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A refinement of the Hille–Wintner comparison theorem and new nonoscillation criteria for half-linear differential equations

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Abstract
A refinement of the Hille–Wintner comparison theorem is obtained for two half-linear differential equations of the second order. As a consequence, some new nonoscillation tests for such equations are derived by means of this improved comparison technique. In most of our results coefficients and their integrals do not need to be nonnegative and are allowed to oscillate in any neighborhood of infinity.

Keywords Half-linear differential equation · Non-oscillatory solutions · Hille–Wintner comparison theorem · Generalized Riccati equation

Mathematics Subject Classification 34C10

1 Introduction

Two basic comparison principles in the theory of linear differential equations which relate oscillation (resp. nonoscillation) of all solutions of a pair of equations

$$y'' + q_1(t)y = 0 \qquad (\mathrm{L}_1)$$

and

$$z'' + q_2(t)z = 0, \qquad (\mathrm{L}_2)$$

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where $q_i : [a, \infty) \to \mathbb{R}$, $i = 1, 2$, are continuos functions, are the Sturm comparison theorem which asserts that nonoscillation of (L_2) implies that of (L_1) (or, equivalently, oscillation of (L_1) implies that of (L_2)) provided that

$$q_1(t) \le q_2(t) \tag{1.1}$$

for all sufficiently large t, and the Hille–Wintner theorem in which the same conclusion is obtained under the condition that q_1 and q_2 are integrable on $[a, \infty)$ and the pointwise inequality (1.1) between coefficients is replaced by the integral inequality

$$\left| \int_t^\infty q_1(s)ds \right| \le \int_t^\infty q_2(s)ds \tag{1.2}$$

holding for all t large enough (see Hille [8] and Wintner [25, 26]).

Both results have been extended to the pair of nonlinear differential equations of the form

$$\left(|y'|^{\alpha-1}y'\right)' + q_1(t)|y|^{\alpha-1}y = 0 \qquad (\mathrm{E}_1)$$

and

$$\left(|z'|^{\alpha-1}z'\right)' + q_2(t)|z|^{\alpha-1}z = 0, \qquad (\mathrm{E}_2)$$

where $\alpha > 0$ is a given constant and q_1 and q_2 are as before. See Mirzov [20] and Elbert [4, 5] for generalization of the Sturm theorem and Kusano et al. [9, 14–16] for extension of the Hille–Wintner integral comparison theorem.

Here, by a solution of (E_i) for a fixed i we understand a real-valued function y which is continuously differentiable on $[t_y, \infty)$ for some $t_y \ge a$ together with $|y'|^{\alpha-1}y'$ and satisfies (E_i) on $[t_y, \infty)$. In this paper we consider only solutions of (E_i) which are not identically zero in any neighborhood of infinity. We call such a solution *oscillatory* if it has arbitrarily large zeros in $[t_y, \infty)$; otherwise we say that it is *nonoscillatory*. Since it is well-known that if one solution of (E_i) is oscillatory (resp. nonoscillatory), then all of them are so, it is natural to call equation (E_i) itself *oscillatory* (resp. *nonoscillatory*) if one (and so all) of its solutions enjoy the respective property.

A variety of sufficient conditions for nonoscillation of (E_1) can be obtained by application of any of the above comparison theorems in particular situations where (E_2) is a suitable nonoscillatory equation. For example, if (E_2) is the nonoscillatory Euler type equation

$$\left(|z'|^{\alpha-1}z'\right)' + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} t^{-\alpha-1}|z|^{\alpha-1}z = 0, \quad t \ge a \ge 1,$$

then (1.2) gives the Hille's nonoscillation criterion

$$t^\alpha \left| \int_t^\infty q_1(s)ds \right| \le \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^\alpha \tag{1.3}$$

which is assumed to hold on some $[T, \infty)$, $T \geq a$.

However, one shortfall of the classical Hille–Wintner theorem and its half-linear extension is that it does not imply another well-known nonoscillation test which improves (1.3), namely, the Hille-Nehari criterion which assumes the satisfaction of the condition

$$-\frac{2\alpha+1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha} \leq t^{\alpha}\int_t^{\infty} q_1(s)ds \leq \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha} \tag{1.4}$$

for all large t (see Došlý [1]).

