gamma(2n+1)\Gamma(n+\alpha+2)}}{\sqrt{\Gamma(n+1)\Gamma(2n+\alpha+2)}} a_j c_{2n} z^{n+j} \\ &\quad + \sum_{j=1}^{\infty} \sum_{n=0}^{j-1} \frac{\sqrt{\Gamma(2n+2)\Gamma(n+\alpha+3)}}{\sqrt{\Gamma(n+2)\Gamma(2n+\alpha+3)}} \cdot \frac{\Gamma(j+1)\Gamma(j-n+\alpha+1)}{\Gamma(j+\alpha+2)\Gamma(j-n)} a_j c_{2n+1} z^{j-n-1} \end{aligned} \tag{8}$$

for any $c_k \in \mathbb{C}$ $(k = 0, 1, 2, \ldots)$. Then, from (8), the coefficient of z^m is

$$\sum_{n=0}^{m} \frac{\sqrt{\Gamma(2n+1)\Gamma(n+\alpha+2)}}{\sqrt{\Gamma(n+1)\Gamma(2n+\alpha+2)}} a_{m-n} c_{2n} + \sum_{n=0}^{\infty} \frac{\sqrt{\Gamma(2n+2)\Gamma(n+\alpha+3)}}{\sqrt{\Gamma(n+2)\Gamma(2n+\alpha+3)}} \cdot \frac{\Gamma(m+n+2)\Gamma(m+\alpha+2)}{\Gamma(m+n+\alpha+3)\Gamma(m+1)} a_{n+m+1} c_{2n+1}.$$

For a fixed $\ell \in \mathbb{N}_0$, set $c_\ell \neq 0$ and $c_k = 0$ for any $k \neq \ell$. We consider the following two cases:

Case 1: If $\ell = 2s$ for any $s \in \mathbb{N}_0$, then

$$B_\varphi k(z) = \sum_{j=0}^{\infty} \frac{\sqrt{\Gamma(2s+1)\Gamma(s+\alpha+2)}}{\sqrt{\Gamma(s+1)\Gamma(2s+\alpha+2)}} a_j c_{2s} z^{s+j}.$$

If B_φ on $A^2_\alpha(\mathbb{D})$ is contractive, then

$$\sum_{j=0}^{\infty} \frac{\Gamma(2s+1)\Gamma(s+\alpha+2)}{\Gamma(s+1)\Gamma(2s+\alpha+2)} \Lambda_\alpha(s+j)|a_j|^2 |c_{2s}|^2 \leq \Lambda_\alpha(2s)|c_{2s}|^2.$$

Thus

$$\sum_{j=0}^{\infty} \Lambda_\alpha(s+j)|a_j|^2 \leq \frac{\Gamma(s+1)\Gamma(2s+\alpha+2)}{\Gamma(2s+1)\Gamma(s+\alpha+2)} \Lambda_\alpha(2s) = \Lambda_\alpha(s) \tag{9}$$

for any $s \in \mathbb{N}_0$. By a direct calculation, $\frac{\Lambda_\alpha(s+j)}{\Lambda_\alpha(s)}$ is increasing for $s \in \mathbb{N}_0$ and

$$\lim_{s \to \infty} \frac{\Lambda_\alpha(s+j)}{\Lambda_\alpha(s)} = 1,$$

and so (9) implies that

$$\sum_{j=0}^{\infty} |a_j|^2 \leq 1.$$

Similarly, if B_φ on $A^2_\alpha(\mathbb{D})$ is expansive, then

$$\sum_{j=0}^{\infty} \Lambda_\alpha(s+j)|a_j|^2 \geq \Lambda_\alpha(s) \tag{10}$$

for any $s \in \mathbb{N}_0$. By setting $s = 0$ in (10), we have the results.

