Monatshefte für Mathematik (2024) 203:175–198
https://doi.org/10.1007/s00605-023-01905-3

Estimates on Bloch constants for certain log-p-harmonic mappings

Ming-Sheng Liu[1] · **Xin Wang**[1,2] · **Kit Ian Kou**[3]

Received: 13 February 2022 / Accepted: 10 September 2023 / Published online: 26 September 2023

Abstract

In this paper, we first provide a brief overview of Landau-type theorems for log-p-harmonic mappings. Next, we establish four new versions of Landau-type theorems for certain bounded p-harmonic mappings F with $J_F(0) = 1$. Then, as applications of these results, the corresponding Landau-type theorems for certain log-p-harmonic mappings f with $J_f(0) = 1$ are provided. In particular, several sharp results of Landau-type theorems for certain bounded p-harmonic mappings or log-p-harmonic mappings with $J_f(0) = 1$ are obtained. Finally, we also establish a Landau-type theorem for a certain bounded log-p-harmonic mappings with $J_f(0) = 1$, which improves the corresponding results of different authors.

Keywords Landau-type theorem · p-harmonic mapping · Log-p-harmonic mapping · Univalent

Mathematics Subject Classification Primary 30C99; Secondary 30C62 · 30C25

Communicated by Adrian Constantin.

Xin Wang is the Co-first author.

✉ Ming-Sheng Liu
liumsh65@163.com

Xin Wang
574149386@qq.com

Kit Ian Kou
kikou@umac.mo

1 School of Mathematical Sciences, South China Normal University, Guangzhou 510631, Guangdong, China

2 Department of Mathematics, Shenzhen Polytechnic University, Shenzhen 518055, Guangdong, China

3 Department of Mathematics, Faculty of Science and Technology, University of Macau, Taipa, Macao, China

1 Introduction

Suppose that $F(z) = u(z) + iv(z)$ is a $2p$ times continuously differentiable complex-valued function in a domain $D \subseteq \mathbb{C}$, where p is a positive integer. For $z = x + iy \in D$, we denote the formal derivatives of f by

$$F_z = \frac{1}{2}(F_x - iF_y) \text{ and } F_{\overline{z}} = \frac{1}{2}(F_x - iF_y).$$

Let Δ denote the complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Then we say that F is p-harmonic in D if F satisfies the p-harmonic equation

$$\Delta^p F = \Delta(\Delta^{p-1})F = 0.$$

Evidently, when $p = 1$ (resp. $p = 2$), F is called harmonic (resp. biharmonic) mapping. For $k \in \{1, \ldots, p\}$, we recall that a mapping F is p-harmonic in a simply connected domain $D \subset \mathbb{C}$ if and only if F has the following representation:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z), \tag{1.1}$$

where each G_{p-k+1} is harmonic in D. For details and the special case of $p = 2$ for biharmonic mappings, we refer to [1, 9], and [25] where one can find characterizations of certain p-harmonic functions.

A mapping f is said to be log-p-harmonic if $\log f$ is a p-harmonic mapping. Then it follows from (1.1) that f is log-p-harmonic in a simply connected domain D if and only if f can be written as

$$f(z) = \prod_{k=1}^{p} (g_{p-k+1}(z))^{|z|^{2(k-1)}}, \tag{1.2}$$

where g_{p-k+1} are log-harmonic mappings in D for each $k \in \{1, \ldots, p\}$. Obviously, when $p = 1$, f is log-harmonic; when $p = 2$, f is the so-called log-biharmonic (cf. [23, 24]).

By [21], it's known that a harmonic mapping f is locally univalent in D if and only if the Jacobian of f satisfies $J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 \neq 0$ for all $z \in D$.

For a continuously differentiable mapping f in D, we may define

$$\Lambda_f = \max_{0 \leqslant \theta \leqslant 2\pi} |f_z + e^{-2i\theta} f_{\overline{z}}| = |f_z| + |f_{\overline{z}}| \text{ and}$$
$$\lambda_f = \min_{0 \leqslant \theta \leqslant 2\pi} |f_z + e^{-2i\theta} f_{\overline{z}}| = ||f_z| - |f_{\overline{z}}||.$$

Then $J_f = \lambda_f \Lambda_f$ if $J_f \geqslant 0$.

Methods of Harmonic mappings have been used to study and solve fluid flow problems (cf. [4, 15]). For example, in 2012, Aleman and Constantin [4] established a connection between harmonic mappings and ideal fluid flows. In fact, they have developed ingenious technique to solve the incompressible two dimensional Euler equations in terms of univalent harmonic mappings (cf. [15]). However, the investigation of harmonic mappings in the context of geometric function theory is a recent one (cf. [7, 8, 16, 17, 19, 27] and the references therein).

The classical Landau's theorem [20] states that if f is an analytic function on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for all $z \in \mathbb{D}$, then f is univalent in the disk $\mathbb{D}_{r_0} := \{z \in \mathbb{C} : |z| < r_0\}$ with $r_0 = M - \sqrt{M^2 - 1}$, and $f(\mathbb{D}_{r_0})$ contains a disk $\mathbb{D}_{R_0}$ with $R_0 = Mr_0^2$. This result is sharp, with the extremal function $f_0(z) = Mz((1 - Mz)/(M - z))$. The Bloch theorem asserts the existence of a positive constant number b such that if f is an analytic function on the unit disk $\mathbb{D}$ with $f'(0) = 1$, then $f(\mathbb{D})$ contains a schlicht disk of radius b, that is, a disk of radius b which is the univalent image of some region on $\mathbb{D}$. The supremum of all such constants b is called the Bloch constant (cf. [7]). For the sake of convenience, we say that $f \in S_{z'}(r, R)$ if f is an univalent log-p-harmonic mapping in the disk $\mathbb{D}_r$ and $f(\mathbb{D}_r)$ contains a schlicht disk $\mathbb{D}(z', R) := \{z \in \mathbb{C} : |z - z'| < R\}$. In particular, we denote $S_0(r, R)$ by $S(r, R)$.

The study on Landau-type theorems for harmonic mappings has attracted much attention. We may refer the interested readers to [7, 8, 10–12, 16, 17, 19, 22, 27–29] for more discussions on harmonic mappings, and refer to [2, 3, 9, 24, 26, 31, 34] for more results of biharmonic and p-harmonic mappings. Recently, Chen et al. generalized the results of planar harmonic mappings to several variables (cf. [6, 8]). In 2011, Li and Wang [23] introduced the log-p-harmonic mappings and derived two versions of Landau-type theorems. We recall the following result which actually improves the result of [23].

Theorem A ([30]) *Let $f(z) = \prod_{k=1}^{p} (g_{p-k+1}(z))^{|z|^{2(k-1)}}$ be a log-p-harmonic mapping of $\mathbb{D}$, where all g_{p-k+1} are log-harmonic on $\mathbb{D}$ with $g_{p-k+1}(0) = g_p(0) = J_f(0) = 1$, $|g_{p-k+1}| < M^*_{p-k+1}$ for $k \in \{2, \ldots, p\}$, and $|g_p| < M^*_p$, where $M^*_i \geqslant 1$, $M_i = \log M^*_i + \pi$ $(i = 1, \ldots, p)$ and $\lambda_0(M)$ is defined by*

$$\lambda_0(M) = \begin{cases} \frac{\sqrt{2}}{\sqrt{M^2-1}+\sqrt{M^2+1}}, & 1 \leqslant M \leqslant M_0, \\ \frac{\pi}{4M}, & M > M_0. \end{cases} \tag{1.3}$$

Then $f \in S_{z_2}(\rho_2, \sigma_2)$, where ρ_2 is the unique root in $(0, 1)$ of the equation:

$$\lambda_0(M_p) - \sum_{k=1}^{p-1} \Big(\frac{4}{\pi(1 - r^2)} + \frac{8k}{\pi(1 - r)} \Big) M_{p-k} r^{2k}$$
$$-\lambda_0(M_p)\sqrt{(M_p)^4 - 1} \cdot \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1 - r^2)^{\frac{3}{2}}} = 0,$$

$$z_2=\cosh\left(\frac{\sigma_2'}{\sqrt{2}}\right),\quad \sigma_2=\min\left\{\sinh\left(\frac{\sigma_2'}{\sqrt{2}}\right),\cosh\left(\frac{\sigma_2'}{\sqrt{2}}\right)\sin\left(\frac{\sigma_2'}{\sqrt{2}}\right)\right\},$$

and

$$\sigma_2'=\lambda_0(M_p)\rho_2-\lambda_0(M_p)\sqrt{(M_p)^4-1}\cdot\frac{\rho_2^2}{(1-\rho_2^2)^{\frac{1}{2}}}-\frac{4\rho_2}{\pi(1-\rho_2)}\sum_{k=1}^{p-1}M_{p-k}\rho_2^{2k}.$$

Theorem B *([30]) Let $f(z)=g(z)^{|z|^{2(p-1)}}$ be a log-p-harmonic of $\mathbb{D}$, where $p>1$, g is log-harmonic, $g(0)=J_g(0)=1$, $|g(z)|\leqslant M^*$ for some $M^*\geqslant 1$, and $M=\log M^*+\pi$. Then $f\in S_{z_3}(\rho_3,\sigma_3)$, where ρ_3 is the unique root in $(0,1)$ of the following equation:*

$$1-2\sqrt{M^4-1}\cdot\frac{r}{(1-r^2)^{\frac{1}{2}}}-\sqrt{M^4-1}\cdot\frac{r\sqrt{r^4-3r^2+4}}{(1-r^2)^{\frac{3}{2}}}=0,$$
$$z_3=\cosh\left(\frac{\sigma_3'}{\sqrt{2}}\right),\quad \sigma_3=\min\left\{\sinh\left(\frac{\sigma_3'}{\sqrt{2}}\right),\cosh\left(\frac{\sigma_3'}{\sqrt{2}}\right)\sin\left(\frac{\sigma_3'}{\sqrt{2}}\right)\right\},$$

and

$$\sigma_3'=\rho_3^{2(p-1)}\lambda_0(M)\left(\rho_3-\sqrt{M^4-1}\cdot\frac{\rho_3^2}{(1-\rho_3^2)^{\frac{1}{2}}}\right).$$