Thus, one of our main purposes here is to extend and refine integral comparison criterion (1.2) so that it would not only imply (1.4), but also unify other earlier results and produce a series of new sufficient conditions for nonoscillation of equation (E_1). We emphasize that in our approach we do not suppose *apriori* that the coefficient $q_1(t)$ and/or its integral $\int_t^{\infty} q_1(s)ds$ (if it exists) is nonnegative in some neighborhood of infinity. Some of our results were motivated by their linear prototypes which were obtained by Kamenev [11–13] and Wong [27].

A survey of generalizations and extensions of the Hille-Wintner theorem in the half-linear settings published before the year 2005 as well as an overwiev of methods and techniques used in their proofs can be found in the monograph Došlý et al [2]. A number of useful nonoscillation criteria for half-linear differential equations of the second order generalizing the classical linear results of Hille, Potter, Moore, Willett and others have been obtained in Li and Yeh [18, 19]. For more recent results concerning this topic we refer to Došlý and Pátiková [3], Fišnarová and Mařik [6], Hasil and Veselý [7], Kandelaki, Lomtatidze and Ugulava [10], Li and Yeah [19], Naito [21], Pátiková [22], Sugie and Wu [24] and Yang and Lo [28].

2 Main results

The following necessary and sufficient condition for nonoscillation of (E_1) which will be used in the proof of our main theorem is a direct consequence of the (generalized) Sturm comparison theorem. For the proof see Skhalykho [23] or Li and Yeh [17].

Lemma 2.1 *Eq.* (E_1) *is nonoscillatory if and only if there exists a function* $u \in C^1([t_1, \infty), \mathbb{R})$ *for some* $t_1 \geq a$ *such that*

$$u'(t) + \alpha\left|u(t)\right|^{1+\frac{1}{\alpha}} + q_1(t) \leq 0$$

for $t \geq t_1$.

Our main result now follows. As can be seen from its subsequent applications in the nonoscillation theory of secod order half-linear differential equations, it significantly improves the classical Hille–Wintner comparison theorem.

Theorem 2.1 *Let $Q_1(t)$ and $Q_2(t)$ be continuously differentiable real-valued functions such that*

$$Q_1'(t) = -q_1(t) \quad and \quad Q_2'(t) = -q_2(t)$$

on $[a, \infty)$ and equation (E_2) be nonoscillatory. If there exists a $T \geq a$ such that

$$\left|v(t) - Q_2(t) + Q_1(t)\right| \leq |v(t)| \tag{2.1}$$

for $t \geq T$, where v is a solution of

$$v' + \alpha|v|^{1+\frac{1}{\alpha}} + q_2(t) = 0 \tag{2.2}$$

on $[T, \infty)$, then equation (E_1) is also nonoscillatory.

Proof . Define

$$u(t) = v(t) - Q_2(t) + Q_1(t), \quad t \geq T \geq a.$$

Then

$$\begin{aligned} &u'(t) + \alpha|u(t)|^{1+\frac{1}{\alpha}} + q_1(t) \\ &\quad = v'(t) + q_2(t) - q_1(t) + \alpha\left|v(t) - Q_2(t) + Q_1(t)\right|^{1+\frac{1}{\alpha}} + q_1(t) \\ &\quad = -\alpha\left|v(t)\right|^{1+\frac{1}{\alpha}} + \alpha\left|v(t) - Q_2(t) + Q_1(t)\right|^{1+\frac{1}{\alpha}} \\ &\quad = \alpha\left[\left|v(t) - Q_2(t) + Q_1(t)\right|^{1+\frac{1}{\alpha}} - \left|v(t)\right|^{1+\frac{1}{\alpha}}\right] \leq 0 \end{aligned}$$

for all large t because of (2.1) and the assertion follows from Lemma 2.1. □

There exists a large class of primitives of $-q_1(t)$ and $-q_2(t)$ which can be used in Theorem 2.1 for $Q_1(t)$ and $Q_2(t)$, respectively. For example, if q_1 and q_2 are integrable on $[a, \infty)$ (possibly only conditionally), we can take

$$Q_1(t) = \int_t^\infty q_1(s)ds, \quad Q_2(t) = \int_t^\infty q_2(s)ds, \quad t \geq a. \tag{2.3}$$