Case 2: If $\ell = 2s + 1$ for any $s \in \mathbb{N}_0$, then

$$B_\varphi k(z) = \sum_{j=s+1}^{\infty} \frac{\sqrt{\Gamma(2s+2)\Gamma(s+\alpha+3)}}{\sqrt{\Gamma(s+2)\Gamma(2s+\alpha+3)}} \cdot \frac{\Gamma(j+1)\Gamma(j-s+\alpha+1)}{\Gamma(j+\alpha+2)\Gamma(j-s)} a_j c_{2s+1} z^{j-s-1}.$$

If B_φ on $A^2_\alpha(\mathbb{D})$ is contractive, then

$$\sum_{j=s+1}^{\infty} \frac{\Gamma(2s+2)\Gamma(s+\alpha+3)}{\Gamma(s+2)\Gamma(2s+\alpha+3)} \cdot \frac{\Gamma(j+1)^2\Gamma(j-s+\alpha+1)^2}{\Gamma(j+\alpha+2)^2\Gamma(j-s)^2} \cdot \frac{\Lambda_\alpha(j-s-1)}{\Lambda_\alpha(2s+1)} |a_j|^2 \leq 1.$$

Thus

$$\sum_{j=s+1}^{\infty} \frac{\Lambda_\alpha(j, s+1)}{\Lambda_\alpha(s+1)} |a_j|^2 \leq 1$$

for any $s \in \mathbb{N}_0$. Similarly, if B_φ on $A^2_\alpha(\mathbb{D})$ is expansive, then

$$\sum_{j=s+1}^{\infty} \frac{\Lambda_\alpha(j, s+1)}{\Lambda_\alpha(s+1)} |a_j|^2 \geq 1$$

for any $s \in \mathbb{N}_0$. This completes the proof. □

Example 3.11 Let $\varphi(z) = \sum_{j=1}^{\infty} \frac{1}{j^{n/2}} z^j$ for any $n \in \mathbb{N}$. Then,

$$\sum_{j=1}^{\infty} \frac{1}{j^n} = \zeta(n) > 1,$$

where $\zeta(n)$ is the Riemann-zeta function for $n \in \mathbb{N}$. Thus B_φ is not contractive from Theorem 3.10.

Example 3.12 Let $\varphi(z) = \sum_{j=0}^{\infty} c^j z^j$ with any $|c| < 1$. Then

$$\sum_{j=0}^{\infty} |c|^{2j} = \frac{1}{1-|c|^2} > 1.$$

Hence B_φ is not contractive from Theorem 3.10.

We give a description on the contractivity and the expansivity of H-Toeplitz operators in terms of the coefficients for the polynomial symbol φ of degree n on the Bergman spaces $A_0^2(\mathbb{D})$.

Corollary 3.13 *Let $\varphi(z) = \sum_{j=1}^{n} a_j z^j$ with any $a_j \in \mathbb{C}$ and $n \geq 1$. If B_φ is contractive on $A_0^2(\mathbb{D})$, then $\sum_{j=1}^{n} |a_j|^2 \leq 1$.*

Proof From the case 1 in the proof of Theorem 3.10, if B_φ is contractive, then

$$\sum_{j=1}^{n} |a_j|^2 \leq 1. \tag{11}$$

Since $\frac{\Lambda_0(j,s+1)}{\Lambda_0(s+1)}$ is increasing for $j \leq 2s$ and decreasing for $j \geq 2s+1$, we have

$$\max_{j \geq s+1} \frac{\Lambda_0(j, s+1)}{\Lambda_0(s+1)} = \max\left\{\frac{\Lambda_0(2s, s+1)}{\Lambda_0(s+1)}, \frac{\Lambda_0(2s+1, s+1)}{\Lambda_0(s+1)}\right\} = \frac{s+2}{4(s+1)}$$

for any $s \in \mathbb{N}_0$, and

$$\max_{s \geq 0}\left\{\frac{s+2}{4(s+1)}\right\} = \frac{1}{2}.$$

Thus, for any $s \in \mathbb{N}_0$, the inequality given by

$$\sum_{j=s+1}^{n} \frac{\Lambda_0(j, s+1)}{\Lambda_0(s+1)} |a_j|^2 \leq 1$$

implies that

$$\sum_{j=1}^{n} \frac{1}{2} |a_j|^2 \leq 1. \tag{12}$$

From (11) and (12), we have complete the proof. □

Next, we consider the necessary and sufficient condition for the contractivity and the expansivity of B_φ with $\varphi(z) = az^N$ for $N \in \mathbb{N}$ and $a \in \mathbb{C}$.