In 2019, Bai and Liu established two new versions of Landau-type theorems as follows:

Theorem C *([5]) Let $f(z)=\prod_{k=1}^{p}(g_{p-k+1}(z))^{|z|^{2(k-1)}}$ be a* log-*p-harmonic mapping in $\mathbb{D}$ satisfying $f(0)=g_p(0)=\lambda_f(0)=1$. Suppose that for each $k\in\{1,\ldots,p\}$, we have*

(i) $g_{p-k+1}(z)$ is log-*harmonic in $\mathbb{D}$, and $G_p(z):=\log g_p(z)$;*
(ii) $|g_{p-k+1}(z)|\leqslant M^*_{p-k+1}$, and $\Lambda_{G_p}(z)\leqslant\Lambda_p$ for all $z\in\mathbb{D}$, where $M^*_{p-k+1}\geqslant 1$, $\Lambda_p\geqslant 1$.
Then $f\in S_{z_4}(\rho_4,\sigma_4)$, where $\rho_4\in(0,1)$ satisfies the following equation:

$$1-\frac{4}{\pi(1-r^2)}\sum_{k=1}^{p-1}r^{2k}M_{p-k}-\frac{8}{\pi(1-r)}\sum_{k=1}^{p-1}kM_{p-k}r^{2k}-\frac{\Lambda_p-1}{\Lambda_p}\frac{r}{1-r}=0, \tag{1.4}$$

where $M_{p-k+1}=\log M^*_{p-k+1}+\pi$, $k=2,3,\ldots,p$,

$$z_4=\cosh(\frac{\sigma_4'}{\sqrt{2}}),\quad \sigma_4=\min\{\sinh(\frac{\sigma_4'}{\sqrt{2}}),\cosh(\frac{\sigma_4'}{\sqrt{2}})\sin(\frac{\sigma_4'}{\sqrt{2}})\}, \tag{1.5}$$

and

$$\sigma_4' = \rho_4 + \frac{\Lambda_p^2 - 1}{\Lambda_p}[\rho_4 + \ln(1-\rho_4)] - \sum_{k=1}^{p-1} \frac{4M_{p-k}\rho_4^{2k+1}}{\pi(1-\rho_4)}. \tag{1.6}$$

In 2021, Liu and Luo obtained the following sharp forms of [5, Theorem 2.9].

Theorem D *([32]) Suppose that p is a positive integer, $\Lambda_1, \Lambda_2, \ldots, \Lambda_{p-1} \geqslant 0$, $\Lambda_p > 1$. Let $f(z) = \prod_{k=1}^{p}(g_{p-k+1}(z))^{|z|^{2(k-1)}}$ be a log-p-harmonic mapping of $\mathbb{D}$, satisfying $f(0) = \lambda_f(0) = 1$. Suppose that for each $k \in \{1, \ldots, p\}$ we have that*

(i) $g_{p-k+1}(z)$ is log-harmonic in $\mathbb{D}$, $g_{p-k+1}(0) = 1$, and $G_{p-k+1} := \log g_{p-k+1}$;
(ii) for each $k \in \{2, \ldots, p\}$, $\Lambda_{G_{p-k+1}}(z) \leqslant \Lambda_{p-k+1}$, and $\Lambda_{G_p}(z) < \Lambda_p$ for all $z \in \mathbb{D}$. Then $f \in S_{z_6}(\rho_6, \sigma_6)$, where ρ_6 is the unique root in $(0, 1)$ of the equation

$$\Lambda_p \frac{1-\Lambda_p r}{\Lambda_p - r} - \sum_{k=1}^{p-1}(2k+1)\Lambda_{p-k} r^{2k} = 0,$$

and $z_6 = \cosh\sigma_6'$, $\sigma_6 = \sinh\sigma_6'$, and $\sigma_6' = \Lambda_p^2\rho_6 - \sum_{k=1}^{p-1}\Lambda_{p-k}\rho_6^{2k+1} + \left(\Lambda_p^3 - \Lambda_p\right)\ln\left(1 - \frac{\rho_6}{\Lambda_p}\right)$. Both of the radii, ρ_6 and $\sigma_6 = \sinh\sigma_6'$ are sharp.

Theorem E *([32]) Suppose that p is a positive integer, $p \geqslant 2$, $\Lambda_1, \Lambda_2, \ldots, \Lambda_{p-1} \geqslant 0$. Let $f(z) = \prod_{k=1}^{p}(g_{p-k+1}(z))^{|z|^{2(k-1)}}$ be a log-p-harmonic mapping in $\mathbb{D}$ satisfying $f(0) = \lambda_f(0) = 1$. Suppose that for each $k \in \{1, \ldots, p\}$, we have that*

(i) $g_{p-k+1}(z)$ is log-harmonic in $\mathbb{D}$, $g_{p-k+1}(0) = 1$, and $G_{p-k+1} := \log g_{p-k+1}$;
(ii) for each $k \in \{2, \ldots, p\}$, $\Lambda_{G_{p-k+1}}(z) \leqslant \Lambda_{p-k+1}$, and $\Lambda_{G_p}(z) \leqslant 1$ for all $z \in \mathbb{D}$. Then (1) $F(z) := \log f(z) = \sum_{k=1}^{p}|z|^{2(k-1)}G_{p-k+1}(z)$ is a p-harmonic mapping in $\mathbb{D}$, and $F(z) \in S(\rho_7, \sigma_7')$, where $\sigma_7' = \rho_7 - \sum_{k=1}^{p-1}\Lambda_{p-k}\rho_7^{2k+1}$,

$$\rho_7 = \begin{cases} 1, & \text{if } \sum_{k=1}^{p-1}(2k+1)\Lambda_{p-k} \leqslant 1, \\ \rho_7', & \text{if } \sum_{k=1}^{p-1}(2k+1)\Lambda_{p-k} > 1, \end{cases}$$

and ρ_7' is the unique root in $(0, 1)$ of the equation: $1 - \sum_{k=1}^{p-1}(2k+1)\Lambda_{p-k}r^{2k} = 0$. Both of the radii, ρ_7 and σ_7'£¬ are sharp, with an extremal function given by

$$F_1(z) = z - \sum_{k=1}^{p-1}\Lambda_{p-k}|z|^{2k}z, \ z \in U. \tag{1.7}$$

(2) $f \in S_{z_7}(\rho_7, \sigma_7)$, where $z_7 = \cosh\sigma_7'$, $\sigma_7 = \sinh\sigma_7'$. Both of the radii, ρ_7 and σ_7, are sharp.

Remark F The number $\sigma_6 = \sinh \sigma_6'$ is the accurate value of the Bloch constant of the subclass of log-p-harmonic mappings in the unit disk $\mathbb{D}$ satisfying the hypotheses of Theorem D, and $\sigma_7 = \sinh \sigma_7'$ is the accurate value of the Bloch constant of the subclass of log-p-harmonic mappings in the unit disk $\mathbb{D}$ satisfying the hypotheses of Theorem E.

It is natural to raise the following problems.

Problem 1: If the condition $\lambda_F(0) = 1$ is replaced by $J_F(0) = 1$ in Theorems D or E, can we obtain the sharp versions of Landau-type theorems for such bounded and normalized log-p-harmonic mappings?

Problem 2: Can we improve the radii in Theorem A?

The paper is organized as follows. In Sect. 2, we recall several lemmas, and prove five new lemmas which play crucial role in the proofs of our Theorems, where Lemma 2.6 is sharp. In Sect. 3, we establish four new versions of Landau-type theorems for certain bounded p-harmonic mappings with $J_f(0) = 1$. In Sect. 4, using these estimates, we present four versions of Landau-type theorems for log-p-harmonic mappings (see Theorems 4.1, 4.3, 4.4 and 4.5), which are the analogues versions of Theorems D, C and B respectively. We also establish a sharp version of Landau-type theorem for a certain log-p-harmonic mappings (see Theorems 4.2), which gives a part of answer to Problem 1. Finally, we improve Theorem A by establishing Theorem 4.6, which gives an affirmative answer to Problem 2.

2 Preliminaries

In order to establish our main results, we need the following key lemmas.

Lemma 2.1 *([14]) Suppose that $f(z)$ is a harmonic mapping of the unit disk $\mathbb{D}$ such that $|f(z)| \leqslant M$ for all $\mathbb{D}$. Then*

$$\Lambda_f(z) \leqslant \frac{4M}{\pi(1-|z|^2)}, \quad z \in \mathbb{D}.$$

The inequality is sharp.

