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On the deductive strength of the Erdős–Dushnik–Miller theorem and two order-theoretic principles

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Abstract

We provide answers to open questions from Banerjee and Gopaulsingh (Bull Pol Acad Sci Math 71: 1–21, 2023) about the relationship between the Erdős–Dushnik–Miller theorem (EDM) and certain weaker forms of the Axiom of Choice (AC), and we properly strengthen some results from Banerjee and Gopaulsingh (2023). We also settle a part of an open question of Lajos Soukup (stated in Banerjee and Gopaulsingh (2023) [Question 6.1]) about the relationship between the following two order-theoretic principles, which [as shown in Banerjee and Gopaulsingh (2023)] are weaker than EDM: (a) "Every partially ordered set such that all of its antichains are finite and all of its chains are countable is countable" (this is known as Kurepa's theorem), and (b) "Every partially ordered set such that all of its antichains are countable and all of its chains are finite is countable". In particular, we prove that (b) does not imply (a) in ZF (i.e., Zermelo–Fraenkel set theory without AC). Moreover, with respect to (b), we answer an open question from Banerjee and Gopaulsingh (2023) about its relationship with the following weak choice form: "Every set is either well orderable or has an amorphous subset"; in particular, we show that (b) follows from, but does not imply, the latter weak choice principle in ZFA (i.e., Zermelo–Fraenkel set theory with atoms).

Keywords Axiom of choice · Weak axioms of choice · Erdős–Dushnik–Miller theorem · Graph · Partially ordered set · Chain · Antichain · Kurepa's theorem · Permutation model

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1 Introduction

The Erdős–Dushnik–Miller theorem (abbreviated here, and in [2], as EDM) asserts that any infinite graph $G = (V, E)$ not containing an infinite independent set contains a complete subgraph of size $|V|$ (definitions of terms will be given in Sect. 2). EDM was established by Ben Dushnik and E. W. Miller in 1941 (see [4, Theorem 5.23]),[1] who credited Paul Erdős with assistance in its proof; in particular, according to the authors [4], Erdős suggested the proof for the case in which $|V|$ is a singular cardinal. It should be mentioned here that for graphs with countably infinite set of vertices, the result was already known since Ramsey [14] had proved (in 1929) that if G is an infinite graph, then either G contains an infinite independent set or G contains an infinite complete subgraph. So, EDM can be explicitly stated in terms of graphs with an uncountable set of vertices.

Banerjee and Gopaulsingh [2] considered, in ZF and in ZFA, the following formally weaker version of EDM: any graph $G = (V, E)$ with an uncountable set V of vertices not containing an infinite independent set contains a complete subgraph $G' = (V', E')$ with V' uncountable.[2] It is this specific graph-theoretic form that we shall henceforth refer to as Erdős–Dushnik–Miller theorem and denote by EDM. The authors in [2] studied the interrelation of EDM with several weaker forms of AC, producing fruitful information via positive and independence results that shed light on the open problem of the placement of EDM in the hierarchy of choice principles.

The research in this paper is motivated by the study in [2] and our aim here is to provide answers to some intriguing open problems stated therein. For example, it was shown in [2] that the Principle of Dependent Choices (DC) does not imply EDM in ZF, and the authors asked whether the stronger $\mathsf{AC}^{\mathsf{LO}}$ (the axiom of choice for linearly ordered families of non-empty sets), and thus whether $\mathsf{AC}^{\mathsf{WO}}$ (the axiom of choice for well-ordered families of non-empty sets, which is also stronger than DC—see Jech [11, Theorems 8.2, 8.3]), implies EDM in ZFA; see [2, Question 6.4]. We will *settle this open problem* by providing a non-trivial negative answer; in particular, we will construct a *new Fraenkel–Mostowski model* and prove that $\mathsf{AC}^{\mathsf{LO}} \wedge \neg\mathsf{EDM}$ is true in the model (see Theorem 3). It should be noted here that in ZF set theory, $\mathsf{AC}^{\mathsf{LO}}$ does imply EDM since $\mathsf{AC}^{\mathsf{LO}}$ is equivalent to AC in ZF (see Howard and Rubin [9]). Whether or not there is a model of ZF satisfying $\mathsf{AC}^{\mathsf{WO}} \wedge \neg\mathsf{EDM}$ is still open; our conjecture is that the answer is in the affirmative.

On the other hand, the following two order-theoretic principles:

(a) Every partially ordered set such that all of its antichains are finite and all of its chains are countable is countable;

(b) Every partially ordered set such that all of its antichains are countable and all of its chains are finite is countable,

[1] In [4], EDM also appears in the following equivalent complementary form: any infinite graph $G = (V, E)$ not containing an infinite complete subgraph contains an independent set of size $|V|$.

[2] More precisely, the authors [2] considered the following formally stronger version: any graph $G = (V, E)$ with an uncountable set V of vertices either contains a countably infinite independent set or contains a complete subgraph $G' = (V', E')$ such that V' is uncountable. However, all results from [2] on the latter version also hold for our formally weaker statement.

for which it seems possible to be (closely) related to EDM, were also addressed in [2]; (a) is known as Kurepa's theorem (see [13]) and has been extensively studied (in set theory without the full power of AC) by Banerjee [1] and Tachtsis [17]. In [2], it was shown that (a) and (b) are strictly weaker than EDM in ZFA, and that (a) does not imply (b) in ZFA. (Whether or not there is a model of ZF in which (a) is true, but (b) is false is—to the best of our knowledge—unknown.) The latter non-implication answers, as mentioned in [2], a question raised by Lajos Soukup about the relationship between (a) and (b). Thus there remains (until now) the open problem whether (b) implies (a); see also [2, Question 6.1]. We will *settle this open problem* by establishing that (b) does not imply (a) in ZF (see Theorem 5). To achieve this goal, we shed more light on the deductive strength of (b) by proving that DC implies (b) (see Theorem 4), which was unknown until now (whereas it is known that DC does not imply (a) in ZF, see [1]).

Last but not least, the authors in [2, Question 6.3] asked whether "Every set is either well orderable or has an amorphous subset" (denoted by WOAM) implies (b) (and we note that, in [2], it was shown that WOAM does not imply EDM in ZFA). We *answer (non-trivially) the above open question* in the affirmative and we observe that the implication is not reversible in ZFA (see Theorem 6). Concluding remarks and open questions are given in Sect. 5.

2 Notation and terminology

Definition 1 Let X and Y be sets. We write:

1. $|X| \leq |Y|$ if there is an injection $f : X \to Y$;
2. $|X| = |Y|$ if there is a bijection $f : X \to Y$;
3. $|X| < |Y|$ if $|X| \leq |Y|$ and $|X| \neq |Y|$.