Theorem 3.14 *For $\varphi(z) = az^N$ with $N \in \mathbb{N}$ and $a \in \mathbb{C}$, B_φ is contractive if and only if $|a| \leq 1$.*

Proof From the proof of Theorem 3.10, for any $k \in A^2_\alpha(\mathbb{D})$, we get that

$$\begin{aligned}B_\varphi k(z) &= P_\alpha M_\varphi K(k(z))\\ &= \sum_{n=0}^{\infty} \frac{\sqrt{\Gamma(2n+1)\Gamma(n+\alpha+2)}}{\sqrt{\Gamma(n+1)\Gamma(2n+\alpha+2)}} ac_{2n} z^{n+N}\\ &\quad + \sum_{n=0}^{N-1} \frac{\sqrt{\Gamma(2n+2)\Gamma(n+\alpha+3)}}{\sqrt{\Gamma(n+2)\Gamma(2n+\alpha+3)}} \cdot \frac{\Gamma(N+1)\Gamma(N-n+\alpha+1)}{\Gamma(N+\alpha+2)\Gamma(N-n)} ac_{2n+1} z^{N-n-1}\end{aligned}$$

and

$$\begin{aligned}\|B_\varphi k(z)\|^2 &= |a|^2 \sum_{n=0}^{\infty} \frac{\Gamma(2n+1)\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(2n+\alpha+2)} \Lambda_\alpha(n+N)|c_{2n}|^2\\ &\quad + |a|^2 \sum_{n=0}^{N-1} \frac{\Gamma(2n+2)\Gamma(n+\alpha+3)}{\Gamma(n+2)\Gamma(2n+\alpha+3)} \Lambda_\alpha(N, n+1)|c_{2n+1}|^2.\end{aligned}$$

Thus the contractivity of B_φ on $A^2_\alpha(\mathbb{D})$ is equivalent to

$$\begin{aligned}&|a|^2 \sum_{n=0}^{\infty} \frac{\Gamma(2n+1)\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(2n+\alpha+2)} \Lambda_\alpha(n+N)|c_{2n}|^2\\ &+ |a|^2 \sum_{n=0}^{N-1} \frac{\Gamma(2n+2)\Gamma(n+\alpha+3)}{\Gamma(n+2)\Gamma(2n+\alpha+3)} \Lambda_\alpha(N, n+1)|c_{2n+1}|^2\\ &\leq \sum_{j=0}^{\infty} \Lambda_\alpha(j)|c_j|^2.\end{aligned} \tag{13}$$

There are two possibilities to consider. The first case is when $c_\ell \neq 0$ for ℓ is even, and $c_\ell = 0$ for ℓ is odd, then by (13), we have

$$|a|^2 \frac{\Gamma(2n+1)\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(2n+\alpha+2)} \Lambda_\alpha(n+N)|c_{2n}|^2 \leq \Lambda_\alpha(2n)|c_{2n}|^2,$$

or equivalently,

$$|a|^2 \leq \frac{\Lambda_\alpha(n)}{\Lambda_\alpha(n+N)}$$

for any $n \in \mathbb{N}_0$. By a direct calculation, $\frac{\Lambda_\alpha(n)}{\Lambda_\alpha(n+N)}$ is decreasing for n, and

$$|a|^2 \leq \min_{n\geq 0} \frac{\Lambda_\alpha(n)}{\Lambda_\alpha(n+N)} = \lim_{n\to\infty} \frac{\Lambda_\alpha(n)}{\Lambda_\alpha(n+N)} = 1. \tag{14}$$

The second case is when $c_\ell \neq 0$ for ℓ is odd, and $c_\ell = 0$ for ℓ is even, then from (13), we have

$$|a|^2 \frac{\Gamma(2n+2)\Gamma(n+\alpha+3)}{\Gamma(n+2)\Gamma(2n+\alpha+3)} \Lambda_\alpha(N, n+1)|c_{2n+1}|^2 \leq \Lambda_\alpha(2n+1)|c_{2n+1}|^2,$$

or equivalently,

$$|a|^2 \leq \frac{\Lambda_\alpha(n+1)}{\Lambda_\alpha(N, n+1)}$$

for any $0 \leq n \leq N-1$. By a simple calculation,

$$\frac{\Lambda_\alpha(n+1)}{\Lambda_\alpha(N, n+1)} = \frac{\Gamma(N+\alpha+2)^2}{\Gamma(N+1)^2} \cdot \frac{\Gamma(N-n)}{\Gamma(N-n+\alpha+1)} \cdot \frac{\Gamma(n+2)}{\Gamma(n+\alpha+3)} > 1, \tag{15}$$

since $(N+j+1)^2 > (N-n+j)(n+j+2)$ for any $j \in \mathbb{R}$ and for all $0 \leq n \leq N-1$. From (14) and (15), B_φ is contractive if and only if $|a| \leq 1$. This completes the proof. □