Lemma 2.2 *([18]) Suppose that $f(z)$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $f(0) = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$. Then*

$$|f(z)| \leqslant \frac{4}{\pi}\arctan|z| \leq \frac{4}{\pi}|z| \text{ for } z \in \mathbb{D}.$$

Lemma 2.3 *([13]) Suppose $\Lambda > 1$. Let $f(z)$ be a harmonic mapping of the unit disk $\mathbb{D}$ with $J_f(0) = 1$ and $\Lambda_f(z) < \Lambda$ for all $z \in \mathbb{D}$. Then:*

(1) for all $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < 1, z_1 \neq z_2)$, we have

$$|f(z_2) - f(z_1)| = \left|\int_{\overline{z_1 z_2}} f_z(z)dz + f_{\bar{z}}(z)d\bar{z}\right| \geqslant \Lambda \frac{\lambda_f(0) - \Lambda r}{\Lambda - \lambda_f(0)r} |z_1 - z_2|.$$

(2) Set $\gamma = f^{-1}(\overline{ow'})$ *with* $w' \in f(\partial\mathbb{D}_r)(0 < r \leqslant 1)$ *and* $\overline{ow'}$ *denotes the closed line segment joining the origin and* w'*, then*

$$\left|\int_\gamma f_\zeta(\zeta)d\zeta + f_{\bar{\zeta}}(\zeta)d\bar{\zeta}\right| \geqslant \Lambda\int_0^r \frac{\lambda_f(0) - \Lambda t}{\Lambda - \lambda_f(0)t}dt.$$

Lemma 2.4 *Suppose* $f(z)$ *is a harmonic mapping of the unit disk* $\mathbb{D}$*. If* $J_f(0) = 1$ *and* $\Lambda_f(z) < \Lambda$ $(\Lambda > 1)$ *for all* $z \in \mathbb{D}$*, then*

(i) for all $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < 1, z_1 \neq z_2)$*, we have*

$$|f(z_1) - f(z_2)| \geq \frac{\Lambda(1 - \Lambda^2 r)}{\Lambda^2 - r}|z_1 - z_2|.$$

(ii) for $w \in \partial f(\mathbb{D}_r)$ $(0 < r \leqslant 1)$*,* $\gamma = f^{-1}(\overline{ow})$ *and* $\overline{ow}$ *denotes the closed line segment joining the origin and* w*, we have*

$$\left|\int_\gamma f_\xi(\xi)d\xi + f_{\bar{\xi}}(\xi)d\bar{\xi}\right| \geqslant \Lambda^3 r + (\Lambda^5 - \Lambda)\ln\left(1 - \frac{r}{\Lambda^2}\right).$$

Proof For any $z_1, z_2 \in \mathbb{D}_r$ with $z_1 \neq z_2$, by Lemma 2.3, we have

$$|f(z_1) - f(z_2)| \geq \Lambda\frac{\lambda_f(0) - \Lambda r}{\Lambda - \lambda_f(0)r}|z_1 - z_2|,$$
$$\left|\int_\gamma f_\xi(\xi)d\xi + f_{\bar{\xi}}(\xi)d\bar{\xi}\right| \geqslant \Lambda\int_0^r \frac{\lambda_f(0) - \Lambda t}{\Lambda - \lambda_f(0)t}dt.$$

Since $J_f(0) = \Lambda_f(0)\lambda_f(0) = 1$, it follows that

$$\lambda_f(0) = \frac{1}{\Lambda_f(0)} > \frac{1}{\Lambda}.$$

As $\Lambda\frac{x - \Lambda r}{\Lambda - xr}$ is an increasing function of x, we obtain that

$$|f(z_1) - f(z_2)| \geq \frac{\Lambda(1 - \Lambda^2 r)}{\Lambda^2 - r}|z_1 - z_2|,$$

and for $w \in \partial f(\mathbb{D}_r)(0 < r \leqslant 1)$ and $\gamma = f^{-1}(\overline{ow})$,

$$\left|\int_\gamma f_\xi(\xi)d\xi + f_{\bar{\xi}}(\xi)d\bar{\xi}\right| \geqslant \Lambda\int_0^r \frac{\lambda_f(0) - \Lambda t}{\Lambda - \lambda_f(0)t}dt \geq \Lambda\int_0^r \frac{1 - \Lambda^2 t}{\Lambda^2 - t}dt$$
$$= \Lambda^3 r + (\Lambda^5 - \Lambda)\ln\left(1 - \frac{r}{\Lambda^2}\right).$$

□

Lemma 2.5 *([13]) Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping in $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n$ and $f(0) = J_f(0) - 1 = 0$. Then $|f(z)| \leqslant 1$ for all $z \in \mathbb{D}$ if and only if $\Lambda_f(z) \leqslant 1$ for all $z \in \mathbb{D}$.*

Moreover, if $\Lambda_f(z) \leqslant \Lambda$ for all $z \in \mathbb{D}$, then $\Lambda \geq 1$, $|a_1| + |b_1| \leqslant \Lambda$, and

$$|a_n| + |b_n| \leqslant \frac{\Lambda^4 - 1}{n\Lambda^3}, \quad n = 2, 3, \ldots, \quad \frac{1}{\Lambda} \leqslant \lambda_f(0) \leqslant 1. \tag{2.1}$$

When $\Lambda = 1$, then $f(z) = a_1 z$ with $|a_1| = 1$, and $\lambda_f(0) = 1$.

Now we establish several sharp coefficient inequalities for harmonic mappings with bounded dilation and $J_f(0) = 1$, which has independent interest.

Lemma 2.6 *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping in $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n$, $f(0) = J_f(0) - 1 = 0$ and $\Lambda_f(0) \leqslant \Lambda$ for some $\Lambda > 1$. Then we have the following sharp inequalities:*

$$|a_1| \leqslant \frac{1}{2}\left(\Lambda + \frac{1}{\Lambda}\right), \quad |b_1| \leqslant \frac{1}{2}\left(\Lambda - \frac{1}{\Lambda}\right), \quad |a_1| + |b_1| \leqslant \Lambda \ \text{and} \ \lambda_f(0) \geqslant \frac{1}{\Lambda}. \tag{2.2}$$

Proof By the assumption, the proofs of the inequalities for $|a_1|$ and $|b_1|$ follow trivially by solving $x + y \leqslant \Lambda$ and $x^2 - y^2 = 1$, where $x = |a_1|$ and $y = |b_1|$. The sharpness follow from the affine mapping

$$f_0(z) = az + b\bar{z}, \quad a = \frac{1}{2}\left(\Lambda + \frac{1}{\Lambda}\right), b = \frac{1}{2}\left(\Lambda - \frac{1}{\Lambda}\right).$$

□

Remark 2.1 It is natural to raise an open problem: under the assumptions of Lemma 2.6, and $\Lambda_f(z) \leqslant \Lambda$ for all $z \in \mathbb{D}$, what is the sharp upper bound of $|a_n| + |b_n|$ for each $n \geqslant 2$?

Lemma 2.7 *([26]) Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping in $\mathbb{D}$ such that $|f(z)| \leqslant M$ for all $z \in \mathbb{D}$ with $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. If $J_f(0) = 1$, then $\lambda_f(0) \geqslant \lambda_0(M)$, where $\lambda_0(M)$ is defined by (1.3).*

Lemma 2.8 *Suppose that $F(z) = |z|^{2k} G(z)$ is p-harmonic in $\mathbb{D}$, where $k \geqslant 0$ is an integer, and $G(z)$ is harmonic in $\mathbb{D}$ with $G(0) = 0$.*

(1) If $\Lambda_G(z) \leqslant \Lambda$ for all $z \in \mathbb{D}$. Then for $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < 1)$, we have

$$|F(z_1) - F(z_2)| \leqslant |z_1 - z_2|(2k + 1)\Lambda r^{2k}. \tag{2.3}$$

(2) If $|G(z)| \leqslant M$ for all $z \in \mathbb{D}$. Then for $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < 1)$, we have

$$|F(z_1) - F(z_2)| \leqslant |z_1 - z_2|\left(\frac{4}{\pi(1 - r^2)} + \frac{8k}{\pi}\right)M r^{2k}. \tag{2.4}$$

Proof The case $k = 0$ is trivial. For $k \geqslant 1$, we choose two distinct points $z_1, z_2 \in \mathbb{D}_r$, and let $\Gamma = \{(z_2 - z_1)t + z_1 : t \in [0, 1]\}$. Elementary calculations show that

$$\begin{aligned} |F(z_1) - F(z_2)| &\leqslant \int_\Gamma |z|^{2k}[|G_z(z)||dz| + |G_{\overline{z}}(z)||d\overline{z}|] \\ &\quad + \int_\Gamma k|G(z)|[|\overline{z}^k z^{k-1}||dz| + |\overline{z}^{k-1} z^k||d\overline{z}|] \\ &\leqslant \int_\Gamma \Lambda_G(z)|z|^{2k}|dz| + \int_\Gamma 2k|G(z)||z|^{2k-1}|dz|, \end{aligned} \tag{2.5}$$

and

$$|G(z)| \leqslant \left|\int_{[0,z]} G_z dz + G_{\overline{z}} d\overline{z}\right| \leqslant \int_{[0,z]} \Lambda_G(z)|dz|. \tag{2.6}$$

(1) Since $\Lambda_G(z) \leqslant \Lambda$ for all $z \in \mathbb{D}$, by (2.5) and (2.6), we have

$$\begin{aligned} |F(z_1) - F(z_2)| &\leqslant \int_\Gamma \Lambda_G(z)|z|^{2k}|dz| + \int_\Gamma 2k\Lambda|z| \cdot |z|^{2k-1}|dz| \\ &\leqslant |z_1 - z_2|(2k+1)\Lambda r^{2k}. \end{aligned}$$

This implies that the inequality (2.3) holds.

(2) Since $|G(z)| \leqslant M$ for all $z \in \mathbb{D}$, by Lemmas 2.1, 2.2, (2.5) and (2.6), we have that

$$\begin{aligned} |F(z_1) - F(z_2)| &\leqslant \int_\Gamma \Lambda_G(z)|z|^{2k}|dz| + \int_\Gamma 2k \cdot \frac{4M|z|}{\pi} \cdot |z|^{2k-1}|dz| \\ &\leqslant |z_1 - z_2|\left(\frac{4}{\pi(1-r^2)} + \frac{8k}{\pi}\right)M r^{2k}. \end{aligned}$$

This implies the inequalities (2.4) hold. The proof of Lemma 2.8 is complete.

□

Lemma 2.9 *([34]) Suppose that* $f(z) = h(z) + \overline{g(z)}$ *is a harmonic mapping in* $\mathbb{D}$ *with* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=1}^{\infty} b_n z^n$. *If* $|f(z)| \leqslant M$ *for all* $z \in \mathbb{D}$ *and* $|J_f(0)| = 1$, *then*

$$\left(\sum_{n=2}^{\infty}(|a_n| + |b_n|)^2\right)^{\frac{1}{2}} \leqslant \sqrt{M^4 - 1} \cdot \lambda_f(0).$$

Lemma 2.10 *Suppose* $0 < t < 1$, *then* $\min\{\sinh t, \cosh t \cdot \sin t\} = \sinh t$.

