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Blow-up, global existence and propagation speed for a modified Camassa–Holm equation both dissipation and dispersion in $H^{s,p}(\mathbb{R})$

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Abstract
In this essay, we investigate the blow-up scenario, global solution and propagation speed for a modified Camassa–Holm (MCH) equation both dissipation and dispersion in Sobolev space $H^{s,p}(\mathbb{R})$, $s \geq 1$, $p \in (1, \infty)$. First of all, by the mathematical induction of index s, we establish the precise blow-up criteria, which extends the result obtained by Gui et al. in article (Comm Math Phys 319: 731–759, 2013). Secondly, we derive the global existence of the strong solution of MCH equation both dissipation and dispersion. Eventually, the propagation speed of the equation is studied when the initial data are compactly supported.

Keywords The modified Camassa–Holm equation · Blow-up scenario · Global existence · Propagation speed

Mathematics Subject Classification 35G25 · 35L05

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1 Introduction

In 1993, by using Hamiltonian methods, Camassa and Holm [3] derived the following new completely integrable dispersive shallow water equation:

$$\begin{cases} y_t + v y_x + 2v_x y + 2kv_x = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ v(t,x)\mid_{t=0} = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

i.e., the classical Camassa-Holm (CH) equation, where k denotes a constant related to the critical shallow water wave speed, and the subscripts of $y = v - v_{xx},\ v$ indicate the partial derivative. Although Fuchssteiner et al. [17] researched the bi-Hamiltonian equation using recursive operators as early as 1981 and derived the CH equation, the work received little attention at that time.

As is known to all, the CH equation has been widely studied. This model simulates the unidirectional propagation of shallow water waves on a flat bottom and the axisymmetric wave propagation in a hyperelastic rod [3, 13]. The CH equation, in contrast to the KdV equation, simulates the breaking phenomenon of shallow water waves. Moreover, scholars have demonstrated the global existence and the blow-up phenomenon of the solution [6, 8, 23]. The remarkable feature of the CH equation is its peaked solitons at the form $v(t,x) = ce^{-|x-ct|}$, where c is a wave speed and $c \in \mathbb{R}$ [4]. At the same time, Constantin et al. [5, 11, 12, 17] not only studied the Hamiltonian structure and integrability of the CH equation, but also proved the orbital stability of peaked solitons. In recent years, many researchers have been greatly interested in the CH equation and have conducted a great deal of research on it [1, 2, 7, 9, 14, 15, 28, 29].

The integrable modified Camassa-Holm (MCH) equation with cubic nonlinearity was first introduced in 1996 by Fuchssteiner [16] and Olver et al. [25] using the bi-Hamiltonian representation of the classical integrable system:

$$\begin{cases} m_t + \left(\omega^2 - \omega_x^2\right) m_x + 2\omega_x m^2 + \alpha\omega_x = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \omega(t,x)\mid_{t=0} = \omega_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

where $m = \omega - \omega_{xx}$, $\omega = \omega(t,x)$ is the fluid velocity and subscripts of $m,\ \omega$ denote the partial derivatives, ω_0 is the initial data. Until now, Fu and Gui et al. [18] demonstrated that the MCH equation is locally well-posed in the Besov space $B^s_{p,r}(\mathbb{R})$, obtained the blow-up scenario and the lower bound of the maximum existence time. Futhermore, they proved that the nonexistence of smooth traveling wave solutions. The creation of singularities and the presence of peaked traveling-wave solutions to MCH equation were studied by Gui and Liu et al. [20]. Moreover, they proved the existence of single peak solution and multi-peak solution. When $\alpha = 0$, Wu and Guo [30] investigated the persistence, infinite propagation and traveling wave solutions of MCH equation. The local well-posedness and asymptotic behavior of MCH equation solution were studied by Wu and Zhang [31, 32] successively.

In this essay, we discuss the Cauchy problem of the modified Camassa-Holm equation both dissipation and dispersion:

$$\begin{cases} m_t + \left(\omega^2 - \omega_x^2\right) m_x + 2\omega_x m^2 + km_x + \lambda m = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \omega(t,x) \mid_{t=0} = \omega_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

where $m = \omega - \omega_{xx}$, $k \in \mathbb{R}$ is a dispersion parameter, and $\lambda > 0$ is a dissipation coefficient. Recently, a number of researchers have investigated nonlinear models with dissipation. Ott and Sudan [24] studied how the KdV equation was modified by the existence of dissipation and what the influence of such a dissipation was on soliton solutions of the KdV equation. Wu and Yin [26] discussed the local well-posedness, blow-up rate and the global solutions of the weakly dissipative periodic CH equation. In addition, the local well-posedness, blow-up rate and decay of solutions to the weakly dissipative Degasperis-Procesi equation were also established by them [27]. Hu [21] investigated the local well-posedness, the global existence and blow-up phenomena for a weakly dissipative periodic two-component CH equation. Thereafter, Hu and Qiao [22] studied the local well-posedness, the precise blow-up scenario, the global existence and propagation speed for a generalized CH model with both dissipation and dispersion.

In this essay, we study the blow-up scenario, global solution and propagation speed of strong solution for the IVP of Eq. (1.3). Our study shows that the blow-up scenario for the solution of Eq. (1.3)($k,\ \lambda \neq 0$) is similar to that of Eq. (1.2)($k = \lambda = 0$). In addition, different from Gui's approach, we used mathematical induction to study the blow-up criterion in Theorem 2.1. It is worth noting that the dissipation term λm and the dissipation term km_x in Eq. (1.3) have an impact on the global existence and the propagation speed of its solution, see Theorem 3.1 and Theorem 4.1. In particular, the propagation speed is heavily influenced by the dispersion parameter k and the dissipation coefficient λ.

The essay is structured as follows. In Sect. 2, we first give three important lemmas and establish the blow-up criterion for the solution of Eq. (1.3). The global existence of strong solutions of Eq. (1.3) is studied in Sect. 3. In the last section, we study the propagation speed of strong solutions to Eq. (1.3) provided the initial data have compact support.

Notation: For convenience, all function spaces are over $\mathbb{R}$, and if there is no ambiguity, we exclude $\mathbb{R}$ from our notation of function spaces. For $1 \leq p \leq \infty$, $\|\cdot\|_{L^p}$ will stand for the norm in the Banach space $L^p(\mathbb{R})$, while the norm in the classical Sobolev spaces $H^{s,p}(\mathbb{R})$ will be written by $\|\cdot\|_{H^{s,p}}$, $s \in \mathbb{R}$. Furthermore, the norm of $H^{s,p}(\mathbb{R})$ is defined as follows

$$\|u\|_{H^{s,p}} = \left(\sum_{0 \leq |\alpha| \leq s} \int_{\mathbb{R}} \left|D^\alpha u\right|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

where $\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$ and $\widehat{f}(\xi)$ represents the Fourier transformation.

