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Hardy's uncertainty principle for Gabor transform on compact extensions of $\mathbb{R}^n$

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Abstract

We prove in this paper a generalization of Hardy's theorem for Gabor transform in the setup of the semidirect product $\mathbb{R}^n \rtimes K$, where K is a compact subgroup of automorphisms of $\mathbb{R}^n$. We also solve the sharpness problem and thus obtain a complete analogue of Hardy's theorem for Gabor transform. The representation theory and Plancherel formula are fundamental tools in the proof of our results.

Keywords Hardy's theorem · Uncertainty principle · Gabor transform · Plancherel formula

Mathematics Subject Classification 22E30 · 43A32 · 43A30

1 Introduction

It is a well-known fact in classical Fourier analysis that an integrable function f defined on the real line and its Fourier transform $\hat{f}$ cannot be simultaneously and sharply localized unless $f = 0$ almost everywhere. This property of functions is widely known as the uncertainty principle in Fourier analysis. The following result of Hardy makes the rather vague statement above precise (see [11]):

Theorem 1 *Let p, q, c be positive real numbers and f a measurable function on $\mathbb{R}^n$ such that:*
(i) $|f(x)| \leq ce^{-p\pi\|x\|^2}, \quad x \in \mathbb{R}^n,$
(ii) $|\hat{f}(y)| \leq ce^{-q\pi\|y\|^2}, \quad k \in \mathbb{R}^n.$

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If $pq > 1$*, then* $f = 0$ *a.e. If* $pq = 1$ *then* $f(t) = Ce^{-p\pi\|x\|^2}$*, for some constant* C*. If* $pq < 1$*, then any finite linear combination of Hermite functions satisfies (i) and (ii).*

Here the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-2i\pi xy}dx, \ \ y \in \mathbb{R}^n,$$

where $xy = \sum_{j=1}^n x_j y_j$, and $\|x\| = \sqrt{x^2}$ is the Euclidean norm.

Naturally, there has been some effort to prove Hardy-like theorems for various connected Lie groups G. Specifically, analogues and variants of Hardy's theorem have been shown for motion groups [22, 23], compact extensions of $\mathbb{R}^n$ [1], non-compact connected semisimple Lie groups G with finite center [6, 18, 20, 21] and nilpotent Lie groups [5, 12, 19, 24].

Unlike the classical Fourier transform, the continuous Gabor transform gives a simultaneous representation of the space and the frequency variables. Let $\psi \in L^2(\mathbb{R}^n)$ be a fixed non-zero function usually called a window function. The Gabor transform of a function $f \in L^2(\mathbb{R}^n)$ with respect to the window function ψ is defined on $\mathbb{R}^n \times \widehat{\mathbb{R}}^n$ by

$$\mathcal{G}_\psi f(x, w) := \int_{\mathbb{R}^n} f(y)\overline{\psi}(y - x)e^{-2i\pi yw}dy.$$

According to [9], we have for all $f_1, f_2, \psi_1, \psi_2 \in L^2(\mathbb{R}^n)$ the functions $\mathcal{G}_{\psi_1} f_1$ and $\mathcal{G}_{\psi_2} f_2$ belong to $L^2(\mathbb{R}^n \times \hat{\mathbb{R}}^n)$ and

$$\langle \mathcal{G}_{\psi_1} f_1, \mathcal{G}_{\psi_2} f_2\rangle_{L^2(\mathbb{R}^n \times \hat{\mathbb{R}}^n)} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^n)} \overline{\langle \psi_1, \psi_2\rangle}_{L^2(\mathbb{R}^n)}. \quad (1)$$

It has been shown in the early 2000 s that many uncertainty principles for the Fourier transform have a counterpart for the continuous Gabor transform (see [2, 10]). We specify that a Hardy-type theorem has been established in [10, Theorem 2.6.2].

Theorem 2 *Let* $f, \psi \in L^2(\mathbb{R}^n)$*. Assume that*

$$\left|\mathcal{G}_\psi f(x, w)\right| \leq ce^{-\frac{\pi}{2}(p\|x\|^2 + q\|w\|^2)},$$

for some constants $p, q, c > 0$*. Then three cases can occur.*
(i) If $pq > 1$*, then either* $f \equiv 0$ *or* $\psi \equiv 0$*.*
(ii) If $pq = 1$ *and* $\mathcal{G}_\psi f$ *is not zero almost everywhere, then both* f *and* ψ *are multiples of some time-frequency shift of the Gaussian* $e^{-p\pi\|x\|^2}$*.*
(iii) If $pq < 1$*, then the decay condition is satisfied whenever* f *and* ψ *are finite linear combinations of Hermite functions.*

In 2012, the continuous Gabor transform for separable locally compact unimodular group of type I has been introduced by Farashahi and Kamyabi-Gol [7]. A brief description is given in Section 2. One should notice that, in the Euclidean setting, the continuous Gabor transform has many symmetries which are lost in the Lie group setting (the dual of G does not identify with G) and this is then a serious obstacle for stating uncertainty principles for the continuous Gabor transform. However, some attempts to extend Theorem 2 on special classes of non-Abelian Lie groups have already been made. Recently, analogues of Hardy's theorem for Gabor transform have been established for locally compact abelian groups having noncompact identity component and groups of the form $\mathbb{R}^n \times K$, where K is a compact group having irreducible representations of bounded dimension (see [3]). On the other hand, the author and K. Abid [17] proved an analogue of Hardy's theorem for Gabor transform on connected, simply connected nilpotent Lie groups. However, in the last two references the results obtained concern only the case $pq > 1$. In this paper, we prove a generalization of Hardy's theorem for Gabor transform on a general compact extension $\mathbb{R}^n \rtimes K$, where K is a compact subgroup of automorphisms of $\mathbb{R}^n$, providing evidence to the three cases cited above. The proof of our result, which is given in Section 3, exploits Hardy's theorem for $\mathbb{R}^n$ and representation theory and the harmonic analysis of $\mathbb{R}^n \rtimes K$.

2 Backgrounds

2.1 Continuous Gabor transform

Let G be a separable locally compact unimodular group of type I, and let dg be its Haar measure. We endow the unitary dual of G with the Mackey Borel structure. We denote by $L^p(G)$ the space of L^p-functions on G for $p \geq 1$, and we define

$$\pi(f) = \int_G f(g)\pi(g)dg, \quad \pi \in \hat{G},\ f \in L^1(G).$$

Then by the abstract Plancherel theorem, there exists a unique Borel measure ρ on $\hat{G}$ such that for any function $f \in L^1(G) \cap L^2(G)$,

$$\int_G |f(g)|^2 dg = \int_{\hat{G}} \|\pi(f)\|_{HS}^2 d\rho(\pi),$$

where $\|\pi(f)\|_{HS} = \big(\mathrm{tr}\big(\pi(f)^*\pi(f)\big)\big)^{1/2}$ denotes the Hilbert-Schmidt norm of $\pi(f)$.

Let $f \in C_c(G)$, the set of all continuous complex-valued functions on G with compact supports, and ψ a fixed nonzero function in $L^2(G)$, usually called window function. For $(x, \pi) \in G \times \hat{G}$, the continuous Gabor transform of f with respect to the window function ψ is defined as a measurable field of operators on $G \times \hat{G}$ by

$$\mathcal{G}_\psi f(x, \pi) := \int_G f(g)\overline{\psi}(x^{-1}g)\pi(g)dg.$$

Let f_ψ^x be the function defined on G by

$$f_\psi^x(g) = f(g)\overline{\psi}(x^{-1}g), \qquad \forall g \in G.$$

Then, $f_\psi^x \in L^1(G) \cap L^2(G)$ and

$$\pi(f_\psi^x) = \int_G f_\psi^x(g)\pi(g)dg = \int_G f(g)\overline{\psi}(x^{-1}g)\pi(g)dg = \mathcal{G}_\psi f(x,\pi). \tag{2}$$

By the Plancherel theorem, $\mathcal{G}_\psi f(x,\pi)$ is a Hilbert-Schmidt operator for all $x \in G$ and for almost all $\pi \in \hat{G}$. Furthermore,

$$\int_G \int_{\hat{G}} \|\mathcal{G}_\psi f(x,\pi)\|_{HS}^2 d\rho(\pi)\, dx = \|\psi\|_2^2 \|f\|_2^2. \tag{3}$$

Thus, the continuous Gabor transform $\mathcal{G}_\psi : f \mapsto \mathcal{G}_\psi f$ ($f \in C_c(G)$) is a multiple of an isometry. So, we can extend $\mathcal{G}_\psi$ uniquely to a bounded linear operator on $L^2(G)$ which we still denote by $\mathcal{G}_\psi$ and this extension satisfies (3) for each $f \in L^2(G)$.