Corollary 3.15 *If $\varphi(z) = az^N$ with $N \in \mathbb{N}$ and $a \in \mathbb{C}$, then B_φ is a neither expansive nor isometric operator.*

Proof It follows from the proof of Theorem 3.14 that if B_φ is expansive, then

$$\begin{aligned} &|a|^2 \sum_{n=0}^{\infty} \frac{\Gamma(2n+1)\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(2n+\alpha+2)} \Lambda_\alpha(n+N)|c_{2n}|^2 \\ &\quad + |a|^2 \sum_{n=0}^{N-1} \frac{\Gamma(2n+2)\Gamma(n+\alpha+3)}{\Gamma(n+2)\Gamma(2n+\alpha+3)} \Lambda_\alpha(N, n+1)|c_{2n+1}|^2 \\ &\geq \sum_{j=0}^{\infty} \Lambda_\alpha(j)|c_j|^2. \end{aligned} \tag{16}$$

If we substitute $c_j = 0$ for $j \neq 2N+1$ in (16), then we obtain that

$$\Lambda_\alpha(2N+1)|c_{2N+1}|^2 \leq 0,$$

which is a contradiction. □

Corollary 3.16 *Let $\varphi(z) = az^N$ with $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Then B_φ is not self-adjoint.*

Proof By the definition of the adjoint of B_φ, we deduce that

$$\begin{aligned}
B_\varphi^* k(z) &= K^* P_{harm} M_{\overline{\varphi}} k(z) \\
&= \overline{a} K^* \Bigg(\sum_{n=0}^{N-1} \frac{\Gamma(N-n+\alpha+2)\Gamma(N+1)}{\Gamma(N-n+1)\Gamma(N+\alpha+2)} c_n \overline{z}^{N-n} \\
&\quad + \sum_{n=N}^{\infty} \frac{\Gamma(n-N+\alpha+2)\Gamma(n+1)}{\Gamma(n-N+1)\Gamma(n+\alpha+2)} c_n z^{n-N} \Bigg) \\
&= \overline{a} \sum_{n=0}^{N-1} \frac{\Gamma(N+1)\sqrt{\Gamma(N-n+\alpha+2)\Gamma(2N-2n+\alpha+1)}}{\Gamma(N+\alpha+2)\sqrt{\Gamma(N-n+1)\Gamma(2N-2n)}} c_n z^{2N-2n-1} \\
&\quad + \overline{a} \sum_{n=N}^{\infty} \frac{\Gamma(n+1)\sqrt{\Gamma(n-N+\alpha+2)\Gamma(2n-2N+\alpha+2)}}{\Gamma(n+\alpha+2)\sqrt{\Gamma(n-N+1)\Gamma(2n-2N+1)}} c_n z^{2n-2N}.
\end{aligned}$$

Comparing constant terms in $B_\varphi k(z)$ and $B_\varphi^* k(z)$, they are

$$\frac{\sqrt{\Gamma(2N)\Gamma(N+1)}\Gamma(\alpha+2)}{\sqrt{\Gamma(2N+\alpha+1)\Gamma(N+\alpha+2)}} a c_{2N-1} \quad \text{and} \quad \frac{\Gamma(N+1)\Gamma(\alpha+2)}{\Gamma(N+\alpha+2)} \overline{a} c_N,$$

respectively. As c_{2N-1} and c_N can be chosen arbitrarily, it follows that the constant terms in $B_\varphi k(z)$ and $B_\varphi^* k(z)$ are different, and hence B_φ is not self-adjoint. □

Corollary 3.17 *For $\varphi(z) = az^N$ with $N \in \mathbb{N}$ and $a \in \mathbb{C}$, B_φ is not normal.*