2 Blow-up scenario

In this section, we will show some significant results in order to achieve our goal. Let the potential $m = \omega - \omega_{xx}$, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, then we rewrite Eq. (1.3) in the form of a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} \omega_t + (\omega^2 - \frac{1}{3}\omega_x^2 + k)\omega_x + \partial_x \left(1 - \partial_x^2\right)^{-1} \left(\frac{2}{3}\omega^3 + \omega\omega_x^2 - \lambda\omega_x\right) \\ \quad + \left(1 - \partial_x^2\right)^{-1} \left(\frac{1}{3}\omega_x^3 + \lambda\omega\right) = 0, \\ \omega(t, x) \mid_{t=0} = \omega_0(x). \end{cases} \tag{2.1}$$

Furthermore, Eq. (2.1) can be reformulated in the following form:

$$\begin{cases} \omega_t + (\omega^2 - \frac{1}{3}\omega_x^2 + k)\omega_x + \partial_x G * \left(\frac{2}{3}\omega^3 + \omega\omega_x^2 - \lambda\omega_x\right) + G * \left(\frac{1}{3}\omega_x^3 + \lambda\omega\right) = 0, \\ \omega(t, x) \mid_{t=0} = \omega_0(x), \end{cases} \tag{2.2}$$

where we used $(1 - \partial_x^2)^{-1} g = G * g$ for all $g \in L^p$, $G(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, namely $(1 - \partial_x^2)^{-1} m = G * m = \omega$, here we represent by $*$ the convolution.

Apparently, similar to [32], the local well-posedness of the Cauchy problem Eq. (1.3) in $H^{s,p}(\mathbb{R})$, $s \geq 1$ can be obtained by applying the Kato's semigroup theorem.

Lemma 2.1 *[32] Suppose that $m_0 \in H^{s,p}(\mathbb{R})$, $s \geq 1$, $p \in (1, \infty)$, in other words, the initial datum $\omega_0 \in H^{s+2,p}(\mathbb{R})$. Then there exist a time $T = T(m_0) > 0$ and a unique strong solution $m(t, x)$ to Eq. (1.3) such that*

$$m(t, x) \in \mathcal{C}([0, T); H^{s,p}) \cap \mathcal{C}^1([0, T); L^p).$$

Moreover, the solution $m(t, x)$ depends continuously on m_0, i.e., the mapping

$$m_0 \to m(t, x) : H^{s,p} \to \mathcal{C}([0, T); H^{s,p}) \cap \mathcal{C}^1([0, T); L^p)$$

is continuous.

In particular, the unique strong solution $m(t, x)$ to Eq. (1.3) satisfies

$$m(t, x) \in \mathcal{C}([0, T); H^{s,p}) \cap \mathcal{C}^1([0, T); H^{s-1,p}).$$

Next, we consider the following ordinary differential system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\rho(t, x) = (\omega^2 - \omega_x^2)(t, \rho(t, x)) + k, & (t, x) \in [0, T) \times \mathbb{R}, \\ \rho(t, x) \mid_{t=0} = \omega_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where ω is the corresponding strong solution to Eq. (1.3). It should be pointed out that a similar system for Camassa-Holm defines the re-expression of that equation as geodesic flow (a detail discussion can be found in [10]). The following several lemmas, which are essential in the demonstration of global existence, can be obtained by applying classical results in the theory of ordinary differential equations.

Lemma 2.2 *Assume that* $\omega_0 \in H^{s,p}(\mathbb{R})$, $s \geq 3$, $p \in (1, \infty)$. *Then there exist a* $T = T(\omega_0) > 0$ *and a unique strong solution* $\rho \in \mathcal{C}([0, T) \times \mathbb{R}; \mathbb{R})$ *to Eq. (2.3) such that the function* $\rho(t, \cdot)$ *is an increasing diffeomorphism of* $\mathbb{R}$ *with*

$$\rho_x(t, x) = \exp\left(2 \int_0^t (m\omega_x)(s, \rho(s, x))\right) ds > 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

Proof. Differentiating Eq.(2.3) with respect to x, it follows that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} \rho_x(t, x) = 2m\omega_x(t, \rho(t, x))\rho_x(t, x), & (t, x) \in [0, T) \times \mathbb{R}, \\ \rho_x(0, x) = 1, & x \in \mathbb{R}. \end{cases} \tag{2.4}$$

Solving the above equation derives the result of Lemma 2.2.

Lemma 2.3 *Suppose that* $\omega_0 \in H^{s,p}(\mathbb{R})$, $s \geq 3$, $p \in (1, \infty)$, *and let* $T = T(\omega_0) > 0$ *be the maximal existence time of the solution* ω *to Eq. (1.3) corresponding to the initial data* ω_0. *Then we have*

$$m(t, \rho(t, x))\rho_x(t, x) = e^{-\lambda t} m_0(x), \quad (t, x) \in [0, T) \times \mathbb{R}. \tag{2.5}$$

For all $(t, x) \in [0, T) \times \mathbb{R}$, *if there exists* $M > 0$ *such that* $-m\omega_x(t, x) \leq M$, *then we obtain*

$$\|m(t)\|_{L^\infty} = \|m(t, \rho(t, \cdot))\|_{L^\infty} \leq e^{(2M-\lambda)t} \|m_0\|_{L^\infty}, \quad \forall t \in [0, T). \tag{2.6}$$

Proof. Differentiating the left-hand side of Eq. (2.5) with respect to t, by virtue of Eqs. (1.3) and (2.3), it follows that

$$\begin{aligned} \frac{d}{dt}[m(t, \rho(t, x))\rho_x(t, x)] &= (m_t + m_x \rho_t)\rho_x + m\rho_{tx} \\ &= \left[m_t + m_x(\omega^2 - \omega_x^2 + k)\right]\rho_x + 2\omega_x m^2 \rho_x \\ &= \left[m_t + m_x(\omega^2 - \omega_x^2 + k) + 2\omega_x m^2\right]\rho_x \\ &= -\lambda m \rho_x. \end{aligned}$$

Solving the above equation, one obtains

$$m(t, \rho(t, x))\rho_x(t, x) = e^{-\lambda t} m_0(x).$$

In addition, it follows that

$$\begin{aligned} \|m(t)\|_{L^\infty} &= \left\| e^{-\lambda t} m_0 \exp\left(-2 \int_0^t (m\omega_x)(s, \rho(s, \cdot))\right) ds \right\|_{L^\infty} \\ &\leq e^{(2M-\lambda)t} \|m_0\|_{L^\infty}. \end{aligned}$$

Therefore, the proof of Lemma 2.3 have been completed.

Lemma 2.4 *[19] For $s \geq 0$, $p \in (1, \infty)$, there exists a constant C_s such that the following estimate holds:*

$$\| f_1 f_2 \|_{H^{s,p}(\mathbb{R})} \leq C_s \left(\| f_1 \|_{H^{s,p}(\mathbb{R})} \| f_2 \|_{L^\infty(\mathbb{R})} + \| f_1 \|_{L^\infty(\mathbb{R})} \| f_2 \|_{H^{s,p}(\mathbb{R})} \right).$$

Next, we will provide the precise blow-up criteria.