2.2 Compact extensions of $\mathbb{R}^n$

Let $G = \mathbb{R}^n \rtimes K$ be a semidirect product of $\mathbb{R}^n$ and a compact subgroup K of the automorphisms group Aut($\mathbb{R}^n$). In the whole paper, $\mathbb{R}^n$ is equipped with an Euclidean scalar product which embeds the compact group K as a subgroup of orthogonal transformations (for details see [15]). The multiplication law in G is given by

$$(a,k)\cdot(b,h) = (a+kb, kh),$$

for $(a,k), (b,h) \in G$. We fixe once for all a Haar measure dg on G by $dg = da\,d\mu(k)$, where da denotes the Lebesgue measure on $\mathbb{R}^n$ and $d\mu(k)$ the normalized Haar measure on K. Let us remark that the compactness of K leads to the proof that da is invariant under the action of K on $\mathbb{R}^n$ given by $\mathbb{R}^n \ni a \mapsto k^{-1}ak$, for $k \in K$.

By Mackey's little group theory [16], the set $\hat{G}$ is given by the following procedure. Let γ be a non-zero real linear form on $\mathbb{R}^n$ and let χ_γ be the unitary character of $\mathbb{R}^n$ defined by $\chi_\gamma(a) = \mathrm{e}^{-2i\pi\langle\gamma,a\rangle}$, $a \in \mathbb{R}^n$. The naturel action $g\cdot\gamma$ on $\mathbb{R}^n$ is given by $\langle g\cdot\gamma, a\rangle = \langle\gamma, g^{-1}ag\rangle$ for $g \in G$ and $a \in \mathbb{R}^n$. If G acts on $\widehat{\mathbb{R}^n}$ by $g\cdot\chi_\gamma(a) = \chi_\gamma(g^{-1}ag)$, then $g\cdot\chi_\gamma = \chi_{g\cdot\gamma}$. We identify $\widehat{\mathbb{R}^n}$ with $\mathbb{R}^n$ by the mapping $\mathbb{R}^n \ni \gamma \mapsto \chi_\gamma \in \widehat{\mathbb{R}^n}$.

Let K_γ be the set of all $k \in K$ such that $k\cdot\chi_\gamma = \chi_\gamma$, then $G_\gamma = \mathbb{R}^n \rtimes K_\gamma$ is the stabilizer of χ_γ in G. Let us take the normalized Haar measure $d\mu_\gamma$ on K_γ and a K-invariant measure $d\dot{\mu}_\gamma$ on K/K_γ such that

$$\int_K \varphi(k)d\mu(k) = \int_{K/K_\gamma}\int_{K_\gamma} \varphi(kk')d\mu_\gamma(k')d\dot{\mu}_\gamma(kK_\gamma).$$

Noting that the measure $d\dot{\mu}_\gamma$ is normalized so that $\int_{K/K_\gamma} d\dot{\mu}_\gamma = 1$. Let $d\overline{\gamma}$ be the image of the Lebesgue measure on $\mathbb{R}^n/K$ by the canonical projection $\mathbb{R}^n \ni \gamma \mapsto \overline{\gamma} = K\cdot\gamma \in \mathbb{R}^n/K$ such that

$$\int_{\mathbb{R}^n} \varphi(\gamma)d\gamma = \int_{\mathbb{R}^n/K}\int_K \varphi(k\cdot\gamma)d\mu(k)d\overline{\gamma}.$$

On the other hand, let $(\sigma, \mathcal{H}_\sigma)$ be a unitary and irreducible representation of K_γ and $\mathcal{H}_{\gamma,\sigma}$ be the completion of the vector space of all continuous mapping functions $\varphi : K \to \mathcal{H}_\sigma$ for which

$$\varphi(kk') = \sigma(k')^{-1}\varphi(k), \ \ \forall k \in K, \ \forall k' \in K_\gamma,$$

with respect to the norm

$$\|\varphi\|_2 = \left(\int_K \|\varphi(k)\|^2_{\mathcal{H}_\sigma} d\mu(k)\right)^{\frac{1}{2}}.$$

The induced representation $\pi_{\gamma,\sigma} := \mathrm{Ind}_{G_\gamma}^G(\chi_\gamma \otimes \sigma)$, realized on $\mathcal{H}_{\gamma,\sigma}$ by

$$\pi_{\gamma,\sigma}(a,k)\varphi(h) = \chi_\gamma(h^{-1}ah)\varphi(k^{-1}h) = \chi_{h\cdot\gamma}(a)\varphi(k^{-1}h), \tag{4}$$

for $\varphi \in \mathcal{H}_{\gamma,\sigma}$, $(a,k) \in G$ and $h \in K$, is an irreducible unitary representation of G. Furthermore, every infinite dimensional irreducible unitary representation of G is equivalent to some representation $\pi_{\gamma,\sigma}$.

We note that, every irreducible unitary representation τ of K extends trivially to an irreducible representation (also denoted by τ) of the entire group G, defined by

$$\tau(a,k) = \tau(k), \quad a \in \mathbb{R}^n \ \text{ and } \ k \in K.$$

According to [13, 14], the Plancherel formula for $f \in L^1(G) \cap L^2(G)$ is given by

$$\int_G |f(a,k)|^2 da d\mu(k) = \int_{\mathbb{R}^n/K} \sum_{\sigma \in \hat{K}_\gamma} \|\pi_{\gamma,\sigma}(f)\|^2_{HS} d\overline{\gamma},$$

where

$$\pi_{\gamma,\sigma}(f) = \int_G f(a,k)\pi_{\gamma,\sigma}(a,k) da d\mu(k)$$

is a kernel operator. Its kernel is defined for $(s,u) \in K/K_\gamma \times K/K_\gamma$ as

$$H(f,\gamma,\sigma)(s,u) = \int_{K_\gamma} f(\cdot, svu^{-1})\widehat{}(s\cdot\gamma)\sigma(v) d\mu_\gamma(v), \tag{5}$$

where $f(\cdot, svu^{-1})\widehat{}$ denotes the partial Fourier transform of the function f with respect to the Euclidean variable.

3 The main result

The main motivation of the present study is to generalize Theorem 2, writing down a generalized analogue of Hardy's uncertainty principle for Gabor transform on $G =$

$\mathbb{R}^n \rtimes K$. Before stating our main result, we need some notations. For every $x, w \in \mathbb{R}^n$, we denote by $\mathcal{M}_w$ and $\mathcal{T}_x$ the modulation and the translation operators defined respectively on $L^2(\mathbb{R}^n)$ by

$$\forall z \in \mathbb{R}^n, \quad \mathcal{M}_w f(z) = e^{2i\pi zw} f(z),$$
$$\forall z \in \mathbb{R}^n, \quad \mathcal{T}_x f(z) = f(z - x).$$

Then we deduce that,

$$\forall z \in \mathbb{R}^n, \quad \mathcal{M}_w(\mathcal{T}_x f)(z) = e^{2i\pi zw} f(z - x),$$

and

$$\forall z \in \mathbb{R}^n, \quad \mathcal{T}_x(\mathcal{M}_w f)(z) = e^{-2i\pi xw} e^{2i\pi zw} f(z - x).$$

On the other hand, for a measurable function φ on G, let

$$\varphi(\cdot, k)(a) := \varphi(a, k), \quad (a, k) \in G.$$

Our main result is the following:

Theorem 3 *Let p and q be positive real numbers. Let $f, \psi \in L^2(G)$ be such that*

$$\|\mathcal{G}_\psi f(g, \pi_{\gamma,\sigma})\|_{HS} \leq \phi_\gamma(k, \sigma) e^{-\frac{\pi}{2}(p\|a\|^2 + q\|\gamma\|^2)}, \tag{6}$$

for all $g = (a, k) \in G$, $\gamma \in \mathbb{R}^n$ and $\sigma \in \hat{K}_\gamma$, with $\|\phi_\gamma\|_{L^2(K \times \hat{K}_\gamma)} \leq C$ for some psitive constant C independent of γ. Then three cases can occur.
(i) If $pq > 1$, then either $f \equiv 0$ or $\psi \equiv 0$.
(ii) If $pq = 1$ and $\mathcal{G}_{\psi(\cdot,k)} f(\cdot, h) \not\equiv 0$ for each $k, h \in K$ such that the Gabor transform of $f(\cdot, h)$ with respect to $\psi(\cdot, k)$ is well defined, then for all $a \in \mathbb{R}^n$ and almost all $h \in K$,

$$f(a, h) = C_1(h)\mathcal{M}_{\lambda_1(h)}\mathcal{T}_{\delta_1(h)} e^{-\pi p\|a\|^2}$$
$$\text{and} \quad \psi(a, h) = C_2(h)\mathcal{M}_{\lambda_2(h)}\mathcal{T}_{\delta_2(h)} e^{-\pi p\|a\|^2},$$

where $C_j \in L^2(K)$ and λ_j, δ_j are functions from K to $\mathbb{R}^n$, $j := 1, 2$.
(iii) If $pq < 1$, then there are infinitely many linearly independent pairs (f, ψ) satisfying (6).

3.1 Some lemmas

The results in the following lemma are quite standard.