Definition 2 A set X is called:

1. *denumerable* if $|X| = \aleph_0$ (where $\aleph_0$ is the first infinite, well-ordered cardinal, i.e., $\aleph_0 = \omega$, the set of natural numbers);
2. *countable* if it is either finite (i.e., $|X| = n$ for some $n \in \omega$) or denumerable;
3. *uncountable* if $|X| \nleq \aleph_0$;
4. *Dedekind-finite* if $\aleph_0 \nleq |X|$. Otherwise, X is called *Dedekind-infinite*;
5. *amorphous* if X is infinite (i.e., for every $n \in \omega$, $|X| \neq n$) and cannot be written as a disjoint union of two infinite subsets; in other words, X is amorphous if it is infinite and the only subsets of X are the finite and the co-finite ones.

Definition 3 A *graph* (or *undirected graph*) G is a pair (V, E) where V is a set and $E \subseteq [V]^2$ $(= \{X \subseteq V : |X| = 2\})$. The elements of V are called *vertices* of G and the elements of E are called *edges* (or *lines*) of G.

Let $G = (V, E)$ be a graph.

1. Two vertices u, v of G are called *adjacent* if $\{u, v\} \in E$.
2. A set $W \subseteq V$ is called *independent*, or an *anticlique*, if no two elements of W are adjacent.

3. G is called a *complete graph*, or a *clique*, if any two vertices of G are adjacent.
4. A graph $H = (W, F)$ is a *subgraph* of G if $W \subseteq V$ and $F \subseteq E$.

Definition 4 Let $(P, \leq)$ be a partially ordered set. (We will henceforth write 'poset' instead of 'partially ordered set'.)

1. A set $C \subseteq P$ is called a *chain* in P, if $(C, \leq\restriction C)$ is linearly ordered.
2. A set $A \subseteq P$ is called an *antichain* in P, if no two distinct elements of A are comparable under $\leq$.
3. An element p of P is called *minimal* if, for all $q \in P$, $(q \leq p) \rightarrow (q = p)$.
4. A set $W \subseteq P$ is called *well founded* if every non-empty subset V of W has a $\leq$-minimal element.

Definition 5 1. AC (*Axiom of Choice*, Form 1 in [10]): Every family of non-empty sets has a choice function.
2. $\mathsf{AC}^{\mathsf{LO}}$ (Form 202 in [10]): Every linearly ordered family of non-empty sets has a choice function.
3. $\mathsf{AC}^{\mathsf{WO}}$ (Form 40 in [10]): Every well-ordered family of non-empty sets has a choice function.
4. LW (Form 90 in [10]): Every linearly ordered set can be well ordered.
5. BPI (*Boolean Prime Ideal Theorem*, Form 14 in [10]): Every Boolean algebra has a prime ideal.
6. CUT (*Countable Union Theorem*, Form 31 in [10]): The union of a countable family of countable sets is countable.
7. DC (*Principle of Dependent Choices*, Form 43 in [10]): Let X be a non-empty set and let R be a binary relation on X such that $(\forall x \in X)(\exists y \in X)(xRy)$. Then, there exists a sequence $(x_n)_{n\in\omega}$ of elements of X such that $x_n R x_{n+1}$ for all $n \in \omega$.
8. WOAM (Form 133 in [10]): Every set is either well orderable or has an amorphous subset.
9. $\mathsf{AC}_{\aleph_0}^{\aleph_0}$ (Form 32A in [10]): Every denumerable family of denumerable sets has a choice function.

Fact 1 *1. Each of* LW *and* $\mathsf{AC}^{\mathsf{LO}}$ *is equivalent to* AC *in* ZF, *but none of them are equivalent to* AC *in* ZFA*; see [9] for the assertion about* $\mathsf{AC}^{\mathsf{LO}}$, *and [11, Theorems 9.1, 9.2]. Furthermore,* $\mathsf{AC}^{\mathsf{LO}} \iff \mathsf{LW} \wedge \mathsf{AC}^{\mathsf{WO}}$*; see [9].*
2. $\mathsf{AC}^{\mathsf{WO}}$ *implies* DC *and the implication is not reversible in neither* ZF *nor* ZFA*; see [11, Theorems 8.2, 8.3].*

Definition 6 1. EDM (*Erdős–Dushnik–Miller theorem*): Any uncountable graph $G = (V, E)$ not containing an infinite independent set contains an uncountable clique.
2. $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$ (*Kurepa's theorem*): Every poset such that all of its antichains are finite and all of its chains are countable is countable.
3. $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$: Every poset such that all of its antichains are countable and all of its chains are finite is countable.
4. RT (*Ramsey's theorem*, Form 17 in [10]): If A is an infinite set and $[A]^2$ is partitioned into two sets X and Y, then there is an infinite subset $B \subseteq A$ such that either $[B]^2 \subseteq X$ or $[B]^2 \subseteq Y$.

5. CAC (*Chain-Antichain Principle*, Form 217 in [10]): Every infinite poset has either an infinite chain or an infinite antichain.

For a study of RT and CAC in set theory without AC, the reader is referred to Tachtsis [16].

2.1 Terminology for permutation models

For the reader's convenience, we provide a concise account of the construction of permutation models; a detailed account can be found in Jech [11, Chapter 4].

One starts with a model M of $\mathsf{ZFA} + \mathsf{AC}$ which has A as its set of atoms. Let G be a group of permutations of A and also let $\mathcal{F}$ be a filter on the lattice of subgroups of G which satisfies the following:

$$(\forall a \in A)(\exists H \in \mathcal{F})(\forall \phi \in H)(\phi(a) = a)$$

and $\mathcal{F}$ is closed under conjugation, that is,

$$(\forall \phi \in G)(\forall H \in \mathcal{F})(\phi H \phi^{-1} \in \mathcal{F}).$$

Such a filter $\mathcal{F}$ of subgroups of G is called a *normal filter* on G. Every permutation of A extends uniquely to an $\in$-automorphism of M by $\in$-induction and, for any $\phi \in G$, we identify ϕ with its (unique) extension. If $x \in M$ and H is a subgroup of G, then $\mathrm{fix}_H(x)$ denotes the (pointwise stabilizer) subgroup $\{\phi \in H : \forall y \in x(\phi(y) = y)\}$ of H and $\mathrm{Sym}_H(x)$ denotes the (stabilizer) subgroup $\{\phi \in H : \phi(x) = x\}$ of H.

An element x of M is called $\mathcal{F}$-*symmetric* if $\mathrm{Sym}_G(x) \in \mathcal{F}$ and it is called *hereditarily* $\mathcal{F}$-*symmetric* if x and all elements of its transitive closure are $\mathcal{F}$-symmetric.

Let $\mathcal{N}$ be the class which consists of all hereditarily $\mathcal{F}$-symmetric elements of M. Then $\mathcal{N}$ is a model of ZFA and $A \in \mathcal{N}$ (see Jech [11, Theorem 4.1, p. 46]); $\mathcal{N}$ is called the *permutation model*, or the *Fraenkel–Mostowski model*, determined by M, G and $\mathcal{F}$.