Theorem 2.1 *Suppose $m_0 \in H^{s,p}(\mathbb{R})$, $s \geq 1$, $p \in (1, \infty)$. Then the solution m of Eq. (1.3) blows up in the finite time $T > 0$ if and only if*

$$\liminf_{t \uparrow T} (m\omega_x)(t, \cdot) = -\infty.$$

Proof. Sufficiency: If $\liminf_{t \uparrow T} (m\omega_x)(t, \cdot) = -\infty$, and $m\omega_x$ is continuous with respect to x, by the Sobolev's embedding theorem, then the solution m of Eq. (1.3) will blow-up in finite time.

Necessity: Without loss of generality, we only prove the necessity when $s \geq 1$, $s \in \mathbb{N}$. If there exists a positive constant $M > 0$ such that $-m\omega_x(t, x) \leq M$ for all $t \in [0, T)$, then we need to prove that $\|m(t, \cdot)\|_{H^{s,p}} < \infty$. For the convenience of writing, let $A = e^{(2M-\lambda)t} \|m_0\|_{L^\infty}$. We will use mathematical induction for s to demonstrate this statement.

(i) Let $s = 1$, we shall estimate $\|m(t, \cdot)\|_{H^{1,p}} < \infty$.

Multiplying Eq. (1.3) by $|m|^{p-2} m$ with $p \geq 2$, and integrating over $\mathbb{R}$ with respect to x, integration by parts, then we have

$$\begin{aligned}
\int_{\mathbb{R}} m_t |m|^{p-2} m dx &= - \int_{\mathbb{R}} \left(\omega^2 - \omega_x^2\right) m_x |m|^{p-2} m dx - 2 \int_{\mathbb{R}} \omega_x m^2 |m|^{p-2} m dx \\
&\quad - k \int_{\mathbb{R}} m_x |m|^{p-2} m dx - \lambda \int_{\mathbb{R}} m |m|^{p-2} m dx \\
&= \frac{1}{p} \int_{\mathbb{R}} \left(\omega^2 - \omega_x^2\right)_x |m|^p dx - 2 \int_{\mathbb{R}} m\omega_x |m|^p dx - \lambda \int_{\mathbb{R}} |m|^p dx \\
&= - \left(2 - \frac{2}{p}\right) \int_{\mathbb{R}} m\omega_x |m|^p dx - \lambda \int_{\mathbb{R}} |m|^p dx \\
&\leq \left(\left(2 - \frac{2}{p}\right) M + \lambda \right) \|m\|_{L^p}^p,
\end{aligned}$$

i.e.,

$$\frac{d}{dt} \int_{\mathbb{R}} |m|^p dx \leq ((2p - 2)M + \lambda p) \|m\|_{L^p}^p. \tag{2.7}$$

Differentiating Eq. (1.3) with respect to x, we get

$$m_{tx} = -2\omega_{xx} m^2 - 6\omega_x m m_x - \left(\omega^2 - \omega_x^2\right) m_{xx} - k m_{xx} - \lambda m_x. \tag{2.8}$$

Applying Eq. (2.8) by $|m_x|^{p-2}m_x$ with $p \geq 2$, and integrating over $\mathbb{R}$ with respect to x, integration by parts leads to

$$\begin{aligned}
&\int_{\mathbb{R}} m_{tx}\,|m_x|^{p-2}\,m_x dx \\
&= -2\int_{\mathbb{R}} \omega_{xx}m^2\,|m_x|^{p-2}\,m_x dx - 6\int_{\mathbb{R}} \omega_x m m_x\,|m_x|^{p-2}\,m_x dx \\
&\quad - \int_{\mathbb{R}} \left(\omega^2-\omega_x^2\right) m_{xx}\,|m_x|^{p-2}\,m_x dx - k\int_{\mathbb{R}} m_{xx}\,|m_x|^{p-2}\,m_x dx \\
&\quad -\lambda\int_{\mathbb{R}} m_x\,|m_x|^{p-2}\,m_x dx \\
&\leq 2\left\|m^2\right\|_{L^\infty}\|\omega_{xx}\|_{L^p}\|m_x\|_{L^p}^{p-1} + \left(6-\frac{2}{p}\right)M\,\|m_x\|_{L^p}^p + \lambda\,\|m_x\|_{L^p}^p \\
&\leq \frac{2A^2}{p}\|m\|_{L^p}^p + \left(\frac{2A^2(p-1)}{p} + \left(6-\frac{2}{p}\right)M + \lambda\right)\|m_x\|_{L^p}^p, \qquad (2.9)
\end{aligned}$$

where the second inequality comes from

$$\begin{aligned}
2\left\|m^2\right\|_{L^\infty}\|\omega_{xx}\|_{L^p}\|m_x\|_{L^p}^{p-1} &\leq 2A^2\,\|m\|_{L^p}\,\|m_x\|_{L^p}^{p-1} \\
&\leq 2A^2\left(\frac{1}{p}\|m\|_{L^p}^p + \frac{p-1}{p}\|m_x\|_{L^p}^p\right).
\end{aligned}$$

By virtue of (2.9), one can easily deduce that

$$\frac{d}{dt}\int_{\mathbb{R}} |m_x|^p\,dx \leq 2A^2\,\|m\|_{L^p}^p + (2A^2(p-1) + (6p-2)M + \lambda p)\,\|m_x\|_{L^p}^p. \qquad (2.10)$$

Add up (2.7) and (2.10), it follows that

$$\frac{d}{dt}\|m\|_{H^{1,p}}^p \leq c_1\,\|m\|_{H^{1,p}}^p, \qquad (2.11)$$

where $c_1 = \max\{2A^2 + (2p-2)M + \lambda p,\ 2A^2(p-1) + (6p-2)M + \lambda p\}$.
By virtue of Gronwall's inequality, one gets

$$\|m\|_{H^{1,p}}^p \leq e^{c_1 t}\,\|m_0\|_{H^{1,p}}^p. \qquad (2.12)$$

(ii) Let $s=2$, we will estimate $\|m(t,\cdot)\|_{H^{2,p}} < \infty$.
Differentiating Eq. (1.3) twice with respect to x, we have

$$\begin{aligned}
m_{txx} &= -2\omega_{xxx}m^2 - 10\omega_{xx}mm_x - 6\omega_x m_x^2 - 8\omega_x m m_{xx} \\
&\quad - \left(\omega^2-\omega_x^2\right)m_{xxx} - km_{xxx} - \lambda m_{xx}.
\end{aligned} \qquad (2.13)$$