Lemma 1 *Let $f, \psi \in L^2(\mathbb{R}^n)$ and $\xi, \lambda, y, z \in \mathbb{R}^n$. Then,*
(i) $\mathcal{G}_{(\mathcal{M}_\xi \mathcal{T}_z \psi)}(\mathcal{M}_\lambda \mathcal{T}_y f)(x, w) = e^{2i\pi x\xi} e^{-2i\pi y(w-\lambda+\xi)} \mathcal{G}_\psi f(x - y + z, w - \lambda + \xi)$.
In particular, $\mathcal{G}_\psi(\mathcal{M}_\lambda \mathcal{T}_y f)(x, w) = e^{-2i\pi yw} e^{2i\pi y\lambda} \mathcal{G}_\psi f(x - y, w - \lambda)$.

(ii) $\mathcal{G}_\psi f(-x,-w) = e^{-2i\pi xw}\overline{\mathcal{G}_f \psi(x,w)}$.
(iii) Let $F(x,w) = \mathcal{G}_\psi f(x,w)\mathcal{G}_\psi f(-x,-w)e^{2i\pi xw}$. *Then,*

$$\hat{F}(\nu,\theta) = F(-\theta,\nu), \quad \nu,\theta \in \mathbb{R}^n.$$

Now, we shall give two lemmas which are required to prove Theorem 3. Let $g = (a,k)$ be an element of G and $f,\psi \in L^2(G)$. For $h \in K$, let $(f_\psi^g)_h$ be the complex valued function defined on $\mathbb{R}^n$ by

$$(f_\psi^g)_h(c) := f_\psi^g(\cdot,h)(c) = f_\psi^g(c,h) = f(c,h)\overline{\psi}\big((a,k)^{-1}(c,h)\big).$$

It is easy to see that $f_\psi^g \in L^1(G)$ for all $g \in G$, it sufficient to use Cauchy–Schwarz inequality. Moreover by [4, Lemma 3.1], we have

$$\mathcal{G}_\psi f(g,\pi_l) = \pi_l(f_\psi^g), \tag{7}$$

for all $g \in G$. We should also mention that $f_\psi^g \in L^2(G)$, for almost all $g \in G$. In fact,

$$\int_G\int_G |f_\psi^g(x)|^2 dx\, dg = \int_G\int_G |f(x)|^2|\psi(g^{-1}x)|^2 dx\, dg = \|f\|_2^2\|\psi\|_2^2 < \infty.$$

Then obviously $\int_G |f_\psi^g(x)|^2 dx < \infty$, for almost all $g \in G$. By setting $f_\psi^{a,k} = f_\psi^g$, we have the following lemma.

Lemma 2 *Let* $f,\psi \in L^2(G)$ *meet the condition (6) of Theorem 3. Then*

$$I(f,\psi,a) := \int_K\int_K \left(\int_{\mathbb{R}^n} \left|(f_\psi^{a,k})_h(c)\right| dc\right)^2 d\mu(h)d\mu(k) < \infty,$$

for all $a \in \mathbb{R}^n$.

Proof By using (6), we have

$$\begin{aligned}
&\int_K\int_{\mathbb{R}^n}\int_{\mathbb{R}^n/K} \sum_{\sigma\in\hat{K}_\gamma} (1+\|a\|^2)\|\mathcal{G}_\psi f\big((a,k),\pi_{\gamma,\sigma}\big)\|_{HS}^2 d\bar{\gamma}\, da\, d\mu(k) \\
&\le \int_K\int_{\mathbb{R}^n}\int_{\mathbb{R}^n/K} \sum_{\sigma\in\hat{K}_\gamma} (1+\|a\|^2)\phi_\gamma^2(k,\sigma)e^{-\pi(p\|a\|^2+q\|\gamma\|^2)} d\bar{\gamma}\, da\, d\mu(k) \\
&\le C\int_{\mathbb{R}^n}(1+\|a\|^2)e^{-\pi p\|a\|^2}da \int_{\mathbb{R}^n/K} e^{-\pi q\|\gamma\|^2}d\bar{\gamma} \\
&= C\int_{\mathbb{R}^n}(1+\|a\|^2)e^{-\pi p\|a\|^2}da \int_{\mathbb{R}^n/K}\int_K e^{-\pi q\|k\cdot\gamma\|^2}d\mu(k)\, d\bar{\gamma} \\
&= C\int_{\mathbb{R}^n}(1+\|a\|^2)e^{-\pi p\|a\|^2}da \int_{\mathbb{R}^n} e^{-\pi q\|\gamma\|^2}d\gamma < \infty.
\end{aligned}$$

By (7) and the Plancherel formula, we obtain

$$\begin{aligned}\infty &> \int_K \int_{\mathbb{R}^n} \int_{\mathbb{R}^n/K} \sum_{\sigma\in\hat{K}_\gamma} (1+\|a\|^2)\big\|\pi_{\gamma,\sigma}\big(f_\psi^{a,k}\big)\big\|_{HS}^2 d\bar{\gamma}\, da\, d\mu(k)\\
&= \int_K \int_{\mathbb{R}^n} \int_K \int_{\mathbb{R}^n} (1+\|a\|^2)\left|f(c,h)\psi\big((a,k)^{-1}(c,h)\big)\right|^2 dc\, d\mu(h)\, da\, d\mu(k)\\
&= \int_K \int_{\mathbb{R}^n} \int_K \int_{\mathbb{R}^n} (1+\|a\|^2)\left|f(c,h)\psi\big(-k^{-1}(a-c),k^{-1}h\big)\right|^2 dc\, d\mu(h)\, da\, d\mu(k)\\
&= \int_K \int_{\mathbb{R}^n} \int_K \int_{\mathbb{R}^n} (1+\|a\|^2)\left|f(c,h)\psi\big(-kh^{-1}a+kh^{-1}c,k\big)\right|^2 dc\, d\mu(h)\, da\, d\mu(k)\\
&= \int_K \int_{\mathbb{R}^n} \int_K \int_{\mathbb{R}^n} (1+\|a-kh^{-1}c\|^2)\,|f(c,h)\psi(a,k)|^2 dc\, d\mu(h)\, da\, d\mu(k).\end{aligned}$$

Therefore,

$$|\psi(a,k)|^2 \int_K \int_{\mathbb{R}^n} (1+\|a-kh^{-1}c\|^2)\,|f(c,h)|^2\, dc\, d\mu(h) < \infty,$$

for almost all $a \in \mathbb{R}^n$ and $k \in K$. As ψ is non identically zero, there exists a_0, k_0 such that $\psi(a_0,k_0) \neq 0$, and

$$\int_K \int_{\mathbb{R}^n} (1+\|a_0-k_0h^{-1}c\|^2)\,|f(c,h)|^2\, dc\, d\mu(h) < \infty. \tag{8}$$

On the other hand, we have

$$\begin{aligned}I(f,\psi,a) &= \int_K \int_K \left(\int_{\mathbb{R}^n} \left|f(c,h)\psi\big((a,k)^{-1}(c,h)\big)\right| dc\right)^2 d\mu(h)\, d\mu(k)\\
&= \int_K \int_K \left(\int_{\mathbb{R}^n} \left|f(c,h)\psi\big(-k^{-1}a+k^{-1}c,k^{-1}h\big)\right| dc\right)^2 d\mu(h)\, d\mu(k)\\
&\leq \int_K \int_K \left(\int_{\mathbb{R}^n} \frac{\left|\psi\big(-k^{-1}a+k^{-1}c,k^{-1}h\big)\right|^2}{1+\|a_0-k_0h^{-1}c\|^2} dc\right)\\
&\quad \times \left(\int_{\mathbb{R}^n} \Big(1+\|a_0-k_0h^{-1}c\|^2\Big)\Big|f(c,h)\Big|^2 dc\right) d\mu(h)\, d\mu(k)\end{aligned}$$

(using Cauchy–Shwartz inequality)

$$\begin{aligned}&\leq \int_K \int_K \left(\int_{\mathbb{R}^n} \left|\psi\big(-k^{-1}a+k^{-1}c,k^{-1}h\big)\right|^2 dc\right)\\
&\quad \times \left(\int_{\mathbb{R}^n} \Big(1+\|a_0-k_0h^{-1}c\|^2\Big)\Big|f(c,h)\Big|^2 dc\right) d\mu(h)\, d\mu(k)\\
&= \int_K \int_K \left(\int_{\mathbb{R}^n} \big|\psi\big(c,k\big)\big|^2 dc\right)\end{aligned}$$

$$\times \left(\int_{\mathbb{R}^n} \left(1 + \|a_0 - k_0 h^{-1} c\|^2\right) \left| f(c,h) \right|^2 dc \right) d\mu(h)\, d\mu(k)$$
$$= \|\psi\|_2^2 \int_K \int_{\mathbb{R}^n} \left(1 + \|a_0 - k_0 h^{-1} c\|^2\right) \left| f(c,h) \right|^2 dc\, d\mu(h),$$

which is finite by (8). □

Lemma 3 *For all $a, \gamma \in \mathbb{R}^n$,*

$$\int_K \int_K \left| \widehat{(f_\psi^{a,k})_h}(\gamma) \right|^2 d\mu(h)\, d\mu(k) \leq C e^{-\pi (p\|a\|^2 + q\|\gamma\|^2)}.$$