Many permutation models of ZFA are constructed via certain ideals of subsets of the set A of atoms. Let M, A and G be as above. A family $\mathcal{I}$ of subsets of A is called a *normal ideal* if it satisfies the following conditions:

(i) $\emptyset \in \mathcal{I}$;
(ii) if $E \in \mathcal{I}$ and $F \subseteq E$, then $F \in \mathcal{I}$;
(iii) if $E, F \in \mathcal{I}$ then $E \cup F \in \mathcal{I}$;
(iv) if $\pi \in G$ and $E \in \mathcal{I}$, then $\pi[E] \in \mathcal{I}$;
(v) for each $a \in A$, $\{a\} \in \mathcal{I}$.

If $\mathcal{I} \subseteq \mathcal{P}(A)$ is a normal ideal, then $\{\mathrm{fix}_G(E) : E \in \mathcal{I}\}$ is a filter base for some normal filter $\mathcal{F}$ on G. Thus, M, G and $\mathcal{I}$ determine a permutation model.

We close this subsection by recalling the following useful fact: If $\mathcal{N}$ is a permutation model determined by A (a set of atoms), G (a group of permutations of A) and $\mathcal{F}$ (a normal filter on G), then, for any $x \in \mathcal{N}$,

$$\mathcal{N} \models \text{`}x \text{ can be well ordered'} \iff \mathrm{fix}_G(x) \in \mathcal{F}$$

(see Jech [11, (4.2), p. 47]).

3 Known results

Theorem 1 *([12, Proposition 8(i)])* WOAM *implies* CUT *and the implication is not reversible in* ZF. *In particular,* WOAM *implies* $\aleph_1$ *is regular.*

Theorem 2 *The following hold:*

1. *([2, Theorem 4.1(2), Theorem 4.4]) Each of* $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$ *and* $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$ *is strictly weaker than* EDM *in* ZFA.
2. *([1, Corollary 4.6], [2, Fact 3.1])* DC *does not imply* $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$ *in* ZF, *and thus (by (1)) neither does it imply* EDM *in* ZF.
3. *([2, Theorem 4.2(1)])* WOAM + RT *implies* EDM.
4. *([2, Theorems 4.1(3), 4.2(3)])* EDM *implies* RT, *but does not imply* WOAM *in* ZFA.
5. *([2, Theorem 4.1(4), Remark 6.1(4)]) None of* WOAM *and* RT *imply* EDM *in* ZFA.
6. *([17, Theorem 8(1)])* WOAM + CAC *implies* $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$.
7. *([2, Theorem 4.2(1)])* WOAM + CAC *implies* $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$.
8. *([2, Corollary 3.6])* $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$ *does not imply* $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$ *in* ZFA.
9. *([2, Proposition 3.3][(1), (2)], Theorem 4.1(2)])"*$\aleph_1$ *is regular"*[3] *implies* EDM *restricted to graphs with uncountable, well-orderable set of vertices, which in turn implies* $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$ *restricted to well-orderable posets.*
10. *([2, Proposition 3.4])* $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$ *implies* $\mathsf{AC}_{\aleph_0}^{\aleph_0}$. *Thus, by (1),* EDM *implies* $\mathsf{AC}_{\aleph_0}^{\aleph_0}$.

Remark 1 To the best of our knowledge, whether or not CAC can be removed from the hypotheses of Theorem 2(6), i.e., whether or not WOAM implies $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$, is still an *open problem*; see also [17, Question 4 of Section 6] for further relative questions. It is also *unknown* whether or not WOAM implies CAC. However, as already mentioned in Sect. 1, we will prove (in the forthcoming Theorem 6) that WOAM implies $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$, and thus CAC can be removed from the hypotheses of Theorem 2(7).

4 Main results

We start by showing that EDM is independent from ZFA + $\mathsf{AC}^{\mathsf{LO}}$, and thus independent from ZFA + LW + $\mathsf{AC}^{\mathsf{WO}}$ (see Fact 1(1)). To achieve our goal, we will construct (in the proof of Theorem 3 below) a *new permutation model* and show that it satisfies $\mathsf{AC}^{\mathsf{LO}} \wedge \neg\mathsf{EDM}$. This will *completely settle Question 6.4 from* [2]. On the other hand, recall (by Fact 1(1)) that each of $\mathsf{AC}^{\mathsf{LO}}$ and LW is equivalent to AC in ZF, so *each of* $\mathsf{AC}^{\mathsf{LO}}$ *and* LW *implies* EDM *in* ZF.

[3] "$\aleph_1$ is regular" is Form 34 in Howard–Rubin [10].

Let us also recall that, by Theorem 2(2), DC does not imply EDM in ZF. Since DC is strictly weaker than AC^{WO} in ZFA (see Fact 1(2)), Theorem 3 below *properly strengthens the above result from* [2] in the setting of ZFA.

Theorem 3 AC^{LO} *does not imply* EDM *in* ZFA. *Hence, neither* LW *nor* AC^{WO} *imply* EDM *in* ZFA.

Proof We start with a model M of ZFA + AC with an $\aleph_1$-sized set A of atoms which is a disjoint union of $\aleph_1$ unordered pairs, so that $A = \bigcup\{A_i : i < \aleph_1\}$, $|A_i| = 2$ for all $i < \aleph_1$, and $A_i \cap A_j = \emptyset$ for all $i, j < \aleph_1$ with $i \neq j$. Let G be the group of all permutations ϕ of A such that:

$$(\forall i < \aleph_1)(\exists j < \aleph_1)(\phi(A_i) = A_j).$$

Let $\mathcal{F}$ be the filter of subgroups of G generated by the pointwise stabilizers $\text{fix}_G(E)$, where $E = \bigcup\{A_i : i \in I\}$ for some $I \in [\aleph_1]^{<\aleph_1} = \{X \in \wp(\aleph_1) : |X| \leq \aleph_0\}$; $\mathcal{F}$ is a normal filter on G since the ideal generated by all subsets E of A of the above form (i.e., the ideal comprising all sets $F \subseteq A$ contained in some set $E \subseteq A$ of the above form) is a normal ideal on A. Let $\mathcal{N}$ be the permutation model determined by M, G and $\mathcal{F}$. Note that, if $x \in \mathcal{N}$, then there exists $E = \bigcup\{A_i : i \in I\}$ for some $I \in [\aleph_1]^{<\aleph_1}$ such that $\text{fix}_G(E) \subseteq \text{Sym}_G(x)$. Any such set $E \subseteq A$ will be called a *support* of x.

Claim In $\mathcal{N}$, the power set of $\mathcal{A} = \{A_i : i < \aleph_1\}$ consists exactly of the countable and the co-countable subsets of $\mathcal{A}$.

