Monatshefte für Mathematik (2024) 204:311–322
https://doi.org/10.1007/s00605-024-01961-3

A measurable spectral decomposition

Bomi Shin[1]

Received: 12 June 2023 / Accepted: 19 February 2024 / Published online: 5 April 2024

Abstract
We introduce the spectral decomposition property for measures and prove that a homeomorphism has the spectral decomposition property if and only if every Borel probability measure has the property too. Furthermore, we show that all shadowable measures for expansive homeomorphisms have the spectral decomposition property. Additionally, we provide illustrative examples relevant to these results.

Keywords Expansive homeomorphism · Probability measure · Shadowable measure · Spectral decomposition property · Compact metric space

Mathematics Subject Classification Primary 37B65; Secondary 28C15

1 Introduction

One of the purpose of the dynamical system is to determinate the asymptotic behavior of the orbits. Since the nonwandering set is the ultimate concept of these orbits, it is important to determinate the structure of the set. A relevant property is if the nonwandering set can be decomposed into finitely many disjoint indecomposable pieces. Inspired from operator theory [10], this is often called *spectral decomposition property*, or SDP for short. SDT in dynamical system has been studied elsewhere in the literature. Let us present some well-known results.

In 1967, Smale [20] proved that any Axiom A diffeomorphism on a compact manifold has the SDP. Bowen improved it in 1975 [4]. In 1983, Aoki [1] proved, on

Communicated by H. Bruin.

This work was supported by the National Research Foundation of Korea(NRF) Grant funded by the Korean Government(MSIP) (NRF-2021R1C1C2005241). We also would like to thank anonymous reviewers for valuable comments and careful reading of the manuscript.

✉ Bomi Shin
bmshin@skku.edu

1 Institute of Basic Science, Sungkyunkwan University, Suwon 16419, Republic of Korea

compact metric spaces, that expansive homeomorphisms with the shadowing property on the nonwandering set have the SDP. It was extended to flows by Komuro [13]. In 2008, Oprocha [19] obtained the SDP for expansive $\mathbb{Z}^d$-actions with a certain type of shadowing property. In 2009, Choi and Kim [6] obtained SDP on G-spaces for G-expansive homeomorphisms with the G-shadowing property. They focused on a G-space, defined as a space with a continuous action of a topological group G, and proved the decomposition theorem in the context of G-action dynamics. In 2013, Das et al. [9] obtained SDP for topologically Anosov homeomorphisms on first countable, locally compact, paracompact, Hausdorff topological spaces. This was improved in 2018 by Das et al. [8]. In 2014, Kim and Lee [12] proved a SDP for expansive $\mathbb{Z}^2$-actions with shadowing on compact metric spaces. In 2016, Thakkar and Das [22] proved the SDP for nonautonomous equicontinuous time varying homeomorphisms with shadowing on compact metric spaces. In 2017, Cordeiro et al.[7] obtained the SDP for strong measure expansive homeomorphisms with the shadowing property on compact metric spaces. In 2018, Lee et al.[17] extended the notions of shadowing and expansivity to noncompact metric spaces to obtain SDP for homeomorphisms on locally compact metric spaces. Furthermore, Jung et al. [11] obtained the SDP for rescaling expansive flows with the shadowing property on locally compact metric spaces. More recently, Le et al.[14] introduced the mild expansiveness and demonstrated that mild expansive homeomorphisms with the shadowing property has the SDP. In 2020, Artigue et al.[3] introduced the L-shadowing property and proved that the chain recurrent classes of homeomorphisms with the L-shadowing property has the SDP into pieces exhibiting the two-sided limit shadowing property. All the versions of the SDP mentioned above are derived from the classical shadowing property and variations of expansiveness.

In this paper, we extend the spectral decomposition property for homeomorphisms to measures. Unlike the aforementioned studies, the spectral decomposition property presented in this work arises from measurable variation of the shadowing property and classical expansiveness. We prove that a homeomorphism has the spectral decomposition property if and only if every Borel probability measure also has this property. Furthermore, we demonstrate that every shadowable measure of an expansive homeomorphism in a compact metric space has the spectral decomposition property. Additionally, we provide some illustrative examples related to these results. This paper is organized as follows. First, we present main results below in this section. In Sect. 2, we provide some examples related to the spectral decomposition for measures. Finally, in Sect. 3, we prove our results.