Proof Let $E(a)$ be the function defined on $\mathbb{R}^n$ by

$$E(a)(c) = \int_K \int_K ((f_\psi^{a,k})_h * (f_\psi^{a,k})_h^\star)(c) d\mu(h)\, d\mu(k),$$

where $c \in \mathbb{R}^n$ and $(f_\psi^{a,k})_h^\star(c) = \overline{(f_\psi^{a,k})_h(-c)}$. Then $E(a) \in L^1(\mathbb{R}^n)$, for all $a \in \mathbb{R}^n$. In fact for all $a \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |E(a)(c)| dc \leq \int_{\mathbb{R}^n} \int_K \int_K \int_{\mathbb{R}^n} \left|(f_\psi^{a,k})_h(t)\right| \left|(f_\psi^{a,k})_h(t-c)\right| dt\, d\mu(h)\, d\mu(k)\, dc$$
$$= \int_K \int_K \left(\int_{\mathbb{R}^n} \left|(f_\psi^{a,k})_h(c)\right| dc \right)^2 d\mu(h)\, d\mu(k) < \infty$$

(using Lemma 2). Thus,

$$\widehat{E(a)}(\gamma) = \int_K \int_K \left| \widehat{(f_\psi^{a,k})_h}(\gamma) \right|^2 d\mu(h)\, d\mu(k), \quad \gamma \in \mathbb{R}^n. \tag{9}$$

For $U \in L^1(\mathbb{R}^n)$, define $U * f_\psi^{a,k}$ on G by

$$U * f_\psi^{a,k}(c,h) = \int_{\mathbb{R}^n} U(t) f_\psi^{a,k}(c-t,h)\, dt$$

and then $E(a)_U : \mathbb{R}^n \to \mathbb{C}$ by

$$E(a)_U(c) = \int_K \int_K \left((U * f_\psi^{a,k})_h * \left((U * f_\psi^{a,k})_h \right)^\star \right)(c)\, d\mu(h)\, d\mu(k).$$

It is not hard to see that

$$E(a)_U(c) = \int_K \int_K \left((U * (f_\psi^{a,k})_h) * (U * (f_\psi^{a,k})_h)^\star \right)(c)\, d\mu(h)\, d\mu(k).$$

Therefore, for every $\eta \in \mathbb{R}^n$

$$\widehat{E(a)_U}(\eta) = \int_K \int_K \left| \left(U * (f_\psi^{a,k})_h\right)\widehat{}(\eta) \right|^2 d\mu(h)\, d\mu(k)$$
$$= |\hat{U}(\eta)|^2 \int_K \int_K \left| \widehat{(f_\psi^{a,k})_h}(\eta) \right|^2 d\mu(h)\, d\mu(k) = |\hat{U}(\eta)|^2 \widehat{E(a)}(\eta). \quad (10)$$

By the inversion formula for $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} \widehat{E(a)_U}(\eta)\, d\eta = E(a)_U(0)$$
$$= \int_K \int_K \int_{\mathbb{R}^n} \left|(U * f_\psi^{a,k})_h(c)\right|^2 dc\, d\mu(h)\, d\mu(k)$$
$$= \int_K \|U * f_\psi^{a,k}\|^2_{L^2(G)} d\mu(k). \quad (11)$$

On the other hand for $d \in \mathbb{N}$ and $\gamma \in \mathbb{R}^n$, let

$$\mathcal{L}_d(\gamma) = \left\{ \eta \in \mathbb{R}^n \;\; \|\gamma\| - \frac{1}{2d} \leq \|\eta\| \leq \|\gamma\| + \frac{1}{2d} \right\}$$

denote the annulus in $\mathbb{R}^n$ and v_d its volume. For every $d \in \mathbb{N}$, there exists a sequence $(U_{d,m})_m$ of L^1-functions on $\mathbb{R}^n$ satisfying following properties:
(i) $0 \leq \widehat{U_{d,m}} \leq 1$.
(ii) $\widehat{(U_{d,m})}_m$ converges pointwise to the characteristic function $\chi_{\mathcal{L}_d(\gamma)}$ of $\mathcal{L}_d(\gamma)$.
As $\widehat{E(a)}$ is continuous and $\mathcal{L}_m(\gamma)$ has volume v_d, we have

$$\widehat{E(a)}(\gamma) = \lim_{d\to\infty} v_d^{-1} \int_{\mathcal{L}_m(\gamma)} \widehat{E(a)}(\eta)\, d\eta = \lim_{d\to\infty} v_d^{-1} \int_{\mathbb{R}^n} \lim_{m\to\infty} \widehat{E(a)}(\eta)\left(\widehat{U_{d,m}}(\eta)\right)^2 d\eta$$
$$= \lim_{d\to\infty} v_d^{-1} \int_{\mathbb{R}^n} \lim_{m\to\infty} \widehat{E(a)_{U_{d,m}}}(\eta)\, d\eta \qquad \text{(using (10))}$$
$$= \lim_{d\to\infty} v_d^{-1} \lim_{m\to\infty} \int_K \|U_{d,m} * f_\psi^{a,k}\|^2_{L^2(G)}\, d\mu(k) \qquad \text{(using (11))}$$
$$= \lim_{d\to\infty} v_d^{-1} \lim_{m\to\infty} \int_K \int_{\mathbb{R}^n/K} \sum_{\sigma\in\hat{K}_\eta} \|\pi_{\eta,\sigma}\left(U_{d,m} * f_\psi^{a,k}\right)\|^2_{HS} d\bar{\eta}\, d\mu(k).$$

From [1, p. 732], one has that

$$\|\pi_{\eta,\sigma}\left(U_{d,m} * f_\psi^{a,k}\right)\|^2_{HS} \leq \widehat{U_{d,m}}(s_\eta \cdot \eta)^2 \|\pi_{\eta,\sigma}\left(f_\psi^{a,k}\right)\|^2_{HS},$$

for some $s_\eta \in K$. It follows, using (6), that

$$\widehat{E(a)}(\gamma) \leq \lim_{d\to\infty} v_d^{-1} \lim_{m\to\infty} \int_K \int_{\mathbb{R}^n/K} \sum_{\sigma\in\hat{K}_\eta} \phi_\eta(k,\sigma)^2 \widehat{U_{d,m}}(s_\eta\cdot\eta)^2 e^{-\pi(p\|a\|^2+q\|\eta\|^2)} d\overline{\eta}\, d\mu(k)$$
$$= C \lim_{d\to\infty} v_d^{-1} \lim_{m\to\infty} \int_{\mathbb{R}^n/K} \widehat{U_{d,m}}(s_\eta\cdot\eta)^2 e^{-\pi(p\|a\|^2+q\|\eta\|^2)} d\overline{\eta}$$
$$= C \lim_{d\to\infty} v_d^{-1} \int_{\mathbb{R}^n/K} \chi_{\mathcal{L}_d(\gamma)}(\eta)^2 e^{-\pi(p\|a\|^2+q\|\eta\|^2)} d\overline{\eta}$$
$$= C \lim_{d\to\infty} v_d^{-1} \int_{\mathbb{R}^n/K} \int_K \chi_{\mathcal{L}_d(\gamma)}(h\cdot\eta) e^{-\pi(p\|a\|^2+q\|\eta\|^2)} d\mu(h)\, d\overline{\eta}$$
$$= C \lim_{d\to\infty} v_d^{-1} \int_{\mathbb{R}^n} \chi_{\mathcal{L}_d(\gamma)}(\eta) e^{-\pi(p\|a\|^2+q\|\eta\|^2)} d\eta$$
$$= C \lim_{d\to\infty} v_d^{-1} \int_{\mathcal{L}_d(\gamma)} e^{-\pi(p\|a\|^2+q\|\eta\|^2)} d\eta = C e^{-\pi(p\|a\|^2+q\|\gamma\|^2)}.$$

Finally, Eq. (9) allows us to conclude. □

3.2 Proof of Theorem 3

For $k, h \in K$, let $f_{k,h}$ and $\psi_{k,h}$ be the complex-valued functions defined on $\mathbb{R}^n$ by

$$f_{k,h}(a) = f(a, kh) \quad \text{and} \quad \psi_{k,h}(a) = \psi(k^{-1}a, h).$$

Then obviously $f_{k,h}, \psi_{k,h} \in L^2(\mathbb{R}^n)$, for almost all $h \in K$ and all $k \in K$.