Proof Set $g(t) = \cosh t \cdot \sin t - \sinh t$, then $g(0) = 0$, and a direct computation yields

$$g'(t) = \sinh t \cdot \sin t + \cosh t\,(\cos t - 1), \quad g'(0) = 0,$$

and

$$g''(t) = \sinh t \cdot (2\cos t - 1) \geqslant \sinh t \cdot (2\cos 1 - 1) \geqslant 0, \quad t \in [0, 1].$$

We conclude that $g'(t)$ is an increasing function of t in [0, 1] and therefore, $g'(t) \geqslant g'(0) = 0$ for $t \in [0, 1]$, which shows that $g(t)$ is an increasing function of t in [0, 1]. Hence we obtain that $g(t) \geqslant g(0) = 0$ for $t \in [0, 1]$, which completes the proof. □

By means of Lemma 2.10, using arguments similar to those in the proof of [32, Lemma 2.4], we may prove the following lemma, and so we omit the details.

Lemma 2.11 *Suppose that p is a positive integer, $0 < t < 1$ and $0 < \rho \leqslant 1$. Let $f(z)$ be a log-p-harmonic mapping in $\mathbb{D}$ with $f(0) = J_f(0) = 1$. Suppose that $f(z)$ is univalent in $\mathbb{D}_\rho$ and $F(\mathbb{D}_\rho) \supset \mathbb{D}_t$, where $F(z) = \log f(z)$. Then the range $f(\mathbb{D}_\rho)$ contains a schlicht disk $\mathbb{D}(w_1, r_1) = \{w \in \mathbb{C} || w - w_1| < r_1\}$, where*

$$w_1 = \cosh t, \quad r_1 = \sinh t. \tag{2.7}$$

Moreover, if ρ is the biggest univalent radius of $f(z)$ and t is the biggest radius of the schlicht disk of $F(z)$, then the radius $r_1 = \sinh t$ is sharp.

Remark 2.2 In Sect. 4, we mainly focus on Landau-type theorems for log-p-harmonic mappings when $p \geq 2$. As for the case of $p = 1$, we may recall [33, Theorem 3.2] as follows:

Suppose $F = H\overline{G}$ is a logharmonic mapping in $\mathbb{D}$. If $F(0) = J_F(0) = 1$ and $M_1 \leq |F(z)| \leq M_2$, where M_1 and M_2 are positive constants, $M^* = \max\{-\log M_1, \log M_2\} + \pi$ and

$$m = \min_{r \in (0,1)} \frac{2 - r^2}{r(1 - r^2)} \approx 4.20.$$

Then F is univalent in the disk $\mathbb{D}_{\xi_0}$ and $F(\mathbb{D}_{\xi_0})$ contains a schlicht disk $\mathbb{D}(z_0, r_0)$, where

$$\xi_0 = \frac{\pi^3}{64 m M^{*2}}, \; z_0 = \cosh(\eta_0/\sqrt{2}), \; \eta_0 = \frac{\pi}{8M^*}\xi_0,$$

and

$$r_0 = \min\left\{\sinh(\eta_0/\sqrt{2}), \cosh(\eta_0/\sqrt{2}) \sin(\eta_0/\sqrt{2})\right\}.$$

It is worth mentioning that the radius r_0 in this theorem is not sharp, while Lemma 2.11 provides a new way to obtain the accurate value of the Bloch constant for log-p-harmonic mappings.

Lemma 2.12 *([30]) Let p be a positive integer. Then for any $z_1 \neq z_2$ in $\mathbb{D}_r$ $(0 < r < 1)$, we have*

$$\int_0^1 |tz_1 + (1-t)z_2|^{2(p-1)} dt \geq \frac{1}{2p-1} \cdot \frac{|z_1|^{2p-1} + |z_2|^{2p-1}}{|z_1| + |z_2|} > 0.$$

3 Landau-type theorems of certain bounded *p*-harmonic mappings

We establish four new versions of Landau-type theorems for bounded p-harmonic mappings.

Theorem 3.1 *Let $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ be a p-harmonic mapping in $\mathbb{D}$ with $F(0) = J_F(0) - 1 = 0$. Suppose that for each $k \in \{2, \ldots, p\}$, $\Lambda_{p-k+1} \geqslant 0$, $\Lambda_p > 1$ and*

(i) for each $k \in \{1, \ldots, p\}$, $G_{p-k+1}(z)$ is harmonic in $\mathbb{D}$ and $G_{p-k+1}(0) = 0$;
(ii) for each $k \in \{2, \ldots, p\}$, $\Lambda_{G_{p-k+1}}(z) \leqslant \Lambda_{p-k+1}$ and $\Lambda_{G_p}(z) < \Lambda_p$ for all $z \in \mathbb{D}$.
Then $F(z) \in S(r_1, R_1)$, where r_1 is the unique root in $(0, 1)$ of the equation

$$\frac{\Lambda_p(1 - \Lambda_p^2 r)}{\Lambda_p^2 - r} - \sum_{k=1}^{p-1} (2k+1) \Lambda_{p-k} r^{2k} = 0, \tag{3.1}$$

and

$$R_1 = \Lambda_p^3 r_1 + (\Lambda_p^5 - \Lambda_p) \ln\left(1 - \frac{r_1}{\Lambda_p^2}\right) - \sum_{k=1}^{p-1} \Lambda_{p-k} r_1^{2k+1}. \tag{3.2}$$

Proof We first observe that

$$J_F(0) = |F_z(0)|^2 - |F_{\overline{z}}(0)|^2 = |(G_p)_z(0)|^2 - |(G_p)_{\overline{z}}(0)|^2 = J_{G_p}(0) = 1.$$

Next, to prove that F is univalent in $\mathbb{D}_{r_1}$, we choose two distinct points $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < r_1)$, and let $\Gamma = \{z_1 + t\,(z_2 - z_1) : t \in [0, 1]\}$. By Lemmas 2.4 and 2.8(1), we find that

$$\begin{aligned} |F(z_1) - F(z_2)| &\geqslant \left|G_p(z_1) - G_p(z_2)\right| - \sum_{k=1}^{p-1} \left|G_{p-k}(z_1)|z_1|^{2k} - G_{p-k}(z_2)|z_2|^{2k}\right| \\ &\geqslant \left[\frac{\Lambda_p(1 - \Lambda_p^2 r)}{\Lambda_p^2 - r} - \sum_{k=1}^{p-1} (2k+1) \Lambda_{p-k} r^{2k}\right] |z_1 - z_2|. \end{aligned}$$

Moreover, an easy calculation shows that the function

$$\mu(r) = \frac{\Lambda_p(1 - \Lambda_p^2 r)}{\Lambda_p^2 - r} - \sum_{k=1}^{p-1} (2k+1) \Lambda_{p-k} r^{2k}$$

Table 1 For the case $p = 2$. The values of r_1, R_1 are in Theorem 3.1

(Λ_1, Λ_2)	(1, 1.1)	(1.2, 1)	(1.5, 2.1)	(3, 4)	(4, 5)
r_1	0.420000	0.394382	0.166761	0.055546	0.036755
R_1	0.254210	0.238504	0.043843	0.007210	0.003769

is continuous and strictly decreasing on [0, 1], and

$$\mu(0) = \frac{1}{\Lambda_p} > 0, \ \ \mu(1) = -\sum_{k=0}^{p-1}(2k+1)\Lambda_{p-k} < 0.$$

Then it follows from the intermediate value theorem that the equation $\mu(r) = 0$ has a unique root r_1 in (0, 1). Hence

$$\begin{aligned} |F(z_1) - F(z_2)| &\geqslant \left(\frac{\Lambda_p(1-\Lambda_p^2 r)}{\Lambda_p^2 - r} - \sum_{k=1}^{p-1}(2k+1)\Lambda_{p-k}r^{2k}\right)|z_1 - z_2| \\ &> |z_1 - z_2|\mu(r_1) = 0, \end{aligned}$$

which shows that $F(z)$ is univalent in $\mathbb{D}_{r_1}$.

Finally, for any $z \in \partial\mathbb{D}_{r_1}$, by Lemmas 2.4, 2.8 and (2.6), we have

$$\begin{aligned} |F(z) - F(0)| &= \left|G_p(z) + \sum_{k=1}^{p-1} G_{p-k}(z)|z|^{2k}\right| \geqslant \left|G_p(z)\right| - \sum_{k=1}^{p-1}|G_{p-k}(z)||z|^{2k} \\ &\geq \Lambda_p^3 r_1 + (\Lambda_p^5 - \Lambda_p)\ln\left(1 - \frac{r_1}{\Lambda_p^2}\right) - \sum_{k=1}^{p-1}\Lambda_{p-k}r_1^{2k+1} = R_1. \end{aligned}$$

This completes the proof of Theorem 3.1. □

Using Computer Algebra System, we list some numerical solutions in Table 1 to Eqs. (3.1)–(3.2).

Remark 3.1 Note that for harmonic mapping $G_p(z)$ of $\mathbb{D}$ with $G_p(0) = J_{G_p}(0) - 1 = 0$, and $\Lambda_{G_p}(z) \leqslant 1$ or $|G_p(z)| \leqslant 1$ for all $z \in \mathbb{D}$, it follows from Lemma 2.5 that $G_p(z) = a_1 z$ with $|a_1| = 1$ and $\lambda_{G_p}(0) = 1$, then by Theorem E(1), we have the following theorem.

Theorem 3.2 *Let* $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ *be a* p*-harmonic mapping in* $\mathbb{D}$ *with* $F(0) = J_F(0) - 1 = 0$. *Suppose that for each* $k \in \{2, \ldots, p\}$, $\Lambda_{p-k+1} \geqslant 0$ *and*

(i) for each $k \in \{1, \ldots, p\}$, $G_{p-k+1}(z)$ *is harmonic in* $\mathbb{D}$ *and* $G_{p-k+1}(0) = 0$;
(ii) for each $k \in \{2, \ldots, p\}$, $\Lambda_{G_{p-k+1}}(z) \leqslant \Lambda_{p-k+1}$, *and* $\Lambda_{G_p}(z) \leqslant 1$ *or* $|G_p(z)| \leqslant 1$ *for all* $z \in \mathbb{D}$.