Proof For any $k \in A^2_\alpha(\mathbb{D})$, the normality of B_φ is equivalent to $B_\varphi^* B_\varphi k(z) = B_\varphi B_\varphi^* k(z)$ or $\|B_\varphi k(z)\|^2 = \|B_\varphi^* k(z)\|^2$. Using the proof of Theorem 3.14 and Corollary 3.16, we get

$$\begin{aligned}
\|B_\varphi k(z)\|^2 &= |a|^2 \sum_{n=0}^{\infty} \frac{\Gamma(2n+1)\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(2n+\alpha+2)} \Lambda_\alpha(n+N)|c_{2n}|^2 \\
&\quad + |a|^2 \sum_{n=0}^{N-1} \frac{\Gamma(2n+2)\Gamma(n+\alpha+3)}{\Gamma(n+2)\Gamma(2n+\alpha+3)} \Lambda_\alpha(N, n+1)|c_{2n+1}|^2
\end{aligned} \tag{17}$$

and

$$\|B_\varphi^* k(z)\|^2 = |a|^2 \sum_{n=0}^{N-1} \frac{\Lambda_\alpha^2(N)}{\Lambda_\alpha(N-n)} |c_n|^2 + \sum_{n=N}^{\infty} \frac{\Lambda_\alpha^2(n)}{\Lambda_\alpha(n-N)} |c_n|^2. \tag{18}$$

If we substitute $c_i = 0$ for $i \neq 2N+1$ in (17) and (18), then we obtain that $\|B_\varphi k(z)\|^2 = 0$ and $\|B_\varphi^* k(z)\|^2 = \frac{\Lambda_\alpha^2(2N+1)}{\Lambda_\alpha(N+1)} |c_{2N+1}|^2 \neq 0$, which gives the results. □

3.3 *H*-Toeplitz operators with coanalytic symbols

In this subsection, we examine the characteristics of H-Toeplitz operators B_φ with coanalytic symbol φ. First, we examine the contractivity and the expansivity of B_φ where φ is of the form $\varphi(z) = \sum_{j=1}^{\infty} b_j \overline{z}^j$ with $b_j \in \mathbb{C}$.

Theorem 3.18 *Let $\varphi(z) = \sum_{j=1}^{\infty} b_j \overline{z}^j$ with $b_j \in \mathbb{C}$. If B_φ is contractive, then*

$$\sum_{j=1}^{s} \frac{1}{\Lambda_\alpha(s-j)} |b_j|^2 \leq \frac{1}{\Lambda_\alpha(s)}$$

for any $s \in \mathbb{N}$.

Proof For any $k \in A_\alpha^2(\mathbb{D})$,

$$\begin{aligned} B_\varphi k(z) &= P_\alpha M_\varphi \left[\sum_{n=0}^{\infty} \left(\frac{\sqrt{\Gamma(2n+1)\Gamma(n+\alpha+2)}}{\sqrt{\Gamma(n+1)\Gamma(2n+\alpha+2)}} c_{2n} z^n + \frac{\sqrt{\Gamma(2n+2)\Gamma(n+\alpha+3)}}{\sqrt{\Gamma(n+2)\Gamma(2n+\alpha+3)}} c_{2n+1} \overline{z}^{n+1} \right) \right] \\ &= P_\alpha \left(\sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \frac{\sqrt{\Gamma(2n+1)\Gamma(n+\alpha+2)}}{\sqrt{\Gamma(n+1)\Gamma(2n+\alpha+2)}} b_j c_{2n} z^n \overline{z}^j \right) \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{\sqrt{\Gamma(2n+1)\Gamma(n+1)}}{\sqrt{\Gamma(n+\alpha+2)\Gamma(2n+\alpha+2)}} \cdot \frac{\Gamma(n-j+\alpha+2)}{\Gamma(n-j+1)} b_j c_{2n} z^{n-j}. \end{aligned} \tag{19}$$

It follows from (19) that, the coefficient of z^m is

$$\sum_{n=m+1}^{\infty} \frac{\sqrt{\Gamma(2n+1)\Gamma(n+1)}}{\sqrt{\Gamma(n+\alpha+2)\Gamma(2n+\alpha+2)}} \cdot \frac{\Gamma(m+\alpha+2)}{\Gamma(m+1)} b_{n-m} c_{2n}.$$

For some $s \in \mathbb{N}$, we set $c_\ell \neq 0$ if $\ell = 2s$ and $c_\ell = 0$ if $\ell \neq 2s$. If B_φ on $A_\alpha^2(\mathbb{D})$ is contractive, then

$$\sum_{j=1}^{s} \frac{\Lambda_\alpha(2s)\Lambda_\alpha(s)}{\Lambda_\alpha(s-j)} |b_j|^2 |c_{2s}|^2 \leq \Lambda_\alpha(2s)|c_{2s}|^2.$$