For fixed $\lambda, y \in \mathbb{R}^n$, let $F_{\lambda,y}(k,h)$ and $K^{\varphi_1,\varphi_2}_{\lambda,y}$ be the functions defined on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$F_{\lambda,y}(k,h)(a,\gamma) = \mathcal{G}_{\psi_{k,h}}(\mathcal{M}_\lambda \mathcal{T}_y f_{k,h})(a,\gamma)\mathcal{G}_{\psi_{k,h}}(\mathcal{M}_\lambda \mathcal{T}_y f_{k,h})(-a,-\gamma)e^{2i\pi a\gamma}.$$

and

$$K^{\varphi_1,\varphi_2}_{\lambda,y}(a,\gamma) = \int_K \int_K F_{\lambda,y}(k,h)(a,\gamma)\varphi_1(k)\varphi_2(h) d\mu(k)\, d\mu(h),$$

where φ_1, φ_2 are bounded functions on K. Noting that, $F_{\lambda,y}(k,h)$ is well defined for almost all $h \in K$ and all $k \in K$ and

$$F_{\lambda,y}(k,h)(a,\gamma) = e^{2i\pi a\gamma} e^{-2i\pi(\gamma-\lambda)y} \mathcal{G}_{\psi_{k,h}} f_{k,h}(a-y, \gamma-\lambda)$$
$$\times e^{-2i\pi(-\gamma-\lambda)y} \mathcal{G}_{\psi_{k,h}} f_{k,h}(-a-y, -\gamma-\lambda) \quad \big(\text{using i) in Lemma 1}\big). \tag{12}$$

Lemma 4 *There exists a positive constant C_1 such that*

$$\left|K^{\varphi_1,\varphi_2}_{\lambda,y}(a,\gamma)\right| \leq C_1\, e^{-\pi\left(p\|a\|^2+q\|\gamma\|^2\right)},$$

for all $a, \gamma \in \mathbb{R}^n$. Moreover, the constant C_1 does not depend on λ and y.

Proof By using Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}\left|K_{\lambda,y}^{\varphi_1,\varphi_2}(a,\gamma)\right| &\leq \int_K\int_K \left|\mathcal{G}_{\psi_{k,h}}(\mathcal{M}_\lambda\mathcal{T}_y f_{k,h})(a,\gamma)\mathcal{G}_{\psi_{k,h}}(\mathcal{M}_\lambda\mathcal{T}_y f_{k,h})(-a,-\gamma)\right| \\ &\quad\times\left|\varphi_1(k)\varphi_2(h)\right| d\mu(h)\,d\mu(k) \\ &\leq \|\varphi_1\|_\infty\|\varphi_2\|_\infty \int_K\int_K \left|\mathcal{G}_{\psi_{k,h}}(\mathcal{M}_\lambda\mathcal{T}_y f_{k,h})(a,\gamma)\mathcal{G}_{\psi_{k,h}}(\mathcal{M}_\lambda\mathcal{T}_y f_{k,h})(-a,-\gamma)\right| \\ &\quad\times d\mu(h)\,d\mu(k) \\ &\leq \|\varphi_1\|_\infty\|\varphi_2\|_\infty\left(\int_K\int_K \left|\mathcal{G}_{\psi_{k,h}}(\mathcal{M}_\lambda\mathcal{T}_y f_{k,h})(a,\gamma)\right|^2 d\mu(h)\,d\mu(k)\right)^{\frac{1}{2}} \\ &\qquad\times\left(\int_K\int_K \left|\mathcal{G}_{\psi_{k,h}}(\mathcal{M}_\lambda\mathcal{T}_y f_{k,h})(-a,-\gamma)\right|^2 d\mu(h)\,d\mu(k)\right)^{\frac{1}{2}}.\end{aligned}$$

Remark that,

$$\begin{aligned}&\int_K\int_K \left|\mathcal{G}_{\psi_{k,h}}(\mathcal{M}_\lambda\mathcal{T}_y f_{k,h})(a,\gamma)\right|^2 d\mu(h)\,d\mu(k) \\ &= \int_K\int_K \left|\mathcal{G}_{\psi_{k,h}} f_{k,h}(a-y,\gamma-\lambda)\right|^2 d\mu(h)\,d\mu(k) \\ &= \int_K\int_K \left|\int_{\mathbb{R}^n} f_{k,h}(c)\overline{\psi}_{k,h}(c-a+y)\mathrm{e}^{-2i\pi c(\gamma-\lambda)}dc\right|^2 d\mu(h)\,d\mu(k) \\ &= \int_K\int_K \left|\int_{\mathbb{R}^n} f(c,kh)\overline{\psi}(-k^{-1}a+k^{-1}(c+y),h)\mathrm{e}^{-2i\pi c(\gamma-\lambda)}dc\right|^2 d\mu(h)\,d\mu(k) \\ &= \int_K\int_K \left|\int_{\mathbb{R}^n} f(c,h)\overline{\psi}(-k^{-1}a+k^{-1}(c+y),k^{-1}h)\mathrm{e}^{-2i\pi c(\gamma-\lambda)}dc\right|^2 d\mu(h)\,d\mu(k) \\ &= \int_K\int_K \left|\int_{\mathbb{R}^n} f(c,h)\overline{\psi}\big((a-y,k)^{-1}(c,h)\big)\mathrm{e}^{-2i\pi c(\gamma-\lambda)}dc\right|^2 d\mu(h)d\mu(k) \\ &= \int_K\int_K \left|(\widehat{f_\psi^{a-y,k}})_h(\gamma-\lambda)\right|^2 d\mu(h)\,d\mu(k).\end{aligned}$$

It results that,

$$\begin{aligned}\left|K_{\lambda,y}^{\varphi_1,\varphi_2}(a,\gamma)\right| &\leq \|\varphi_1\|_\infty\|\varphi_2\|_\infty\left(\int_K\int_K \left|(\widehat{f_\psi^{a-y,k}})_h(\gamma-\lambda)\right|^2 d\mu(h)\,d\mu(k)\right)^{\frac{1}{2}} \\ &\quad\times\left(\int_K\int_K \left|(\widehat{f_\psi^{-a-y,k}})_h(-\gamma-\lambda)\right|^2 d\mu(h)\,d\mu(k)\right)^{\frac{1}{2}} \\ &\leq C\|\varphi_1\|_\infty\|\varphi_2\|_\infty\left(e^{-\pi(p\|a-y\|^2+q\|\gamma-\lambda\|^2)}\right)^{\frac{1}{2}} \\ &\quad\left(e^{-\pi(p\|a+y\|^2+q\|\gamma+\lambda\|^2)}\right)^{\frac{1}{2}} \quad \text{(using Lemma 3)} \\ &\leq C\|\varphi_1\|_\infty\|\varphi_2\|_\infty e^{-\pi(p\|a\|^2+q\|\gamma\|^2)},\end{aligned}$$

which is the desired result. □

Lemma 5 *For all* $w, \theta \in \mathbb{R}^n$,

$$\left|\hat{K}_{\lambda,y}^{\varphi_1,\varphi_2}(w,\theta)\right| \le C_1\, e^{-\pi\left(p\|\theta\|^2+q\|w\|^2\right)}.$$

Proof By using (iii) in Lemma 1, we have

$$\begin{aligned}\hat{K}_{\lambda,y}^{\varphi_1,\varphi_2}(w,\theta) &= \int_K\int_K \hat{F}_{\lambda,y}(w,\theta)\varphi_1(k)\varphi_2(h)d\mu(k)\,d\mu(h)\\ &= \int_K\int_K F_{\lambda,y}(-\theta,w)\varphi_1(k)\varphi_2(h)d\mu(k)\,d\mu(h) = K_{\lambda,y}^{\varphi_1,\varphi_2}(-\theta,w).\end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned}\left|\hat{K}_{\lambda,y}^{\varphi_1,\varphi_2}(w,\theta)\right| \le \|\varphi_1\|_\infty\|\varphi_2\|_\infty \int_K\int_K &\left|\mathcal{G}_{\psi_{k,h}}(\mathcal{M}_\lambda\mathcal{T}_y f_{k,h})(-\theta,w)\right|\\ &\times\left|\mathcal{G}_{\psi_{k,h}}(\mathcal{M}_\lambda\mathcal{T}_y f_{k,h})(\theta,-w)\right|d\mu(h)\,d\mu(k).\end{aligned}$$

As in the proof of the Lemma 4 we can show that,

$$\left|\hat{K}_{\lambda,y}^{\varphi_1,\varphi_2}(w,\theta)\right| \le C\|\varphi_1\|_\infty\|\varphi_2\|_\infty e^{-\pi\left(p\|\theta\|^2+q\|w\|^2\right)},$$

which allows us to conclude. □

(i) For fixed λ, y and θ in $\mathbb{R}^n$, let $R_{\lambda,y,\theta}$ be the function defined on $\mathbb{R}^n$ by

$$R_{\lambda,y,\theta}(a) = \widehat{K_{\lambda,y}^{\varphi_1,\varphi_2}(a,.)}(\theta),$$

where $\widehat{K_{\lambda,y}^{\varphi_1,\varphi_2}(a,.)}$ is the partial Fourier transform of $K_{\lambda,y}^{\varphi_1,\varphi_2}$ with respect to the second variable γ. It follows, using (13), that

$$\widehat{R}_{\lambda,y,\theta}(w) = \hat{K}_{\lambda,y}^{\varphi_1,\varphi_2}(w,\theta) = K_{\lambda,y}^{\varphi_1,\varphi_2}(-\theta,w). \tag{14}$$