Proof First, note that $\mathcal{A} \in \mathcal{N}$ since $\text{Sym}_G(\mathcal{A}) = G \in \mathcal{F}$, and that every countable or co-countable subset of $\mathcal{A}$ is an element of $\mathcal{N}$. Indeed, if $\mathcal{V}$ is a countable subset of $\mathcal{A}$ in $\mathcal{N}$, then $E = \bigcup\{A_i : i \in I\}$, where $I = \{i \in \aleph_1 : A_i \in \mathcal{V}\}$, is a support of $\mathcal{V}$. If $\mathcal{W}$ is a co-countable subset of $\mathcal{A}$ in $\mathcal{N}$, then $E' = \bigcup\{A_j : j \in J\}$, where $J = \{j \in \aleph_1 : A_j \in \mathcal{A} \setminus \mathcal{W}\}$, is a support of $\mathcal{W}$. Second, the set

$$\mathcal{U} = \{\mathcal{Z} \in \wp(\mathcal{A})^{\mathcal{N}} : |\mathcal{Z}| < \aleph_1 \text{ or } |\mathcal{A} \setminus \mathcal{Z}| < \aleph_1\}$$

is an element of $\mathcal{N}$ since $\text{Sym}_G(\mathcal{U}) = G \in \mathcal{F}$.

Now, we show that $\wp(\mathcal{A})^{\mathcal{N}} = \mathcal{U}$. Assuming the contrary, there exists $\mathcal{B} \in \wp(\mathcal{A})^{\mathcal{N}} \setminus \mathcal{U}$. Then neither $\mathcal{B}$ nor $\mathcal{A} \setminus \mathcal{B}$ is countable. Let $E = \bigcup\{A_i : i \in I\}$, $I \in [\aleph_1]^{<\aleph_1}$, be a support of $\mathcal{B}$. Since I is countable, whereas $\mathcal{B}$ and $\mathcal{A}\setminus\mathcal{B}$ are not, it follows that there exist $k, m \in \aleph_1 \setminus I$ such that $A_k \in \mathcal{B}$ and $A_m \in \mathcal{A} \setminus \mathcal{B}$. Then $A_k \cap A_m \cap E = \emptyset$. Consider a permutation ϕ of A which interchanges A_k and A_m and fixes $A\setminus(A_k \cup A_m)$ pointwise. Then $\phi \in \text{fix}_G(E)$, so $\phi(\mathcal{B}) = \mathcal{B}$. However,

$$A_k \in \mathcal{B} \Rightarrow \phi(A_k) \in \phi(\mathcal{B}) \Rightarrow A_m \in \mathcal{B}.$$

This is a contradiction since $A_m \in \mathcal{A} \setminus \mathcal{B}$ and $\mathcal{B} \cap (\mathcal{A} \setminus \mathcal{B}) = \emptyset$. Thus $\wp(\mathcal{A})^{\mathcal{N}} = \mathcal{U}$, finishing the proof of the claim. □

Claim No co-countable subset of $\mathcal{A}$ has a choice function in $\mathcal{N}$.[4] In particular, $\mathcal{A}$ has no choice function in $\mathcal{N}$. Thus, by the first claim, every uncountable subset of A in $\mathcal{N}$ is a union of some co-countable subset of $\mathcal{A}$.

Proof Towards a contradiction, we assume that $\mathcal{A}$ has a co-countable subset, $\mathcal{B}$ say, with a choice function in $\mathcal{N}$, f say. Let $E = \bigcup\{A_i : i \in I\}$, $I \in [\aleph_1]^{<\aleph_1}$, be a support of f. Let $k \in \aleph_1 \setminus I$ such that $A_k \in \mathcal{B}$ (and hence $A_k \cap E = \emptyset$), let $a = f(A_k)$ and also let $b \in A_k \backslash \{a\}$. Consider the transposition $\phi = (a, b)$, that is, ϕ interchanges a and b and fixes all the other atoms. Then $\phi \in \text{fix}_G(E)$, so $\phi(f) = f$. However,

$$(A_k, a) \in f \Rightarrow (\phi(A_k), \phi(a)) \in \phi(f) \Rightarrow (A_k, b) \in f,$$

contradicting f's being a function, and finishing the proof of the claim. □

Claim EDM is false in $\mathcal{N}$.

Proof Let $\mathfrak{G} = (V_{\mathfrak{G}}, E_{\mathfrak{G}})$ be the graph defined by: $V_{\mathfrak{G}} = A$ and

$$E_{\mathfrak{G}} = \{\{a, b\} \in [A]^2 : (\forall i < \aleph_1)(|\{a, b\} \cap A_i| \in \{0, 1\})\}.$$

In other words, two distinct $a, b \in A$ are joined by an edge if and only if there exist distinct $i, j \in \aleph_1$ such that $a \in A_i$ and $b \in A_j$. We have $\mathfrak{G} \in \mathcal{N}$ since $\text{Sym}_G(\mathfrak{G}) = G \in \mathcal{F}$.

It is clear that $\mathfrak{G}$ does not contain an infinite independent set; in particular, a set $W \subseteq V_{\mathfrak{G}}$ is independent if and only if $W \subseteq A_i$ for some $i < \aleph_1$. Furthermore, $\mathfrak{G}$ does not contain an uncountable clique; otherwise, by the definition of $\mathfrak{G}$ and the first claim, there would exist a co-countable subset of $\mathcal{A}$ with a choice function in $\mathcal{N}$, contrary to the second claim.[5] Therefore, EDM is false in $\mathcal{N}$ as required. □

Claim LW is true in $\mathcal{N}$.

Proof Let $(X, \leq)$ be a linearly ordered set in $\mathcal{N}$ and let E be a support of $(X, \leq)$. We will show that $\text{fix}_G(E) \subseteq \text{fix}_G(X)$; this will yield X is well orderable in $\mathcal{N}$ (see the last paragraph of Subsection 2.1). By way of contradiction, we assume $\text{fix}_G(E) \nsubseteq \text{fix}_G(X)$. There exist $\eta \in \text{fix}_G(E)$ and $y \in X$ such that $\eta(y) \neq y$. Let $E' \subseteq A$ be a support of y. Since E is not a support of y, $E' \nsubseteq E$, and, without loss of generality, we assume that $E \subsetneq E'$; otherwise, we may work with $E \cup E'$.

Our first step is to construct a permutation $\phi \in \text{fix}_G(E)$ such that $\{a \in A : \phi(a) \neq a\}$ is countable and $\phi(e) = \eta(e)$ for all $e \in E'$, so that $\phi(y) = \eta(y)$ (since $\eta^{-1}\phi \in \text{fix}_G(E')$ and E' is a support of y), and therefore $\phi(y) \neq y$ (since $\phi(y) = \eta(y)$ and $\eta(y) \neq y$). To this end, first note that, for every $a \in E'$, the set $\{\eta^n(a) : n \in \mathbb{Z}\}$ is countable. Therefore, since E' is countable (being a countable union of pairs—recall the definition of support), the set $D = \bigcup_{a \in E'}\{\eta^n(a) : n \in \mathbb{Z}\}$ is countable.

