




On some special subspaces of a Banach space, from the perspective of best coapproximation

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Abstract

We study the best coapproximation problem in Banach spaces, by using Birkhoff–James orthogonality techniques. We introduce two special types of subspaces, christened the anti-coproximinal subspaces and the strongly anti-coproximinal subspaces. We obtain a necessary condition for the strongly anti-coproximinal subspaces in a reflexive Banach space whose dual space satisfies the Kadets–Klee Property. On the other hand, we provide a sufficient condition for the strongly anti-coproximinal subspaces in a general Banach space. We also characterize the anti-coproximinal subspaces of a smooth Banach space. Further, we study these special subspaces in a finite-dimensional polyhedral Banach space and find some interesting geometric structures associated with them.

Keywords Approximate Birkhoff–James orthogonality · Best coapproximations · Polyhedral Banach spaces

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1 Introduction

Franchetti and Furi introduced the best coapproximation problem in the framework of Banach spaces in [7], as a complementary notion to the well-known best approximation problem. Unlike its counterpart, the best coapproximation problem remains a less explored area of research, especially from the computational point of view. Indeed, the difficulty of the problem arises from the fact that the existence of best coapproximation is not guaranteed even in finite-dimensional Banach spaces. Interested readers are referred to [13, 15–17, 22] for more information on this topic. The concept of contractive mapping is closely connected with the idea of best coapproximation, to delve deeper the readers can go through [2, 3, 12]. Recently, an algorithmic approach has been given in [20] and [21] to study the best coapproximation in ℓ_∞^n and ℓ_1^n , respectively. The purpose of this article is to identify some special types of subspaces of a Banach space, which may be regarded as the least favorable candidates for the purpose of the best coapproximation problem. We also illustrate that such subspaces enjoy several geometric properties unique to them. Having described the motivation behind this study, let us now introduce the relevant notations and terminologies.

Let \mathbb{X}, \mathbb{Y} denote real Banach spaces and let \mathbb{H} denote a real Hilbert space. We use the notations $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$, for the unit ball $B_{\mathbb{X}} := \{x \in \mathbb{X} : \|x\| \leq 1\}$ and the unit sphere $S_{\mathbb{X}} := \{x \in \mathbb{X} : \|x\| = 1\}$ of \mathbb{X} , respectively. The dual space of \mathbb{X} is denoted by \mathbb{X}^* . The annihilator of a subspace \mathbb{Y} of \mathbb{X} is defined as $\mathbb{Y}^\perp := \{f \in \mathbb{X}^* : f(y) = 0, \text{ for each } y \in \mathbb{Y}\}$. For a subspace \mathbb{Z} of \mathbb{X}^* , the pre-annihilator of \mathbb{Z} is defined as ${}^\perp\mathbb{Z} := \{x \in \mathbb{X} : f(x) = 0, \text{ for each } f \in \mathbb{Z}\}$. For a non-empty convex subset A of \mathbb{X} , an element $z \in A$ is said to be an extreme point of A if whenever $z = (1 - t)x + ty$, for some $t \in (0, 1)$ and some $x, y \in A$, then $x = y = z$. The collection of all the extreme points of A is denoted as $Ext(A)$. A Banach space \mathbb{X} is said to be strictly convex if $Ext(B_{\mathbb{X}}) = S_{\mathbb{X}}$. It is easy to observe that strict convexity of \mathbb{X} is equivalent to the geometric condition that $S_{\mathbb{X}}$ does not contain any non-trivial straight line segment. We recall that a point $x \in S_{\mathbb{X}}$ is said to be a rotund point [8] of $B_{\mathbb{X}}$ if $\|y\| = \|\frac{x+y}{2}\| = 1$ implies that $x = y$. Clearly, if every point of $S_{\mathbb{X}}$ is rotund then \mathbb{X} is strictly convex. Given any non-zero $x \in \mathbb{X}$, $f \in S_{\mathbb{X}^*}$ is said to be a support functional of x if $f(x) = \|x\|$. The set of all support functional(s) of a non-zero $x \in \mathbb{X}$ is written as $J(x) := \{f \in S_{\mathbb{X}^*} : f(x) = \|x\|\}$. For any non-zero $x \in \mathbb{X}$, x is said to be smooth if $J(x)$ is singleton. A Banach space \mathbb{X} is said to be smooth if each $x \in S_{\mathbb{X}}$ is smooth. The collection of all smooth points in \mathbb{X} is denoted by $Sm(\mathbb{X})$. For a subspace \mathbb{Y} of \mathbb{X} , let $\mathcal{J}_{\mathbb{Y}} = \{f \in S_{\mathbb{X}^*} : f(y) = 1, \text{ for some } y \in Sm(\mathbb{X}) \cap S_{\mathbb{Y}}\}$. It is easy to check that $\mathcal{J}_{\mathbb{Y}} \subseteq Ext(B_{\mathbb{X}^*})$. Whenever $Sm(\mathbb{X}) \cap S_{\mathbb{Y}} = \emptyset$, we define $\mathcal{J}_{\mathbb{Y}} = \emptyset$. Let us recall the well known definition of the modulus of smoothness of a Banach space $\mathbb{X}(\neq \{0\})$, which is denoted by $\rho_{\mathbb{X}}(t)$, and is defined as follows:

$$\rho_{\mathbb{X}}(t) = \sup \left\{ \frac{\|x + ty\| + \|x - ty\|}{2} - 1 : x, y \in S_{\mathbb{X}} \right\},$$

where $t \in (0, +\infty)$. The space $\mathbb{X}(\neq \{0\})$ is said to be uniformly smooth ([14, Def. 5.5.2]) if $\frac{\rho_{\mathbb{X}}(t)}{t} \rightarrow 0$, as $t \rightarrow 0^+$.

Given any $f \in \mathbb{X}^*$, the kernel of f is denoted by $\ker f := \{x \in \mathbb{X} : f(x) = 0\}$. A finite-dimensional Banach space \mathbb{X} is said to be a polyhedral Banach space if $\text{Ext}(B_{\mathbb{X}})$ is finite. A convex set $F \subset S_{\mathbb{X}}$ is said to be a face of $B_{\mathbb{X}}$ if for some $x_1, x_2 \in S_{\mathbb{X}}$, $(1 - t)x_1 + tx_2 \in F$ implies that $x_1, x_2 \in F$, where $0 < t < 1$. A maximal face of $B_{\mathbb{X}}$ is said to be a facet. We use the notation $\text{int}(F)$ to denote the interior of a face F endowed with the usual subspace topology of F . Given a subset M of \mathbb{X}^* , \overline{M}^{w^*} denotes the closure of M with respect to the weak*- topology defined on \mathbb{X}^* . We also recall that \mathbb{X} satisfies the Kadets–Klee Property if whenever $\{x_n\}$ is a sequence in \mathbb{X} and $x \in \mathbb{X}$ such that $x_n \xrightarrow{w} x$ and $\|x_n\| \rightarrow \|x\|$, it follows that $x_n \rightarrow x$. Next we recall the definition of best coapproximation in Banach spaces.

Definition 1.1 [7] Let \mathbb{Y} be a subspace of Banach space \mathbb{X} . Given any $x \in \mathbb{X}$, we say that $y_0 \in \mathbb{Y}$ is a best coapproximation to x out of \mathbb{Y} if $\|y_0 - y\| \leq \|x - y\|$, for all $y \in \mathbb{Y}$.

Given any $x \in \mathbb{X}$ and a subspace \mathbb{Y} of \mathbb{X} , $\mathcal{R}_{\mathbb{Y}}(x)$ denotes the (possibly empty) set of all best coapproximations to x out of \mathbb{Y} . A subspace \mathbb{Y} of the Banach space \mathbb{X} is said to be coproximal if for any $x \in \mathbb{X}$, $\mathcal{R}_{\mathbb{Y}}(x) \neq \emptyset$. The concept of Birkhoff–James orthogonality plays a vital role in the study of the best coapproximation problem. Given $x, y \in \mathbb{X}$, we say that x is Birkhoff–James orthogonal [1, 10] to y , written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$, for all $\lambda \in \mathbb{R}$. As mentioned in [7], it is easy to observe that given any subspace \mathbb{Y} of a Banach space \mathbb{X} and an element $x \in \mathbb{X}$, $y_0 \in \mathbb{Y}$ is a best coapproximation to x out of \mathbb{Y} if and only if $\mathbb{Y} \perp_B (x - y_0)$, i.e., $y \perp_B (x - y_0)$, for all $y \in \mathbb{Y}$. We also need the notion of approximate Birkhoff–James orthogonality in our study. In [6], Dragomir first introduced the approximate Birkhoff–James orthogonality in the following way:

Let $\epsilon \in [0, 1)$. Then for $x, y \in \mathbb{X}$, x is said to be approximate ϵ -Birkhoff–James orthogonal to y if for each $\lambda \in \mathbb{R}$, the following holds:

$$\|x + \lambda y\| \geq (1 - \epsilon)\|x\|.$$

Later on Chmieliński [4] introduced another version of the approximate Birkhoff–James orthogonality. Let $\epsilon \in [0, 1)$. Given any $x, y \in \mathbb{X}$, we say that x is said to be approximate orthogonal to y , written as $x \perp_B^\epsilon y$, if

$$\|x + \lambda y\|^2 \geq \|x\|^2 - 2\epsilon\|x\|\|\lambda y\|.$$

Recently [5], an equivalent definition of the approximate orthogonality has been obtained:

$$x \perp_B^\epsilon y \iff \|x + \lambda y\| \geq \|x\| - \epsilon\|\lambda y\|, \text{ for every } \lambda \in \mathbb{R}.$$

In view of this characterization of approximate orthogonality, it is natural to consider the following approximate version of the best coapproximation problem:

Definition 1.2 Let \mathbb{Y} be a subspace of Banach space \mathbb{X} . Let $\epsilon \in [0, 1)$. Given any $x \in \mathbb{X}$, we say that $y_0 \in \mathbb{Y}$ is an ϵ -best coapproximation to x out of \mathbb{Y} if $\mathbb{Y} \perp_B^\epsilon x - y_0$, i.e., $y \perp_B^\epsilon (x - y_0)$, for all $y \in \mathbb{Y}$.