There exists a positive constant C_2 such that

$$\left|R_{\lambda,y,\theta}(a)\right| \le C_2\, e^{-p\pi\|a\|^2}.$$

In fact, from Lemma 4 we have,

$$\begin{aligned}|R_{\lambda,y,\theta}(a)| = \left|\widehat{K_{\lambda,y}^{\varphi_1,\varphi_2}(a,.)}(\theta)\right| &\le \int_{\mathbb{R}^n}\left|K_{\lambda,y}^{\varphi_1,\varphi_2}(a,\gamma)\right|d\gamma\\ &\le C_1\int_{\mathbb{R}^n} e^{-\pi\left(p\|a\|^2+q\|\gamma\|^2\right)}d\gamma = C_2 e^{-p\pi\|a\|^2},\end{aligned}$$

where $C_2 = C_1 \int_{\mathbb{R}^n} e^{-\pi q\|\gamma\|^2} d\gamma$. On the other hand, by (14) and Lemma 4, we have

$$\left|\hat{R}_{\lambda,y,\theta}(w)\right| = \left|K_{\lambda,y}^{\varphi_1,\varphi_2}(-\theta, w)\right| \leq C_1 e^{-q\pi\|w\|^2}.$$

By Hardy's theorem, this implies $R_{\lambda,y,\theta} \equiv 0$ and $\hat{R}_{\lambda,y,\theta} = 0$ for all $\lambda, y, \theta \in \mathbb{R}^n$. We then obtain

$$K_{\lambda,y}^{\varphi_1,\varphi_2}(-\theta, w) = \int_K \int_K F_{\lambda,y}(k,h)(-\theta, w)\varphi_1(k)\varphi_2(h)d\mu(k)\,d\mu(h) = 0,$$

for any bounded function φ_1 and φ_2 on K. Therefore, $F_{\lambda,y}(k,h)(-\theta, w) = 0$ for all λ, y, θ in $\mathbb{R}^n$ and almost all $w \in \mathbb{R}^n$. As $F_{-\lambda,-y}(k,h)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$,

$$|F_{-\lambda,-y}(k,h)(0,0)| = |G_{\psi_{k,h}} f_{k,h}(y,\lambda)|^2 = 0 \quad \text{(using (12))}.$$

Hence, $\mathcal{G}_{\psi_{k,h}} f_{k,h} \equiv 0$. By using (1), we have

$$\|\psi_{k,h}\|_2^2 \|f_{k,h}\|_2^2 = 0,$$

which implies either $\psi_{k,h} \equiv 0$ or $f_{k,h} \equiv 0$. Observe that,

$$\begin{aligned}
&\int_K \int_K \|\psi_{k,h}\|_2^2 \|f_{k,h}\|_2^2 d\mu(k)\,d\mu(h) \\
&= \int_K \int_K \left(\int_{\mathbb{R}^n} |f(c,kh)|^2 dc\right)\left(\int_{\mathbb{R}^n} |\psi(k^{-1}t,h)|^2 dt\right) d\mu(k)\,d\mu(h) \\
&= \int_K \int_K \left(\int_{\mathbb{R}^n} |f(c,h)|^2 dc\right)\left(\int_{\mathbb{R}^n} |\psi(t,k^{-1}h)|^2 dt\right) d\mu(k)\,d\mu(h) \\
&= \|f\|_2^2 \int_K \int_{\mathbb{R}^n} |\psi(t,k)|^2 dt\,d\mu(k) = \|f\|_2^2 \|\psi\|_2^2.
\end{aligned}$$

This allow us to achieve this case.

(ii) We start by treat the case $p = q = 1$. By using Lemmas 4 and 5, the function $K_{\lambda,y}^{\varphi_1,\varphi_2}$ verifies the decay conditions of Hardy's theorem on $\mathbb{R}^n \times \mathbb{R}^n$. Then,

$$K_{\lambda,y}^{\varphi_1,\varphi_2}(a,\gamma) = C_{\lambda,y}^{\varphi_1,\varphi_2} e^{-\pi\left(\|a\|^2+\|\gamma\|^2\right)},$$

where $C_{\lambda,y}^{\varphi_1,\varphi_2}$ is a positive constant.

For $\tau \in \hat{K}$, let u_{ij}^τ be the matrix coefficients of τ in an orthonormal basis $\{e_j^\tau\ 1 \leq j \leq d_\tau\}$ of its associated Hilbert space $\mathcal{H}_\tau$ of dimension d_τ. In other words, $u_{ij}^\tau(k) = \langle \tau(k)e_i^\tau, e_j^\tau\rangle$, for each $k \in K$. Peter-Weyl Theorem asserts that the set of functions $\left\{\sqrt{d_\tau}u_{ij}^\tau\ \ \tau \in \hat{K},\ 1 \leq i, j \leq d_\tau\right\}$ is an orthonormal basis of $L^2(K)$. Allowing now

φ_1 and φ_2 to vary over this base, we obtain

$$\int_K \int_K F_{\lambda,y}(k,h)(a,\gamma)\overline{u_{ij}^{\tau}(k)u_{i'j'}^{\tau'}(h)}d\mu(k)\,d\mu(h) = C_{\lambda,y}^{\tau,\tau',i,j,i',j'} e^{-\pi\left(\|a\|^2+\|\gamma\|^2\right)}.$$

This entails that

$$F_{\lambda,y}(k,h)(a,\gamma) = C_{\lambda,y}(k,h)e^{-\pi\left(\|a\|^2+\|\gamma\|^2\right)}, \tag{15}$$

where $C_{\lambda,y}(k,h) = \sum_{\tau\in\hat{K}} \sum_{1\le i,j\le d_\tau} \sum_{\tau'\in\hat{K}} \sum_{1\le i',j'\le d_{\tau'}} C_{\lambda,y}^{\tau,\tau',i,j,i',j'} u_{ij}^{\tau}(k)u_{i'j'}^{\tau'}(h)$. Moreover by using (12),

$$C_{\lambda,y}(k,h) = F_{\lambda,y}(k,h)(0,0) = e^{4\pi i\lambda y}\left(\mathcal{G}_{\psi_{k,h}} f_{k,h}\right)^2(-y,-\lambda). \tag{16}$$

□

Lemma 6 *There exist $\lambda_0, y_0 \in \mathbb{R}^n$ such that $C_{\lambda_0,y_0}(k,h)$ is different to zero whenever it exists.*

Proof There exist $\lambda_0, y_0 \in \mathbb{R}^n$ such that $C_{\lambda_0,y_0} \neq 0$, otherwise using (16), we have $\mathcal{G}_{\psi_{k,h}} f_{k,h} = 0$, for almost all $h \in K$ and all $k \in K$. Hence, $\psi_{k,h} \equiv 0$ or $f_{k,h} \equiv 0$, for almost all $h \in K$ and all $k \in K$. By taken $k = \mathrm{Id}$, we obtain $\psi(\cdot,s) \equiv 0$ or $f(\cdot,t) \equiv 0$, for almost all $s,t \in K$. It results that, $\mathcal{G}_{\psi(\cdot,s)} f(\cdot,t) \equiv 0$, contradicting the assumption of the theorem. Now, if there exist $k_0, h_0 \in K$ such that $C_{\lambda_0,y_0}(k_0,h_0) = 0$, then $|F_{\lambda_0,y_0}(k_0,h_0)(a,\gamma)| = 0$, for all $a,\gamma \in \mathbb{R}^n$. By using (12), we have

$$\left|\mathcal{G}_{\psi_{k_0,h_0}} f_{k_0,h_0}(a-y_0,\gamma-\lambda_0)\right| \left|\mathcal{G}_{\psi_{k_0,h_0}} f_{k_0,h_0}(-a-y_0,-\gamma-\lambda_0)\right| = 0,$$

for all $a,\gamma \in \mathbb{R}^n$. Thus, $\mathcal{G}_{\psi_{k_0,h_0}} f_{k_0,h_0} = 0$ and $\psi_{k_0,h_0} \equiv 0$ or $f_{k_0,h_0} \equiv 0$. As the Lebesgue measure da is invariant under the action of K on $\mathbb{R}^n$, we obtain $\psi(\cdot,h_0) \equiv 0$. Therefore for almost all $k,h \in K$, $\mathcal{G}_{\psi(\cdot,h_0)} f(\cdot,k) \equiv 0$ or $\mathcal{G}_{\psi(\cdot,h)} f(\cdot,k_0h_0) \equiv 0$. This contradicts again the hypothesis of the theorem. □