[4] In contrast, note that every countable subset of $\mathcal{A}$ has a choice function in $\mathcal{N}$.

[5] Note that $\mathfrak{G}$ contains denumerable cliques in $\mathcal{N}$.

Furthermore, D contains E' (and thus contains E, since $E \subseteq E'$) and is closed under η. We define $\phi : A \to A$ by

$$\phi(a) = \begin{cases} \eta(a), & \text{if } a \in D; \\ a, & \text{otherwise.} \end{cases}$$

Then, ϕ:

1. is an element of G (since D is closed under η and $\eta \in G$);
2. moves only countably many atoms (since D is countable);
3. fixes E pointwise (since $E \subseteq D$ and η fixes E pointwise); and
4. agrees with η on E'.

Therefore, ϕ has all the required properties.

Our second step is to construct (using properties (2)-(4) of ϕ) a permutation $\psi \in \mathrm{fix}_G(E)$ such that $\psi(y) \neq y$ but ψ^2 is the identity mapping, so that $\psi^2(y) = y$. This will contradict the fact that E is a support of the linear order $\leq$ on X. Indeed, first note that $\psi(y) \in X$, since $y \in X$ and $\psi \in \mathrm{fix}_G(E) \subseteq \mathrm{Sym}_G(X)$. Secondly, since $\psi(y) \neq y$ and $\leq$ is a linear order on X, either $\psi(y) < y$ or $y < \psi(y)$. Suppose that $\psi(y) < y$; then $\psi^2(y) < \psi(y)$, so $y < \psi(y)$, a contradiction. Similarly, we reach a contradiction if we assume $y < \psi(y)$.

Let $W = \{a \in A : \phi(a) \neq a\}$. By (2), W is countable (and non-empty since, by (4) and the fact that E' is a support of y, $\phi(y) = \eta(y) \neq y$), and, by (3), $W \cap E = \emptyset$. Furthermore, note that, by the fact that $|A_i| = 2$ for all $i \in \aleph_1$ and the definition of the group G, W is a countable union of A_i's. Let U be a countable union of A_i's which is disjoint from $E' \cup W$ and is such that there exists a bijection $H : tr(U) \to tr((E' \cup W) \setminus E)$ (where, for a set $x \subseteq A$, $tr(x)$ is the *trace* of x, i.e., $tr(x) = \{i \in \aleph_1 : A_i \cap x \neq \emptyset\}$) with the property that, if $i \in tr((E' \cup W) \setminus E)$, which means that $A_i \subseteq (E' \cup W) \setminus E$, then $A_{H^{-1}(i)} \subseteq U$. Let $f : U \to (E' \cup W) \setminus E$ be a bijection such that, for every $i \in tr(U)$, $f \restriction A_i$ $(= U \cap A_i)$ is a one-to-one function from A_i onto $A_{H(i)}$ $(= ((E' \cup W) \setminus E) \cap A_{H(i)})$.

We define a permutation ψ of A by

$$\psi = \prod_{u \in U} (u, f(u)),$$

that is, ψ is a product of disjoint transpositions. It is clear that $\psi \in \mathrm{fix}_G(E)$ and that ψ^2 is the identity mapping on A, and thus $\psi^2(y) = y$. On the other hand, $\psi(y) \neq y$. To see this, assume on the contrary that $\psi(y) = y$. Note that

$$\psi(E' \cup W) = \psi(((E' \cup W) \setminus E) \cup E) = \psi((E' \cup W) \setminus E) \cup \psi(E) = U \cup E,$$

and since $E' \cup W$ is a support of y, we have $\psi(E' \cup W) = U \cup E$ is a support of $\psi(y) = y$. Furthermore, since $\phi \in \mathrm{fix}_G(U) \cap \mathrm{fix}_G(E)$ (recall that $U \cap W = \emptyset$), we have ϕ fixes $U \cup E$ pointwise, and thus fixes a support of $\psi(y)$ pointwise. Therefore, $\phi(\psi(y)) = \psi(y)$ and, since $\psi(y) = y$, we conclude that $\phi(y) = y$. This is a contradiction since (by property (4) of ϕ) $\phi(y) = \eta(y) \neq y$. Hence, $\mathrm{fix}_G(E) \subseteq \mathrm{fix}_G(X)$, i.e., X is well orderable in $\mathcal{N}$ as required. □

Claim $\mathsf{AC}^{\mathsf{LO}}$ (and thus $\mathsf{AC}^{\mathsf{WO}}$) is true in $\mathcal{N}$.

Proof Let $\mathcal{Z}$ be a linearly ordered family of non-empty sets in $\mathcal{N}$. By the fourth claim $\mathcal{Z}$ is well orderable. Let E be a support of a well ordering of $\mathcal{Z}$. Then, for every $Z \in \mathcal{Z}$, $\mathrm{fix}_G(E) \subseteq \mathrm{Sym}_G(Z)$ (see Subsection 2.1). Let

$$i_0 = \sup\{i \in \aleph_1 : A_i \subseteq E\}.$$

Then $i_0 \in \aleph_1$ since $\aleph_1$ is a regular cardinal in the model M. Let

$$E' = \bigcup\{A_j : j < i_0 + \omega\}.$$

Clearly, $E \subseteq E'$. To complete the proof, it suffices to show that, for every $Z \in \mathcal{Z}$, there exists $y \in Z$ such that y has a support which is a subset of E'.

In the model M, which satisfies AC, choose, for each $Z \in \mathcal{Z}$, an element z of Z and a support E_z of z. Fix $Z \in \mathcal{Z}$. If $E_z \subseteq E'$, then there is nothing to prove (since $z \in Z$ and $E_z \subseteq E'$), so we assume $E_z \nsubseteq E'$. Since $E' \setminus E$ is a disjoint union of denumerably many A_i's and $E_z \setminus E'$ is a disjoint union of countably many A_i's, it is easy to see that there exists $\gamma_{(z,Z)} \in \mathrm{fix}_G(E)$ such that $\gamma_{(z,Z)}(E_z) \subseteq E'$. Then $\gamma_{(z,Z)}(Z) = Z$ since E is a support of Z and $\gamma_{(z,Z)} \in \mathrm{fix}_G(E)$. Hence,

$$y_Z := \gamma_{(z,Z)}(z) \in Z$$

and $\gamma_{(z,Z)}(E_z)$ is a support of y_Z contained in E'. Let

$$f = \{(Z, y_Z) : Z \in \mathcal{Z}\}.$$

Then f is a choice function for $\mathcal{Z}$ and $f \in \mathcal{N}$ since E' is a support of (every element of) f. Thus, $\mathsf{AC}^{\mathsf{LO}}$ is true in $\mathcal{N}$, finishing the proof of the claim. □