The primary purpose of our study is to investigate the least favorable scenario that can arise in studying the best coapproximation problem. Accordingly, we introduce the following two types of subspaces of a Banach space from the perspective of best coapproximation and ϵ -best coapproximation.

Definition 1.3 Let \mathbb{Y} be a subspace of Banach space \mathbb{X} . Then

- (i) \mathbb{Y} is said to be an *anti-coproximinal subspace* of \mathbb{X} if for any given $x \in \mathbb{X} \setminus \mathbb{Y}$, there does not exist any best coapproximation to x out of \mathbb{Y} .
- (ii) \mathbb{Y} is said to be a *strongly anti-coproximinal subspace* of \mathbb{X} if for any given $x \in \mathbb{X} \setminus \mathbb{Y}$ and for any $\epsilon \in [0, 1)$, there does not exist ϵ -best coapproximation to x out of \mathbb{Y} .

It is easy to observe that if \mathbb{Y} is (strongly) anti-coproximinal in \mathbb{X} and \mathbb{Z} is a subspace of \mathbb{X} containing \mathbb{Y} then \mathbb{Y} is (strongly) anti-coproximinal in \mathbb{Z} . However, it is quite obvious that if \mathbb{Y} is (strongly) anti-coproximinal in \mathbb{Z} , it does not imply that \mathbb{Y} is (strongly) anti-coproximinal in \mathbb{X} (see Remark 2.27). Therefore, it is important to specify the mother space whenever we consider the anti-coproximinal and the strongly anti-coproximinal subspaces.

We note from [20, Example 2.13] that if a subspace \mathbb{Y} is not coproximinal in \mathbb{X} then it is not necessarily true that \mathbb{Y} is an anti-coproximinal subspace of \mathbb{X} . On the other hand, if \mathbb{Y} is a strongly anti-coproximinal subspace of \mathbb{X} , it implies that \mathbb{Y} is an anti-coproximinal subspace of \mathbb{X} . However, as we will observe in Example 2.21, the converse of this fact is not true. It should be noted that every one-dimensional subspace is coproximinal in any Banach space [7, Lemma 1]. Whenever the anti-coproximinal and the strongly anti-coproximinal subspaces are concerned, we only consider the proper subspaces of dimension strictly greater than one. In order to facilitate the understanding of these two special types of subspaces of a Banach space, let us consider the following simple example:

Example 1.4 Let $\tilde{v}_1 = (3, 0, 2)$, $\tilde{v}_2 = (0, 3, 2) \in \ell_\infty^3$ and let $\mathbb{Y} = \text{span}\{\tilde{v}_1, \tilde{v}_2\}$. Applying [20, Th. 2.10] along with some straightforward calculations, we observe that for any $x \in \ell_\infty^3 \setminus \mathbb{Y}$, there does not exist a best coapproximation to x out of \mathbb{Y} . In other words, \mathbb{Y} is an anti-coproximinal subspace of ℓ_∞^3 . We will see that the same subspace happens to be a strongly anti-coproximinal subspace of ℓ_∞^3 as well. On the other hand, we will also present several explicit examples of the anti-coproximinal subspaces of a Banach space, which are not strongly anti-coproximinal in that Banach space.

This article is divided into two sections including the introductory one. The main results are separated in two subsections, viz., Sections I and II. In Section I, we characterize the anti-coproximinal subspaces in a smooth Banach space and using that we characterize the Hilbert spaces among smooth Banach spaces. We also present a necessary condition for a subspace to be strongly anti-coproximinal in a reflexive Banach space, whose dual space satisfies the Kadets–Klee Property. On the other

hand, a sufficient condition for the same is obtained in any Banach space. In Section II, we exclusively consider the finite-dimensional polyhedral Banach spaces. We characterize the strongly anti-coproximinal subspaces in a finite-dimensional polyhedral Banach space and show that the strongly anti-coproximinal subspaces of such spaces possess some nice geometric structures. We study the spaces ℓ_∞^n and ℓ_1^n separately and characterize the anti-coproximinal and the strongly anti-coproximinal subspaces of these distinguished polyhedral Banach spaces, which is especially advantageous from the computational point of view.

2 Main results

Section-I

We begin this section with the following two well-known observations.

Theorem 2.1 [16, Th. 2.1] *Let \mathbb{Y} be a subspace of a Banach space \mathbb{X} and let $x \in \mathbb{X}$. Then $y_0 \in \mathbb{Y}$ is a best coapproximation to x out of \mathbb{Y} if and only if for any $y \in \mathbb{Y}$, there exists $f_y \in J(y)$ such that $f_y(x - y_0) = 0$.*

Theorem 2.2 [5, Th. 2.3] *Let \mathbb{X} be a Banach space. Let $x, y \in \mathbb{X}$ and let $\epsilon \in [0, 1)$. Then the following conditions are equivalent:*

- (i) $x \perp_B^\epsilon y$
- (ii) *there exists $f \in J(x)$ such that $|f(y)| \leq \epsilon \|y\|$*

Using Theorem 2.1, a simple characterization of the anti-coproximinal subspace follows immediately.

Proposition 2.3 *Let \mathbb{Y} be a subspace of Banach space \mathbb{X} . Then \mathbb{Y} is an anti-coproximinal subspace of \mathbb{X} if and only if for any $x \in \mathbb{X} \setminus \mathbb{Y}$, there exists a $y_0 \in \mathbb{Y}$ such that $x \notin \ker f$, for any $f \in J(y_0)$.*

We next obtain a characterization of the ϵ -best coapproximation, by applying Theorem 2.2.

Proposition 2.4 *Let \mathbb{Y} be a subspace of a Banach space \mathbb{X} . Then the following statements are equivalent:*

- (i) y_0 is an ϵ -best coapproximation to x out of \mathbb{Y}
- (ii) *for any $y \in \mathbb{Y}$, $\|x - y\| \geq \|y_0 - y\| - \epsilon \|x - y_0\|$*
- (iii) *for any $y \in \mathbb{Y}$, there exists $f_y \in J(y)$ such that $|f_y(x - y_0)| \leq \epsilon \|x - y_0\|$.*

Proof (i) \Rightarrow (ii): Let y_0 be an ϵ -best coapproximation to x out of \mathbb{Y} . Therefore, $\mathbb{Y} \perp_B^\epsilon (x - y_0)$, i.e., $y \perp_B^\epsilon (x - y_0)$, for any $y \in \mathbb{Y}$. Thus for any $y \in \mathbb{Y}$, $\|y + \lambda(x - y_0)\| \geq \|y\| - \epsilon \|\lambda(x - y_0)\|$, for all $\lambda \in \mathbb{R}$. Putting $\lambda = 1$, we get $\|y + (x - y_0)\| \geq \|y\| - \epsilon \|(x - y_0)\|$, from which the desired inequality follows easily.

(ii) \Rightarrow (iii): For any $y \in \mathbb{Y}$ and for any non-zero $\lambda \in \mathbb{R}$,

$$\begin{aligned} \|y + \lambda(x - y_0)\| &= |\lambda| \|x - (y_0 - \frac{1}{\lambda}y)\| \\ &\geq |\lambda| \{ \|y_0 - (y_0 - \frac{1}{\lambda}y)\| - \epsilon \|x - y_0\| \} \\ &= |\lambda| (\| \frac{1}{\lambda}y \| - \epsilon \|x - y_0\|) \\ &= \|y\| - \epsilon \|\lambda(x - y_0)\|. \end{aligned}$$

For $\lambda = 0$, the above inequality holds trivially. This implies $y \perp_B^\epsilon (x - y_0)$, for any $y \in \mathbb{Y}$ and so the result follows from Theorem 2.2.

(iii) \Rightarrow (i): Following Theorem 2.2, we get $y \perp_B^\epsilon x - y_0$, for all $y \in \mathbb{Y}$. Thus we obtain $\mathbb{Y} \perp_B^\epsilon x - y_0$. □

Throughout the next part of this section, we focus on the anti-coproximality and the strong anti-coproximality of the closed subspaces in a Banach space. In the following proposition, we observe that every dense subspace of a Banach space is strongly anti-coproximal in that space and consequently, anti-coproximal too.

Proposition 2.5 *Let \mathbb{Y} be a dense subspace of a Banach space \mathbb{X} . Then \mathbb{Y} is a strongly anti-coproximal subspace of \mathbb{X} .*

Proof Suppose on the contrary that \mathbb{Y} is not a strongly anti-coproximal subspace of \mathbb{X} . Then there exists an $\epsilon \in [0, 1)$ and an $x \in \mathbb{X} \setminus \mathbb{Y}$ such that $\mathbb{Y} \not\perp_B^\epsilon x$. Since \mathbb{Y} is dense in \mathbb{X} , it follows that there exists a sequence $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{Y}$ such that $y_n \rightarrow x$. Also, we note that $y_n \perp_B^\epsilon x$. Then for each $\lambda \in \mathbb{R}$, we have $\|y_n + \lambda x\| \geq \|y_n\| - \epsilon \|\lambda x\|$. Since the norm function is continuous, letting $n \rightarrow \infty$, we get that $\|x + \lambda x\| \geq \|x\| - \epsilon \|\lambda x\|$, for each $\lambda \in \mathbb{R}$. This implies that $x \perp_B^\epsilon x$, and therefore $x = 0$, which contradicts the fact that $x \in \mathbb{X} \setminus \mathbb{Y}$. □

In the next theorem we characterize the anti-comproximal subspaces in a smooth Banach space.