Multiplying Eq. (2.13) by $|m_{xx}|^{p-2} m_{xx}$ with $p \geq 2$, and integrating over $\mathbb{R}$ with respect to x, then we obatin

$$\begin{aligned}
&\int_{\mathbb{R}} m_{txx} |m_{xx}|^{p-2} m_{xx} dx \\
&= -2\int_{\mathbb{R}} \omega_{xxx} m^2 |m_{xx}|^{p-2} m_{xx} dx - 10\int_{\mathbb{R}} \omega_{xx} m m_x |m_{xx}|^{p-2} m_{xx} dx \\
&\quad -6\int_{\mathbb{R}} \omega_x m_x^2 |m_{xx}|^{p-2} m_{xx} dx - 8\int_{\mathbb{R}} \omega_x m m_{xx} |m_{xx}|^{p-2} m_{xx} dx \\
&\quad -\int_{\mathbb{R}} (\omega^2 - \omega_x^2) m_{xxx} |m_{xx}|^{p-2} m_{xx} dx - k\int_{\mathbb{R}} m_{xxx} |m_{xx}|^{p-2} m_{xx} dx \\
&\quad -\lambda\int_{\mathbb{R}} m_{xx} |m_{xx}|^{p-2} m_{xx} dx \\
&\leq 2\left\|m^2\right\|_{L^\infty} \|\omega_{xxx}\|_{L^p} \|m_{xx}\|_{L^p}^{p-1} + 10\|\omega_{xx}\|_{L^\infty} \|m\|_{L^\infty} \|m_x\|_{L^p} \|m_{xx}\|_{L^p}^{p-1} \\
&\quad +6\|\omega_x\|_{L^\infty} \left\|m_x^2\right\|_{L^p} \|m_{xx}\|_{L^p}^{p-1} + \left(8 - \frac{2}{p}\right) M \|m_{xx}\|_{L^p}^p + \lambda \|m_{xx}\|_{L^p}^p \\
&\leq \frac{12A^2}{p} \|m\|_{H^{1,p}}^p + \frac{c_2}{p} \|m\|_{H^{2,p}}^p + \left(\frac{p-1}{p}(12A^2 + c_2) + \left(8 - \frac{2}{p}\right)M + \lambda\right) \|m_{xx}\|_{L^p}^p,
\end{aligned} \tag{2.14}$$

where the constant $c_2 = 6Ae^{\frac{c_1 t}{p}} \|m_0\|_{H^{1,p}}$ and the third inequality comes from

$$\begin{aligned}
&2\left\|m^2\right\|_{L^\infty} \|\omega_{xxx}\|_{L^p} \|m_{xx}\|_{L^p}^{p-1} + 10\|\omega_{xx}\|_{L^\infty} \|m\|_{L^\infty} \|m_x\|_{L^p} \|m_{xx}\|_{L^p}^{p-1} \\
&\quad +6\|\omega_x\|_{L^\infty} \left\|m_x^2\right\|_{L^p} \|m_{xx}\|_{L^p}^{p-1} \\
&\leq 12A^2 \|m\|_{H^{1,p}} \|m_{xx}\|_{L^p}^{p-1} + 6A\|m\|_{H^{1,p}} \|m\|_{H^{2,p}} \|m_{xx}\|_{L^p}^{p-1} \\
&\leq 12A^2 \|m\|_{H^{1,p}} \|m_{xx}\|_{L^p}^{p-1} + c_2 \|m\|_{H^{2,p}} \|m_{xx}\|_{L^p}^{p-1} \\
&\leq 12A^2 \left(\frac{1}{p}\|m\|_{H^{1,p}}^p + \frac{p-1}{p}\|m_{xx}\|_{L^p}^p\right) + c_2\left(\frac{1}{p}\|m\|_{H^{2,p}}^p + \frac{p-1}{p}\|m_{xx}\|_{L^p}^p\right) \\
&\leq \frac{12A^2}{p} \|m\|_{H^{1,p}}^p + \frac{c_2}{p}\|m\|_{H^{2,p}}^p + \frac{p-1}{p}\left(12A^2 + c_2\right) \|m_{xx}\|_{L^p}^p.
\end{aligned} \tag{2.15}$$

Then we have

$$\frac{d}{dt}\int_{\mathbb{R}} |m_{xx}|^p dx \leq 12A^2 \|m\|_{H^{1,p}}^p + c_2 \|m\|_{H^{2,p}}^p + c_3 \|m_{xx}\|_{L^p}^p, \tag{2.16}$$

where the constant $c_3 = (p-1)(12A^2 + c_2) + (8p-2)M + \lambda p$.

Combining (2.11) with (2.16), it yields that

$$\frac{d}{dt}\|m\|_{H^{2,p}}^p \leq (c_1 + c_2 + c_3 + 12A^2) \|m\|_{H^{2,p}}^p. \tag{2.17}$$

Thanks to Gronwall's inequality, one has

$$\|m\|_{H^{2,p}}^{p} \leq e^{(c_1+c_2+c_3+12A^2)t} \|m_0\|_{H^{2,p}}^{p} . \tag{2.18}$$

(iii) Suppose that $\|m(t,\cdot)\|_{H^{s-1,p}} < \infty$, thus we will prove $\|m(t,\cdot)\|_{H^{s,p}} < \infty$. In other words, we only need to prove $\left\|\partial_x^s m(t,\cdot)\right\|_{L^p} < \infty$.

Differentiating Eq. (1.3) with respect to x variable s times, applying the result by $\left|\partial_x^s m\right|^{p-2} \partial_x^s m$ with $p \geq 2$, and integrating by parts, we obtains

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}} \left|\partial_x^s m\right|^p dx \; &= \sum_{i=0}^{s} \int_{\mathbb{R}} C_s^i \partial_x^i (\omega^2 - \omega_x^2) \partial_x^{s-i+1} m \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx \\ &\quad -2\sum_{j=0}^{s} \int_{\mathbb{R}} C_s^j \partial_x^{j+1} \omega \partial_x^{s-j} m^2 \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx \\ &\quad -k \int_{\mathbb{R}} \partial_x^{s+1} m \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx - \lambda \int_{\mathbb{R}} \partial_x^s m \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx.\end{aligned} \tag{2.19}$$

Obviously, one gets

$$k \int_{\mathbb{R}} \partial_x^{s+1} m \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx = 0,$$

and

$$\lambda \int_{\mathbb{R}} \partial_x^s m \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx \leq \lambda \left\|\partial_x^s m\right\|_{L^p}^{p} .$$

Note that in the following inequalities, we applied $\|m\|_{H^{2,p}} < \infty$ and the assumption $\|m\|_{H^{s-1,p}} < \infty$.