Consider a compact metric space X, and $f : X \to X$ be a homeomorphism. The *nonwandwering set* $\Omega(f)$ of f is the set of points $x \in X$ such that

$$U \cap \bigcup_{n=1}^{\infty} f^n(U) \neq \emptyset$$

for every neighborhood U of x. We say that $C \subseteq X$ is *invariant* if $f(C) = C$. Moreover, C is *chain transitive* if for all $a, b \in C$ and $\epsilon > 0$, there are $x_1, \ldots, x_n \in X$ such that $x_1 = a$, $x_n = b$ and $d(f(x_i), x_{i+1}) < \epsilon$, for every $1 \leq i \leq n-1$.

Definition 1.1 We say that f *has the spectral decomposition property* (abbrev. *SDP*) if there are finitely many disjoint chain transitive compact invariant sets $C_1, \dots, C_l$ such that

$$\Omega(f) = \bigcup_{i=1}^{l} C_i. \tag{1}$$

It is important to determine when a given homeomorphism of a compact metric space has the SDP. As previously mentioned, this is the case for Axiom A diffeomorphisms of a closed manifold or for expansive, or more generally, for strong measure expansive homeomorphisms with the shadowing property. On the other hand, there are homeomorphisms without this property, such as the identity of an infinite metric space or even the N-expansive homeomorphisms with the shadowing property as discussed in [5].

In our first result, we provide a measure-theoretical characterization of the SDP for homeomorphisms: A homeomorphism of a compact metric space has the SDP if and only if every Borel probability measure does. Recall that a *Borel probability measure* is a σ-finite measure μ of X defined on the Borel σ-algebra of X such that $\mu(X) = 1$.

Theorem 1.2 *A homeomorphism of a compact metric space $f : X \to X$ has the spectral decomposition property if and only if for every Borel probability measure μ of X there are finitely many disjoint chain transitive compact invariant sets $C_1, \dots, C_l \subset \Omega(f)$ such that*

$$\mu(\Omega(f)) = \sum_{i=1}^{l} \mu(C_i). \tag{2}$$

This theorem suggests the study of those measures μ for which there are a finite collection of chain transitive sets $C_1, \dots, C_l \subset \Omega(f)$ satisfying (2). Note that we can assume, without loss of generality, that $\mu(C_i) \neq 0$ for all $i = 1, \dots, l$. More precisely, we consider measures as defined in the following.

Definition 1.3 Let $f : X \to X$ be a homeomorphism of a compact metric space. We say that a Borel probability measure μ of X has the *spectral decomposition property* (abbrev. *SDP*) if there are finitely many chain transitive sets $C_1, \dots, C_l \subset \Omega(f)$ such that

$$\mu(\Omega(f)) = \sum_{i=1}^{l} \mu(C_i).$$

In such a case, we say that f has the *μ-spectral decomposition property* (abbrev. μ-SDP).

This highlights the significance of measures with the spectral decomposition property in the study of homeomorphisms of compact metric spaces. We will present some pertinent examples in the next section. In the meantime, we will establish sufficient conditions for a Borel probability measure to possess this property. So, we introduce some fundamental concepts.

Consider a compact metric space X and a homeomorphism $f : X \to X$. We say f is *expansive* if there is δ, called *expansivity constant*, such that $\Gamma_\delta(x) = \{x\}$

for all $x \in X$ where $\Gamma_\delta(x) = \{y \in X : d(f^i(x), f^i(y) \leq \delta, \quad \forall i \in \mathbb{Z}\}$. Now, consider a Borel probability measure μ of X. We say that μ is *shadowable* if for every $\epsilon > 0$, there exist $\delta > 0$ and a Borelian set B with $\mu(X \setminus B) = 0$ which is (ϵ, δ)-*shadowable*. In other words, for every bi-infinite sequence $(x_i)_{i\in\mathbb{Z}}$ satisfying $x_0 \in B$ and $d(f(x_i), x_{i+1}) \leq \delta$, there is $\xi \in X$ such that $d(f^i(\xi), x_i) \leq \epsilon$ for all $i \in \mathbb{Z}$.

With these definitions we can state our second result.