Theorem 2.6 *Let \mathbb{Y} be a subspace of a smooth Banach space \mathbb{X} . Then \mathbb{Y} is an anti-coproximal subspace of \mathbb{X} if and only if $\overline{\text{span } \mathcal{J}_{\mathbb{Y}}^{w*}} = \mathbb{X}^*$.*

Proof Let us first prove the sufficient part of the theorem. Suppose on the contrary that \mathbb{Y} is not an anti-coproximal subspace of \mathbb{X} . This implies that for some $x \in \mathbb{X} \setminus \mathbb{Y}$, there exists a $y_0 \in \mathbb{Y}$ such that y_0 is a best coapproximation to x out of \mathbb{Y} . Since \mathbb{X} is smooth, it follows from Theorem 2.1 that for any $y \in \mathbb{Y}$, $f_y(x - y_0) = 0$, where $J(y) = \{f_y\}$. It is easy to see that $\mathcal{J}_{\mathbb{Y}} = \{f_y : y \in \mathbb{Y}\}$. This implies, $g(x - y_0) = 0$, for any $g \in \text{span } \mathcal{J}_{\mathbb{Y}}$. Now let us consider $f \in \mathbb{X}^*$. Since $\overline{\text{span } \mathcal{J}_{\mathbb{Y}}^{w*}} = \mathbb{X}^*$, it follows that there exists a net $\{f_\alpha\}_{\alpha \in \Lambda} \subset \text{span } \mathcal{J}_{\mathbb{Y}}$ such that f_α is weak*-convergent to f . Since $f_\alpha(x - y_0) = 0$, for each $\alpha \in \Lambda$, it follows that $f(x - y_0) = 0$. Note that $f \in \mathbb{X}^*$ is taken arbitrarily. Thus we obtain that $f(x - y_0) = 0$, for all $f \in \mathbb{X}^*$, i.e., $x - y_0 = 0$, which contradicts the fact that $x \in \mathbb{X} \setminus \mathbb{Y}$.

Now we prove the necessary part of the theorem. Suppose on the contrary that $\overline{\text{span } \mathcal{J}_{\mathbb{Y}}^{w*}} \subsetneq \mathbb{X}^*$. Clearly, ${}^\perp(\text{span } \mathcal{J}_{\mathbb{Y}}) = \cap_{f \in \mathcal{J}_{\mathbb{Y}}} \ker f$. We note from [18, Th. 4.7] that $({}^\perp(\text{span } \mathcal{J}_{\mathbb{Y}}))^\perp = \overline{\text{span } \mathcal{J}_{\mathbb{Y}}^{w*}}$. This implies that $(\cap_{f \in \mathcal{J}_{\mathbb{Y}}} \ker f)^\perp = \overline{\text{span } \mathcal{J}_{\mathbb{Y}}^{w*}}$. Following [18, Th.4.7], we obtain that $(\cap_{f \in \mathcal{J}_{\mathbb{Y}}} \ker f)^*$ is isometrically isomorphic to $\mathbb{X}^*/\overline{\text{span } \mathcal{J}_{\mathbb{Y}}^{w*}}$. Therefore, $(\cap_{f \in \mathcal{J}_{\mathbb{Y}}} \ker f)^* \neq 0$, which implies that $\cap_{f \in \mathcal{J}_{\mathbb{Y}}} \ker f \neq 0$. Moreover, it is straightforward to see that $(\cap_{f \in \mathcal{J}_{\mathbb{Y}}} \ker f) \cap \mathbb{Y} = \{0\}$. So, there exists a $z \in \mathbb{X} \setminus \mathbb{Y}$ such that $z \in \cap_{f \in \mathcal{J}_{\mathbb{Y}}} \ker f$. Since \mathbb{X} is smooth, applying [10, Th. 2.1], we obtain that $\mathbb{Y} \perp_B z$. Thus 0 is a best coapproximation to z out of \mathbb{Y} . This contradicts the fact that \mathbb{Y} is an anti-coproximinal subspace of \mathbb{X} . This proves the necessary part and completes the proof of the theorem. \square

If \mathbb{X} is finite-dimensional then we have the following corollary.

Corollary 2.7 *Let \mathbb{Y} be a subspace of an n -dimensional smooth Banach space \mathbb{X} . Then \mathbb{Y} is an anti-coproximinal subspace of \mathbb{X} if and only if $\dim(\text{span } \mathcal{J}_{\mathbb{Y}}) = n$.*

Next we give an example of an anti-coproximinal subspace in ℓ_p^n , where $2 < p < \infty$. We require the following well-known result which explicitly describes the support functional of an element of ℓ_p^n . Suppose that ϕ is the isometric isomorphism between $(\ell_p^n)^*$ and ℓ_q^n , where $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.8 *Let $\tilde{x} = (x_1, x_2, \dots, x_n) \in \ell_p^n$, where $1 < p < \infty$. Then $J(\tilde{x}) = \{\tilde{f}\}$, where $\phi(\tilde{f}) = \left(\frac{x_1|x_1|^{p-2}}{\|\tilde{x}\|_p^{p/q}}, \frac{x_2|x_2|^{p-2}}{\|\tilde{x}\|_p^{p/q}}, \dots, \frac{x_n|x_n|^{p-2}}{\|\tilde{x}\|_p^{p/q}} \right) \in \ell_q^n$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\|\tilde{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$.*

Example 2.9 Let us consider the space ℓ_p^n , where $p \in (1, \infty) \setminus \{2\}$ and $n \geq 3$ with $\{e_1, e_2, \dots, e_n\}$ as the standard ordered basis. Suppose that \mathbb{Y} is a hyper-space of ℓ_p^n , where $\mathbb{Y} = \text{span}\{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{n-1}\}$, $\tilde{x}_1 = (1, 1, 1, 0, \dots, 0)$, $\tilde{x}_2 = (1, 2, 3, 0, \dots, 0)$, $\tilde{x}_3 = e_4, \dots, \tilde{x}_{n-1} = e_n$. Let $J(\tilde{x}_i) = \{\tilde{f}_i\}$, for any $1 \leq i \leq n$. Clearly, $\tilde{f}_i \in \mathcal{J}_{\mathbb{Y}}$. Applying Lemma 2.8, we obtain the following:

$$\phi(\tilde{f}_1) = \left(\frac{1}{3^{1-\frac{1}{p}}}, \frac{1}{3^{1-\frac{1}{p}}}, \frac{1}{3^{1-\frac{1}{p}}}, 0, \dots, 0 \right) \in \ell_q^n,$$

$$\phi(\tilde{f}_2) = \left(\frac{1}{(1+2^p+3^p)^{1-\frac{1}{p}}}, \frac{2^{p-1}}{(1+2^p+3^p)^{1-\frac{1}{p}}}, \frac{3^{p-1}}{(1+2^p+3^p)^{1-\frac{1}{p}}}, 0, \dots, 0 \right) \in \ell_q^n,$$

and $\phi(\tilde{f}_k) = e_{k+1} \in \ell_q^n$, for all $3 \leq k \leq n - 1$. Consider the element $\tilde{x}_n = 3\tilde{x}_1 - \tilde{x}_2 \in \mathbb{Y}$ and let $J(\tilde{x}_n) = \{\tilde{f}_n\}$. Again using Lemma 2.8, we get

$$\phi(\tilde{f}_n) = \left(\frac{2^{p-1}}{(2^p+1)^{1-\frac{1}{p}}}, \frac{1}{(2^p+1)^{1-\frac{1}{p}}}, 0, \dots, 0 \right) \in \ell_q^n.$$

A straightforward computation shows that $\{f_1, f_2, \dots, f_n\}$ is a linearly independent set. Therefore, $\dim(\text{span } \mathcal{J}_{\mathbb{Y}}) = n$. Thus from Corollary 2.7, we conclude that \mathbb{Y} is an anti-coproximinal subspace of ℓ_p^n , where $p \in (1, \infty) \setminus \{2\}$. By a similar computation, it can be shown that $\mathbb{W} = \underbrace{\mathbb{Y} \oplus \mathbb{Y} \oplus \dots \oplus \mathbb{Y}}_{r\text{-times}}$ is an anti-coproximinal subspace of ℓ_p^{rn} .

It is easy to note from Example 2.9 that for every $p \in (1, \infty) \setminus \{2\}$ and for every $n > 2$, there exists an anti-coproximinal subspace of ℓ_p^n . Combining with the well

known fact that every closed subspace of a Hilbert space is coproximal ([7, Lemma. 1]), we can characterize the Hilbert space ℓ_2^n among the ℓ_p^n spaces.

Theorem 2.10 *Let $\mathbb{X} = \ell_p^n$, where $1 < p < \infty$ and $n \geq 3$. Then $p = 2$ if and only if there does not exist any anti-coproximal subspace in \mathbb{X} .*

The previous result can be further extended to characterize the Hilbert space among smooth Banach spaces having dimension at least 3.

Theorem 2.11 *Let \mathbb{X} be a smooth Banach space and let $\dim(\mathbb{X}) \geq 3$. Then \mathbb{X} is Hilbert space if and only if there does not exist any anti-coproximal closed hyperspace in \mathbb{X} .*

Proof Since the necessary part follows from [7, Th. 1], we only prove the sufficient part. Suppose that \mathbb{Y} is a closed hyperspace of \mathbb{X} . Since \mathbb{Y} is not an anti-coproximal subspace of \mathbb{X} , it follows from Theorem 2.6 that $\overline{\text{span } \mathcal{F}_{\mathbb{Y}}^{w*}} \subsetneq \mathbb{X}^*$. Now following similar arguments as given in the proof of the necessary part of Theorem 2.6, we observe that there exists an element $z \in \mathbb{X} \setminus \mathbb{Y}$ such that $\mathbb{Y} \perp_B z$. Following [11, Th.4], we conclude that \mathbb{X} is a Hilbert space. □

Remark 2.12 We recall that every one-dimensional subspace of a Banach space is coproximal. Therefore, it is clear that the previous result is not valid for $n = 2$.