As $i = 0$ and $i = 1$, it follows that

$$\begin{aligned}&\int_{\mathbb{R}} \left((\omega^2 - \omega_x^2)\partial_x^{s+1} m + C_s^1 \partial_x (\omega^2 - \omega_x^2) \partial_x^s m\right) \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx \\ &= -\frac{1}{p}\int_{\mathbb{R}} (\omega^2 - \omega_x^2)_x \left|\partial_x^s m\right|^p dx + 2s \int_{\mathbb{R}} \omega_x m \left|\partial_x^s m\right|^p dx \\ &= \left(2s - \frac{2}{p}\right) \int_{\mathbb{R}} m\omega_x \left|\partial_x^s m\right|^p dx \\ &\leq c\, \|m\omega_x\|_{L^\infty} \left\|\partial_x^s m\right\|_{L^p}^{p} \\ &\leq c \left\|\partial_x^s m\right\|_{L^p}^{p} .\end{aligned}$$

When $i = s$, we have

$$
\begin{aligned}
\int_{\mathbb{R}} C_s^s \partial_x^s(\omega^2-\omega_x^2)\partial_x m \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx &= \int_{\mathbb{R}} \partial_x^{s-1}(2\omega_x m)\partial_x m \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx \\
&\le c \left\|\partial_x m\right\|_{L^\infty} \left\|\partial_x^{s-1}(\omega_x m)\right\|_{L^p} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\
&\le c \left\|m\right\|_{H^{2,p}} \left\|\omega_x m\right\|_{H^{s-1,p}} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\
&\le c \left\|\partial_x^s m\right\|_{L^p}^{p},
\end{aligned}
$$

where

$$
\begin{aligned}
\left\|\omega_x m\right\|_{H^{s-1,p}} &\le \left\|\omega_x\right\|_{H^{s-1,p}} \left\|m\right\|_{L^\infty} + \left\|\omega_x\right\|_{L^\infty} \left\|m\right\|_{H^{s-1,p}} \\
&\le \left\|m\right\|_{H^{s-2,p}} \left\|m\right\|_{H^{1,p}} + \left\|m\right\|_{L^p} \left\|m\right\|_{H^{s-1,p}} < \infty.
\end{aligned}
$$

For $2 \le i \le s-1,\ i \in \mathbb{N}$, it yields that

$$
\begin{aligned}
&\int_{\mathbb{R}} C_s^i \partial_x^i(\omega^2-\omega_x^2)\partial_x^{s-i+1} m \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx \\
&\quad \le c \left\|\partial_x^i(\omega^2-\omega_x^2)\right\|_{L^\infty} \left\|\partial_x^{s-i+1} m\right\|_{L^p} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\
&\quad \le c \left\|\omega^2-\omega_x^2\right\|_{H^{s,p}} \left\|m\right\|_{H^{s-1,p}} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\
&\quad \le c \left\|\partial_x^s m\right\|_{L^p}^{p},
\end{aligned}
$$

where the above inequality comes from

$$
\begin{aligned}
\left\|\omega^2-\omega_x^2\right\|_{H^{s,p}} &\le \left\|(\omega+\omega_x)(\omega-\omega_x)\right\|_{H^{s,p}} \\
&\le \left\|\omega+\omega_x\right\|_{H^{s,p}} \left\|\omega-\omega_x\right\|_{L^\infty} + \left\|\omega+\omega_x\right\|_{L^\infty} \left\|\omega-\omega_x\right\|_{H^{s,p}} < \infty.
\end{aligned}
$$

When $j = 0$, one shows that

$$
\begin{aligned}
2\int_{\mathbb{R}} \partial_x \omega \partial_x^s m^2 \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx &\le c \int_{\mathbb{R}} \partial_x \omega \partial_x^{s-1}(2mm_x) \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx \\
&\le c \left\|\partial_x \omega\right\|_{L^\infty} \left\|\partial_x^{s-1}(mm_x)\right\|_{L^p} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\
&\le c \left\|m\right\|_{L^p} \left\|mm_x\right\|_{H^{s-1,p}} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\
&\le c \left\|\partial_x^s m\right\|_{L^p}^{p},
\end{aligned}
$$

where

$$
\begin{aligned}
\left\|mm_x\right\|_{H^{s-1,p}} &\le \left\|m\right\|_{H^{s-1,p}} \left\|m_x\right\|_{L^\infty} + \left\|m\right\|_{L^\infty} \left\|m_x\right\|_{H^{s-1,p}} \\
&\le \left\|m\right\|_{H^{s-1,p}} \left\|m\right\|_{H^{2,p}} + \left\|m\right\|_{H^{1,p}} \left\|m\right\|_{H^{s,p}} < \infty.
\end{aligned}
$$

As $j = 1$, one gets that

$$\begin{aligned} 2\int_{\mathbb{R}} C_s^1 \partial_x^2 \omega \partial_x^{s-1} m^2 \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx &\leq c \left\|\partial_x^2 \omega\right\|_{L^\infty} \left\|\partial_x^{s-1} m^2\right\|_{L^p} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\ &\leq c \left\|m\right\|_{H^{1,p}} \left\|m^2\right\|_{H^{s-1,p}} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\ &\leq c \left\|m\right\|_{H^{1,p}} \left\|m\right\|_{L^\infty} \left\|m\right\|_{H^{s-1,p}} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\ &\leq c \left\|\partial_x^s m\right\|_{L^p}^{p}. \end{aligned}$$

When $j = s$, it follows that

$$\begin{aligned} 2\int_{\mathbb{R}} C_s^s \partial_x^{s+1} \omega m^2 \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx &\leq c \left\|m^2\right\|_{L^\infty} \left\|\partial_x^{s+1} \omega\right\|_{L^p} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\ &\leq c \left\|m^2\right\|_{H^{1,p}} \left\|m\right\|_{H^{s-1,p}} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\ &\leq c \left\|\partial_x^s m\right\|_{L^p}^{p}. \end{aligned}$$

For $2 \leq j \leq s-1,\ j \in \mathbb{N}$, we have

$$\begin{aligned} 2\int_{\mathbb{R}} C_s^j \partial_x^{j+1} \omega \partial_x^{s-j} m^2 \left|\partial_x^s m\right|^{p-2} \partial_x^s m dx &\leq c \left\|\partial_x^{j+1} \omega\right\|_{L^\infty} \left\|\partial_x^{s-j} m^2\right\|_{L^p} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\ &\leq c \left\|m\right\|_{H^{s-1,p}} \left\|m^2\right\|_{H^{s-2,p}} \left\|\partial_x^s m\right\|_{L^p}^{p-1} \\ &\leq c \left\|\partial_x^s m\right\|_{L^p}^{p}. \end{aligned}$$

Plugging the above inequalities into (2.19) yields that

$$\frac{d}{dt} \left\|\partial_x^s m\right\|_{L^p}^{p} \leq c \left\|\partial_x^s m\right\|_{L^p}^{p}. \tag{2.20}$$

In view of Gronwall's inequality, there exists a constant $c(M,\ p,\ s,\ \lambda) > 0$ such that

$$\left\|\partial_x^s m\right\|_{L^p}^{p} \leq e^{ct} \left\|\partial_x^s m_0\right\|_{L^p}^{p}. \tag{2.21}$$

In summary, we have completed the proof of Theorem 2.1.