The previous lemma and (15) imply that $\left|F_{\lambda_0,y_0}(k,h)(a,\gamma)\right| \neq 0$, for all $a,\gamma \in \mathbb{R}^n$. This in turn will imply that

$$\left|\mathcal{G}_{\psi_{k,h}} f_{k,h}(a-y_0,\gamma-\lambda_0)\right| \left|\mathcal{G}_{\psi_{k,h}} f_{k,h}(-a-y_0,-\gamma-\lambda_0)\right| \neq 0,$$

for all $a,\gamma \in \mathbb{R}^n$. In particular, $\mathcal{G}_{\psi_{k,h}} f_{k,h}(a,\gamma) \neq 0$, for all $a,\gamma \in \mathbb{R}^n$. Thus we may define $H(k,h)(a,\gamma) = \log\left(\mathcal{G}_{\psi_{k,h}} f_{k,h}(-a,-\gamma)\right)$. Combining (12), (15) and (16), we get

$$\begin{aligned} &e^{4\pi i\lambda y}\left(\mathcal{G}_{\psi_{k,h}} f_{k,h}\right)^2(-y,-\lambda)e^{-\pi\left(\|a\|^2+\|\gamma\|^2\right)} \\ &\quad= e^{2i\pi a\gamma} e^{-2i\pi(\gamma-\lambda)y} \mathcal{G}_{\psi_{k,h}} f_{k,h}(a-y,\gamma-\lambda) \\ &\qquad e^{-2i\pi(-\gamma-\lambda)y} \mathcal{G}_{\psi_{k,h}} f_{k,h}(-a-y,-\gamma-\lambda). \end{aligned}$$

By taking log on both sides, we obtain

$$
\begin{aligned}
&4\pi i\lambda y + 2H(k,h)(y,\lambda) - \pi\left(\|a\|^2 + \|\gamma\|^2\right) \\
&\quad = 2i\pi a\gamma - 2i\pi(\gamma-\lambda)y + H(k,h)(y-a,\lambda-\gamma) - 2i\pi(-\gamma-\lambda)y \\
&\qquad + H(k,h)(a+y,\gamma+\lambda) + 2i\pi m,
\end{aligned}
$$

for some $m \in \mathbb{Z}$. Letting $a = \gamma = 0$ shows that $m = 0$, hence we get the following difference equation,

$$
\begin{aligned}
&H(k,h)\big((y,\lambda)+(a,\gamma)\big) - 2H(k,h)(y,\lambda) + H(k,h)\big((y,\lambda)-(a,\gamma)\big) \\
&\quad = -\pi\left(\|a\|^2 + \|\gamma\|^2\right) - 2i\pi a\gamma.
\end{aligned}
$$

The solution of the above equation can be written as

$$
H(k,h)(a,\gamma) = -\pi\left(\|a\|^2 + \|\gamma\|^2\right)/2 - i\pi a\gamma + \alpha(k,h)a + \beta(k,h)\gamma + \varsigma(k,h),
$$

where $\alpha(k,h), \beta(k,h) \in \mathbb{C}^n$ and $\varsigma(k,h) \in \mathbb{C}$. This shows that

$$
\mathcal{G}_{\psi_{k,h}} f_{k,h}(a,\gamma) = \tilde{C}(k,h) e^{-\frac{\pi}{2}\left(\|a\|^2+\|\gamma\|^2\right) - i\pi a\gamma - \alpha(k,h)a - \beta(k,h)\gamma}, \tag{17}
$$

where $\tilde{C}$ is complex valued function on $K \times K$. Hence,

$$
\left|\mathcal{G}_{\psi_{k,h}} f_{k,h}(a,\gamma)\right| = \left|\tilde{C}(k,h)\right| e^{-\frac{\pi}{2}\left(\|a\|^2+\|\gamma\|^2\right) - \mathrm{Re}(\alpha(k,h))a - \mathrm{Re}(\beta(k,h))\gamma}. \tag{18}
$$

The computation in the proof of Lemma 4 shows that,

$$
\begin{aligned}
&\int_K \int_K \left|\mathcal{G}_{\psi_{k,h}} f_{k,h}(a,\gamma)\right|^2 d\mu(h)\, d\mu(k) \\
&\quad = \int_K \int_K \left|\widehat{\left(f_\psi^{a,k}\right)}_h(\gamma)\right|^2 d\mu(h)\, d\mu(k) \le C e^{-\pi(\|a\|^2+\|\gamma\|^2)},
\end{aligned}
$$

for all $a, \gamma \in \mathbb{R}^n$. It follows, using (18), that

$$
\int_K \int_K \left|\tilde{C}(k,h)\right|^2 e^{-2\mathrm{Re}(\alpha(k,h))a - 2\mathrm{Re}(\beta(k,h))\gamma} d\mu(h)\, d\mu(k) \le C,
$$

for all $a, \gamma \in \mathbb{R}^n$. Hence for all a and all γ in the countable set $\mathbb{Z}^n$,

$$
\left|\tilde{C}(k,h)\right|^2 e^{-2\mathrm{Re}(\alpha(k,h))a - 2\mathrm{Re}(\beta(k,h))\gamma} < \infty,
$$

for almost all $h, k \in K$. This implies that $\left|\tilde{C}(k,h)\right|$ is finite for almost all $h, k \in K$, $\mathrm{Re}(\alpha(k,h)) \equiv 0$ and $\mathrm{Re}(\beta(k,h)) \equiv 0$. By choosing $\lambda(k,h) = -\alpha(k,h)/2\pi i$ and

$\delta(k,h) = \beta(k,h)/2\pi i$ we have for all $a \in \mathbb{R}^n$ and almost all $h, k \in K$,

$$\mathcal{G}_{\psi_{k,h}} f_{k,h}(a,\gamma) = \tilde{C}(k,h)e^{2\pi i(\lambda(k,h)a-\delta(k,h)\gamma)}e^{-\frac{\pi}{2}\left(\|a\|^2+\|\gamma\|^2\right)}e^{-i\pi a\gamma} \quad \text{(using (17))}.$$

Then it follows from Theorem 1.2 in [8], that for all $a \in \mathbb{R}^n$ and almost all $h, k \in K$,

$$f_{k,h}(a) = f(a,kh) = \tilde{C}_1(k,h)e^{2\pi i\lambda(k,h)a}e^{-\pi\|a-\delta(k,h)\|^2},$$
$$\text{and} \quad \psi_{k,h}(a) = \psi(k^{-1}a,h) = \tilde{C}_2(k,h)e^{2\pi i\lambda(k,h)a}e^{-\pi\|a-\delta(k,h)\|^2},$$

where $\tilde{C}_2(k,h)$, $\tilde{C}_2(k,h)$ are multiplicative constants depending on k and h. Fix k_0 in K such that for almost all $h \in K$, $\mathrm{Re}(\alpha(k_0,h)) = \mathrm{Re}(\beta(k_0,h)) = 0$. We then obtain

$$\begin{aligned}\psi(a,h) &= \tilde{C}_2(k_0,h)e^{2\pi i\lambda(k_0,h)k_0a}e^{-\pi\|k_0a-\delta(k_0,h)\|^2}\\ &= \tilde{C}_2(k_0,h)e^{2\pi ik_0^{-1}\lambda(k_0,h)a}e^{-\pi\|a-k_0^{-1}\delta(k_0,h)\|^2},\end{aligned}$$

for all $a \in \mathbb{R}^n$ and almost all $h \in K$. Therefore, we may define $C_2(h) = \tilde{C}_2(k_0,h)$, $\lambda_2(h) = k_0^{-1}\lambda(k_0,h)$ and $\delta_2(h) = k_0^{-1}\delta(k_0,h)$ and obtain for ψ the form claimed in the theorem. It is obvious that $C_2 \in L^2(K)$, simce $\psi \in L^2(G)$. On the other hand, we have

$$f(a,k_0h) = \tilde{C}_1(k_0,h)e^{2\pi i\lambda(k_0,h)a}e^{-\pi\|a-\delta(k_0,h)\|^2},$$

for all $a \in \mathbb{R}^n$ and almost all $h \in K$. As $d\mu$ is a Haar measure on K, we get

$$f(a,h) = \tilde{C}_1(k_0,k_0^{-1}h)e^{2\pi i\lambda(k_0,k_0^{-1}h)a}e^{-\pi\|a-\delta(k_0,k_0^{-1}h)\|^2}.$$

By setting $C_1(h) = \tilde{C}_1(k_0,k_0^{-1}h)$, $\lambda_1(h) = \lambda(k_0,k_0^{-1}h)$ and $\delta_1(h) = \delta(k_0,k_0^{-1}h)$, we have

$$f(a,h) = C_1(h)\mathcal{M}_{\lambda_1(h)}\mathcal{T}_{\delta_1(h)}\mathrm{e}^{-\pi\|a\|^2},$$

for all $a \in \mathbb{R}^n$ and almost all $h \in K$.