Our next goal is to separately present a necessary condition and a sufficient condition for strongly anti-coproximal subspaces of a Banach space. First we give the sufficient condition.

Theorem 2.13 *Let \mathbb{Y} be a subspace of a Banach space \mathbb{X} . Then \mathbb{Y} is a strongly anti-coproximal subspace of \mathbb{X} if for each $x \in \mathbb{X} \setminus \mathbb{Y}$, there exists a $y \in \mathbb{Y}$ such that $J(y) \subseteq J(x) \cup J(-x)$.*

Proof Suppose on the contrary that \mathbb{Y} is not a strongly anti-coproximal subspace of \mathbb{X} . Therefore, there exists an $x \in \mathbb{X} \setminus \mathbb{Y}$ such that $y_1 \in \mathbb{Y}$ is an ϵ -best coapproximation to x out of \mathbb{Y} , for some $\epsilon \in [0, 1)$. Applying Proposition 2.4, we obtain that for each $y \in \mathbb{Y}$, there exists an $f_y \in J(y)$ such that $|f_y(x - y_1)| \leq \epsilon \|x - y_1\| < \|x - y_1\|$. Therefore, for each $y \in \mathbb{Y}$, there exists an $f_y \in J(y)$, such that $f_y \notin J(x - y_1) \cup J(-(x - y_1))$. This contradicts the hypothesis of the theorem, thereby finishing the proof. □

Let us now present a necessary condition for strongly anti-coproximal subspaces of a Banach space under some additional nice conditions.

Theorem 2.14 *Let \mathbb{X} be a reflexive Banach space and let \mathbb{X}^* satisfies the Kadets–Klee Property. Let \mathbb{Y} be a closed subspace of \mathbb{X} . If \mathbb{Y} is a strongly anti-coproximal subspace of \mathbb{X} then for each $x \in \mathbb{X}$, there exists an element $y \in \mathbb{Y}$ such that $J(y) \cap J(x) \neq \emptyset$.*

Proof Note that whenever $x \in \mathbb{Y}$, we have nothing to prove. Take $x \in \mathbb{X} \setminus \mathbb{Y}$. Since \mathbb{Y} is a strongly anti-coproximal subspace of \mathbb{X} , it follows that for any $\epsilon \in [0, 1)$, $\mathbb{Y} \not\perp_B^\epsilon x$. Let us take a sequence $\{\epsilon_n\}_{n \in \mathbb{N}} \subset [0, 1) \rightarrow 1$, as $n \rightarrow \infty$. Suppose that for each $n \in \mathbb{N}$, there exists $y_n \in S_{\mathbb{Y}}$ such that $y_n \not\perp_B^{\epsilon_n} x$. From Theorem 2.2 we

obtain that for any $f_n \in J(y_n)$, $|f_n(x)| > \epsilon_n \|x\|$. Since \mathbb{X} is reflexive, it follows that \mathbb{X}^* is reflexive and therefore without loss of generality we may and do assume that f_n is weakly convergent to f , for some $f \in B_{\mathbb{X}^*}$. So, $f_n(x) \rightarrow f(x)$. Taking limit on the both sides of the above inequality, we obtain $|f(x)| \geq \|x\|$. Since $f \in B_{\mathbb{X}^*}$, $|f(x)| = \|x\|$, and therefore, $\|f\| = 1$. Thus either $f \in J(x)$ or $-f \in J(x)$. Also $\|f_n\| \rightarrow \|f\|$ as $n \rightarrow \infty$. Since \mathbb{X}^* satisfies the Kadets–Klee Property, it follows that $f_n \rightarrow f$ as $n \rightarrow \infty$. As \mathbb{X} is reflexive and \mathbb{Y} is a closed subspace of \mathbb{X} , then \mathbb{Y} is also reflexive, and therefore $B_{\mathbb{Y}}$ is weakly compact. So, y_n weakly converges to y , for some $y \in B_{\mathbb{Y}}$. Now as $f_n \rightarrow f$ and $y_n \xrightarrow{w} y$, it is straightforward to see that $f(y) = 1$. Therefore, $f \in J(y)$ and consequently, either $J(y) \cap J(x) \neq \emptyset$ or $J(-y) \cap J(x) \neq \emptyset$. This completes the theorem. \square

Remark 2.15 Observe that the above condition is only necessary but not sufficient, see Example 2.28.

Applying Theorem 2.14, it is possible to give examples of Banach spaces which do not contain any strongly anti-coproximal closed subspaces.

Theorem 2.16 *Let \mathbb{X} be a reflexive Banach space and let \mathbb{X}^* satisfies the Kadets–Klee Property. Suppose that \mathbb{Y} is a closed subspace of \mathbb{X} such that there exists a rotund point in $S_{\mathbb{X}} \setminus S_{\mathbb{Y}}$. Then \mathbb{Y} is not a strongly anti-coproximal subspace of \mathbb{X} . In particular, every reflexive strictly convex Banach space, whose dual satisfies the Kadets–Klee Property, does not contain any strongly anti-coproximal closed subspaces.*

Proof Suppose that $x \in S_{\mathbb{X}} \setminus S_{\mathbb{Y}}$ is a rotund point. It is straightforward to see that for any $y \in \mathbb{Y}$, $J(x) \cap J(y) = \emptyset$. Indeed, if possible let $y_0 \in S_{\mathbb{Y}}$ be such that $f \in J(x) \cap J(y_0)$. This implies that $f(\frac{x+y_0}{2}) = 1 \implies \|\frac{x+y_0}{2}\| = 1$. Since x is rotund, it follows that $x = y_0$, a contradiction. Now applying Theorem 2.14, it can be concluded that \mathbb{Y} is not a strongly anti-coproximal subspace of \mathbb{X} . This completes the proof of the first part. The second part follows trivially from the first part. \square

Our next result shows that the condition of strict convexity in the previous theorem can be replaced by the condition of smoothness.

Theorem 2.17 *Let \mathbb{X} be a reflexive smooth Banach space and let \mathbb{X}^* satisfies the Kadets–Klee Property. Suppose that \mathbb{Y} is a closed subspace of \mathbb{X} . Then \mathbb{Y} is not a strongly anti-coproximal subspace of \mathbb{X} .*

Proof Suppose that \mathbb{Y} is a closed subspace of \mathbb{X} . Let us consider $g \in \mathbb{Y}^\perp$ such that $\|g\| = 1$. Since \mathbb{X} is reflexive, there exists a $z \in S_{\mathbb{X}}$ such that $|g(z)| = 1$. Therefore, $J(z) = \{g\}$ or $J(z) = \{-g\}$, as \mathbb{X} is smooth. We claim that for any $y \in \mathbb{Y}$, $J(y) \cap J(z) = \emptyset$. Otherwise, take a non-zero element $y_1 \in \mathbb{Y}$ such that $J(y_1) \cap J(z) \neq \emptyset$. Since \mathbb{X} is smooth, it follows that either $J(y_1) = \{g\}$ or $J(y_1) = \{-g\}$. Then $|g(y_1)| = \|y_1\|$, which contradicts the fact that $g \in \mathbb{Y}^\perp$. Therefore, applying Theorem 2.14, we conclude that \mathbb{Y} is not a strongly anti-coproximal subspace of \mathbb{X} . This completes the theorem. \square

We end this section with the following result, which is an immediate corollary of Theorem 2.14.

Corollary 2.18 *Let \mathbb{Y} be a closed subspace of a Banach space \mathbb{X} . Suppose that \mathbb{X} satisfies either of the following properties:*

- (i) \mathbb{X} is a finite-dimensional smooth Banach space
- (ii) \mathbb{X} is a finite-dimensional strictly convex Banach space
- (iii) \mathbb{X} is a uniformly smooth Banach space.

Then \mathbb{Y} is not a strongly anti-coproximinal subspace of \mathbb{X} .

Section-II

In this section we study the anti-coproximinal and the strongly anti-coproximinal subspaces in finite-dimensional polyhedral Banach spaces. In particular, we obtain characterizations of the said subspaces in ℓ_∞^n and ℓ_1^n , which are computationally effective. We begin with the following characterization of the anti-coproximinal subspaces of a finite-dimensional polyhedral Banach space, provided that $Sm(\mathbb{X}) \cap \mathbb{Y}$ is dense in \mathbb{Y} .

Theorem 2.19 *Let \mathbb{Y} be a subspace of an n -dimensional polyhedral Banach space \mathbb{X} such that $Sm(\mathbb{X}) \cap \mathbb{Y}$ is dense in \mathbb{Y} . Then \mathbb{Y} is an anti-coproximinal subspace of \mathbb{X} if and only if there are n -linearly independent elements in $\mathcal{J}_\mathbb{Y}$.*

Proof We first prove the sufficient part. Suppose on the contrary that \mathbb{Y} is not an anti-coproximinal subspace of \mathbb{X} . Then there exists an element $x \in \mathbb{X} \setminus \mathbb{Y}$ and a $y_0 \in \mathbb{Y}$ such that $y_0 \in \mathcal{R}_\mathbb{Y}(x)$. Therefore, from Theorem 2.1, it follows that $x - y_0 \in \ker f$, for all $f \in \mathcal{J}_\mathbb{Y}$. Since $\mathcal{J}_\mathbb{Y}$ contains n linearly independent elements and $\dim(\mathbb{X}^*) = n$, it follows that $\cap_{f \in \mathcal{J}_\mathbb{Y}} \ker f = \{0\}$. Therefore, $x - y_0 = 0$, i.e., $x = y_0$, which is a contradiction. This completes the proof of the sufficient part.