Or, if there exist finite limits

$$c_i = \lim_{t\to\infty} \frac{\alpha}{t^\alpha} \int_a^t s^{\alpha-1} \int_a^s q_i(\tau)d\tau ds, \quad i = 1, 2, \tag{2.4}$$

we can take

$$Q_i(t) = c_i - \int_a^t q_i(s)ds, \quad t \geq a, \quad i = 1, 2. \tag{2.5}$$

We remark that if $\lim_{t\to\infty}\int_a^t q_i(s)ds$ exist and are finite, then

$$c_i = \lim_{t\to\infty}\int_a^t q_i(s)ds = \int_a^\infty q_i(t)dt \quad i = 1, 2. \tag{2.6}$$

On the other hand, there are functions for which the limit in (2.6) does not exist as a finite number, but at the same time (2.4) is satisfied. An example of this type of functions is given, for example, in [10].

It is known that if (2.4) holds for a fixed i and equation (E_i) is nonoscillatory, then the solution v_i of the associated Riccati equation $v_i' + \alpha|v_i|^{1+\frac{1}{\alpha}} + q_i(t) = 0$ can be expressed as

$$v_i(t) = Q_i(t) + \alpha\int_t^\infty |v_i(s)|^{1+\frac{1}{\alpha}}ds \tag{2.7}$$

on $[T, \infty)$ for some $T \geq a$, where Q_i is given by (2.5) (see Kandelaki et al. [10]).

Remark 2.1 Clearly, if both limits in (2.4) exist as finite numbers and the functions $Q_1(t)$ and $Q_2(t)$ defined by (2.5) satisfy

$$\big|Q_1(t)\big| \leq Q_2(t)$$

for all large t, then from (2.7) it follows that $v(t) - Q_2(t) \geq 0$ and

$$\begin{aligned}\big|v(t) - Q_2(t) + Q_1(t)\big| &\leq \big[v(t) - Q_2(t)\big] + \big|Q_1(t)\big| \\ &\leq v(t) - Q_2(t) + Q_2(t) = v(t)\end{aligned}$$

for any (nonnegative) solution v of (2.2), so that the Hille-Wintner criterion is contained in the new result.

Corollary 2.1 *Let $k > 0$ be a given constant. If Q_1 is a continuously differentiable function such that $Q_1'(t) = -q_1(t)$ on $[a, \infty)$ and*

$$-k^{\frac{\alpha}{\alpha+1}} - k \;\leq\; t^\alpha Q_1(t) \;\leq\; k^{\frac{\alpha}{\alpha+1}} - k \tag{2.8}$$

for all sufficiently large t, then equation (E_1) is nonoscillatory.

Proof Compare (E_1) through (2.1) with the nonoscillatory equation

$$\big(|z'|^{\alpha-1}z'\big)' + \alpha\big(k^{\frac{\alpha}{\alpha+1}} - k\big)t^{-\alpha-1}|z|^{\alpha-1}z = 0$$

for which $v(t) = k^{\alpha/(\alpha+1)}t^{-\alpha}$ is the exact solution of the corresponding Riccati equation. $\square$

Remark 2.2 The right-hand side of the second inequality in (2.8) as the function of k assumes its maximum at $k = \big(\alpha/(\alpha+1)\big)^{\alpha+1}$. With this value of k the condition (2.8) becomes the Hille–Nehari criterion

$$-\frac{2\alpha+1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha} \le t^{\alpha}Q_1(t) \le \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha}$$

which extends (1.4) (where $Q_1(t) = \int_t^\infty q_1(s)ds$) to the larger class of coefficients satisfying (2.4) (cf. with Theorem 1.6 from Kandelaki [10]).