Theorem 1.4 *Every shadowable measure of an expansive homeomorphism of a compact metric space has the spectral decomposition property.*

In particular, these theorems can be used to obtain a probabilistic proof of the well known spectral decomposition theorem below.

Corollary 1.5 *Every expansive homeomorphism with the shadowing property of a compact metric space has the spectral decomposition theorem.*

Proof Due to the shadowing property, we have that every Borel probability measure is shadowable. Since the homeomorphism is expansive, every Borel probability measure has the SDP as proven in Theorem 1.4. Consequently, the homeomorphism also has the SDP by Theorem 1.2. □

A natural question arising from our results is whether a property of a homeomorphism can be extended to the context of flows. In this regard, we can cite the work of Lee and Nguyen [16], where the SDP was established for invariantly measure expanding flows with the invariant shadowing property on the chain recurrent set.

2 Examples

In this section, we present some relevant examples related to measures with the spectral decomposition property. Recall that for every $x, y \in X$ and $\delta > 0$, we say that $x \sim_\delta y$ if there are $x_0, \ldots, x_l$ such that $x_0 = x$, $x_l = y$, and $d(f(x_i), x_{i+1}) \leq \delta$ for every $0 \leq i \leq l-1$. We say that $x \sim y$ if $x \sim_\delta y$ for every $\delta > 0$. We define the *chain recurrent set* by $CR(f) = \{x \in X : x \sim x\}$. The relation $\sim$ is an equivalent relation when restricted to $CR(f)$, and the equivalent classes of this relation are called the *chain recurrent classes* of f. It turns out that $CR(f) = \bigcup_{i\in\Lambda} C_i$ where $\{C_i : i \in \Lambda\}$ is the set of chain recurrent classes of f. We also note that every chain recurrent class is chain transitive. Cleary, $\Omega(f) \subseteq CR(f)$.

The spectral decomposition property for measures is related to the chain recurrent set, as demonstrated in the following example. For every $p \in X$, denote by δ_p the Dirac measure supported on $p \in X$.

Example 2.1 Let $f : X \to X$ be a homeomorphism of a compact metric space. If $p \in \Omega(f) \cup (X \setminus CR(f))$, then δ_p has the SDP.

Proof First, suppose $p \in \Omega(f)$. Since $\Omega(f) \subset CR(f)$, $p \in CR(f)$ and so the chain recurrent class C_p containing p is well defined. We have that C_p is a chain transitive compact invariant set of f. Also $\delta_p(\Omega(f)) = 1 = \delta_p(C_p)$ so δ_p has the

SDP with respect to f (by taking $l = 1$ and $C_1 = C_p$ in the corresponding definition). Second, suppose that $p \notin CR(f)$. Take any chain recurrent class C intersecting $\Omega(f)$, so $\delta_p(\Omega(f)) = 0 = \delta_p(C \cap \Omega(f))$. Thus, δ_p has the SDP with $l = 1$ and $C_1 = C \cap \Omega(f)$. □

The converse in the above example is false, by the following counterexample.

Example 2.2 There exists a homeomorphism of a compact metric space $f : X \to X$ and a point $p \in X$ such that δ_p has the SDP, but $p \notin \Omega(f) \cup (X \setminus CR(f))$.

Proof Let $f_1 : [0, 1] \to [0, 1]$ be defined by $f(x) = x^2$. By identifying $0 \sim 1$, we obtain a homeomorphism still denoted by $f : S \to S$ of the unit circle S. In this case, $\Omega(f) = \{0\}$ and $\lim_{n\to\pm\infty} f^n(x) = 0$ for every $x \in S$, which implies that $CR(f) = S$. Take any point $p \in S \setminus \{0\}$. It follows that $p \in CR(f)\setminus\Omega(f)$, and thus, $p \notin \Omega(f) \cup (X \setminus CR(f))$. Furthermore, $\delta_p(\Omega(f)) = 0 = \delta_p(\{0\})$ which indicates that δ_p has the SDP with $l = 1$ and $C_1 = \{0\}$. This concludes the proof. □

Recall that a Borel probability measure μ is *invariant* under a homeomorphism $f : X \to X$ if $\mu(f^{-1}(A)) = \mu(A)$ for every measurable set $A \subset X$. We say that μ is *ergodic* if $\mu(I) = 0$ or 1 for every measurable set $I \subset X$ which is *invariant* (i.e. $f^{-1}(I) = I$).