To prove the necessary part, we first show that for any $y \in S_\mathbb{Y}$, $\mathcal{J}_\mathbb{Y} \cap J(y) \neq \emptyset$. Since $Sm(\mathbb{X}) \cap \mathbb{Y}$ is dense in \mathbb{Y} , it is immediate that $Sm(\mathbb{X}) \cap S_\mathbb{Y}$ is dense in $S_\mathbb{Y}$, and therefore, for any $y \in S_\mathbb{Y}$, there exists a sequence $\{y_n\}_{n \in \mathbb{N}} \subset Sm(\mathbb{X}) \cap S_\mathbb{Y}$ such that $y_n \rightarrow y$. Suppose that $J(y_n) = \{f_n\} \subset \mathcal{J}_\mathbb{Y}$, for each $n \in \mathbb{N}$. Therefore, it is easy to see that $f_n(y) \rightarrow 1$. Since \mathbb{X} is polyhedral and $\mathcal{J}_\mathbb{Y} \subset Ext(B_{\mathbb{X}^*})$, it follows that $\mathcal{J}_\mathbb{Y}$ is finite. Moreover, since for each $n \in \mathbb{N}$, $f_n \in \mathcal{J}_\mathbb{Y}$ and $\mathcal{J}_\mathbb{Y}$ is finite, we get that for some $f \in \mathcal{J}_\mathbb{Y}$, $f(y) = 1$. Thus $f \in \mathcal{J}_\mathbb{Y} \cap J(y)$. If possible, suppose that $\mathcal{J}_\mathbb{Y}$ contains exactly k linearly independent elements such that $k < n$. Let us assume that $(\neq 0)x \in \cap_{f \in \mathcal{J}_\mathbb{Y}} \ker f$. Clearly, $x \notin \mathbb{Y}$. Otherwise, from the above arguments $f(x) = 1$, for some $f \in \mathcal{J}_\mathbb{Y}$. Since for each $y \in S_\mathbb{Y}$, $\mathcal{J}_\mathbb{Y} \cap J(y) \neq \emptyset$, it follows that there exists an $f \in J(y)$ such that $f(x) = 0$. It is immediate from Theorem 2.1 that 0 is a best coapproximation to x out of \mathbb{Y} , which is not possible since \mathbb{Y} is an anti-coproximinal subspace of \mathbb{X} . This completes the proof of the necessary part. Hence the theorem. □

We note that the sufficient part of the above theorem holds true for any finite-dimensional Banach space \mathbb{X} without any additional assumptions. In particular, this implies that *if \mathbb{Y} is a subspace of an n -dimensional Banach space \mathbb{X} such that there are n -linearly independent elements in $\mathcal{J}_\mathbb{Y}$, then \mathbb{Y} is an anti-coproximinal subspace of \mathbb{X} .*

We next characterize the strongly anti-coproximal subspaces in finite-dimensional polyhedral Banach spaces.

Theorem 2.20 *Let \mathbb{Y} be a subspace of a finite-dimensional polyhedral Banach space \mathbb{X} . Then the following statements are equivalent:*

- (i) \mathbb{Y} is a strongly anti-coproximal subspace of \mathbb{X}
- (ii) \mathbb{Y} intersects the interior of every facet of $B_{\mathbb{X}}$
- (iii) $\mathcal{J}_{\mathbb{Y}} = Ext(B_{\mathbb{X}^*})$.

Proof Suppose that $\pm F_1, \pm F_2, \dots, \pm F_r$ are the facets of $B_{\mathbb{X}}$. Following from [19, Lemma 2.1], assume that $\pm f_1, \pm f_2, \dots, \pm f_r$ are the corresponding extreme points of $B_{\mathbb{X}^*}$, respectively. Clearly, $Ext(B_{\mathbb{X}^*}) = \{\pm f_1, \pm f_2, \dots, \pm f_r\}$. We complete the proof in the following three steps:

- (i) \implies (ii): Suppose on the contrary that \mathbb{Y} does not intersect the interior of the facet F_i , for some $i \in \{1, 2, \dots, r\}$. Take $x \in int(F_i)$. Clearly, $x \notin \mathbb{Y}$. Moreover, x is smooth and $J(x) = \{f_i\}$. Since \mathbb{Y} does not intersect the interior of F_i , it is easy to observe that for any $y \in \mathbb{Y}$, $J(y) \cap (Ext(B_{\mathbb{X}^*}) \setminus \{\pm f_i\}) \neq \emptyset$, otherwise, $J(y) = \{f_i\}$. Take $\epsilon_0 = \max\{|f(x)| : f \in Ext(B_{\mathbb{X}^*}) \setminus \{\pm f_i\}\}$. As $J(x) = \{f_i\}$, it is clear that $\epsilon_0 < 1$. Since for any $y \in \mathbb{Y}$, $J(y) \cap (Ext(B_{\mathbb{X}^*}) \setminus \{\pm f_i\}) \neq \emptyset$, it is easy to see that for any $y \in \mathbb{Y}$ there exists an $f \in J(y)$ such that $|f(x)| \leq \epsilon_0$. Following Theorem 2.2, $y \perp_B^{\epsilon_0} x$, for any $y \in \mathbb{Y}$. In other words, 0 is an ϵ_0 -best coapproximation to x out of \mathbb{Y} . This contradicts that \mathbb{Y} is a strongly anti-coproximal subspace of \mathbb{X} .
- (ii) \implies (iii): Suppose that $y_i \in int(F_i) \cap \mathbb{Y}$, for each $i \in \{1, 2, \dots, r\}$. Clearly, y_i is smooth and $J(y_i) = \{f_i\}$. Therefore, $f_i \in \mathcal{J}_{\mathbb{Y}}$, for each $i \in \{1, 2, \dots, r\}$, this implies that $Ext(B_{\mathbb{X}^*}) \subset \mathcal{J}_{\mathbb{Y}}$. So, $Ext(B_{\mathbb{X}^*}) = \mathcal{J}_{\mathbb{Y}}$.
- (iii) \implies (i): Let $x \in \mathbb{X}$. Without loss of generality we assume that $x \in F_i$, for some $i \in \{1, 2, \dots, r\}$. Clearly, $f_i \in J(x)$. Since $\mathcal{J}_{\mathbb{Y}} = Ext(B_{\mathbb{X}^*})$, there exists a $y \in Sm(\mathbb{X}) \cap \mathbb{Y}$ such that $J(y) = \{f_i\}$. Therefore, $J(y) = \{f_i\} \subseteq J(x)$. By applying Theorem 2.13, we obtain that \mathbb{Y} is a strongly anti-coproximal subspace of \mathbb{X} .

□

In the following example we explicitly show the applicability of the previous two theorems.

Example 2.21 Let \mathbb{X} be the 3-dimensional Banach space, where $B_{\mathbb{X}}$ is a hexagonal right prism with vertices $\pm(1, 0, 1), \pm(-1, 0, 1), \pm(\frac{1}{2}, \frac{1}{2}, 1), \pm(-\frac{1}{2}, \frac{1}{2}, 1), \pm(-\frac{1}{2}, -\frac{1}{2}, 1), \pm(\frac{1}{2}, -\frac{1}{2}, 1)$. Let $f_1(x, y, z) = x + y, f_2(x, y, z) = x - y, f_3(x, y, z) = y, f_4(x, y, z) = z$, for any $(x, y, z) \in \mathbb{X}$. It is easy to observe that $Ext(B_{\mathbb{X}^*}) = \{\pm f_1, \pm f_2, \pm f_3, \pm f_4\}$. Let us assume that $\pm F_1, \pm F_2, \pm F_3$ and $\pm F_4$ are the corresponding facets of $B_{\mathbb{X}}$ of the extreme functionals $\pm f_1, \pm f_2, \pm f_3, \pm f_4$, respectively [19, Lemma 2.1]. Clearly, $\pm F_1, \pm F_2, \pm F_3$ are the rectangular facets and $\pm F_4$ are the hexagonal facets of $B_{\mathbb{X}}$.

Coproximal subspace: Let $\mathbb{Y} = span\{(1, 0, 0), (0, 1, 0)\}$. It is now easy to observe that for any $y \in \mathbb{Y}$, the elements of $J(y)$ can be written as a convex combination

of $\pm f_1, \pm f_2, \pm f_3$. Otherwise, there exists an element $z \in \mathbb{Y}$ such that $f_4 \in J(z)$, which is clearly not possible. Let $\tilde{x} = (a, b, c) \in \mathbb{X}$. It is straightforward to observe that $f_i((a, b, c) - (a, b, 0)) = 0$, for any $i \in \{1, 2, 3\}$. Therefore for any $y \in \mathbb{Y}$, there exists $f_y \in J(y)$ such that $f_y((a, b, c) - (a, b, 0)) = 0$, and consequently, $\mathbb{Y} \perp_B (a, b, c) - (a, b, 0)$. This implies $(a, b, 0)$ is a best coapproximation to (a, b, c) out of \mathbb{Y} . Therefore, \mathbb{Y} is a coproximal subspace of \mathbb{X} .

Anti-coproximal subspace: Let us now take $\mathbb{Y} = \text{span}\{(\frac{3}{4}, -\frac{1}{4}, 1), (-\frac{3}{4}, -\frac{1}{4}, 1)\}$. It is easy to observe that $S_m(\mathbb{X}) \cap S_{\mathbb{Y}}$ is dense in $S_{\mathbb{Y}}$ and \mathbb{Y} intersects four rectangular facets, $\pm F_1, \pm F_2$ and two hexagonal facets, $\pm F_4$ of $B_{\mathbb{X}}$. It is evident that $f_1, f_2, f_4 \in \mathcal{J}_{\mathbb{Y}}$. As f_1, f_2, f_4 are linearly independent, by applying Theorem 2.19 we get that \mathbb{Y} is an anti-coproximal subspace of \mathbb{X} .