Theorem 2.2 *Suppose that* (2.4) *holds and Eq.* (E_2) *is nonoscillatory. If*

$$\left|\alpha\int_t^\infty \big|v(s)\big|^{1+\frac{1}{\alpha}}ds + Q_1(t)\right| \le \left|\alpha\int_t^\infty \big|v(s)\big|^{1+\frac{1}{\alpha}}ds + Q_2(t)\right| \tag{2.9}$$

for $t \ge T \ge a$, where v is the solution of (2.2) *on* $[T,\infty)$*, then Eq.* (E_1) *is also nonoscillatory.*

Proof Since (2.4) holds and (E_2) is supposed to be nonoscillatory, we can express the solution v of (2.2) as $v(t) = Q_2(t) + \alpha\int_t^\infty |v(s)|^{1+\frac{1}{\alpha}}ds$. Inserting this integral expression for v into the left-hand side of (2.1) and using (2.9), we find that all conditions of Theorem 2.1 are satisfied, and so equation (E_1) is nonoscillatory. □

A class of explicitly solvable Riccati equations of type (2.2) which can be used in Theorems 2.1 and 2.2 includes equations

$$v' + \alpha|v|^{1+\frac{1}{\alpha}} - \alpha|f(t)|^{1+\frac{1}{\alpha}} - f'(t) = 0 \tag{2.10}$$

and

$$v' + \alpha|v|^{1+\frac{1}{\alpha}} + f'(t) - \alpha|f(t)|^{1+\frac{1}{\alpha}} = 0$$

with the exact solutions $v = f$ and $v = -f$, respectively. In particular, if in (2.9) we use (2.10) and define Q_2 by (2.3) where $q_2(t) = -\alpha|f(t)|^{1+\frac{1}{\alpha}} - f'(t)$, we obtain

Theorem 2.3 *Suppose that for q_1 the limit in* (2.4) *exists as a finite number and define Q_1 by* (2.5)*. If there exists a function $f \in C^1([a,\infty),\mathbb{R})$ such that* $\lim_{t\to\infty} f(t) = 0$, $\int_a^\infty |f(t)|^{1+\frac{1}{\alpha}}dt < \infty$ *and*

$$\left|\alpha\int_t^\infty \big|f(s)\big|^{1+\frac{1}{\alpha}}ds + Q_1(t)\right| \le |f(t)|$$

for all large t, then Eq. (E_1) *is nonoscillatory.*

Remark 2.3 . Similarly as in the linear case (see Kamenev [12]) it can be shown that the existence of a function f with the properties stated in Theorem 2.3 is also a necessary condition for nonoscillation of (E_1).

Corollary 2.2 *Suppose that* (2.4) *holds. If*

$$\int_a^{\infty} \left|Q_1(t)\right|^{(\alpha+1)/\alpha} dt < \infty$$

and for all sufficiently large t *the inequality*

$$\int_t^{\infty} |Q_1(s)|^{1+\frac{1}{\alpha}} ds \le \frac{(\alpha+1)^{-\frac{\alpha+1}{\alpha}}}{\alpha} \big[(\alpha+1)|Q_1(t)| - Q_1(t)\big], \tag{2.11}$$

holds, then Eq. (E_1) *is nonoscillatory.*

Proof It follows from Theorem 2.3 where $f(t) = (\alpha+1)\left|Q_1(t)\right|$. □

Remark 2.4 . Under the additional restriction $Q_1(t) \ge 0$ on $[a, \infty)$, condition (2.11) in Corollary 2.2 reduces to the Opial type nonoscillation criterion

$$\int_t^{\infty} Q_1(s)^{1+\frac{1}{\alpha}} ds \le \big(\alpha+1\big)^{-\frac{\alpha+1}{\alpha}} Q_1(t) \tag{2.12}$$

for all sufficiently large t. This results can be generalized further as follows.