Example 2.3 Every ergodic invariant measure of a homeomorphism of a compact metric space has the SDP.

Proof The *support* of a Borel probability measure is defined as the complement of the union of those open sets with measure zero. It is well known that the support of every invariant measure is compact, invariant, and contained in the nonwandering set. If the measure is additionally ergodic, then the support is also chain transitive. From this, we derive the assertion. □

It is natural to ask whether the conclusion of Theorem 1.4 holds with general notions of expansivity. For example, recall that a Borel probability measure μ of a homeomorphism $f : X \to X$ is *expansive* if there is $\delta > 0$ (called expansivity constant) such that

$$\mu(\{y \in X : d(f^n(x), f^n(y)) \leq \delta, \quad \forall n \in \mathbb{Z}\}) = 0, \qquad \forall x \in X.$$

On the other hand, we say that a homeomorphim $f : X \to X$ is *measure expansive* [7] if every non-atomic invariant measure is expansive with a common expansivity constant. Every expansive homeomorphism is measure expansive, but the converse is not true. The following example illustrates that the context of Theorem 1.4 is not valid with measure expansive homeomorphism instead of expansive ones.

Example 2.4 There exists a shadowable measure of a measure expansive homeomorphism of a compact metric space which doesn't have the spectral decomposition property.

Proof Consider an N-expansive homeomorphism with the shadowing property in Theorem A in [5]. Since N-expansive homeomorphisms are measure expansive, it has the shadowing property but not has the SDP. Consequently, this exhibits a Borel probability measure without the SDP by Theorem 1.2. □

This example also shows that there are homeomorphisms of compact metric spaces exhibiting shadowable expansive measures without the SDP. This observation contrasts with the fact that these measures exhibit topological stability [15].

On the other hand, we say that a homeomorphism $f : X \to X$ is *strong measure expansive* [7] if there is $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = \mu\{x\}$ for every $x \in X$ and every Borel probabilitty measure μ of x. They have proven that every strong measure expansive homeomorphism with the shadowing property of a compact metric space has the SDP. This also motivates the following question if the conclusion of Theorem 1.4 true for strong measure expansive homeomorphisms instead of expansive ones. The question is still open and we leave the answer for other study.

Question 2.5 *Does every shadowable measure of a strong measure expansive homeomorphism on a compact metric space have the spectral decomposition property?*

3 Proof of the theorems

Proof of Theorem 1.2 First, suppose that f has the SDP. Take the corresponding invariant chain transitive sets $C_1, \ldots, C_l \subset \Omega(f)$ satisfying (1). Then,

$$\Omega(f)\setminus\bigcup_{i=1}^{l} C_i = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{l} C_i\setminus\Omega(f) = \emptyset$$

so

$$\mu\left(\Omega(f)\setminus\bigcup_{i=1}^{l} C_i\right) = 0 \quad \text{and} \quad \mu\left(\bigcup_{i=1}^{l} C_i\setminus\Omega(f)\right) = 0,$$

for every Borel probability measure μ. But, the latter two identities together are equivalent to

$$\mu(\Omega(f)) = \mu\left(\bigcup_{i=1}^{l} C_i\right)$$

and the union is disjoint. So we have

$$\mu(\Omega(f)) = \sum_{i=1}^{l} \mu(C_i).$$

Therefore, μ has the SDP.

Conversely, suppose that all Borel probability measures μ have the SDP, but f has not. If $\Omega(f)$ intersects only a finite number of chain recurrent classes $C_1, \ldots, C_k$ of f, then we have

$$\Omega(f) = \bigcup_{i=1}^{k} (C_i \cap \Omega(f)).$$

Then, we are done since each $C_i \cap \Omega(f)$ is compact invariant chain transitive and disjoint each other.