Strongly anti-coproximal subspace: Let $\mathbb{Y} = \text{span}\{(\frac{7}{8}, \frac{1}{8}, 1), (\frac{7}{8}, -\frac{1}{8}, 1)\}$. By a straightforward computation we obtain that $(\frac{11}{16}, \frac{5}{16}, \frac{11}{14}) \in \text{int}(F_1), (\frac{11}{16}, -\frac{5}{16}, \frac{11}{14}) \in \text{int}(F_2), (0, \frac{1}{2}, 0) \in \text{int}(F_3)$ and $(\frac{7}{8}, 0, 1) \in \text{int}(F_4)$. Therefore, it is easy to see that the subspace \mathbb{Y} intersects the interior of each facet of $B_{\mathbb{X}}$. Applying Theorem 2.20, we conclude that \mathbb{Y} is a strongly anti-coproximal subspace of \mathbb{X} .

We now study the anti-coproximal and the strongly anti-coproximal subspaces of ℓ_{∞}^n . The best coapproximation problem in ℓ_{∞}^n was studied in [20] using the concept of the $*$ -Property which plays a crucial role in the whole scheme of things. For the convenience of the readers, let us recall the definition of the $*$ -Property.

Definition 2.22 [20] Let $\mathcal{A} = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m\}$ be a set of linearly independent elements in ℓ_{∞}^n , where $1 \leq m \leq n$ and $\tilde{a}_k = (a_1^k, a_2^k, \dots, a_n^k)$, for each $1 \leq k \leq m$.

- (i) For each $i \in \{1, 2, \dots, n\}$, the i -th component of \mathcal{A} is defined as $(a_i^1, a_i^2, \dots, a_i^m)$.
- (ii) The positively associative set $P_i^+(\mathcal{A})$ of the i -th component is defined as:

$$P_i^+(\mathcal{A}) := \{j \in \{1, 2, \dots, n\} : (a_i^1, a_i^2, \dots, a_i^m) = (a_j^1, a_j^2, \dots, a_j^m)\}.$$

Similarly, the negatively associated set $P_i^-(\mathcal{A})$ is defined as:

$$P_i^-(\mathcal{A}) := \{j \in \{1, 2, \dots, n\} : (a_i^1, a_i^2, \dots, a_i^m) = -(a_j^1, a_j^2, \dots, a_j^m)\}.$$

We write $P_i^+(\mathcal{A}) = P_i^+$ and $P_i^-(\mathcal{A}) = P_i^-$.

- (iii) The i -th component of \mathcal{A} is said to satisfy the $*$ -Property if there exist $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{R}$ such that the following holds true:

$$\left| \sum_{k=1}^m \beta_k a_i^k \right| > \max \left\{ \left| \sum_{k=1}^m \beta_k a_j^k \right| : j \in \{1, 2, \dots, n\} \setminus P_i^+ \cup P_i^- \right\}.$$

We next recall the characterization of smooth points of the unit sphere of ℓ_{∞}^n and the extreme points of the unit ball of $(\ell_{\infty}^n)^*$.

Proposition 2.23 $\tilde{x} = (x_1, x_2, \dots, x_n) \in S_{\ell_{\infty}^n}$ is a smooth point if and only if there exists $i_0 \in \{1, 2, \dots, n\}$ such that $|x_{i_0}| = 1$ and $|x_j| < 1$, for each

$j \in \{1, 2, \dots, n\} \setminus \{i_0\}$. Moreover, $f \in (\ell_\infty^n)^*$ is an extreme point of $B(\ell_\infty^n)^*$ if and only if there exists $i_0 \in \{1, 2, \dots, n\}$ such that $f(x_1, x_2, \dots, x_n) = x_{i_0}$, for any $(x_1, x_2, \dots, x_n) \in \ell_\infty^n$.

The following lemma is essential to characterize the anti-coproximal and the strongly anti-coproximal subspaces in ℓ_∞^n .

Lemma 2.24 *Let $\mathcal{A} = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m\}$ be a set of linearly independent elements in ℓ_∞^n , where $1 < m < n$ and $\tilde{a}_k = (a_1^k, a_2^k, \dots, a_n^k)$, for each $1 \leq k \leq m$. Let $\mathbb{Y} = \text{span } \mathcal{A}$. If every component of \mathcal{A} satisfies the $*$ -Property and $|P_i^+ \cup P_i^-| = 1$, for all $1 \leq i \leq n$ then for every $f \in \text{Ext}(B(\ell_\infty^n)^*)$, there exists a $\tilde{y} \in \text{Sm}(\mathbb{X}) \cap S_{\mathbb{Y}}$ such that $J(\tilde{y}) = \{f\}$.*

Proof Let $f_i \in (\ell_\infty^n)^*$ be such that $f_i(x_1, x_2, \dots, x_n) = x_i$. Clearly, $\text{Ext}(B(\ell_\infty^n)^*) = \{\pm f_1, \pm f_2, \dots, \pm f_n\}$. Since the i -th component satisfies the $*$ -Property and $|P_i^+ \cup P_i^-| = 1$ then there exist $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{R}$ such that

$$\left| \sum_{k=1}^m \beta_k a_i^k \right| > \max \left\{ \left| \sum_{k=1}^m \beta_k a_j^k \right| : j \in \{1, 2, \dots, n\} \setminus \{i\} \right\}.$$

Clearly, $\|\sum_{k=1}^m \beta_k \tilde{a}_k\| = |\sum_{k=1}^m \beta_k a_i^k|$. Take $\tilde{y} = \frac{\sum_{k=1}^m \beta_k \tilde{a}_k}{\|\sum_{k=1}^m \beta_k \tilde{a}_k\|}$. Clearly, $|f_i(\tilde{y})| = 1$. It is easy to observe from Proposition 2.23 that $\tilde{y} \in \text{Sm}(\mathbb{X}) \cap S_{\mathbb{Y}}$. Thus either $J(\tilde{y}) = \{f_i\}$ or $J(-\tilde{y}) = \{f_i\}$. \square

We are now ready to present the desired characterization.

Theorem 2.25 *Let $\mathcal{A} = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m\}$ be a set of linearly independent elements in ℓ_∞^n , where $1 < m < n$ and $\tilde{a}_k = (a_1^k, a_2^k, \dots, a_n^k)$, for each $1 \leq k \leq m$. Let $\mathbb{Y} = \text{span } \mathcal{A}$. Then the following statements are equivalent:*

- (i) \mathbb{Y} is a strongly anti-coproximal subspace of ℓ_∞^n .
- (ii) \mathbb{Y} is an anti-coproximal subspace of ℓ_∞^n .
- (iii) Every component of \mathcal{A} satisfies the $*$ -Property and $|P_i^+ \cup P_i^-| = 1$, for all $1 \leq i \leq n$.

Proof We begin the proof by noting that (i) \implies (ii) follows trivially.

Next we prove that (ii) \implies (iii). Let $\text{Ext}(B(\ell_\infty^n)^*) = \{\pm f_1, \pm f_2, \dots, \pm f_n\}$, where $f_i(x_1, x_2, \dots, x_n) = x_i$, for any $(x_1, x_2, \dots, x_n) \in \ell_\infty^n$. Suppose on the contrary that for some $j \in \{1, 2, \dots, n\}$, the j -th component does not satisfy the $*$ -Property. It is easy to observe that there does not exist any $y \in \mathbb{Y}$ such that $J(y) = \{f_j\}$. Otherwise, considering $y = \sum_{k=1}^m \beta_k \tilde{a}_k$, for some $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{R}$, we obtain that $|\sum_{k=1}^m \beta_k a_j^k| = |f_j(y)| > |f_i(y)| = |\sum_{k=1}^m \beta_k a_i^k|$, for any $t \in \{1, 2, \dots, n\} \setminus \{j\}$, which contradicts that the j -th component does not satisfy the $*$ -Property. Let us consider $\tilde{b} = (b_1, b_2, \dots, b_n) \in \ell_\infty^n$, where $b_j = 1$ and $b_k = 0$, for all $k \neq j$. Clearly, $\tilde{b} \notin \mathbb{Y}$. Since $J(y)$ is a face of $B(\ell_\infty^n)^*$ and there does not exist any $y \in \mathbb{Y}$ such that $J(y) = \{f_j\}$, we conclude that for any $y \in \mathbb{Y}$, there exists an $f \in \text{Ext}(B(\ell_\infty^n)^*) \setminus \{\pm f_j\}$

such that $f \in J(y)$. As $f(\tilde{b}) = 0$, for any $f \in Ext(B_{\ell_\infty^n}^*) \setminus \{\pm f_j\}$, by using Theorem 2.1 we obtain that 0 is a best coapproximation to \tilde{b} out of \mathbb{Y} . This contradicts that \mathbb{Y} is an anti-coproximinal subspace of ℓ_∞^n .

To obtain (iii), we now need to show that $|P_i^+ \cup P_i^-| = 1$, for all $1 \leq i \leq n$. Clearly, $i \in P_i^+ \cup P_i^-$. Suppose on the contrary that $l \in P_i^+ \cup P_i^-$, for some $l \in \{1, 2, \dots, n\} \setminus \{i\}$. Consider the element $\tilde{b} = (b_1, b_2, \dots, b_n) \in \ell_\infty^n$, where $b_l = 1$ and $b_k = 0$, for all $k \neq l$. Clearly, $\tilde{b} \notin \mathbb{Y}$ and $f_k(\tilde{b}) = 0$, for any $k \in \{1, 2, \dots, n\} \setminus \{l\}$. As, $l \in P_i^+ \cup P_i^-$, for any $y \in \mathbb{Y}$, $|f_l(y)| = |f_i(y)|$. Therefore, if $f_l \in J(y)$, for some $y \in \mathbb{Y}$, then either $f_i \in J(y)$ or $-f_i \in J(y)$. We conclude that for any $y \in \mathbb{Y}$, there exists $f \in Ext(B_{\ell_\infty^n}^*) \setminus \{\pm f_l\}$ such that $f \in J(y)$. As, $f(\tilde{b}) = 0$, for any $f \in Ext(B_{\ell_\infty^n}^*) \setminus \{\pm f_l\}$, by using Theorem 2.1 we obtain that 0 is a best coapproximation to \tilde{b} out of \mathbb{Y} . This again contradicts our hypothesis that \mathbb{Y} is an anti-coproximinal subspace of ℓ_∞^n . This completes the proof.