Then $F(z) \in S(r_2, R_2)$, *where*

$$r_2 = \begin{cases} 1, & \text{if } \sum_{k=1}^{p-1} (2k+1)\Lambda_{p-k} \leqslant 1, \\ r_2', & \text{if } \sum_{k=1}^{p-1} (2k+1)\Lambda_{p-k} > 1, \end{cases} \tag{3.3}$$

and r_2' *is the unique root in* $(0, 1)$ *of the equation*

$$1 - \sum_{k=1}^{p-1} (2k+1)\Lambda_{p-k} r^{2k} = 0, \tag{3.4}$$

and $R_2 = r_2 - \sum_{k=1}^{p-1} \Lambda_{p-k} r_2^{2k+1}$. *Moreover, both of radii,* r_2 *and* R_2 *are sharp, with an extremal function given by (1.7).*

Theorem 3.3 *Let* $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ *be a* p*-harmonic mapping in* $\mathbb{D}$ *with* $F(0) = J_F(0) - 1 = 0$. *Suppose that for each* $k \in \{2, \ldots, p\}$, $M_{p-k+1} \geqslant 0$, $\Lambda_p > 1$ *and*

(i) for each $k \in \{1, \ldots, p\}$, $G_{p-k+1}(z)$ *is harmonic in* $\mathbb{D}$ *and* $G_{p-k+1}(0) = 0$;
(ii) for each $k \in \{2, \ldots, p\}$, $|G_{p-k+1}(z)| \leqslant M_{p-k+1}$ *and* $\Lambda_{G_p}(z) < \Lambda_p$ *for all* $z \in \mathbb{D}$.

Then $F(z) \in S(r_3, R_3)$, *where* r_3 *is the unique root in* $(0, 1)$ *of the equation*

$$\frac{\Lambda_p(1 - \Lambda_p^2 r)}{\Lambda_p^2 - r} - \sum_{k=1}^{p-1} \left(\frac{4}{\pi(1 - r^2)} + \frac{8k}{\pi} \right) M_{p-k} r^{2k} = 0, \tag{3.5}$$

and

$$R_3 = \Lambda_p^3 r_3 + (\Lambda_p^5 - \Lambda_p) \ln\left(1 - \frac{r_3}{\Lambda_p^2}\right) - \sum_{k=1}^{p-1} \frac{4M_{p-k}}{\pi} r_3^{2k+1}. \tag{3.6}$$

Proof We first prove that $F(z)$ is univalent in $\mathbb{D}_{r_3}$. In fact, for any two distinct points $z_1, z_2 \in \mathbb{D}_r$ $(0 < r < r_3)$, let $\Gamma = \{z_1 + t\,(z_2 - z_1) : t \in [0, 1]\}$. Then we have

$$|F(z_1) - F(z_2)| \geqslant \left|G_p(z_1) - G_p(z_2)\right| - \sum_{k=1}^{p-1} \left| G_{p-k}(z_1)|z_1|^{2k} - G_{p-k}(z_2)|z_2|^{2k} \right|.$$

Note that $J_F(0) = 1$ implies $J_{G_p}(0) = 1$, and thus, by using Lemma 2.4(i), we obtain

$$|G_p(z_1) - G_p(z_2)| \geq \frac{\Lambda_p(1 - \Lambda_p^2 r)}{\Lambda_p^2 - r} |z_1 - z_2|.$$

Table 2 For the case $p = 2$. The values of r_3, R_3 are in Theorem 3.3

(M_1, Λ_2)	$(1, 1.1)$	$(1.2, 1)$	$(1.5, 2.1)$	$(3, 4)$	$(4, 5)$
r_3	0.379287	0.365639	0.157884	0.054096	0.036029
R_3	0.233366	0.218269	0.042042	0.007074	0.003724

Consequently, by the hypotheses of Theorem 3.3 and Lemma 2.8(2), we have

$$|F(z_1) - F(z_2)| \geq \left(\frac{\Lambda_p\,(1 - \Lambda_p^2 r)}{\Lambda_p^2 - r} - \sum_{k=1}^{p-1} \left(\frac{4}{\pi(1 - r^2)} + \frac{8k}{\pi} \right) M_{p-k} r^{2k} \right) |z_1 - z_2| \\ > 0,$$

which shows that $F(z)$ is univalent in $\mathbb{D}_{r_3}$.

Next, we denote any $z \in \partial \mathbb{D}_{r_3}$ by $r_3 e^{i\theta}$. By Lemmas 2.2 and 2.4(ii), we have

$$|F(z) - F(0)| = |F(z)| = \left| G_p(z) + \sum_{k=1}^{p-1} G_{p-k}(z)|z|^{2k} \right| \geqslant \left| G_p(z) \right| - \sum_{k=1}^{p-1} |G_{p-k}(z)||z|^{2k} \\ \geqslant \Lambda_p^3 r_3 + (\Lambda_p^5 - \Lambda_p) \ln\left(1 - \frac{r_3}{\Lambda_p^2}\right) - \sum_{k=1}^{p-1} \frac{4M_{p-k}}{\pi} r_3^{2k+1} = R_3.$$

This completes the proof. □

Using Computer Algebra System, we list some numerical solutions in Table 2 to Eqs. (3.5)–(3.6).

By means of Remark 3.1, using the same method as in our proof of Theorem 3.3, we have the following theorem.

Theorem 3.4 *Let $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ be a p-harmonic mapping of $\mathbb{D}$ satisfying $F(0) = J_F(0) - 1 = 0$. Suppose that for each $k \in \{2, \ldots, p\}$, $M_{p-k+1} \geqslant 0$, and*

(i) for each $k \in \{1, \ldots, p\}$, $G_{p-k+1}(z)$ is harmonic in $\mathbb{D}$ and $G_{p-k+1}(0) = 0$;
(ii) for each $k \in \{2, \ldots, p\}$, $|G_{p-k+1}(z)| \leqslant M_{p-k+1}$, and $\Lambda_{G_p}(z) \leqslant 1$ or $|G_p(z)| \leqslant 1$ for all $z \in \mathbb{D}$.

Then $F(z) \in S(r_4, R_4)$, where $r_4 = 1$ for $M_{p-k+1} = 0\,(k = 2, \cdots, p)$ and r_4', otherwise. Here r_4' is the unique root in $(0, 1)$ of the equation

$$1 - \sum_{k=1}^{p-1} \left(\frac{4}{\pi(1 - r^2)} + \frac{8k}{\pi} \right) M_{p-k} r^{2k} = 0, \tag{3.7}$$

$$R_4 = r_4 - \sum_{k=1}^{p-1} \frac{4M_{p-k}}{\pi} r_4^{2k+1}. \tag{3.8}$$

When $M_{p-k+1} = 0\,(k = 2, \cdots, p)$, $R_4 = r_4 = 1$, which is sharp.

4 Landau-type theorems for log-p-harmonic mappings

By means of Theorem 3.1 and Lemma 2.11, we may establish the following Landau-type theorem of log-p-harmonic mappings, which is the analogues version of Theorem D.

Theorem 4.1 *Suppose that p is a positive integer, $p \geqslant 2$, $\Lambda_1, \Lambda_2, \ldots, \Lambda_{p-1} \geqslant 0$ and $\Lambda_p > 1$. Let $f(z) = \prod_{k=1}^{p} (g_{p-k+1}(z))^{|z|^{2(k-1)}}$ be a log-p-harmonic mapping of $\mathbb{D}$ with $J_f(0) = 1$. Assume that for each $k \in \{1, \ldots, p\}$, we have*

(i) $g_k(z)$ is log-harmonic in $\mathbb{D}$ and $g_k(0) = 1$, and $G_k(z) := \log g_k(z)$;
(ii) for each $k \in \{1, \ldots, p-1\}$, $\Lambda_{G_k}(z) \leqslant \Lambda_k$ and $\Lambda_{G_p}(z) < \Lambda_p$ for all $z \in \mathbb{D}$. Then $f(z) \in S_{w_1}(r_1, R_1')$, where r_1 is the unique root in $(0, 1)$ of Eq. (3.1), R_1 is defined by (3.2), $w_1 = \cosh R_1$ and $R_1' = \sinh R_1$.

Proof Since $G_k(z) = \log g_k(z)$ for each $k \in \{1, \cdots, p\}$, we obtain that

$$F(z) = \log f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$$

is p-harmonic in $\mathbb{D}$. A direct computation yields

$$J_f(0) = |f_z(0)|^2 - |f_{\overline{z}}(0)|^2 = |f(0)|^2(|F_z(0)|^2 - |F_{\overline{z}}(0)|^2) = J_F(0).$$

Thus the condition $g_p(0) = J_f(0) = 1$ leads to $G_p(0) = J_F(0) - 1 = J_f(0) - 1 = 0$.

Therefore, for $z_1 \neq z_2$ in $\mathbb{D}_r$ $(0 < r < r_1)$, by Theorem 3.1, we have

$$\begin{aligned} |\log f(z_1) - \log f(z_2)| &= |F(z_1) - F(z_2)| \\ &\geqslant |z_1 - z_2| \left(\frac{\Lambda_p(1 - \Lambda_p^2 r)}{\Lambda_p^2 - r} - \sum_{k=1}^{p-1} (2k+1) \Lambda_{p-k} r^{2k} \right) > 0. \end{aligned}$$

This implies that f is univalent in $\mathbb{D}_{r_1}$.