So, we can assume that there is an infinite collection of distinct chain recurrent classes $(C_i)_{i\in\mathbb{N}}$ such that $\Omega(f) \cap C_i \neq \emptyset$ for every $i \in \mathbb{N}$. For every $i \in \mathbb{N}$, we choose $y_i \in \Omega(f) \cap C_i$ and define

$$\mu = \sum_{i=1}^{\infty} \frac{1}{2^i} \delta_{y_i}.$$

It follows from the hypothesis that μ has the SDP. So, there are finitely many disjoint compact invariant chain transitive sets $F_1, \ldots F_l \subset \Omega(f)$ such that

$$\mu(\Omega(f)) = \sum_{j=1}^{l} \mu(F_j).$$

Then, from the definition of μ that $\mu(\Omega(f)) = 1$, so

$$\sum_{j=1}^{l} \mu(F_j) = 1.$$

Then,

$$\mu\left(\bigcup_{j=1}^{l} F_j\right) = 1.$$

Hence

$$y_i \in \bigcup_{j=1}^{l} F_j, \qquad \forall i = 1, 2, 3, \ldots.$$

Therefore, there exists $j = 1, \ldots, l$ such that $y_i, y_r \in F_j$ for some $i \neq r$. Since F_j is chain transitive, we would have $C_i = C_r$ a contradiction. This completes the proof.

We split the proof of Theorem 1.4 into the following lemmas. The first one is well known by [2]. Given a homeomorphism of a compact metric space $f : X \to X$ and $x \in X$ we define

$$W^s(x) = \{y \in X : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}$$

and

$$W^u(x) = \{y \in X : d(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to \infty\}.$$

If additionally $\epsilon > 0$, we define

$$W^s_\epsilon(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \epsilon \text{ for every } n \in \mathbb{N}\},$$

and

$$W^u_\epsilon(x) = \{y \in X : d(f^{-n}(x), f^{-n}(y)) \le \epsilon \text{ for every } n \in \mathbb{N}\}.$$

Lemma 3.1 [2] *If $f : X \to X$ is an expansive homeomorphism with expansivity constant ϵ of a compact metric space, then $W^s_\epsilon(x) \subset W^s(x)$ and $W^u_\epsilon(x) \subset W^u(x)$ for every $x \in X$.*

Now, we define a subset to be open relative to a Borel probability measure μ.

Definition 3.2 Let μ be a Borel probability measure of a compact metric space X. Given $R \subset X$ we say that $A \subset R$ is *μ-open* in R if there is $O \subset X$ open such that

$$O \supset A \qquad \text{and} \qquad \mu(O \cap R) = \mu(A).$$

We use this definition to obtain a sufficient condition for a Borel probability measure to have the SDP.

Lemma 3.3 *Let $f : X \to X$ a homeomorphism of a compact metric space and μ be a Borel probability measure of X. Suppose that the following two conditions hold:*

(i) $\mu(\Omega(f)) = \mu(CR(f))$.
(ii) *There is a closed subset B of full measure such that $C \cap B$ is μ-open in $CR(f)$ for every chain recurrent component C of f.*

Then, μ has the SDP.

Proof We have $\Omega(f) \subset CR(f)$, so $\Omega(f) = \bigcup_{i\in\Lambda}(C_i \cap \Omega(f))$ where $\{C_i : i \in \Lambda\}$ is the collection of chain recurrent classes of f. Then,

$$\Omega(f) \cap B = \bigcup_{i\in\Lambda}(C_i \cap \Omega(f) \cap B). \tag{3}$$

By the μ-openess hypothesis (ii), there is a collection of open sets O_i for $i \in \Lambda$ such that

$$O_i \supset C_i \cap B \qquad \text{and} \qquad \mu(O_i \cap CR(f)) = \mu(C_i \cap B), \qquad \forall i \in \Lambda. \tag{4}$$

Since $\Omega(f) \subset CR(f)$, and $\mu(\Omega(f)) = \mu(CR(f))$ by (i), we obtain from the second identity of (4) that

$$\mu(O_i \cap \Omega(f)) = \mu(O_i \cap CR(f)) = \mu(C_i \cap B) = \mu(C_i \cap \Omega(f) \cap B), \qquad \forall i \in \Lambda.$$

On the other hand, the first inclusion in (4), together with (3), implies that $\{O_i : i \in \Lambda\}$ is an open covering of $\Omega(f) \cap B$. Since X is compact, so does $\Omega(f) \cap B$. Then, there is a finite open covering

$$\Omega(f) \cap B = \bigcup_{i=1}^{l}(O_i \cap \Omega(f)).$$

So

$$\mu(\Omega(f)) = \mu(\Omega(f) \cap B) \leq \sum_{i=1}^{l} \mu(O_i \cap \Omega(f))$$
$$= \sum_{i=1}^{l} \mu(C_i \cap \Omega(f) \cap B) = \sum_{i=1}^{l} \mu(C_i \cap \Omega(f)).$$