Theorem 2.4 *Let* $Q_1 : [a, \infty) \to [0, \infty)$ *be a continuously differentiable function such that* $Q_1'(t) = -q_1(t)$ *on* $[a, \infty)$. *If there exists a function* $\beta(t)$ *such that*

$$\alpha \int_t^{\infty} \left|Q_1(s) + \beta(s)\right|^{1+\frac{1}{\alpha}} ds \le \beta(t) \tag{2.13}$$

for all large t, *then equation* (E_1) *is nonoscillatory.*

Proof Define

$$u(t) = Q_1(t) + \alpha \int_t^{\infty} \left|Q_1(s) + \beta(s)\right|^{1+\frac{1}{\alpha}} ds \quad \big(\ge 0\big).$$

Then

$$u'(t) = -q_1(t) - \alpha\left|Q_1(t) + \beta(t)\right|^{1+\frac{1}{\alpha}}$$

and since $|u(t)|^{1+\frac{1}{\alpha}} \le |Q_1(t) + \beta(t)|^{1+\frac{1}{\alpha}}$ by (2.13), we finally obtain

$$u'(t) + \alpha|u(t)|^{1+\frac{1}{\alpha}} + q_1(t) \le -\alpha|Q_1(t) + \beta(t)|^{1+\frac{1}{\alpha}} + \alpha|Q_1(t) + \beta(t)|^{1+\frac{1}{\alpha}} = 0$$

for all t large enough. Conclusion now follows from Lemma 2.1. □

Remark 2.5 . If, in Theorem 2.4, the function $\beta(t)$ is set equal to $\alpha Q_1(t)$, then (2.13) reduces to Opial's criterion (2.12). The advantage of more general condition (2.13) is illustrated by the following example.

Example 2.1 Consider equation (E_1) with the coefficient

$$q_1(t) = \frac{1}{ct^\alpha}\left(\alpha \frac{1+\sin t}{t} - \cos t\right), \quad c > 0, \tag{2.14}$$

which changes its sign infinitely often on any interval of the form $[T, \infty)$, $T \geq 1$. It is easy to verify that in this case we can take

$$Q_1(t) = \frac{1+\sin t}{ct^\alpha} \quad (\geq 0), \quad t \geq 1.$$

To apply Theorem 2.4, we choose

$$\beta(t) = \frac{1}{ct^\alpha},$$

and show without difficulty that (2.13) holds for all large t if $c \geq 3^{\alpha+1}$, so that equation (E_1) with q_1 given by (2.14) is nonoscillatory in this case. It is to be remarked that Opial's condition (2.12) is not satisfied for any $c > 0$.

The proof of the following theorem is similar to that of Theorem 2.4 and is omitted.

Theorem 2.5 *Let Q_1 be a continuously differentiable real-valued function such that $Q_1'(t) = -q_1(t)$ and $Q_1(t) \leq 0$ on $[a, \infty)$. If there exists a function $\beta(t)$ such that*

$$\alpha \int_t^\infty \left| -Q_1(s) + \beta(s)\right|^{1+\frac{1}{\alpha}} ds \leq -2Q_1(t) + \beta(t)$$

for all large t, then equation (E_1) is nonoscillatory.

Example 2.2 Consider equation (E_1) with the oscillatory coefficient as in (2.14), but with the opposite sign, i.e.

$$q_1(t) = \frac{1}{ct^\alpha}\left(\cos t - \alpha \frac{1+\sin t}{t}\right), \quad c > 0. \tag{2.15}$$

In this case $Q_1(t) \leq 0$ on $[1, \infty)$ and neither (2.12) nor Theorem 2.4 are applicable. However, if we take $\beta(t) = 1/(ct^\alpha)$ with $c \geq 3^{\alpha+1}$, then it is not difficult to verify that all conditions of Theorem 2.5 are satisfied, and so equation (E_1) with coefficient q_1 given by (2.15) must be nonoscillatory.

Motivated by Wong's Theorem 4 in [27] we unify the above two theorems in a way which enables to handle also the case where $Q_1(t)$ is not eventually nonnegative or eventually nonpositive.

Theorem 2.6 *Let $Q_1 \in C\big([a, \infty), \mathbb{R}\big)$ be such that $Q_1'(t) = -q_1(t)$ on $[a, \infty)$. If there exists a function $\beta(t)$ such that*

$$\left|Q_1(t) + \alpha \int_t^\infty \left|Q_1(s) + \beta(s)\right|^{1+\frac{1}{\alpha}} ds\right| \leq \left|Q_1(t) + \beta(t)\right|$$

for all sufficiently large t, then equation (E_1) is nonoscillatory.

The proof of this theorem is left to the reader.

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