For any $z \in \partial \mathbb{D}_{r_1}$, it also follows from Theorem 3.1 that

$$|\log f(z)| = |F(z)| \geqslant \Lambda_p^3 r_1 + (\Lambda_p^5 - \Lambda_p) \ln\left(1 - \frac{r_1}{\Lambda_p^2}\right) - \sum_{k=1}^{p-1} \Lambda_{p-k} r_1^{2k+1} = R_1.$$

Hence, by Lemma 2.11, we get that the range $f(\mathbb{D}_{r_1})$ contains a schlicht disk $\mathbb{D}(w_1, R_1') = \{w \in \mathbb{C} \mid |w - w_1| < R_1'\}$, where $w_1 = \cosh R_1$, $R_1' = \sinh R_1$. This completes the proof. □

Remark 4.1 Note that for harmonic mapping $G_p(z)$ of $\mathbb{D}$ with $G_p(0) = J_{G_p}(0) - 1 = 0$, and $\Lambda_{G_p}(z) \leqslant \Lambda_p$ for all $z \in \mathbb{D}$, it follows from Lemma 2.5 that $\Lambda_p \geqslant$

1. Theorem 4.1 provides a version of the Landau-type theorem of certain log-p-harmonic mappings with $J_f(0) = 1$ for the cases $\Lambda_1, \ldots, \Lambda_{p-1} \geqslant 0$ and $\Lambda_p > 1$. If $\Lambda_1, \ldots, \Lambda_{p-1} \geqslant 0$, and $\Lambda_p = 1$, then we will prove the following precise form of the Landau-type theorem of certain log-p-harmonic mappings by using Lemma 2.5 and Theorem E(2).

Theorem 4.2 *Suppose that p is a positive integer, $p \geqslant 2$, $\Lambda_1, \Lambda_2, \ldots, \Lambda_{p-1} \geqslant 0$. Let $f(z) = \prod_{k=1}^{p} (g_{p-k+1}(z))^{|z|^{2(k-1)}}$ be a log-p-harmonic mapping of $\mathbb{D}$ satisfying $f(0) = J_f(0) = 1$. Suppose that for each $k \in \{1, \ldots, p\}$, we have that*

(i) $g_{p-k+1}(z)$ is log-harmonic in $\mathbb{D}$, $g_{p-k+1}(0) = 1$, and $G_{p-k+1} := \log g_{p-k+1}$;
(ii) for each $k \in \{2, \ldots, p\}$, $\Lambda_{G_{p-k+1}}(z) \leqslant \Lambda_{p-k+1}$, and $\Lambda_{G_p}(z) \leqslant 1$ or $|G_p(z)| \leqslant 1$ for all $z \in \mathbb{D}$.
Then $f(z) \in S_{w_2}(r_2, R_2')$, where r_2 is defined by (3.3), R_1 is defined by (3.4) and

$$w_2 = \cosh R_2, \quad R_2' = \sinh R_2, \tag{4.1}$$

Both of the radii, r_2 and $R_2' = \sinh R_2$, are sharp.

Proof Since $f(0) = J_f(0) = 1$, it is easy to verify that $J_{G_p}(0) = J_f(0) = 1$.

Since $G_p(0) = 0$, and $\Lambda_{G_p}(z) \leqslant 1$ or $|G_p(z)| \leqslant 1$ for all $z \in \mathbb{D}$, it follows from Lemma 2.5 that $\lambda_{G_p}(0) = 1$. Hence $\lambda_f(0) = \lambda_{G_p}(0) = 1$, and the conclusion of Theorem 4.2 follows from Theorem E(2). □

By means of Theorem 3.3 and Lemma 2.11, we may establish the following Landau-type theorem of log-p-harmonic mappings, which is the analogues version of Theorem C.

Theorem 4.3 *Suppose that p is a positive integer, $p \geqslant 2$, $M_1^*, M_2^*, \ldots, M_{p-1}^* \geqslant 1$ and $\Lambda_p > 1$. Let $f(z) = \prod_{k=1}^{p} (g_{p-k+1}(z))^{|z|^{2(k-1)}}$ be a log-p-harmonic mapping of $\mathbb{D}$ such that $J_f(0) = 1$. Suppose that for each $k \in \{1, \ldots, p\}$, we have*

(i) $g_{p-k+1}(z)$ is log-*harmonic in $\mathbb{D}$ and $g_{p-k+1}(0) = 1$, $G_p(z) = \log g_p(z)$;*
(ii) for each $k \in \{2, \ldots, p\}$, $|g_{p-k+1}(z)| \leqslant M_{p-k+1}^*$ and $\Lambda_{G_p}(z) < \Lambda_p$ for all $z \in \mathbb{D}$.
Then $f(z) \in S_{w_3}(r_3, R_3')$, where $M_i = \log M_i^* + \pi$ $(i = 1, \ldots, p-1)$, and r_3 is the unique root in $(0, 1)$ of Eq.(3.5), R_3 is defined by (3.6), $w_3 = \cosh R_3$ and $R_3' = \sinh R_3$.

Proof Since $G_k(z) = \log\, g_k(z)$ for each $k \in \{1, \ldots, p\}$, we obtain that

$$F(z) = \log f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$$

is p-harmonic in $\mathbb{D}$. Also, it is easy to see that

$$J_f(0) = |f_z(0)|^2 - |f_{\overline{z}}(0)|^2 = |f(0)|^2 (|F_z(0)|^2 - |F_{\overline{z}}(0)|^2) = J_F(0).$$

Thus the condition $g_p(0) = J_f(0) = 1$ leads to $G_p(0) = J_F(0) - 1 = 0$. For $k \in \{1, \dots, p-1\}$, we have $\arg g_{p-k+1} \in (-\pi, \pi]$ and

$$|G_{p-k+1}| = |\log g_{p-k-1}| = |\log |g_{p-k-1}| + i \arg g_{p-k-1}| \leqslant |\log |g_{p-k-1}|| + \pi.$$

This implies that $|G_{p-k+1}| \leqslant \log M^*_{p-k+1} + \pi := M_{p-k+1}, k \in \{1, \dots, p-1\}$.

Therefore, for $z_1 \neq z_2$ in $\mathbb{D}_r$ $(0 < r < r_3)$, by Theorem 3.3, we have

$$\begin{aligned} |\log f(z_1) - \log f(z_2)| &= |F(z_1) - F(z_2)| = \left| \int_\Gamma F_z(z)dz + F_{\overline{z}}(z)d\overline{z} \right| \\ &\geqslant |z_1 - z_2| \left(\frac{\Lambda_p(1 - \Lambda_p^2 r)}{\Lambda_p^2 - r} - \sum_{k=1}^{p-1} \left(\frac{4}{\pi(1-r^2)} + \frac{8k}{\pi} \right) M_{p-k} r^{2k} \right) > 0. \end{aligned}$$

This implies that f is univalent in $\mathbb{D}_{r_3}$.

For any $z \in \partial\mathbb{D}_{r_3}$, it also follows from Theorem 3.3 that

$$|\log f(z)| = |F(z)| \geqslant \Lambda_p^3 r_3 + (\Lambda_p^5 - \Lambda_p)\ln\left(1 - \frac{r_3}{\Lambda_p^2}\right) - \sum_{k=1}^{p-1} \frac{4M_{p-k}}{\pi} r_3^{2k+1} = R_3.$$

Hence, by Lemma 2.11, we get that the range $f(\mathbb{D}_{r_3})$ contains a schlicht disk $\mathbb{D}(w_3, R_3) = \{w \in \mathbb{C} | |w - w_3| < R_3\}$, where

$$w_3 = \cosh R_3, \quad R_3' = \sinh R_3.$$

This completes the proof of Theorem 4.3. □

By means of Theorem 3.4 and Lemma 2.11, using the same method as in our proof of Theorem 4.3, we have the following theorem.

Theorem 4.4 *Suppose that p is a positive integer, $p \geqslant 2$, $M_1^*, M_2^*, \dots, M_{p-1}^* \geqslant 1$. Let $f(z) = \prod_{k=1}^{p}(g_{p-k+1}(z))^{|z|^{2(k-1)}}$ be a log-p-harmonic mapping of $\mathbb{D}$ such that $J_f(0) = 1$. Suppose that for each $k \in \{1, \dots, p\}$, we have*

(i) $g_{p-k+1}(z)$ is log-*harmonic in $\mathbb{D}$ and $g_{p-k+1}(0) = 1$, $G_p(z) = \log g_p(z)$;*
(ii) for each $k \in \{2, \dots, p\}$, $|g_{p-k+1}(z)| \leqslant M^*_{p-k+1}$, and $\Lambda_{G_p}(z) \leqslant 1$ or $|G_p(z)| \leqslant 1$ for all $z \in \mathbb{D}$.

Then $f(z) \in S_{w_4}(r_4, R_4')$, where $M_i = \log M_i^* + \pi$ $(i = 1, \dots, p-1)$, and r_4 is the unique root in $(0, 1)$ of Eq. (3.7), R_4 is defined by (3.8), $w_4 = \cosh R_4$ and $R_4' = \sinh R_4$.

Next, we establish the following result, which is the analogues version of Theorem B.