Let us now prove (iii) \implies (i). Suppose on the contrary that y_0 is an ϵ -best coapproximation to $\tilde{b} \in \ell_\infty^n \setminus \mathbb{Y}$ out of \mathbb{Y} . Following Proposition 2.4(iii), we note that for each $y \in \mathbb{Y}$ there exists an $f \in J(y)$ such that $|f(\tilde{b} - y_0)| \leq \epsilon \|\tilde{b} - y_0\|$, for some $\epsilon \in [0, 1)$. Since each component of \mathcal{A} satisfying the $*$ -Property and $|P_i^+ \cup P_i^-| = 1$, it follows from Lemma 2.24 that for every $f \in Ext(B_{(\ell_\infty^n)^*})$ there exists a $\tilde{y} \in Sm(\mathbb{X}) \cap S_{\mathbb{Y}}$ such that $J(\tilde{y}) = \{f\}$. Therefore, $|f(\tilde{b} - y_0)| \leq \epsilon \|\tilde{b} - y_0\|$, for any $f \in Ext(B_{(\ell_\infty^n)^*})$. This implies that $|f(\tilde{b} - y_0)| < \|\tilde{b} - y_0\|$, for any $f \in Ext(B_{(\ell_\infty^n)^*})$. Therefore, it is easy to observe that

$$\|\tilde{b} - y_0\| = \sup_{\|f\| \leq 1} \{|f(\tilde{b} - y_0)|\} = \max_{f \in Ext(B_{(\ell_\infty^n)^*})} \{|f(\tilde{b} - y_0)|\} < \|\tilde{b} - y_0\|,$$

a contradiction. Hence (iii) \implies (i). □

The following example illustrates the computational effectiveness of Theorem 2.25 in verifying the strong anti-coproximality of a given subspace of ℓ_∞^n .

Example 2.26 Let $\tilde{a}_1 = (-4, 2, 3, 1, 3)$, $\tilde{a}_2 = (1, -5, 4, 2, -3)$, $\tilde{a}_3 = (1, 3, -7, 4, 6) \in \ell_\infty^5$ and let $\mathbb{Y} = span\{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3\}$. The 1st, 2nd, 3rd, 4th and the 5th components are $(-4, 1, 1)$, $(2, -5, 3)$, $(3, 4, -7)$, $(1, 2, 4)$ and $(3, -3, 6)$, respectively. It is immediate that $|P_i^+ \cup P_i^-| = 1$, for all $1 \leq i \leq 5$. It is straightforward to verify that every component satisfies the $*$ -Property. Therefore, following Theorem 2.25, \mathbb{Y} is a strongly anti-coproximinal subspace of ℓ_∞^5 .

In light of the above example, we make the following remark that emphasizes the importance of the mother space in deciding whether a given subspace is strongly anti-coproximinal or not.

Remark 2.27 Let $\mathbb{X} = \ell_\infty^6$ and let $\{e_1, e_2, \dots, e_6\}$ be the standard ordered basis of \mathbb{X} . Suppose that $\mathbb{Z} = span\{e_1, e_2, \dots, e_5\}$ is a subspace of \mathbb{X} . Let us now define $\psi : \ell_\infty^5 \rightarrow \mathbb{Z}$, as $\psi(x_1, x_2, \dots, x_5) = (x_1, x_2, \dots, x_5, 0)$, for any $(x_1, x_2, \dots, x_5) \in \ell_\infty^5$. Clearly, ψ is an isometric isomorphism. Let \mathbb{Y} be the same subspace given in Example 2.26. As \mathbb{Y} is strongly anti-coproximinal in ℓ_∞^5 , clearly we observe that $\psi(\mathbb{Y})$ is

also strongly anti-coproximal in \mathbb{Z} . However, $\psi(\mathbb{Y})$ is not an anti-coproximal subspace of \mathbb{X} , as $\psi(\tilde{a}_1) = (-4, 2, 3, 1, 3, 0)$, $\psi(\tilde{a}_2) = (1, -5, 4, 2, -3, 0)$, $\psi(\tilde{a}_3) = (1, 3, -7, 4, 6, 0)$ and the 6th component does not satisfy the $*$ -Property (see Theorem 2.25).

As promised earlier, here we present an example of a subspace which is not strongly anti-coproximal but satisfies the necessary condition of Theorem 2.14.

Example 2.28 Let $\tilde{a}_1 = (1, 1, 2), \tilde{a}_2 = (2, 2, 1) \in \ell_\infty^3$ and let $\mathbb{Y} = \text{span}\{\tilde{a}_1, \tilde{a}_2\}$. The 1st, 2nd, 3rd components are $(1, 2), (1, 2), (2, 1)$, respectively. It is immediate that $|P_1^+ \cup P_1^-| = 2$. Therefore, following Theorem 2.25, \mathbb{Y} is not a strongly anti-coproximal subspace of ℓ_∞^3 . However, by a straightforward observation we can verify that for any $x \in \ell_\infty^3$, there exists a $y \in \mathbb{Y}$ such that $J(y) \cap J(x) \neq \emptyset$.

In [21], the authors introduced the notions ‘the zero set (\mathcal{Z}_A) ’ and ‘the minimal norming set (\mathcal{N}) ’ to study the best coapproximation problem in ℓ_1^n . Using these notions, we study the anti-coproximal subspaces and the strongly anti-coproximal subspaces in ℓ_1^n . Let us now recall the definitions of the zero set and the minimal norming set.

Definition 2.29 [21] Let $\mathcal{A} = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m\}$ be a set of linearly independent elements in ℓ_1^n , where $1 \leq m \leq n$ and $\tilde{a}_k = (a_1^k, a_2^k, \dots, a_n^k)$, for each $1 \leq k \leq m$. The zero set \mathcal{Z}_A of \mathcal{A} is defined as

$$\mathcal{Z}_A = \left\{ i \in \{1, 2, \dots, n\} : (a_i^1, a_i^2, \dots, a_i^m) = (0, 0, \dots, 0) \right\}.$$

Definition 2.30 [21] A set S in a Banach space \mathbb{X} is said to be symmetric if $x \in S$ implies $-x \in S$. Let \mathbb{Y} be a subspace of ℓ_1^n . A symmetric set \mathcal{N} is said to be a norming set of \mathbb{Y} if $(M_g \cap \text{Ext}(B_{\ell_\infty^n})) \cap \mathcal{N} \neq \emptyset$, for each $g \in \psi(\mathbb{Y})$, where ψ is the canonical isometric isomorphism between ℓ_1^n and $(\ell_\infty^n)^*$. A norming set \mathcal{N} is said to be a minimal norming set of \mathbb{Y} if for any norming set \mathcal{M} of \mathbb{Y} , $\mathcal{M} \subset \mathcal{N}$ implies that $\mathcal{M} = \mathcal{N}$.

In the following theorem we completely characterize the anti-coproximal subspaces in ℓ_1^n in terms of the minimal norming set. As we will observe, this characterization turns out to be particularly helpful for explicitly describing the anti-coproximal subspaces in ℓ_1^n .

Theorem 2.31 Let \mathbb{Y} be a subspace of ℓ_1^n and let $\mathcal{A} = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m\}$ be a basis of \mathbb{Y} , where $1 < m < n$ and $\tilde{a}_k = (a_1^k, a_2^k, \dots, a_n^k)$, for each $1 \leq k \leq m$. Then the following statements hold true:

- (i) If $\mathcal{Z}_A \neq \emptyset$, then \mathbb{Y} is not an anti-coproximal subspace of ℓ_1^n .
- (ii) If $\mathcal{Z}_A = \emptyset$, then \mathbb{Y} is an anti-coproximal subspace of ℓ_1^n if and only if $\dim(\text{span } \mathcal{N}) = n$, where \mathcal{N} is the minimal norming set.

Proof (i) Suppose that $j \in \mathcal{Z}_A$, i.e., $a_j^k = 0$, for all $k \in \{1, 2, \dots, m\}$. Let $\tilde{b} = (b_1, b_2, \dots, b_n) \in \ell_1^n$, where $b_j = 1$ and $b_i = 0$, for any $i \in \{1, 2, \dots, n\} \setminus \{j\}$.

Clearly, $\tilde{b} \notin \mathbb{Y}$. For any $\beta_1, \beta_2, \dots, \beta_n \in \mathbb{R}$, it is easy to see that

$$\|\tilde{b} - \sum_{i=1}^m \beta_i \tilde{a}_i\| = \|\sum_{i=1}^m \beta_i \tilde{a}_i\| + 1 > \|\sum_{i=1}^m \beta_i \tilde{a}_i - 0\|.$$

Therefore, 0 is a best coapproximation to \tilde{b} out of \mathbb{Y} and consequently \mathbb{Y} is not an anti-coproximinal subspace of ℓ^n .

(ii) Since $\mathcal{Z}_A = \emptyset$, it follows from [21, Th 2.2] that the minimal norming set \mathcal{N} is unique. Suppose that $\mathcal{N} = \{\pm\tilde{x}_1, \pm\tilde{x}_2, \dots, \pm\tilde{x}_q\}$, where $\tilde{x}_k = (x_1^k, x_2^k, \dots, x_n^k)$. Without loss of generality we assume that $\{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_q\}$ is linearly independent.