To prove the general case where $pq = 1$, we apply the following dilation. Let $\varepsilon = (q/p)^{1/4}$, $f_\varepsilon(a,h) = \varepsilon^{n/2}f(\varepsilon a,h)$ and $\psi_\varepsilon(a,h) = \varepsilon^{n/2}\psi(\varepsilon a,h)$. Noting that,

$$\begin{aligned}\mathcal{G}_{\psi_\varepsilon} f_\varepsilon\big((a,k),\pi_{\gamma,\sigma}\big) &= \int_K\int_{\mathbb{R}^n} f_\varepsilon(c,h)\overline{\psi}_\varepsilon(-k^{-1}(a-c),k^{-1}h)\pi_{\gamma,\sigma}(c,h)dc\,d\mu(h)\\ &= \varepsilon^n\int_K\int_{\mathbb{R}^n} f(\varepsilon c,h)\overline{\psi}(-k^{-1}(\varepsilon a-\varepsilon c),k^{-1}h)\pi_{\gamma,\sigma}(c,h)dc\,d\mu(h)\\ &= \int_K\int_{\mathbb{R}^n} f(c,h)\overline{\psi}(-k^{-1}(\varepsilon a-c),k^{-1}h)\pi_{\gamma,\sigma}(c/\varepsilon,h)dc\,d\mu(h)\\ &= \mathcal{G}_\psi f\big((\varepsilon a,k),\pi_{\gamma/\varepsilon,\sigma}\big) \quad \text{(using(4))}.\end{aligned}$$

Therefore for all $(a,k)\in G$, $\gamma\in\mathbb{R}^n$ and $\sigma\in\hat{K}_l$,

$$\begin{aligned}\left\|\mathcal{G}_{\psi_\varepsilon} f_\varepsilon\big((a,k),\pi_{\gamma,\sigma}\big)\right\|_{HS} &= \left\|\mathcal{G}_{\psi} f\big((\varepsilon a,k),\pi_{\gamma/\varepsilon,\sigma}\big)\right\|_{HS}\\ &\le \phi_{\gamma/\varepsilon}(k,\sigma)e^{-\frac{\pi}{2}(p\|\varepsilon a\|^2+q\|\gamma/\varepsilon\|^2)}\\ &= \phi_{\gamma/\varepsilon}(k,\sigma)e^{-\frac{\pi}{2}\sqrt{pq}(\|a\|^2+\|\gamma\|^2)}.\end{aligned}$$

This implies that f_ε and ψ_ε have the required form, as well as f and ψ.

(iii) We show in this case that the functions $f_{\zeta_1,r}$ and $\psi_{\zeta_2,r}$ defined on G by

$$f_{\zeta_1,r}(c,h)=\zeta_1(h)e^{-\pi r\|c\|^2}\quad\text{and}\quad \psi_{\zeta_2,r}(c,h)=\zeta_2(h)e^{-\pi r\|c\|^2}$$

satisfy condition (6) of Theorem 3 for any $r\in[p,1/q]$ and any $\zeta_1,\zeta_2\in L^2(K)$. Indeed, for $g=(a,k)\in G$, we have

$$\begin{aligned}\big(f_{\zeta_1,r}\big)^g_{\psi_{\zeta_2,r}}(c,h) &= f_{\zeta_1,r}(c,h)\overline{\psi_{\zeta_2,r}}\big((a,k)^{-1}(c,h)\big)\\ &= f_{\zeta_1,r}(c,h)\overline{\psi_{\zeta_2,r}}\big(-k^{-1}(a-c),k^{-1}h\big)\\ &= \zeta_1(h)\overline{\zeta_2}(k^{-1}h)e^{-\pi r\|c\|^2}e^{-\pi r\|c-a\|^2}.\end{aligned}$$

Thus,

$$\begin{aligned}\Big(\big(f_{\zeta_1,r}\big)^g_{\psi_{\zeta_2,r}}(\cdot,h)\Big)^{\widehat{}}(\gamma) &= \zeta_1(h)\overline{\zeta_2}(k^{-1}h)\int_{\mathbb{R}^n} e^{-\pi r\|c\|^2}e^{-\pi r\|c-a\|^2}e^{-2i\pi\gamma c}dc\\ &= \zeta_1(h)\overline{\zeta_2}(k^{-1}h)e^{-i\pi\gamma a}\int_{\mathbb{R}^n} e^{-\pi r\|c+a/2\|^2}e^{-\pi r\|c-a/2\|^2}e^{-2i\pi\gamma c}dc\\ &= \zeta_1(h)\overline{\zeta_2}(k^{-1}h)e^{-i\pi\gamma a}e^{-\frac{\pi}{2}r\|a\|^2}\int_{\mathbb{R}^n} e^{-2\pi r\|c\|^2}e^{-2i\pi\gamma c}dc\\ &= (2r)^{-n/2}\zeta_1(h)\overline{\zeta_2}(k^{-1}h)e^{-i\pi\gamma a}e^{-\frac{\pi}{2}r\|a\|^2}e^{-\frac{\pi\|\gamma\|^2}{2r}}.\end{aligned}$$

By using (5), the kernel of the operator $\pi_{\gamma,\sigma}\left(\big(f_{\zeta_1,r}\big)^g_{\psi_{\zeta_2,r}}\right)$ is defined on $K/K_\gamma\times K/K_\gamma$ by

$$\begin{aligned}H\left(\big(f_{\zeta_1,r}\big)^g_{\psi_{\zeta_2,r}},\gamma,\sigma\right)(s,u) &= (2r)^{-n/2}e^{-i\pi s\cdot\gamma a}e^{-\frac{\pi}{2}(r\|a\|^2+\frac{1}{r}\|\gamma\|^2)}\\ &\quad\times\int_{K_\gamma}\zeta_1(svu^{-1})\overline{\zeta_2}(k^{-1}svu^{-1})\sigma(v)d\mu_\gamma(v)\\ &= (2r)^{-n/2}e^{-i\pi s\cdot\gamma a}e^{-\frac{\pi}{2}(r\|a\|^2+\frac{1}{r}\|\gamma\|^2)}\sigma\left(\zeta(s,u,k)\right),\end{aligned}$$

where $\zeta(s,u,k)(v)=\zeta_1(svu^{-1})\overline{\zeta_2}(k^{-1}svu^{-1})$, for all $v\in K_\gamma$. It follows, using (7), that

$$\begin{aligned} &\left\|\mathcal{G}_{\psi_{\zeta_2,r}} f_{\zeta_1,r}(g,\pi_{\gamma,\sigma})\right\|_{HS} = \left\|\pi_{\gamma,\sigma}\Big(\big(f_{\zeta_1,r}\big)^{g}_{\psi_{\zeta_2,r}}\Big)\right\|_{HS} \\ &= (2r)^{-n/2} e^{-\frac{\pi}{2}(r\|a\|^2+\frac{1}{r}\|\gamma\|^2)} \left(\int_{K/K_\gamma}\int_{K/K_\gamma} \|\sigma\left(\zeta(s,u,k)\right)\|^2_{HS}\, d\dot{\mu}_\gamma(sK_\gamma) d\dot{\mu}_\gamma(uK_\gamma)\right)^{\frac{1}{2}} \\ &\leq (2r)^{-n/2}\tilde{\phi}_\gamma(k,\sigma) e^{-\frac{\pi}{2}(p\|a\|^2+q\|\gamma\|^2)}, \end{aligned}$$

where $\tilde{\phi}_\gamma(k,\sigma) = \left(\int_{K/K_\gamma}\int_{K/K_\gamma} \|\sigma\left(\zeta(s,u,k)\right)\|^2_{HS}\, d\dot{\mu}_\gamma(sK_\gamma) d\dot{\mu}_\gamma(uK_\gamma)\right)^{\frac{1}{2}}$. Finally, notice that

$$\begin{aligned} &\int_K \sum_{\sigma\in\hat{K}_\gamma} \tilde{\phi}_\gamma(k,\sigma)^2 \\ &= \int_K \int_{K/K_\gamma}\int_{K/K_\gamma} \sum_{\sigma\in\hat{K}_\gamma} \|\sigma\left(\zeta(s,u,k)\right)\|^2_{HS}\, d\dot{\mu}_\gamma(sK_\gamma) d\dot{\mu}_\gamma(uK_\gamma)\, d\mu(k) \\ &= \int_K \int_{K/K_\gamma}\int_{K/K_\gamma}\int_{K_\gamma} \left|\zeta_1(svu^{-1})\overline{\zeta_2}(k^{-1}svu^{-1})\right|^2 d\mu_\gamma(v) \\ &\quad d\dot{\mu}_\gamma(sK_\gamma) d\dot{\mu}_\gamma(uK_\gamma)\, d\mu(k) \end{aligned}$$

(using the Plancherel formula forK_γ)

$$\begin{aligned} &= \|\zeta_2\|_2^2 \int_{K/K_\gamma}\int_{K/K_\gamma}\int_{K_\gamma} \left|\zeta_1(svu^{-1})\right|^2 d\mu_\gamma(v)\, d\dot{\mu}_\gamma(sK_\gamma) d\dot{\mu}_\gamma(uK_\gamma) \\ &= \|\zeta_2\|_2^2 \int_{K/K_\gamma}\int_K \left|\zeta_1(ku^{-1})\right|^2 d\mu(k)\, d\dot{\mu}_\gamma(uK_\gamma) \leq \|\zeta_2\|_2^2\|\zeta_1\|_2^2, \end{aligned}$$

which is independent of γ. This completes the proof.

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