The above arguments complete the proof of the theorem. □

Remark 2 The model $\mathcal{N}$ of the proof of Theorem 3 is equal to the model $\mathcal{N}^*$ determined by the same set A of atoms as in $\mathcal{N}$, the (smaller than G) group G^* comprising all permutations ϕ of A with the following two properties:

(a) $(\forall i < \aleph_1)(\exists j < \aleph_1)(\phi(A_i) = A_j)$;

(b) ϕ moves only countably many elements of A,

and the corresponding (normal) filter $\mathcal{F}^*$ on G^* generated by the subgroups $\mathrm{fix}_{G^*}(E)$, where $E = \bigcup\{A_i : i \in I\}$ for some $I \in [\aleph_1]^{<\aleph_1}$. To establish that $\mathcal{N} = \mathcal{N}^*$, we prove by $\in$-induction that, for every $x \in M$ (the model of $\mathsf{ZFA} + \mathsf{AC}$ used for the construction of $\mathcal{N}$ and $\mathcal{N}^*$), $\Phi(x)$ is true, where $\Phi(x)$ is the following formula:

$$x \in \mathcal{N} \Longleftrightarrow x \in \mathcal{N}^*.$$

Clearly, $\Phi(x)$ is true if $x = \emptyset$ or if $x \in A$. Assume that $y \in M$ and that, for every $x \in y$, $\Phi(x)$ is true. We argue that $\Phi(y)$ is true.

Assume first that $y \in \mathcal{N}^*$. Then we have the following about y:

1. y has a support, say E, relative to the group G^*.
2. For every $x \in y$, $x \in \mathcal{N}^*$.
3. For every $x \in y$, $x \in \mathcal{N}$ (by item (2) and the induction hypothesis).

We will show that E is a support of y relative to the group G. In other words, we will argue that, for every $\eta \in \text{fix}_G(E)$, $\eta(y) = y$. This will follow from "$(\forall \eta \in \text{fix}_G(E))(\forall x \in y)(\eta(x) \in y)$" (since $\eta(y) = y$ follows from "$\eta(y) \subseteq y$ and $\eta^{-1}(y) \subseteq y$"). Therefore, we assume that $\eta \in \text{fix}_G(E)$ and $x \in y$. We will prove that $\eta(x) \in y$.

By item (3) above, x has a support E' relative to G. The permutation η may not be in G^*, but working exactly as in the second paragraph of the proof of the fourth claim (of the proof of Theorem 3) we can construct a permutation $\phi \in \text{fix}_{G^*}(E)$ which agrees with η on E'. For such a permutation ϕ, we have $\phi(y) = y$ (because $\phi \in \text{fix}_{G^*}(E)$ and, by item (1), E is a support of y relative to G^*) and $\phi(x) \in y$ (because $x \in y$ and $\phi(y) = y$). Furthermore, since $\phi(x) \in y$ and $\phi(x) = \eta(x)$ (because ϕ and η agree on E', which is a support of x relative to G), we obtain $\eta(x) \in y$. Therefore, $y \in \mathcal{N}$.

Secondly, assume that $y \in \mathcal{N}$ and has a support E' relative to G. Then E' is a support of y relative to G^* since $G^* \subseteq G$. By the induction hypothesis, every element of y is in $\mathcal{N}^*$, so $y \in \mathcal{N}^*$. This completes the inductive step and proves that $\mathcal{N} = \mathcal{N}^*$.

Question 1 Is there a model of ZF which satisfies $\mathsf{AC}^{\mathsf{WO}} \wedge \neg\mathsf{EDM}$?

As mentioned in [2], Lajos Soukup raised the question about the relationship between $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$ and $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$; recall (by Theorem 2(1)) that both of these principles are weaker than EDM in ZFA. By Theorem 2(8), we know that $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$ does not imply $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$ in ZFA. There remains the question (until now) whether or not $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$ implies $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$ in ZF or in ZFA; this question is posed in [2, Question 6.1]. We *settle this open problem* by showing that

$$\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0) \nRightarrow \mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0) \text{ in ZF.}$$

First, we prove the following theorem which provides *new information* on the set-theoretic strength of $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$.

Theorem 4 DC *implies* $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$.

Proof Assume DC is true. Fix a poset $(P, \leq)$ such that all of its antichains are countable and all of its chains are finite. By way of contradiction, we assume that P is uncountable.

Claim $(P, \leq)$ is well founded.

Proof If not, then there is a non-empty subset P_1 of P with no $\leq$-minimal elements. We define

$$S = \{(p_0, p_1, \ldots, p_n) : n \in \omega \setminus \{0\}, p_i \in P_1 \text{ for all } i \leq n, \text{and } p_0 > p_1 > \cdots > p_n\}.$$

Since P_1 has no minimal elements, we have $S \neq \emptyset$. We define a binary relation R on S by stipulating, for every $s, t \in S$,

$$sRt \iff s \subsetneq t.$$

Again, as P_1 has no minimal elements, it follows that $\mathrm{dom}(R) = S$. Thus, by DC applied to the relational system (S, R), we obtain a sequence $(s_n)_{n\in\omega}$ of elements of S such that, for every $n \in \omega$, $s_n R s_{n+1}$. This readily yields a strictly decreasing sequence of elements of P (namely, the sequence $\bigcup_{n\in\omega} s_n$), and thus a denumerable chain in P, contradicting P's having no infinite chains. Therefore, P is well founded, finishing the proof of the claim. □

Claim $(P, \leq)$ has a strictly increasing sequence.

Proof By the first claim, we obtain

$$P = \bigcup\{P_\alpha : \alpha < \kappa\} \tag{1}$$

for some well-ordered cardinal number κ, where P_0 is the set of all minimal elements of P and, for every $\alpha < \kappa$ with $\alpha > 0$, P_α is the set of all minimal elements of $P \setminus \bigcup\{P_\beta : \beta < \alpha\}$. Note that, for every $\alpha < \kappa$, P_α is an antichain in P, and thus (by our hypothesis on P) P_α is countable for all $\alpha < \kappa$. Furthermore, $\{P_\alpha : \alpha < \kappa\}$ is pairwise disjoint and

$$(\forall\alpha < \beta < \kappa)(\forall x \in P_\beta)(\exists y \in P_\alpha)(y < x) \tag{2}$$

(and note that, for $\alpha < \beta < \kappa$ and $x \in P_\beta$, there is no $z \in P_\alpha$ such that $x \leq z$).