Theorem 4.5 *Suppose that $f(z) = g(z)^{|z|^{2(p-1)}}$ is a log-p-harmonic of $\mathbb{D}$, where $p > 1$, g is log-harmonic and $g(0) = J_g(0) = 1$. Let $G(z) = \log g(z)$, and $\Lambda_G \leq \Lambda$.*

Then $f(z) \in S_{z_5}(r_5, R_5')$, *where* $r_5 = 1$ *for* $\Lambda = 1$ *and* r_5', *otherwise, and* r_5' *is the unique root in* $(0, 1)$ *of the equation*

$$\frac{1}{\Lambda} + 2\frac{\Lambda^4 - 1}{\Lambda^3}\Big[\frac{\ln(1-r)}{r} + 1\Big] - \frac{\Lambda^4 - 1}{\Lambda^3}\frac{r}{1-r} = 0, \tag{4.2}$$

$$z_5 = \cosh(R_5),\ R_5' = \sinh(R_5) \text{ and}$$

$$R_5 = r_5^{2p-1}\left\{\frac{1}{\Lambda} + \frac{\Lambda^4 - 1}{\Lambda^3}\Big[\frac{\ln(1-r_5)}{r_5} + 1\Big]\right\}.$$

When $\Lambda = 1$, *the radii* $r_5 = 1$ *and* $R_5' = \sinh 1$ *are sharp.*

Proof Let

$$F(z) := \log f(z) = |z|^{2(p-1)} \log g(z) = |z|^{2(p-1)} G(z).$$

Then F is p-harmonic in $\mathbb{D}$, $G(z) = \log g(z)$ is harmonic in $\mathbb{D}$, and it has the series expansion:

$$G(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n.$$

Note that

$$g(0) = J_g(0) = |g_z(0)|^2 - |g_{\overline{z}}(0)|^2 = |g(0)|^2(|G_z(0)|^2 - |G_{\overline{z}}(0)|^2) = J_G(0) = 1.$$

So, when $\Lambda > 1$, for $z_1 \neq z_2$ in $\mathbb{D}_r (0 < r < r_5')$, we adopt the same method as in [9]. By Lemmas 2.5 and 2.11, we have

$$\begin{aligned}
&|\log f(z_1) - \log f(z_2)| = |F(z_1) - F(z_2)| \\
&\quad \geqslant |z_1 - z_2|\Big(\int_0^1 |tz_1 + (1-t)z_2|^{2(p-1)} dt\Big) \\
&\qquad \times \Big[||a_1| - |b_1|| - 2\sum_{n=2}^{\infty}(|a_n| + |b_n|)r^{n-1} - \sum_{n=2}^{\infty} n(|a_n| + |b_n|)r^{n-1}\Big] \\
&\quad \geqslant |z_1 - z_2|\left(\int_0^1 |tz_1 + (1-t)z_2|^{2(p-1)} dt\right) \\
&\qquad \times \Big(\lambda_G(0) - 2\sum_{n=2}^{\infty}\frac{\Lambda^4 - 1}{n\Lambda^3} r^{n-1} - \sum_{n=2}^{\infty}\frac{\Lambda^4 - 1}{\Lambda^3} r^{n-1}\Big) \\
&\quad \geqslant |z_1 - z_2|\Big(\int_0^1 |tz_1 + (1-t)z_2|^{2(p-1)} dt\Big) \\
&\qquad \times \left\{\frac{1}{\Lambda} + 2\frac{\Lambda^4 - 1}{\Lambda^3}\Big[\frac{\ln(1-r)}{r} + 1\Big] - \frac{\Lambda^4 - 1}{\Lambda^3}\frac{r}{1-r}\right\}
\end{aligned}$$

Table 3 For the case $p = 2$. The values of r_5, R'_5 are in Theorem 4.5

Λ	1.1	1.5	2	2.5	3
r_5	0.605505	0.224286	0.119898	0.076818	0.053722
R'_5	$5.8812e^{-2}$	$2.8795e^{-4}$	$9.3588e^{-6}$	$8.0610e^{-7}$	$1.1222e^{-7}$

$$> |z_1 - z_2|\left(\int_0^1 |tz_1 + (1-t)z_2|^{2(p-1)}dt\right)$$
$$\times \left\{\frac{1}{\Lambda} + 2\frac{\Lambda^4-1}{\Lambda^3}\left[\frac{\ln(1-r'_5)}{r'_5} + 1\right] - \frac{\Lambda^4-1}{\Lambda^3}\frac{r'_5}{1-r'_5}\right\} = 0.$$

This implies f is univalent in $\mathbb{D}_{r'_5}$.

When $\Lambda = 1$, it follows from Lemmas 2.5 that $G(z) = a_1 z$ with $|a_1| = 1$. Thus, $F(z) = a_1|z|^{2(p-1)}z$, and for $z_1 \neq z_2$ in $\mathbb{D}$, by Lemma 2.12, we have that

$$|\log f(z_1) - \log f(z_2)| = |F(z_1) - F(z_2)| = \left|\int_{\overline{z_1z_2}} F_z(z)dz + F_{\overline{z}}(z)d\overline{z}\right|$$
$$= \left|\int_{[z_1,z_2]} p|z|^{2(p-1)}dz + (p-1)|z|^{2(p-2)}z^2 d\overline{z}\right|$$
$$\geqslant |z_1 - z_2|\left(\int_0^1 |tz_1 + (1-t)z_2|^{2(p-1)}dt\right) > 0.$$

This implies f is univalent in $\mathbb{D}$, and the radius $r_5 = 1$ is sharp.

Finally, for any $z \in \partial\mathbb{D}_{r_5}$, by Lemma 2.5, we obtain

$$|\log f(z)| = |F(z)| \geqslant r_5^{2(p-1)}\left(|a_1 z + \overline{b}_1\overline{z}| - |\sum_{n=2}^{\infty}(a_n z^n + \overline{b}_n\overline{z}^n)|\right)$$
$$\geqslant r_5^{2p-1}\left\{\frac{1}{\Lambda} + \frac{\Lambda^4-1}{\Lambda^3}\left[\frac{\ln(1-r_5)}{r_5} + 1\right]\right\} = R_5.$$

Thus it follows from Lemma 2.11 that $f(\mathbb{D}_{r_5})$ contains a schlicht disk $\mathbb{D}(z_5, R'_5)$, where Table 3

$$z_5 = \cosh(R_5) \text{ and } R'_5 = \sinh(R_5).$$

In particular, when $\Lambda = 1$, it follows from Lemma 2.11 and the sharpness of $r_5 = 1$ that the radius $R'_5 = \sinh 1$ is sharp. This completes the proof of Theorem 4.5. □

Finally, we establish the following result, which improves Theorem A.

Theorem 4.6 *Let $f(z) = \prod_{k=1}^{p}(g_{p-k+1}(z))^{|z|^{2(k-1)}}$ be a log-p-harmonic mapping of $\mathbb{D}$ such that $J_f(0) = 1$. Suppose that for each $k \in \{1, \ldots, p\}$, we have that*

(i) $g_k(z)$ *is* log*-harmonic in* $\mathbb{D}$*,* $g_k(0) = 1$*, and* $G_k(z) := \log\, g_k(z)$*;*
(ii) $|g_k(z)| \leqslant M_k^*$ *for all* $z \in \mathbb{D}$*, where* $M_k^* > 1$ *and* $M_k := \log M_k^* + \pi$*.*
Then $f(z) \in S_{z_6}(r_6, R_6')$*, where* r_6 *is the unique root in* $(0, 1)$ *of the equation*

$$\lambda_0(M_p) - \sum_{k=1}^{p-1} \left(\frac{4}{\pi(1-r^2)} + \frac{8k}{\pi} \right) M_{p-k} r^{2k} - \lambda_0(M_p)\sqrt{M_p^4 - 1} \cdot \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1-r^2)^{\frac{3}{2}}} = 0, \tag{4.3}$$

$$z_6 = \cosh\left(R_6\right), \quad R_6' = \sinh\left(R_6\right), \tag{4.4}$$

and

$$R_6 = \lambda_0(M_p) r_6 - \lambda_0(M_p)\sqrt{M_p^4 - 1} \cdot \frac{r_6^2}{(1-r_6^2)^{\frac{1}{2}}} - \frac{4}{\pi} \sum_{k=1}^{p-1} M_{p-k} r_6^{2k+1}. \tag{4.5}$$

Proof For all $k \in \{1, \ldots, p\}$, assume $G_k = \log g_k$ have the following series expansions:

$$G_k(z) = \sum_{n=1}^{\infty} a_{n,k} z^n + \sum_{n=1}^{\infty} \overline{b}_{n,k} \overline{z}^n.$$

Then $F(z) = \log f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ is p-harmonic in $\mathbb{D}$. By the proof of Theorem 4.1, we can prove that $G_p(0) = J_F(0) - 1 = J_f(0) - 1 = 0$, and for any $k \in \{1, \ldots, p\}$, we have

$$|G_{p-k+1}| \leqslant |\log|g_{p-k-1}|| + \pi.$$

This implies $|G_{p-k+1}| \leqslant \log M_{p-k+1}^* + \pi = M_{p-k+1}$ for $k \in \{1, \ldots, p\}$.

Now, we prove that f is univalent in $\mathbb{D}_{r_6}$. To this end, for any $z_1 \neq z_2$ in $\mathbb{D}_r$ $(0 < r < r_6)$, let

$$I_1 = \left| G_p(z_1) - G_p(z_2) \right| \text{ and } I_2 = \left| \sum_{k=1}^{p-1} G_{p-k}(z_1)|z_1|^{2k} - \sum_{k=1}^{p-1} G_{p-k}(z_2)|z_2|^{2k} \right|.$$

By Lemmas 2.2, 2.9 and 2.8(2), simple calculation yields

$$I_1 \geqslant \left| \int_{[z_1,z_2]} (G_p)_z(0) dz + (G_p)_{\overline{z}}(0) d\overline{z} \right| - \int_{[z_1,z_2]} \Big(|(G_p)_z(z) - (G_p)_z(0)||dz| + |(G_p)_{\overline{z}}(z) - (G_p)_{\overline{z}}(0)||d\overline{z}| \Big)$$

$$
\begin{aligned}
&\geqslant \int_{[z_1,z_2]} \lambda_{G_p}(0)|dz| - |z_1 - z_2| \sum_{n=2}^{\infty} n(|a_{n,p}| + |b_{n,p}|) r^{n-1} \\
&\geqslant |z_1 - z_2| \left(\lambda_{G_p}(0) - \left(\sum_{n=2}^{\infty} (|a_{n,p}| + |b_{n,p}|)^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{n=2}^{\infty} n^2 r^{2(n-1)} \right)^{\frac{1}{2}} \right) \\
&\geqslant |z_1 - z_2| \lambda_{G_p}(0) \left(1 - \sqrt{M_p^4 - 1} \cdot \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1-r^2)^{\frac{3}{2}}} \right) \\
&\geqslant |z_1 - z_2| \lambda_0(M_p) \left(1 - \sqrt{M_p^4 - 1} \cdot \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1-r^2)^{\frac{3}{2}}} \right),
\end{aligned}
$$

$$
\begin{aligned}
I_2 &\leqslant \sum_{k=1}^{p-1} \left| G_{p-k}(z_1)|z_1|^{2k} - G_{p-k}(z_2)|z_2|^{2k} \right| \\
&\leqslant |z_1 - z_2| \sum_{k=1}^{p-1} \left(\frac{4}{\pi(1-r^2)} + \frac{8k}{\pi} \right) M_{p-k}\, r^{2k}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|\log f(z_1) - \log f(z_2)| = |F(z_1) - F(z_2)| &\geqslant I_1 - I_2 \\
&> |z_1 - z_2| \Bigg[\lambda_0(M_p) - \sum_{k=1}^{p-1} \left(\frac{4}{\pi(1-r_6^2)} + \frac{8k}{\pi} \right) M_{p-k} r_6^{2k} \\
&\quad - \lambda_0(M_p) \sqrt{M_p^4 - 1} \cdot \frac{r_6\sqrt{r_6^4 - 3r_6^2 + 4}}{(1-r_6^2)^{\frac{3}{2}}} \Bigg] = 0.
\end{aligned}
$$

This implies that f is univalent in $\mathbb{D}_{r_6}$.