However, as $\mu(\Omega(f)) = \mu(CR(f))$, we obtain

$$\sum_{i=1}^{l} \mu(C_i \cap \Omega(f)) = \mu\left(\bigcup_{i=1}^{l}(C_i \cap \Omega(f))\right) \leq \mu(CR(f)) = \mu(\Omega(f)).$$

This implies

$$\mu(\Omega(f)) = \sum_{i=1}^{l} \mu(C_i \cap \Omega(f)).$$

Therefore, we are done, as each $C_i \cap \Omega(f)$ is compact invariant chain transitive and contained in $\Omega(f)$. $\square$

Let f be a homeomorphism of a metric space X. Recall that a bi-infinite sequence $(x_i)_{i\in\mathbb{Z}}$ of X is a δ-pseudo-orbit if $d(f(x_i), x_{i+1}) \leq \delta$ for every $i \in \mathbb{Z}$. And we say $(x_i)_{i\in\mathbb{Z}}$ can be ϵ-shadowed if there is $\xi \in X$ such that $d(f^i(\xi), x_i) \leq \epsilon$ for every $i \in \mathbb{Z}$. We say that a point $x \in X$ is a *shadowable* if for every $\epsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit $(x_i)_{i\in\mathbb{Z}}$ with $x_0 = x$ can be ϵ-shadowed. Denote by $Sh(f)$ the set of shadowable points.

Lemma 3.4 *If μ is a shadowable measure of a homeomorphism of a compact metric space $f : X \to X$, then $\mu(Sh(f)) = 1$.*

Proof Since μ is a shadowable measure, there are δ and Borelian B_n with $\mu(X \setminus B_n) = 0$ such that every δ-pseudo-orbit through B_n can be $\frac{1}{n}$-shadowed for each $n \in \mathbb{N}^+$. Now we put $B = \bigcap_{n\in\mathbb{N}^+} B_n$. Since

$$\mu(X \setminus B) = \mu(X \setminus \bigcap_{n\in\mathbb{N}^+} B_n)$$
$$= \mu(\bigcup_{n\in\mathbb{N}^+} (X \setminus B_n))$$
$$\leq \sum_{n\in\mathbb{N}^+} \mu(X \setminus B_n) = 0,$$

we have $\mu(B) = 1$. Now take $x \in B$ and $\epsilon > 0$. Fix $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \epsilon$. Define $\delta = \delta_N$ and take a δ-pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ through x. As $x \in B = \bigcap_{n\in\mathbb{N}^+} B_n$,

$x \in B_N$ and so the δ-pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ intersects with the set B_N. That is, such $(x_k)_{k\in\mathbb{Z}}$ can be $\frac{1}{N}$-shadowed. Since $\frac{1}{N} < \epsilon$, we have that $(x_k)_{k\in\mathbb{Z}}$ can be ϵ-shadowed and then $x \in Sh(f)$. Therefore $B \subset Sh(f)$ and so $\mu(Sh(f)) \geq \mu(B) = 1$. □

Now we state a lemma related to the hypotheses of Lemma 3.3.

Lemma 3.5 *If μ is a shadowable measure of a homeomorphism of a compact metric space $f : X \to X$, then $\mu(\Omega(f)) = \mu(CR(f))$.*

Proof It suffices to show $\mu(CR(f)) \leq \mu(\Omega(f))$. By Lemma 2.5 in [18] one has $CR(f) \cap Sh(f) \subset \Omega(f)$. So $\mu(CR(f) \cap Sh(f)) \subset \mu(\Omega(f))$. But, $Sh(f)$ has full measure by Lemma 3.4. Therefore, $\mu(CR(f)) = \mu(CR(f) \cap Sh(f))$ and then $\mu(CR(f)) \leq \mu(\Omega(f))$ as desired. □

One more lemma is as follows.

Lemma 3.6 *Let C and $\bar{C}$ be chain recurrent classes of a homeomorphism of a compact metric space $f : X \to X$. If there are $a \in C$ and $\bar{a} \in \bar{C}$ such that*

$$W^s(a) \cap W^u(\bar{a}) \neq \emptyset \neq W^u(a) \cap W^s(\bar{a}), \tag{5}$$

then $C = \bar{C}$.