Therefore,

$$\sum_{j=1}^{s} \frac{1}{\Lambda_\alpha(s-j)} |b_j|^2 \leq \frac{1}{\Lambda_\alpha(s)}.$$

This completes the proof. ◻

Corollary 3.19 *For $\varphi(z) = \sum_{j=1}^{\infty} b_j \overline{z}^i$ with $b_j \in \mathbb{C}$, B_φ is not an expansive operator.*

Proof From the Eq. (19), if we substitute $c_j = 0$ for j is even, then we obtain that $B_\varphi k(z) = 0$. Thus B_φ on $A^2_\alpha(\mathbb{D})$ is not an expansive operator. □

Example 3.20 For $\varphi(z) = \sqrt{\alpha+3}\bar{z} + \sqrt{\alpha+2}\bar{z}^2$, we have

$$\sum_{j=1}^{2} \frac{1}{\Lambda_\alpha(2-j)}|b_j|^2 = (\alpha+2)(\alpha+4) > \frac{(\alpha+2)(\alpha+3)}{2} = \frac{1}{\Lambda_\alpha(2)}.$$

Hence by Theorem 3.18, B_φ is not contractive.

Next, we study the necessary and sufficient condition for the contractivity and the expansivity of B_φ with $\varphi = b\bar{z}^N$ for $N \in \mathbb{N}$ and $b \in \mathbb{C}$.

Theorem 3.21 *Let $\varphi(z) = b\bar{z}^N$ with $N \in \mathbb{N}$ and $b \in \mathbb{C}$. Then B_φ is contractive if and only if $|b| \leq 1$.*

Proof From the proof of Theorem 3.18, for any $k \in A^2_\alpha(\mathbb{D})$,

$$B_\varphi k(z) = \sum_{n=N}^{\infty} \frac{\sqrt{\Gamma(2n+1)\Gamma(n+1)}}{\sqrt{\Gamma(n+\alpha+2)\Gamma(2n+\alpha+2)}} \cdot \frac{\Gamma(n-N+\alpha+2)}{\Gamma(n-N+1)} b c_{2n} z^{n-N}.$$

Thus

$$\|B_\varphi k(z)\|^2 = |b|^2 \sum_{n=N}^{\infty} \frac{\Lambda_\alpha(2n)\Lambda_\alpha(n)}{\Lambda_\alpha(n-N)}|c_{2n}|^2.$$

Hence the contractivity of B_φ is equivalent to

$$|b|^2 \sum_{n=N}^{\infty} \frac{\Lambda_\alpha(2n)\Lambda_\alpha(n)}{\Lambda_\alpha(n-N)}|c_{2n}|^2 \leq \sum_{n=0}^{\infty} \Lambda_\alpha(n)|c_n|^2.$$

If we compare the terms involving $|c_{2n}|^2$, then we have

$$\frac{\Lambda_\alpha(2n)\Lambda_\alpha(n)}{\Lambda_\alpha(n-N)}|b|^2|c_{2n}|^2 \leq \Lambda_\alpha(2n)|c_{2n}|^2,$$

and so

$$|b|^2 \leq \frac{\Lambda_\alpha(n-N)}{\Lambda_\alpha(n)}$$

for any $n \geq N$. Since $\frac{\Lambda_\alpha(n-N)}{\Lambda_\alpha(n)}$ is decreasing for $n \geq N$, B_φ is contractive if and only if

$$|b|^2 \leq \min_{n\geq 0} \frac{\Lambda_\alpha(n-N)}{\Lambda_\alpha(n)} = \lim_{n\to\infty} \frac{\Lambda_\alpha(n-N)}{\Lambda_\alpha(n)} = 1.$$

This completes the proof. □

Corollary 3.22 *For $\varphi(z) = b\overline{z}^N$ with $N \in \mathbb{N}$ and $b \in \mathbb{C}$, B_φ is neither expansive nor isometric.*

Proof Using the result as in the proof of Theorem 3.21, the expansivity of B_φ is equivalent to

$$|b|^2 \sum_{n=N}^{\infty} \frac{\Lambda_\alpha(2n)\Lambda_\alpha(n)}{\Lambda_\alpha(n-N)}|c_{2n}|^2 \geq \sum_{n=0}^{\infty} \Lambda_\alpha(n)|c_n|^2.$$

If we substitute $c_n = 0$ for $n \geq 2N$, then we deduce that $\sum_{n=0}^{2N-1} \Lambda_\alpha(n)|c_n|^2 \leq 0$, which is a contradiction. □

3.4 *H*-Toeplitz operators with harmonic symbols

Finally, we analyze the properties of H-Toeplitz operators B_φ that have harmonic symbols of the form $\varphi(z) = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} b_j \overline{z}^j$ with $a_j, b_j \in \mathbb{C}$. Our focus is on determining the necessary and sufficient conditions for the contractivity and the expansivity of B_φ.