Remark 2.1 Theorem 2.1 shows that the dispersion coefficient k and the dissipation parameter λ have no effect on the blow-up criterion of solution of Eq. (1.3). That is to say, when $k = \lambda = 0$, Theorem 2.1 also holds.

Lemma 2.5 *Let $m_0 \in H^{s,p}(\mathbb{R})$, $s \geq 1$, $p \in (1, \infty)$. Then as long as the solution $\omega(t)$ given by Lemma 2.1 exists for any $t \in [0, T)$, we have*

$$\|\omega(t)\|_{H^1}^2 = e^{-2\lambda t} \|\omega_0\|_{H^1}^2.$$

Proof. Multiplying Eq. (1.3) by ω and integration by parts, it yields that

$$\int_{\mathbb{R}} \omega m_t dx + \int_{\mathbb{R}} \omega \left(\omega^2 - \omega_x^2\right) m_x dx \\ + 2\int_{\mathbb{R}} \omega\omega_x m^2 dx + k\int_{\mathbb{R}} \omega m_x dx + \lambda \int_{\mathbb{R}} \omega m dx = 0.$$

Owning to

$$k\int_{\mathbb{R}} \omega m_x dx = 0,$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} \omega \left(\omega^2 - \omega_x^2\right) m_x dx + 2\int_{\mathbb{R}} \omega\omega_x m^2 dx \\
&= \int_{\mathbb{R}} \omega \left(\omega^2 - \omega_x^2\right) m_x dx + \int_{\mathbb{R}} \omega \left(\omega^2 - \omega_x^2\right)_x m dx \\
&= -\int_{\mathbb{R}} \omega_x \left(\omega^2 - \omega_x^2\right) (\omega - \omega_{xx})\, dx \\
&= -\int_{\mathbb{R}} \left(\omega^3\omega_x - \omega^2\omega_x\omega_{xx} - \omega\omega_x^2 + \omega_x^3\omega_{xx}\right) dx \\
&= -\int_{\mathbb{R}} \left(\omega^3\omega_x + \omega\omega_x^2 - \omega\omega_x^2 + \omega_x^3\omega_{xx}\right) dx \\
&= 0,
\end{aligned}$$

one can easily check that

$$\begin{aligned}
\int_{\mathbb{R}} \omega m_t dx + \lambda \int_{\mathbb{R}} \omega m dx &= \int_{\mathbb{R}} \omega(\omega_t - \omega_{txx}) dx + \lambda \int_{\mathbb{R}} \omega(\omega - \omega_{xx}) dx \\
&= \int_{\mathbb{R}} (\omega\omega_t + \omega_t\omega_{tx}) dx + \lambda \int_{\mathbb{R}} (\omega^2 + \omega_x^2) dx \\
&= \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} (\omega^2 + \omega_x^2) dx + \lambda \int_{\mathbb{R}} (\omega^2 + \omega_x^2) dx \\
&= \frac{1}{2}\frac{d}{dt} \|\omega(t)\|_{H^1}^2 + \lambda \|\omega(t)\|_{H^1}^2 = 0.
\end{aligned}$$

Applying Gronwall's inequality, we have

$$\|\omega(t)\|_{H^1}^2 = e^{-2\lambda t} \|\omega_0\|_{H^1}^2 .$$

This completes the proof of Lemma 2.5.

3 Global existence of solution

In this section, we provide global existence result for strong solutions to Eq. (1.3).

Theorem 3.1 *Let $m_0 \in H^{s,p}(\mathbb{R})$, $s \geq 1$, $p \in (1, \infty)$, and $m_0 = \omega_0 - \omega_{0,xx}$. If $m_0(x) \neq 0$, $x \in \mathbb{R}$ and $\|m_0\|_{H^1} \leq (\frac{2\lambda}{c})^{\frac{1}{2}}$, then the solution $m(t, x)$ of Eq. (1.3) globally exists in time.*

Proof. Multiplying Eq. (1.3) by m, and integrating over $\mathbb{R}$ with respect to x, integration by parts, then we have

$$\frac{d}{dt}\int_{\mathbb{R}} m^2 dx = -2\int_{\mathbb{R}} \omega_x m^3 dx - 2\lambda \int_{\mathbb{R}} m^2 dx. \tag{3.1}$$

Applying Eq. (2.8) by m_x, and integrating over $\mathbb{R}$ with respect to x, integration by parts leads to

$$\frac{d}{dt}\int_{\mathbb{R}} m_x^2 dx = \frac{4}{3}\int_{\mathbb{R}} \omega_x m^3 dx - 10\int_{\mathbb{R}} \omega_x m m_x^2 dx - 2\lambda \int_{\mathbb{R}} m_x^2 dx. \tag{3.2}$$

Add up (3.1) and (3.2), it yields that

$$\frac{d}{dt}\int_{\mathbb{R}} (m^2 + m_x^2) dx = -\frac{2}{3}\int_{\mathbb{R}} \omega_x m^3 dx - 10\int_{\mathbb{R}} \omega_x m m_x^2 dx - 2\lambda \int_{\mathbb{R}} (m^2 + m_x^2) dx. \tag{3.3}$$

Applying both side of Eq. (3.3) by $e^{2\lambda t}$, one gets

$$\frac{d}{dt}\left(e^{2\lambda t}\int_{\mathbb{R}} (m^2 + m_x^2) dx\right) = -\frac{2}{3}e^{2\lambda t}\int_{\mathbb{R}} \omega_x m^3 dx - 10e^{2\lambda t}\int_{\mathbb{R}} \omega_x m m_x^2 dx. \tag{3.4}$$

By using Sobolev embedding theorem $H^1 \hookrightarrow L^\infty$, we have

$$\|m\|_{L^\infty} \leq c\,\|m\|_{H^1}, \tag{3.5}$$

and

$$\|m\omega_x\|_{L^\infty} \leq \|m\|_{L^\infty}\,\|\omega_x\|_{L^\infty} \leq c\,\|m\|_{L^\infty}\,\|m\|_{L^\infty} \leq c\,\|m\|_{H^1}^2, \tag{3.6}$$

where constant $c > 0$ and (3.6) comes from

$$\|\omega_x\|_{L^\infty} \leq \|G_x\|_{L^1}\,\|m\|_{L^\infty} \leq \|m\|_{L^\infty}.$$

So, plugging (3.5) and (3.6) into (3.4), it follows that

$$\begin{aligned}\frac{d}{dt}\left(e^{2\lambda t}\int_{\mathbb{R}}(m^2+m_x^2)dx\right) &\le ce^{2\lambda t}\,\|m\|_{H^1}^2\int_{\mathbb{R}}(m^2+m_x^2)dx\\ &= ce^{-2\lambda t}\left(e^{2\lambda t}\int_{\mathbb{R}}(m^2+m_x^2)dx\right)^2.\end{aligned} \tag{3.7}$$

Set $g(t) \doteq e^{2\lambda t}\int_{\mathbb{R}}(m^2+m_x^2)dx$. For all $t \in \mathbb{R}$ and $m_0(x) \neq 0,\ x \in \mathbb{R}$, then we have $g(t) > 0$. If $m_0(x) \neq 0,\ x \in \mathbb{R}$, then $m(t, \rho(t, x)) \neq 0,\ x \in \mathbb{R}$ from (2.5). Obviously, $g(t) > 0$.