For every $p \in P$, we let

$$P_{\geq p} = \{q \in P : p \leq q\}.$$

By (2), we obtain

$$P = \bigcup\{P_{\geq p} : p \in P_0\} \tag{3}$$

and since P_0 is countable (being an antichain in P) and P is uncountable, there exists $p \in P_0$ such that $P_{\geq p}$ is uncountable; otherwise, by DC (which implies CUT), we would have P is countable, which is a contradiction.

Now, we define

$$T = \{(p_0, \ldots, p_n) : n \in \omega,\ P_{\geq p_i} \text{ is uncountable for all } i \leq n, \text{ and if } n > 0, \text{ then } p_0 < p_1 < \cdots < p_n\}.$$

By the observation of the previous paragraph, we have $T \neq \emptyset$ (since for some $p_0 \in P_0$, $P_{\geq p_0}$ is uncountable, and thus $(p_0) \in T$). We define a binary relation Q on T by stipulating, for every $s, t \in T$,

$$sQt \iff s = (p_0, \dots, p_n) \text{ and } t = (p_0, \dots, p_n, p_{n+1}) \; (= s^\frown(p_{n+1})).$$

Similarly to the previous argument (for '$T \neq \emptyset$'), it can be shown that $\mathrm{dom}(Q) = T$. Indeed, let $s = (p_0, \dots, p_n) \in T$; then, by the definition of T, $P_{\geq p_n}$ is uncountable. By analogous (1)–(3) written for the poset $P_{>p_n} = P_{\geq p_n} \setminus \{p_n\} = \{q \in P : p_n < q\}$ in place of P, we infer (in the same way as with the poset P) that for some minimal element of $P_{>p_n}$, p_{n+1} say, the set $(P_{>p_n})_{\geq p_{n+1}} = P_{\geq p_{n+1}}$ (the latter equality holds since $p_n < p_{n+1}$) is uncountable. It follows that the finite sequence

$$t = s^\frown(p_{n+1}) = (p_0, \dots, p_n, p_{n+1})$$

is an element of T and, clearly, sQt. Therefore, $\mathrm{dom}(Q) = T$.

Applying DC to the relational system (T, Q), we obtain a sequence $(t_n)_{n \in \omega}$ of elements of T such that $t_n Q t_{n+1}$ for all $n \in \omega$. It is evident that $(t_n)_{n \in \omega}$ yields a strictly increasing sequence $(p_n)_{n \in \omega}$ of elements of P. This completes the proof of the claim. □

By the second claim, we obtain a contradiction to the hypothesis that $(P, \leq)$ has no infinite chains. Therefore, P is countable, finishing the proof of the theorem. □

Having established Theorem 4, we are now in position to provide a *complete answer to Soukup's question, and thus to* [2, Question 6.1], about the relationship between the order-theoretic principles $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$ and $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$.

Theorem 5 $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$ *does not imply* $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$ *in* ZF.

Proof By Theorem 2(2), we know that DC does not imply $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$ in ZF, whereas, by Theorem 4, DC implies $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$. The latter two observations yield the required independence result in ZF. □

Question 2 Is there a model of ZF in which $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$ is true but $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$ is false?

By Theorem 2(7), $\mathsf{WOAM} + \mathsf{CAC}$ implies $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$. On the other hand, in [2, Question 6.3], it was asked whether WOAM implies $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$. We *answer this question* in the affirmative, and thus *strengthening the above result* from [2].

Theorem 6 WOAM *implies* $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$. *The implication is not reversible in* ZFA.

Proof Assume WOAM is true. Fix a poset $(P, \leq)$ such that all of its antichains are countable and all of its chains are finite. We will prove by contradiction that P is well orderable. So, suppose that P is not well orderable. Then, by WOAM, P has an amorphous subset, A say. Since A is amorphous, the poset $(A, \leq)$ cannot be well founded. Otherwise, as in the proof of the second claim of Theorem 4, A would have a well-ordered partition $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ (κ a well-ordered cardinal) such that, for every $\alpha < \kappa$, A_α is an antichain in $(A, \leq)$. As A is amorphous, $\wp(A)$ is Dedekind-finite, and so $\mathcal{A}$ is finite. Furthermore, since all antichains in P are countable and A is Dedekind-finite (being amorphous), A_α is finite for all $\alpha < \kappa$. But then A is finite, which is a contradiction. Therefore, A is not well founded, i.e., there exists a non-empty set $B \subseteq A$ with no $\leq$-minimal elements. It follows that B is infinite, and thus co-finite in A. Without loss of generality, we assume $B = A$; so $(A, \leq)$ has no minimal elements.

Since $(A, \leq)$ has no minimal elements, the family

$$\mathcal{C} = \{X \in [A]^2 : X \text{ is a chain}\}$$

is infinite. If not, then the set $C = \bigcup \mathcal{C}$ is finite. Suppose $|C| = k$ for some positive integer k. Fix $a \in A \backslash C$. Since A has no minimal elements, we may find a chain $c : a > a_1 > \cdots > a_{k+1}$ in A. As $|C| = k < k + 1 = |c \backslash \{a\}|$, there exists $j \in \{1, \ldots, k + 1\}$ such that $a_j \notin C$ and, as $a > a_j$, we have $\{a, a_j\} \in \mathcal{C}$. However, $\{a, a_j\} \cap C = \emptyset$, which is a contradiction. Therefore, $\mathcal{C}$ is infinite.

Now, we define a choice function f for $\mathcal{C}$ by

$$f(X) = \leq -\min(X), \text{ for every } X \in \mathcal{C}.$$

Since A is amorphous and $\operatorname{ran}(f) \subseteq A$, $\operatorname{ran}(f)$ is either finite or co-finite.

Case 1: $\operatorname{ran}(f)$ *is finite*. For every $x \in \operatorname{ran}(f)$, we let

$$\mathcal{C}_x = \{X \in \mathcal{C} : x \in X\}.$$

Since f is a choice function for $\mathcal{C}$, we have

$$\mathcal{C} = \bigcup \{\mathcal{C}_x : x \in \operatorname{ran}(f)\}$$

and, since $\mathcal{C}$ is infinite and $\operatorname{ran}(f)$ is finite, there exists $x_0 \in \operatorname{ran}(f)$ such that $\mathcal{C}_{x_0}$ is infinite. It follows that $(\bigcup \mathcal{C}_{x_0}) \setminus \{x_0\}$ is a co-finite subset of A. Let

$$\mathcal{U} = \{U : U \text{ is a} \subseteq \text{-maximal chain in } (A, \leq) \text{ with } \min(U) = x_0\}.$$

By the definition of f and the fact that $\operatorname{ran}(f)$ is finite, it follows that $\mathcal{U} \neq \emptyset$. Moreover, as $\mathcal{C}_{x_0}$ is infinite, we have $\mathcal{U}$ is infinite. Since all chains in P are finite, we can unambiguously define

$$V = \{\leq -\max(U) : U \in \mathcal{U}\}.$$

Since every $U \in \mathcal{U}$ is a $\subseteq$-maximal chain with $\min(U) = x_0$, and $\mathcal{U}$ is infinite, it readily follows that V is an infinite antichain in $(A, \leq)$. As all antichains in P are countable, V is a denumerable subset of A, i.e., A is Dedekind-infinite, contrary to the fact that A is amorphous.