Theorem 3.23 *Let $\varphi(z) = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} b_j \overline{z}^j$ and $a_j, b_j \in \mathbb{C}$.*
(i) *If B_φ is contractive, then*

$$\sum_{j=0}^{\infty} \Lambda_\alpha(j)|a_j|^2 \leq 1, \quad \sum_{j=0}^{\infty} \frac{\Lambda_\alpha(s+j)}{\Lambda_\alpha(s)}|a_j|^2 + \sum_{j=1}^{s} \frac{\Lambda_\alpha(s)}{\Lambda_\alpha(s-j)}|b_j|^2 \leq 1$$

and

$$\sum_{j=s}^{\infty} \Lambda_\alpha(j,s)|a_j|^2 \leq \Lambda_\alpha(s)$$

for any $s \in \mathbb{N}$.
(ii) *If B_φ is expansive, then*

$$\sum_{j=0}^{\infty} \Lambda_\alpha(j)|a_j|^2 \geq 1, \quad \sum_{j=0}^{\infty} \frac{\Lambda_\alpha(s+j)}{\Lambda_\alpha(s)}|a_j|^2 + \sum_{j=1}^{s} \frac{\Lambda_\alpha(s)}{\Lambda_\alpha(s-j)}|b_j|^2 \geq 1$$

and

$$\sum_{j=s}^{\infty} \Lambda_\alpha(j,s)|a_j|^2 \geq \Lambda_\alpha(s)$$

for any $s \in \mathbb{N}$.

Proof By the similar arguments as in the proof of Theorems 3.10 and 3.18, for any $k \in A^2_\alpha(\mathbb{D})$,

$$\begin{aligned} B_\varphi k(z) &= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{\sqrt{\Gamma(2n+1)\Gamma(n+\alpha+2)}}{\sqrt{\Gamma(n+1)\Gamma(2n+\alpha+2)}} a_j c_{2n} z^{n+j} \\ &+ \sum_{j=1}^{\infty}\sum_{n=0}^{j-1} \frac{\sqrt{\Gamma(2n+2)\Gamma(n+\alpha+3)}}{\sqrt{\Gamma(n+2)\Gamma(2n+\alpha+3)}} \cdot \frac{\Gamma(j+1)\Gamma(j-n+\alpha+1)}{\Gamma(j+\alpha+2)\Gamma(j-n)} a_j c_{2n+1} z^{j-n-1} \\ &+ \sum_{n=1}^{\infty}\sum_{j=1}^{n} \frac{\sqrt{\Gamma(2n+1)\Gamma(n+1)}}{\sqrt{\Gamma(n+\alpha+2)\Gamma(2n+\alpha+2)}} \cdot \frac{\Gamma(n-j+\alpha+2)}{\Gamma(n-j+1)} b_j c_{2n} z^{n-j} \end{aligned}$$

for any $c_j \in \mathbb{C}$ $(j = 0, 1, 2, \ldots)$. For some $\ell \in \mathbb{N}_0$, set $c_\ell \neq 0$ and $c_j = 0$ for any $j \neq \ell$. Next, we examine the two cases below:

Case 1: If $\ell = 0$, then $B_\varphi k(z) = \sum_{j=0}^{\infty} a_j c_0 z^j$. Thus if B_φ on $A^2_\alpha(\mathbb{D})$ is contractive, then

$$\sum_{j=0}^{\infty} \Lambda_\alpha(j)|a_j|^2|c_0|^2 \leq \Lambda_\alpha(0)|c_0|^2,$$

or equivalently $\sum_{j=0}^{\infty} \Lambda_\alpha(j)|a_j|^2 \leq 1$. Similarly, if B_φ on $A^2_\alpha(\mathbb{D})$ is expansive, then $\sum_{j=0}^{\infty} \Lambda_\alpha(j)|a_j|^2 \geq 1$.