Proof We know that the chain recurrent classes are compact invariant sets. For every $x \in X$, we define the omega-limit and alpha-limit sets by

$$\omega(x) = \{y \in X : \exists \text{ sequence } n_k \to \infty \text{ such that } f^{n_k}(x) \to y\}$$

and

$$\alpha(x) = \{y \in X : \exists \text{ sequence } n_k \to \infty \text{ such that } f^{-n_k}(x) \to y\}.$$

Take $b \in \alpha(a)$, $d \in \omega(a)$, $\bar{b} \in \alpha(\bar{a})$ and $\bar{d} \in \omega(\bar{a})$. Since C and $\bar{C}$ are compact invariant, $b, d \in C$ and $\bar{b}, \bar{d} \in \bar{C}$. By (5) we can choose

$$x \in W^s(a) \cap W^u(\bar{(a)}) \quad \text{and} \quad \bar{x} \in W^u(a) \cap W^s(\bar{a}).$$

Fix $\lambda > 0$. By using the negative and positive orbits of $\bar{x}$, we can construct λ-pseudo-orbit from b to $\bar{C}$. Similarly, using x, we can construct a λ-pseudo-orbit from $\bar{b}$ to C. Since both c and b belong to C, we obtain a δ-pseudo-orbit from C to b. Additionally, we can construct a λ-pseudo-orbit from $\bar{C}$ to b. By connecting these pseudo-orbits together, we construct a λ-pseudo-orbit from $\bar{c}$ to b. As λ is arbitrary, we have $b \sim \bar{C}$ (see p. 120 in [2]). Since $\bar{c} \in \bar{C}$ and $b \in C$, we conclude that $C = \bar{C}$, thus proving the result. □

Proof *(Proof of Theorem 1.4)* Let μ be a shadowable measure of an expansive homeomorphism of a compact metric space $f : X \to X$. Using Lemma 3.5, we have $\mu(\Omega(f)) = \mu(CR(f))$. Then, to prove the result, it suffices to find B satisfying item (ii) of Lemma 3.3. We proceed to find such a set as follows.

According to Lemma 3.1, there is $\epsilon > 0$ such that

$$W^s_\epsilon(z) \subset W^s(z) \qquad \text{and} \qquad W^u_\epsilon(z) \subset W^u(z), \qquad \forall z \in X. \tag{6}$$

For this ϵ, we let $\delta > 0$ and B be the (ϵ, δ)-shadowable set of full measure, which is given by the shadowability of μ. By [21], we can assume that B is closed.

Now, we prove that this B satisfies the condition (ii) in Lemma 3.3, specifically, that $B \cap C$ is μ-open in $CR(f)$ for every chain recurrent class C.

Take a chain recurrent class C. Define $U_\delta(C)$ the δ-neighborhood of C in $CR(f)$. We claim that

$$U_\delta(C \cap B) \subset C. \tag{7}$$

Take $p \in U_\delta(C)$. Then there is $y \in C \cap B$ such that $d(y, p) < \delta$.

Define the sequence x_i by

$$x_i = \begin{cases} f^i(y), & \text{if } i \leq 0, \\ f^i(p), & \text{if } i > 0. \end{cases}$$

It follows that $(x_i)_{i \in \mathbb{Z}}$ is a δ-pseudo-orbit with $x_0 = y \in B$. So there is $x \in X$ such that

$$d(f^i(x), x_i) < \epsilon, \qquad \forall i \in \mathbb{Z}.$$

Taking $i \leq 0$ we get $x \in W^u_\epsilon(y)$ and with $i > 0$ we get $x \in W^s(p)$. So

$$x \in W^u_\epsilon(y) \cap W^s_\epsilon(p).$$

By the choice of ϵ and (6), we get $x \in W^u(y) \cap W^s(p)$ proving

$$W^u(y) \cap W^s(p) \neq \emptyset. \tag{8}$$

Repeating the argument but with the sequence $(\bar{x}_i)_{i \in \mathbb{Z}}$ defined by

$$x_i = \begin{cases} f^i(p), & \text{if } i \leq 0, \\ f^i(y), & \text{if } i > 0, \end{cases}$$

we get

$$W^u(p) \cap W^s(y) \neq \emptyset. \tag{9}$$

Therefore, the point p belongs to C. As $p \in U_\delta(C \cap B)$ is arbitrary, we get (7).