Let us first prove the necessary part. Suppose on the contrary that $q < n$. It is straightforward to see that there exists a non-zero element $\tilde{b} = (b_1, b_2, \dots, b_n) \in \ell_1^n$ such that $\sum_{i=1}^n b_i x_i^p = 0$, for any $1 \leq p \leq q$. Observe that $\tilde{b} \notin \mathbb{Y}$. Otherwise, there exists an $\tilde{x}_k \in \mathcal{N}$ such that $\tilde{x}_k \in M_\psi(\tilde{b})$, for some $1 \leq k \leq q$, i.e.,

$$\psi(\tilde{b})(\tilde{x}_k) = \sum_{i=1}^n b_i x_i^k \neq 0,$$

where $\psi : \ell_1^n \rightarrow (\ell_\infty^n)^*$ is the canonical isometric isomorphism. Therefore, for $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$, the following system of linear equations holds true:

$$\alpha_1 \sum_{i=1}^n a_i^1 x_i^p + \alpha_2 \sum_{i=1}^n a_i^2 x_i^p + \dots + \alpha_m \sum_{i=1}^n a_i^m x_i^p = 0 = \sum_{i=1}^n b_i x_i^p,$$

where $p \in \{1, 2, \dots, q\}$. Now applying Theorem [21, Th. 2.4], it is easy to observe that 0 is a best coapproximation to \tilde{b} out of \mathbb{Y} . This contradicts the fact that \mathbb{Y} is an anti-coproximinal subspace of ℓ_1^n . Thus we obtain $q = n$. This proves the necessary part.

We now prove the sufficient part. Let \mathcal{N} be the minimal norming set and let $|\mathcal{N}| = q$. Suppose on the contrary that \mathbb{Y} is not an anti-coproximinal subspace of ℓ_1^n . Then there exists $\tilde{b} = (b_1, b_2, \dots, b_n) \in \ell_1^n \setminus \mathbb{Y}$ such that y_0 is a best coapproximation to \tilde{b} out of \mathbb{Y} . It follows that 0 is a best coapproximation to $\tilde{b} - y_0$ out of \mathbb{Y} . Assume that $\tilde{b} - y_0 = \tilde{d} = (d_1, d_2, \dots, d_n)$. Clearly, $\tilde{d} \in \ell_1^n \setminus \mathbb{Y}$. Applying Theorem [21, Th. 2.4], we obtain that $\sum_{i=1}^n d_i x_i^p = 0$, for each $1 \leq p \leq q$. Since $\dim(\text{span } \mathcal{N}) = \dim(\text{span}\{(x_1^p, x_2^p, \dots, x_n^p) : 1 \leq p \leq q\}) = n$, it follows that $\tilde{d} = (d_1, d_2, \dots, d_n) = 0$. This contradicts that $\tilde{d} \in \ell_1^n \setminus \mathbb{Y}$. This completes the proof. \square

Next we give an example of an anti-coproximinal subspace of ℓ_1^3 by applying Theorem 2.31.

Example 2.32 Let $\mathcal{A} = \{(0, 1, 1), (-1, 0, 1)\} \subset \ell_1^3$ and let $\mathbb{Y} = \text{span } \mathcal{A}$. It is easy to observe that $\mathcal{Z}_A = \emptyset$. Then following the same technique as given in the proof of [21, Th. 2.2], we obtain that the minimal norming set \mathcal{N} of \mathbb{Y} as

$\{\pm(1, 1, 1), \pm(-1, 1, 1), \pm(-1, -1, 1)\}$. Thus we get $\dim(\text{span } \mathcal{N}) = 3$. Applying Theorem 2.31(ii) we conclude that \mathbb{Y} is an anti-coproximal subspace of ℓ_1^n .

Using Theorem 2.20 we show that there does not exist any strongly anti-coproximal subspace in ℓ_1^n . Before proving the result, we note the following lemma.

Lemma 2.33 *Let \mathbb{Y} be a strongly anti-coproximal subspace of ℓ_1^n . Then $\phi(\mathcal{J}_{\mathbb{Y}}) = \mathcal{N}$, where \mathcal{N} is the minimal norming set and ϕ is the canonical isometric isomorphism between $(\ell_1^n)^*$ and ℓ_∞^n .*

Proof Since \mathbb{Y} is a strongly anti-coproximal subspace of ℓ_1^n , it follows from Theorem 2.31 that $\mathcal{Z}_{\mathcal{A}} = \emptyset$. Suppose that $\psi : \ell_1^n \rightarrow (\ell_\infty^n)^*$ is the canonical isometric isomorphism. Let us first assume that $f \in \mathcal{J}_{\mathbb{Y}}$. This implies that there exists a $y \in \text{Sm}(\ell_1^n) \cap S_{\mathbb{Y}}$ such that $f(y) = 1$. Now observe that $\psi(y)(\phi(f)) = f(y) = 1$. As $\psi(y)$ is a smooth point in $(\ell_\infty^n)^*$, we have $M_{\psi(y)} = \{\pm\phi(f)\}$. Therefore, $\phi(\mathcal{J}_{\mathbb{Y}}) \subset \mathcal{N}$. On the other hand, since \mathbb{Y} is a strongly anti-coproximal subspace of ℓ_1^n , it follows from Theorem 2.20 that $\mathcal{N} \subset \text{Ext}(B_{\ell_\infty^n}) = \phi(\text{Ext}(B_{(\ell_1^n)^*})) = \phi(\mathcal{J}_{\mathbb{Y}})$. This proves our lemma. □

Let us now present the desired result.

Theorem 2.34 *There is no strongly anti-coproximal subspace in ℓ_1^n .*

Proof Suppose on the contrary we assume that \mathbb{Y} is a strongly anti-coproximal subspace in ℓ_1^n . Clearly, \mathbb{Y} is a proper subspace of ℓ_1^n . Let $\mathcal{A} = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m\}$ be a basis of \mathbb{Y} , where $\tilde{a}_k = (a_1^k, a_2^k, \dots, a_n^k)$, for all $k \in \{1, 2, \dots, m\}$, where $1 < m < n$. It follows from Theorem 2.31 that $\mathcal{Z}_{\mathcal{A}} = \emptyset$. Also from Theorem 2.20, we have $|\mathcal{J}_{\mathbb{Y}}| = |\text{Ext}(B_{(\ell_1^n)^*})| = 2^m$. By virtue of Lemma 2.33, we note that the cardinality of the minimal norming set of \mathbb{Y} is equal to the cardinality of the set $\mathcal{J}_{\mathbb{Y}}$. Let us now consider the following hyperspaces in \mathbb{R}^m

$$H_i = \left\{ (\beta_1, \beta_2, \dots, \beta_m) \in \mathbb{R}^m : \sum_{k=1}^m \beta_k a_i^k = 0 \right\},$$

where $i \in \{1, 2, \dots, n\}$. Now it is easy to observe from [21, Th. 2.2] that there is an one-one correspondence between the minimal norming set of \mathbb{Y} and the set of regions \mathbb{R}^m formed by H_i 's for all $i \in \{1, 2, \dots, n\}$. Applying [9, Th.1], we obtain that these hyperspaces divide \mathbb{R}^m into at most $2\left[\binom{n-1}{0} + \binom{n-1}{1} + \dots + \binom{n-1}{m-1}\right]$ regions. Therefore the cardinality of the minimal norming set of \mathbb{Y} is at most $2\left[\binom{n-1}{0} + \binom{n-1}{1} + \dots + \binom{n-1}{m-1}\right]$. Since $m < n$, it is immediate that

$$2\left[\binom{n-1}{0} + \binom{n-1}{1} + \dots + \binom{n-1}{m-1}\right] < 2^n.$$

This contradicts that the cardinality of the minimal norming set of \mathbb{Y} is 2^m . This completes the proof of the theorem. □

We would like to end this article with the following remarks regarding the anti-coproximal subspaces and the strongly anti-coproximal subspaces.

- Remark 2.35** (i) We have already shown that there exists a strongly anti-coproximal subspace in ℓ_∞^n ($n \geq 3$), whereas there does not exist any strongly anti-coproximal subspace in ℓ_1^n ($n \geq 3$). Using Theorem 2.20, it is possible to give a geometric interpretation of this phenomenon in a visually appealing manner. Indeed, we observe that there is a subspace in ℓ_∞^n which intersects the interior of each of the facet of $B_{\ell_\infty^n}$, whereas ℓ_1^n does not contain any such subspaces (see Theorem 2.34).
- (ii) We note that the anti-coproximal subspaces are the least favorable subspaces from the perspective of best coapproximation. In this study we have observed that in general there may be many subspaces in a polyhedral Banach spaces which are anti-coproximal. This further illustrates the non-triviality and the computational difficulty associated with the best coapproximation problem, even in finite-dimensional Banach spaces. We note from [11, Th. 4] that a Banach space \mathbb{X} having three or more dimension is an inner product space if and only if for each hyperspace H of \mathbb{X} , there exists an $x \in \mathbb{X}$ such that $H \perp_B x$. In any Banach space, it is easy to verify that the anti-coproximal hyperspaces are precisely those which are not orthogonal to any element of \mathbb{X} . Moreover, a strongly anti-coproximal hyperspace H is precisely those for which there does not exist any element $x \in \mathbb{X}$ such that $H \perp_B^\epsilon x$, for some $\epsilon \in [0, 1)$. From Theorem 2.34, for any hyperspace H of ℓ_1^n , we note that there exists an $\epsilon \in [0, 1)$ and an $x \in \ell_1^n$ such that $H \perp_B^\epsilon x$.
- (iii) It is known [13, 16] that given a subspace \mathbb{Y} of a Banach space \mathbb{X} and an element $x \notin \mathbb{Y}$, y_0 is a best coapproximation to x out of \mathbb{Y} if and only if there exists a norm one projection from $\text{span}\{x, \mathbb{Y}\}$ to \mathbb{Y} . It is clear that \mathbb{Y} is an anti-coproximal subspace in \mathbb{X} if and only if given any $x \notin \mathbb{Y}$, there does not exist any norm one projection from $\text{span}\{x, \mathbb{Y}\}$ to \mathbb{Y} . In other words, \mathbb{Y} is an anti-coproximal subspace in \mathbb{X} if and only if for any subspace \mathbb{Z} which properly contains \mathbb{Y} , there does not exist any norm one projection from \mathbb{Z} to \mathbb{Y} .

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