Replacing $e^{2\lambda t}\int_{\mathbb{R}}(m^2+m_x^2)dx$ in (3.7) with $g(t)$, one obtains

$$\frac{d}{dt}(g(t)) \le ce^{-2\lambda t}(g(t))^2. \tag{3.8}$$

Solving the above equation implies that

$$\frac{d}{dt}(g(t)^{-1}) \ge -ce^{-2\lambda t}. \tag{3.9}$$

Integrating Eq. (3.9) with respect to t, we can derive

$$g(t)^{-1} - g(0)^{-1} \ge -c\int_0^t e^{-2\lambda s}ds = \frac{c(e^{-2\lambda t}-1)}{2\lambda},$$

namely, it follows that

$$g(t)^{-1} \ge g(0)^{-1} - \frac{c}{2\lambda} \ge 0,$$

where $g(0)^{-1} - \frac{c}{2\lambda} \ge 0$ is guaranteed by the assumption $\|m_0\|_{H^1} \le (\frac{2\lambda}{c})^{\frac{1}{2}}$.

In addition, we have

$$\left[g(0)^{-1} - \frac{c}{2\lambda}\right]^{-1} \ge g(t),$$

i.e.,

$$\left[\left(\int_{\mathbb{R}}(m_0^2+m_{0,x}^2)dx\right)^{-1} - \frac{c}{2\lambda}\right]^{-1} \ge e^{2\lambda t}\int_{\mathbb{R}}(m^2+m_x^2)dx.$$

Thus, it yields that

$$\|m\|_{H^1}^2 \le e^{-2\lambda t}\left[\|m_0\|_{H^1}^{-2} - \frac{c}{2\lambda}\right]^{-1}. \tag{3.10}$$

Combining (3.5), (3.6) with (3.10), we can easily obtain that m and $m\omega_x$ are bounded, namely,

$$\|m\|_{L^\infty} \le c\,\|m\|_{H^1} \le e^{-\lambda t}\left[\|m_0\|_{H^1}^{-2} - \frac{c}{2\lambda}\right]^{-\frac{1}{2}},$$
$$\|m\omega_x\|_{L^\infty} \le c\,\|m\|_{H^1}^2 \le e^{-2\lambda t}\left[\|m_0\|_{H^1}^{-2} - \frac{c}{2\lambda}\right]^{-1}.$$

Therefore, we can obtain the solution $m(t,x)$ of Eq. (1.3) globally exists in time. This completes the proof of Theorem 3.1.

Remark 3.1 From Theorem 3.1, one can easily check that the dissipation term λm affects the global existence of the strong solution of Eq. (1.3), however the dispersion term km_x does not affect the global solution.

4 Propagation speed

The effect of the dispersion coefficient k and the dissipation parameter λ on the propagation speed of the strong solutions to Eq. (1.3) will be examined in this section.

Theorem 4.1 *Let $m_0 \in H^{s,p}(\mathbb{R})$, $s \ge 1$, $p \in (1,\infty)$. The maximal existence time that the solution $\omega(t,x)$ to Eq. (1.3) with the initial data ω_0 can exist is given by $T = T(\omega_0)$. If the initial data ω_0 are compactly supported in $[a_{\omega_0}, b_{\omega_0}]$, for all $t \in [0,T)$, then we have*

$$\omega(t,x) = \begin{cases} \frac{1}{2}e^{-x}E^+(t), & \text{as} \quad x \ge q(t,b), \\ \frac{1}{2}e^{x}E^-(t), & \text{as} \quad x < q(t,a), \end{cases} \tag{4.1}$$

where the compact support $[q(t,a), q(t,b)]$ of $m(t,\cdot)$ is contained in the interval $[q(t,a_{\omega_0}), q(t,b_{\omega_0})]$. Furthermore, if the initial potential $m_0 = \omega_0 - \omega_{0,xx}$ does not change sign on $\mathbb{R}$ and for any $t \in [0,T)$ the solution $\omega \not\equiv 0$, then for $m_0 \ge 0$, we have $E^+(t) > 0$ and $E^-(t) < 0$; on the contrary, for $m_0 \le 0$, one obtains $E^+(t) < 0$ and $E^-(t) > 0$, where $E^+(t)$, $E^-(t)$ are continuous non-vanishing functions, and $E^+(0) = E^-(0) = 0$.

Proof. If ω_0 is compactly supported in the closed interval $[a_{\omega_0}, b_{\omega_0}]$, then $m(t,\cdot)$ has compact support with its support contained in the interval $[q(t,a), q(t,b)] \subseteq [q(t,a_{\omega_0}), q(t,b_{\omega_0})]$ according to Lemma 2.3. We define the following two necessary functions:

$$E^+(t) = \int_{q(t,a)}^{q(t,b)} e^y m(t,y)dy, \quad E^-(t) = \int_{q(t,a)}^{q(t,b)} e^{-y} m(t,y)dy. \tag{4.2}$$

In view of $\omega(t,x) = G * m = \frac{1}{2}e^{-|x|} * m$, it yields that

$$\omega(t,x) = \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}m(t,y)dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}m(t,y)dy. \tag{4.3}$$

Due to (4.2) and (4.3), we can derive

$$\omega(t,x) = \begin{cases} \frac{1}{2}e^{-x}E^{+}(t) = \frac{1}{2}e^{-x}\int_{q(t,a)}^{q(t,b)} e^{y}m(t,y)dy, & \text{if} \quad x \geq q(t,b), \\ \frac{1}{2}e^{x}E^{-}(t) = \frac{1}{2}e^{x}\int_{q(t,a)}^{q(t,b)} e^{-y}m(t,y)dy, & \text{if} \quad x < q(t,a). \end{cases} \tag{4.4}$$

Combining (4.2) with (4.4), we have

$$\begin{cases} \omega(t,x) = -\omega_x(t,x) = \omega_{xx}(t,x) = \frac{1}{2}e^{-x}E^{+}(t), & \text{as} \quad x \geq q(t,b), \\ \omega(t,x) = \omega_x(t,x) = \omega_{xx}(t,x) = \frac{1}{2}e^{x}E^{-}(t), & \text{as} \quad x < q(t,a). \end{cases} \tag{4.5}$$