Case 2: If $\ell = 2s$ for any $s \in \mathbb{N}$ and $c_{2s} \neq 0$, then

$$\begin{aligned} B_\varphi k(z) &= \sum_{j=0}^{\infty} \frac{\sqrt{\Gamma(2s+1)\Gamma(s+\alpha+2)}}{\sqrt{\Gamma(s+1)\Gamma(2s+\alpha+2)}} a_j c_{2s} z^{s+j} \\ &\quad + \sum_{j=1}^{s} \frac{\sqrt{\Gamma(2s+1)\Gamma(s+1)}}{\sqrt{\Gamma(s+\alpha+2)\Gamma(2s+\alpha+2)}} \cdot \frac{\Gamma(s-j+\alpha+2)}{\Gamma(s-j+1)} b_j c_{2s} z^{s-j}. \end{aligned}$$

If B_φ on $A^2_\alpha(\mathbb{D})$ is contractive, then

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{\Gamma(2s+1)\Gamma(s+\alpha+2)}{\Gamma(s+1)\Gamma(2s+\alpha+2)} \Lambda_\alpha(s+j)|a_j|^2|c_{2s}|^2 \\ &\quad + \sum_{j=1}^{s} \frac{\Lambda_\alpha(2s)\Lambda_\alpha(s)}{\Lambda_\alpha(s-j)} |b_j|^2|c_{2s}|^2 \leq \Lambda_\alpha(2s)|c_{2s}|^2 \end{aligned}$$

or equivalently

$$\sum_{j=0}^{\infty} \frac{\Lambda_\alpha(s+j)}{\Lambda_\alpha(s)}|a_j|^2 + \sum_{j=1}^{s} \frac{\Lambda_\alpha(s)}{\Lambda_\alpha(s-j)}|b_j|^2 \leq 1.$$

Similarly, if B_φ on $A^2_\alpha(\mathbb{D})$ is expansive, then

$$\sum_{j=0}^{\infty} \frac{\Lambda_\alpha(s+j)}{\Lambda_\alpha(s)}|a_j|^2 + \sum_{j=1}^{s} \frac{\Lambda_\alpha(s)}{\Lambda_\alpha(s-j)}|b_j|^2 \geq 1.$$

Case 3: If $\ell = 2s-1$ for any $s \in \mathbb{N}$ and $c_{2s-1} \neq 0$, then by the case 2 of Theorem 3.10, we have the results. This completes the proof. □

The following results can be easily derived from Theorem 3.23.

Corollary 3.24 *Let $\varphi(z) = a_1 z + b_1 \overline{z}$ and $a_1, b_1 \in \mathbb{C}$. If B_φ is contractive, then*

$$|a_1|^2 \leq \alpha + 2, \quad \frac{2}{\alpha+3}|a_1|^2 + \frac{1}{\alpha+2}|b_1|^2 \leq 1$$

and

$$\frac{s+1}{s+\alpha+2}|a_1|^2 + \frac{s}{s+\alpha+1}|b_1|^2 \leq 1$$

for any $s \geq 2$.

Example 3.25 For $\varphi(z) = \frac{\sqrt{\alpha+2}}{\sqrt{2}}z + \frac{\sqrt{(\alpha+3)(3\alpha+7)}}{4}z^2 - \frac{3\sqrt{(\alpha+1)(\alpha+2)}}{2\sqrt{2(\alpha+3)}}\overline{z}$, we have

$$\sum_{j=0}^{\infty} \Lambda_\alpha(j)|a_j|^2 = \frac{1}{2} + \frac{3\alpha+7}{8(\alpha+2)} < 1 = \Lambda_\alpha(0),$$

$$\sum_{j=1}^{\infty} \Lambda_\alpha(j,1)|a_j|^2 = \frac{1}{2(\alpha+2)} + \frac{3\alpha+7}{4(\alpha+2)(\alpha+3)} > \frac{1}{\alpha+2} = \Lambda_\alpha(1),$$

and

$$\sum_{j=0}^{\infty} \frac{\Lambda_\alpha(1+j)}{\Lambda_\alpha(1)}|a_j|^2 + \frac{\Lambda_\alpha(1)}{\Lambda_\alpha(0)}|b_1|^2 = \frac{\alpha+2}{\alpha+3} + \frac{3(3\alpha+7)}{8(\alpha+4)} - \frac{9(\alpha+1)}{8(\alpha+3)} < 1.$$

Hence, by the Theorem 3.23, B_φ is neither contractive nor expansive.

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Declarations

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