Case 2: ran(f) *is co-finite*. Without loss of generality, we assume ran$(f) = A$. By the first paragraph of the proof, we know that $(A, \leq)$ has no $\leq$-minimal elements.

By the definition of f and the fact that ran$(f) = A$, we easily obtain that, for every $a \in A$, the family $\mathcal{C}_a = \{X \in \mathcal{C} : a \in X\}$ is infinite, and thus the set

$$A_a = \{x \in A : a < x\}$$

is infinite for all $a \in A$. On the other hand, as A has no minimal elements, we conclude that, for every $a \in A$, the set

$$B_a = \{x \in A : x < a\}$$

is infinite. Fix any $a \in A$. Then A_a and B_a are infinite disjoint subsets of A, contrary to the fact that A is amorphous.

In view of the above arguments, we conclude that P is well orderable. Since (by Theorem 1) WOAM implies '$\aleph_1$ is regular', the latter principle, together with the fact that P is well orderable, yields (by Theorem 2(9)) P is countable. Thus $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$ is true as required.

The second assertion of the theorem follows from the fact that the principle $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$ is true in the Mostowski Linearly Ordered Model $\mathcal{N}3$ of [10] (see [2, Theorems 4.1(2), 4.2(3)]), whereas WOAM is false in $\mathcal{N}3$ (see [10], [11]).[6] This completes the proof of the theorem. □

5 Concluding remarks and open questions

Remark 3 1. As noted in [2, Remark (3) of Subsection 6.1], BPI does not imply EDM in ZF. [By Theorem 2(4), EDM implies RT. On the other hand, Blass [3] showed that RT is false in the Basic Cohen Model of ZF (Model $\mathcal{M}1$ of [10]) in which BPI is true; see Halpern–Levy [8]. Therefore, EDM is false in $\mathcal{M}1$.] However, we do not know of a specific permutation model which satisfies BPI $\wedge$ $\neg$EDM. Let us also recall here that, in [2, Theorem 4.2(3)], it was shown that EDM is true in the Mostowski Linearly Ordered Model of ZFA (Model $\mathcal{N}3$ of [10]) in which BPI is true (see Halpern [6]).

2. In [2, Question 6.5], it is asked whether EDM is true in the Brunner–Pincus permutation model $\mathcal{N}26$ of [10], whose description is as follows: We start with a model M of ZFA + AC with a denumerable set A of atoms which is a denumerable disjoint union of denumerable sets, so that $A = \bigcup\{P_n : n \in \omega\}$, where $\{P_n : n \in \omega\}$ is disjoint and $|P_n| = \aleph_0$ for all $n \in \omega$. Let G be the group of all permutations ϕ of A such that $\phi(P_n) = P_n$ for all $n \in \omega$. Let $\mathcal{I}$ be the ideal of all finite subsets of A; $\mathcal{I}$ is a normal ideal on A. Then, $\mathcal{N}26$ is the permutation model determined by M, G and $\mathcal{I}$.

   The *answer to the above question from* [2] *is in the affirmative*. First, note that, for every $n \in \omega$, P_n is amorphous in $\mathcal{N}26$; this can be proved exactly as with the

[6] The second assertion of the theorem also follows from Theorem 2[(1), (4)].

Basic Fraenkel Model (Model $\mathcal{N}1$ in [10]) in which its set of atoms is amorphous, see Jech [11]. Second, working in much the same way as in Blass [3], one shows that every set in $\mathcal{N}26$ is either well orderable or contains a copy of an infinite (and thus co-finite) subset of P_n for some $n \in \omega$; so WOAM is true in $\mathcal{N}26$ (this fact is mentioned in [10]). Third, by the latter observation and the following facts: (a) RT holds for infinite subsets of P_n for any $n \in \omega$ (this can be proved exactly as in Blass [3] for $\mathcal{N}1$); (b) if RT holds for X then it holds for all supersets of X, we conclude that RT is true in $\mathcal{N}26$ (the status of RT is not specified in [10]). Therefore, by Theorem 2(3), EDM is true in $\mathcal{N}26$.

3. The statement "For every infinite cardinal $\mathfrak{p}$, $\mathfrak{p} + \mathfrak{p} = \mathfrak{p}$" (Form 3 in [10]) does not imply $\mathsf{KT}(a \leq \aleph_0, c < \aleph_0)$ in ZF, and thus (by Theorem 2(1)) does not imply EDM in ZF either. Indeed, we consider Sageev's ZF-model $\mathcal{M}6$ of [10]. In this model, Sageev [15] proved that Form 3 is true, but the axiom of choice for denumerable collections of denumerable sets of reals is false, i.e., $\mathsf{AC}_{\aleph_0}^{\aleph_0}$ is false in $\mathcal{M}6$. This, together with Theorem 2(10), yields $\mathsf{KT}(a \leq \aleph_0, c < \aleph_0)$ (and thus EDM) is false in $\mathcal{M}6$.

Question 3 1. Is there a model of ZF which satisfies $\mathsf{AC}^{\mathsf{WO}} \wedge \neg\mathsf{EDM}$? (This is Question 1 of Sect. 4.)

2. Is EDM false in Feferman's ZF-model $\mathcal{M}2$ of [10] (also see [5] and [11, Problem 24, p. 82] for the construction of this model), in which $\mathsf{AC}^{\mathsf{WO}}$ is true (see [5], [18] for the latter fact)?
3. Is EDM false in the permutation model $\mathcal{N}12(\aleph_1)$ of [10] in which $\mathsf{AC}^{\mathsf{LO}}$ (and thus $\mathsf{AC}^{\mathsf{WO}}$) is true (see [9] for the latter fact)? We note that an affirmative answer to this question would yield, via the Jech–Sochor transfer techniques of the proof of [11, Theorem 8.9], a ZF-model satisfying $\mathsf{AC}^{\mathsf{WO}} \wedge \neg\mathsf{EDM}$.
4. Is there a model of ZF satisfying $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0) \wedge \neg\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$? (This is Question 2 of Sect. 4.)
5. Is EDM, or any of $\mathsf{KT}(\mathsf{a} < \aleph_0, \mathsf{c} \leq \aleph_0)$ and $\mathsf{KT}(\mathsf{a} \leq \aleph_0, \mathsf{c} < \aleph_0)$, false in the Halpern–Howard permutation model $\mathcal{N}9$ of [10] in which "For every infinite cardinal $\mathfrak{p}$, $\mathfrak{p} + \mathfrak{p} = \mathfrak{p}$" is true (see [7] for the latter fact)?

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