Now, since $\mu(B) = 1$, we have

$$\mu(U_\delta(C \cap B)) \leq \mu(C) = \mu(C \cap B).$$

This proves that $C \cap B$ is μ-open in $CR(f)$, thus completing the proof. □

References

1. Aoki, N.: On homeomorphisms with pseudo-orbit tracing property. Tokyo J. Math. **6**(2), 329–334 (1983)
2. Aoki, N., Hiraide, K.: Topological theory of dynamical systems. Recent advances, North-Holland Mathematical Library, 52. North-Holland Publishing Co., Amsterdam (1994)
3. Artigue, A., Carvalho, B., Cordeiro, W., Vieitez, J.: Beyond topological hyperbolicity: the L-shadowing property. J. Differ. Equ. **268**(6), 3057–3080 (2020)
4. Bowen, R.: Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Mathematics, vol. 470. Springer-Verlag, Berlin-New York (1975)
5. Carvalho, B., Cordeiro, W.: N-expansive homeomorphisms with the shadowing property. J. Differ. Equ. **261**(6), 3734–3755 (2016)
6. Choi, T., Kim, J.: Decomposition theorem on G-spaces. Osaka J. Math. **46**(1), 87–104 (2009)
7. Cordeiro, W., Denker, M., Zhang, X.: On specification and measure expansiveness. Discrete Contin. Dyn. Syst. **37**(4), 1941–1957 (2017)
8. Das, R., Das, T., Shah, S.: Bowen's decomposition theorem for topologically Anosov homeomorphisms on noncompact and non-metrizable spaces. Commun. Korean Math. Soc. **33**(1), 337–344 (2018)
9. Das, T., Lee, K., Richeson, D., Wiseman, J.: Spectral decomposition for topologically Anosov homeomorphisms on noncompact and non-metrizable spaces. Topol. Appl. **160**(1), 149–158 (2013)
10. Erdelyi, I., Shengwang, W.: A local spectral theory for closed operators, London Mathematical Society Lecture Notes Series 105
11. Jung, W., Nguyen, N.-T., Yang, Y.: Spectral decomposition for rescaling expansive flows with rescaled shadowing. Discrete Contin. Dyn. Syst. **40**(4), 2267–2283 (2020)
12. Kim, D., Lee, S.: Spectral decomposition of k-type nonwandering sets for $\mathbb{Z}^2$-actions. Bull. Korean Math. Soc. **51**(2), 387–400 (2014)
13. Komuro, M.: One-parameter flows with the pseudo-orbit tracing property. Monatsh. Math. **98**(3), 219–253 (1984)
14. Le, H.-T., Lee, K., Nguyen, N.: Spectral decomposition and stability of mild expansive systems. Topol. Methods Nonlinear Anal. **56**(1), 63–81 (2020)
15. Lee, K., Morales, C.A.: Topological stability and pseudo-orbit tracing property for expansive measures. J. Differ. Equ. **262**, 3467–3487 (2017)
16. Lee, K., Nguyen, N.: Spectral decomposition and Ω-stability of flows with expanding measures. J. Differ. Equ. **269**(9), 7574–7604 (2020)
17. Lee, K., Nguyen, N.-T., Yang, Y.: Topological stability and spectral decomposition for homeomorphisms on noncompact spaces. Discrete Contin. Dyn. Syst. **38**(5), 2487–2503 (2018)
18. Morales, C.A.: Shadowable points. Dyn. Syst. **31**(3), 347–356 (2016)
19. Oprocha, P.: Shadowing in multi-dimensional shift spaces. Colloq. Math. **110**(2), 451–460 (2008)
20. Smale, S.: Differentiable dynamical systems. Bull. Am. Math. Soc. **73**, 747–817 (1967)
21. Shin, B.: On the set of shadowable measures. J. Math. Anal. Appl. **469**(2), 872–881 (2019)
22. Thakkar, D., Das, R.: Spectral decomposition theorem in equicontinuous nonautonomous discrete dynamical systems. J. Differ. Equ. Appl. **22**(5), 676–686 (2016)