Apparently, $\omega_0(x)$ is compactly supported in the interval $[a_{\omega_0}, b_{\omega_0}]$, with $E^{+}(0) = E^{-}(0) = 0$. Thanks to

$$\begin{aligned} E^{-}(0) &= \int_{q(t,a)}^{q(t,b)} e^{-y}m_0(y)dy = \int_{\mathbb{R}} e^{-x}m_0(x)dx \\ &= \int_{\mathbb{R}} e^{-x}(\omega_0(x) - \omega_{0,xx}(x))dx \\ &= \int_{\mathbb{R}} e^{-x}\omega_0(x)dx - \int_{\mathbb{R}} e^{-x}\omega_0(x)dx = 0. \end{aligned}$$

Since $m(t,\cdot)$ is compactly support in the interval $[q(t,a), q(t,b)]$ and $\omega(t,x) = \omega_x(t,x)$ as $x < q(t,a)$. Differentiating Eq. (4.2) with respect to t, combining with Eq. (1.3) yields

$$\begin{aligned} \frac{d}{dt}E^{-}(t) &= \int_{q(t,a)}^{q(t,b)} e^{-y}m_t(t,y)dy = \int_{\mathbb{R}} e^{-y}m_t(t,y)dy \\ &= -\int_{\mathbb{R}} \left[\left(\omega^2 - \omega_x^2\right)m_x + 2\omega_x m^2 + km_x + \lambda m\right]e^{-x}dx \\ &= -\int_{\mathbb{R}} \left[\left(\omega^2 - \omega_x^2\right)m\right]_x e^{-x}dx - (k+\lambda)\int_{\mathbb{R}} me^{-x}dx. \end{aligned} \tag{4.6}$$

It follows from (4.6) that

$$\frac{d}{dt}E^{-}(t) + (k+\lambda)E^{-}(t) = f, \tag{4.7}$$

where $f \doteq -\int_{\mathbb{R}} \left[\left(\omega^2-\omega_x^2\right)m\right]_x e^{-x}dx$. We can easily check that

$$\begin{aligned} f &\doteq -\int_{\mathbb{R}} \left[\left(\omega^2-\omega_x^2\right)m\right]_x e^{-x}dx \\ &= -\int_{\mathbb{R}} \left(\omega^2-\omega_x^2\right)me^{-x}dx \\ &= -\int_{\mathbb{R}} \left(\omega^3-\omega^2\omega_{xx}-\omega\omega_x^2+\omega_x^2\omega_{xx}\right)e^{-x}dx \\ &= -\int_{\mathbb{R}} \left(\frac{2}{3}\omega^3+\omega\omega_x^2+\frac{1}{3}\omega_x^3\right)e^{-x}dx. \end{aligned} \tag{4.8}$$

Since m_0 does not change sign on $\mathbb{R}$, we have

$$\begin{cases} 0 \le |\omega_x| \le \omega, & \text{if} \quad m_0 \ge 0, \\ 0 \le |\omega_x| \le -\omega, & \text{if} \quad m_0 \le 0. \end{cases} \tag{4.9}$$

Now we consider the case $m_0 \ge 0$. As $\omega_x \ge 0$, then $0 \le \omega_x \le \omega$, we can easily get

$$\frac{2}{3}\omega^3+\omega\omega_x^2+\frac{1}{3}\omega_x^3 \ge 0,$$

otherwise, as $\omega_x \le 0$, then $0 \le -\omega_x \le \omega$, i.e., $\omega^3+\omega_x^3 \ge 0$, one has

$$\frac{2}{3}\omega^3+\omega\omega_x^2+\frac{1}{3}\omega_x^3 = \frac{1}{3}\omega^3+\frac{1}{3}(\omega^3+\omega_x^3)+\omega\omega_x^2 \ge 0.$$

Combining the above two inequalities and (4.8), it follows that

$$f = -\int_{\mathbb{R}} \left(\frac{2}{3}\omega^3+\omega\omega_x^2+\frac{1}{3}\omega_x^3\right)e^{-x}dx \le 0, \tag{4.10}$$

which means

$$\frac{d}{dt}E^-(t)+(k+\lambda)E^-(t) \le 0, \tag{4.11}$$

namely,

$$\frac{d}{dt}\left[e^{(k+\lambda)t}E^-(t)\right] \le 0, \tag{4.12}$$

then from (4.12) we obtain

$$e^{(k+\lambda)t}E^-(t) \le 0. \tag{4.13}$$

Thus it follows that $E^-(t) \le 0$ for the case $m_0 \ge 0$. Analogous to the above process, if $m_0 \le 0$, we can get $E^-(t) \ge 0$.

Obviously, similar to (4.6), it follows that

$$\begin{aligned} \frac{d}{dt}E^+(t) &= \int_{q(t,a)}^{q(t,b)} e^y m(t,y)dy = \int_{\mathbb{R}} e^y m_t(t,y)dy \\ &= -\int_{\mathbb{R}} \left[\left(\omega^2 - \omega_x^2\right) m\right]_x e^x dx - (\lambda - k)\int_{\mathbb{R}} m e^x dx, \end{aligned} \tag{4.14}$$

From the above relation (4.14), we deduce

$$\frac{d}{dt}E^+(t) + (\lambda - k)E^+(t) = g, \tag{4.15}$$

where $g \doteq -\int_{\mathbb{R}} \left[\left(\omega^2 - \omega_x^2\right) m\right]_x e^x dx$. Thus one can get

$$\begin{aligned} g &\doteq -\int_{\mathbb{R}} \left[\left(\omega^2 - \omega_x^2\right) m\right]_x e^x dx \\ &= -\int_{\mathbb{R}} \left(\omega^2 - \omega_x^2\right) m e^x dx \\ &= -\int_{\mathbb{R}} \left(\omega^3 - \omega^2\omega_{xx} - \omega\omega_x^2 + \omega_x^2\omega_{xx}\right) e^x dx \\ &= -\int_{\mathbb{R}} \left(\frac{2}{3}\omega^3 + \omega\omega_x^2 - \frac{1}{3}\omega_x^3\right) e^x dx. \end{aligned} \tag{4.16}$$

When $m_0 \geq 0$, one has $g \leq 0$, similar to (4.10–4.13), it yields that $E^+(t) \geq 0$. For the same reason, as $m_0 \leq 0$, we can get $E^+(t) \leq 0$. In summary,

$$\begin{cases} E^+(t) \geq 0, \; E^-(t) \leq 0, & \text{if} \quad m_0 \geq 0, \\ E^+(t) \leq 0, \; E^-(t) \geq 0, & \text{if} \quad m_0 \leq 0. \end{cases} \tag{4.17}$$

This completes the proof of Theorem 4.1.

Remark 4.1 From the proof of Theorem 4.1, it follows that the dispersion coefficient k and the dissipation parameter λ have an effect on the propagation speed of the solution to Eq. (1.3). When $k = \lambda = 0$ in [30], $E^+(t)$ and $E^-(t)$ are both non-vanishing functions. For $m_0 \geq 0$, $E^+(t)$ is strictly increasing function and $E^-(t)$ is strictly decreasing function; conversely, for $m_0 \leq 0$, $E^+(t)$ is strictly decreasing function and $E^-(t)$ is strictly increasing function.

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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