For any $z \in \partial\mathbb{D}_{r_6}$, again by Lemmas 2.2 and 2.9, we obtain

$$
\begin{aligned}
|\log f(z)| = |F(z)| &= \left| \sum_{n=1}^{\infty} (a_{n,p} z^n + \overline{b}_{n,p} \overline{z}^n) + \sum_{k=1}^{p-1} |z|^{2k} G_{p-k}(z) \right| \\
&\geqslant |a_{1,p} z + \overline{b}_{1,p} \overline{z}| - \left| \sum_{n=2}^{\infty} (a_{n,p} z^n + \overline{b}_{n,p} \overline{z}^n) \right| - \left| \sum_{k=1}^{p-1} |z|^{2k} G_{p-k}(z) \right| \\
&\geqslant \lambda_{G_p}(0) r_6 - \left(\sum_{n=2}^{\infty} (|a_{n,p}| + |b_{n,p}|)^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{n=2}^{\infty} r_6^{2n} \right)^{\frac{1}{2}} - \frac{4r_6}{\pi} \sum_{k=1}^{p-1} M_{p-k} r_6^{2k} \\
&\geqslant \lambda_{G_p}(0) \left(r_6 - \sqrt{M_p^4 - 1} \cdot \frac{r_6^2}{(1-r_6^2)^{\frac{1}{2}}} \right) - \frac{4r_6}{\pi} \sum_{k=1}^{p-1} M_{p-k} r_6^{2k} \\
&\geqslant \lambda_0(M_p) r_6 - \lambda_0(M_p) \sqrt{M_p^4 - 1} \cdot \frac{r_6^2}{(1-r_6^2)^{\frac{1}{2}}} - \frac{4}{\pi} \sum_{k=1}^{p-1} M_{p-k} r_6^{2k+1} = R_6.
\end{aligned}
$$

Table 4 For the case $p = 2$. The values of r_6, R_6 are in Theorem 4.6 and the values of ρ_2, σ_2 are in Theorem A

(M_1, M_2)	(1.2, 1)	(1.5, 1)	(2, 1.5)	(3, 2.5)	(4, 3.5)
ρ_2	0.050352	0.050324	0.039588	0.030300	0.025854
r_6	0.050366	0.050339	0.039593	0.030302	0.025855
σ_2	0.071575	0.071548	0.063423	0.055462	0.051221
R_6	0.101337	0.101301	0.089767	0.078480	0.072472

Also, this together with Lemma 2.11 imply that $f(\mathbb{D}_{r_6})$ contains a schlicht disk $\mathbb{D}(z_6, R_6')$, where

$$z_6 = \cosh R_6 \text{ and } R_6' = \sinh R_6.$$

This completes the proof of Theorem 4.6. □

Remark 4.2 Note that for $r = r_6$, we have

$$\sum_{k=1}^{p-1}\left(\frac{4}{\pi(1-r^2)}+\frac{8k}{\pi}\right)M_{p-k}r^{2k} < \sum_{k=1}^{p-1}\left(\frac{4}{\pi(1-r^2)}+\frac{8k}{\pi(1-r)}\right)M_{p-k}r^{2k} \text{ and}$$
$$\frac{4}{\pi}\sum_{k=1}^{p-1}M_{p-k}r^{2k+1} < \frac{4}{\pi(1-r)}\sum_{k=1}^{p-1}M_{p-k}r^{2k+1}.$$

It is easy to verify that $r_6 > \rho_2$ and $R_6 > \sigma_2$, where ρ_2, σ_2 are given in Theorem A.

Using Computer Algebra System, we list some numerical solutions to Eqs. (4.3)–(4.4). By Table 4, we know that the results of Theorem 4.6 are better than that of Theorem A.

Acknowledgements The work of the first two authors are supported by Natural Science Foundation of Guangdong Province (Grant No. 2021A1515010058). The third author was supported by University of Macau (MYRG2022-00108-FST, MYRG-CRG2022–00010-ICMS), The Science and Technology Development Fund, Macau S.A.R (0036/2021/AGJ). The authors are grateful to the anonymous referee for making many suggestions that improved the readability of this paper.

Data availability The authors declare that this research is purely theoretical and does not associate with any data.

Declarations

Conflict of interest The authors declare that they have no conflict of interest, regarding the publication of this paper.

References

1. Abdulhadi, Z., Abu Muhanna, Y., Khuri, S.: On univalent solutions of the biharmonic equation. J. Inequal. Appl. **5**, 469–478 (2005)
2. Abdulhadi, Z., Abu Muhanna, Y., Khuri, S.: On some properties of solutions of the biharmonic equation. Appl. Math. Comput. **177**, 346–351 (2006)
3. Abdulhadi, Z., Abu Muhanna, Y.: Landau's theorem for biharmonic mappings. J. Math. Anal. Appl. **338**, 705–709 (2008)
4. Aleman, A., Constantin, A.: Harmonic maps and ideal fluid flows. Arch. Ration. Mech. Anal. **204**, 479–513 (2012)
5. Bai, X.X., Liu, M.S.: Landau-type theorems of ployharmonic mappings and log-p-harmonic mappings. Complex Anal. Oper. Theory **13**, 321–340 (2019)
6. Chen, H.H., Gautheir, P.M.: Bloch constants in several variables. Trans. Am. Math. Soc. **353**, 1371–1386 (2000)
7. Chen, H.H., Gautheir, P.M., Hengartner, W.: Bloch constants for planar harmonic mappings. Proc. Am. Math. Soc. **128**, 3231–3240 (2000)
8. Chen, H.H., Gautheir, P.M.: The Landau theorem and Bloch theorem for planar harmonic and pluri-harmonic mappings. Proc. Am. Math. Soc. **139**(2), 583–595 (2011)
9. Chen, Sh., Ponnusamy, S., Wang, X.: Bloch constant and Landau's theorem for planar p-harmonic mappings. J. Math. Appl. Appl. **373**, 102–110 (2011)
10. Chen, Sh., Ponnusamy, S., Wang, X.: Coefficient estimates and Landau-Bloch's constant for harmonic mappings. Bull. Malays. Math. Sci. Soc. **34**(2), 255–265 (2011)
11. Chen, Sh., Ponnusamy, S., Wang, X.: Properties of some classes of planar harmonic and planar biharmonic mappings. Complex Anal. Oper. Theory **5**, 901–916 (2011)
12. Chen, Sh., Ponnusamy, S., Rasila, A.: Coefficient estimates, Landau's theorem and Lipschitz-type spaces on planar harmonic mappings. J. Aust. Math. Soc. **96**(2), 198–215 (2014)
13. Chen, S.F., Liu, M.S.: Landau-type theorems and bi-Lipschitz theorems for bounded biharmonic mappings. Monatshefte Math. **193**, 783–806 (2020)
14. Colonna, F.: The Bloch constant of bounded harmonic mappings. Indiana Univ. Math. J. **38**(4), 829–840 (1989)
15. Constantin, O., Martin, M.J.: A harmonic maps approach to fluid flows. Math. Ann. **369**, 1–16 (2017)
16. Dorff, M., Nowark, M.: Landau's theorem for planar harmonic mappings. Comput. Methods Funct. Theory **4**, 151–158 (2004)
17. Grigoryan, A.: Landau and Bloch theorems for planar harmonic mappings. Complex Var. Elliptic Eq. **51**, 81–87 (2006)
18. Heinz, E.: On one-to-one harmonic mappings. Pacific J. Math. **9**, 101–105 (1959)
19. Huang, X.Z.: Estimates on Bloch constants for planar harmonic mappings. J. Math. Anal. Appl. **337**, 880–887 (2008)
20. Landau, E.: Der Picard-Schottysche Satz und die Blochsche Konstanten. Sitzungsber Press Akad, Wiss. Berlin Phys.-Math, Kl 467–474 (1929)
21. Lewy, H.: On the non-vanishing of the Jacobian in certain one-to-one mappings. Bull. Am. Math. Soc. **42**, 689–692 (1936)
22. Li, D.Z., Chen, X.D.: Bloch constant of harmonic mappings. J. Huaqiao Univ. (Nat. Sci.) **33**(1), 103–106 (2012)
23. Li, P., Wang, X.: Landau's theorem for log-p-harmonic mappings. Appl. Math. Comput. **218**(9), 4806–4812 (2012)
24. Li, P., Ponnusamy, S., Wang, X.: Some properties of planar p-harmonic and log-p-harmonic mappings. Bull. Malays. Math. Sci. Soc. **36**(3), 595–609 (2013)
25. Liu, G., Ponnusamy, S.: Compositions of polyharmonic mappings: in complex analysis and dynamical systems VII. Contemp. Math. (AMS) **699**, 209–221 (2017)
26. Liu, M.S.: Landau's theorems for biharmonic mappings. Complex Var. Elliptic Eq. **53**, 843–855 (2008)
27. Liu, M.S.: Estimates on Bloch constants for planar harmonic mappings. Sci. China Ser. A-Math. **52**(1), 87–93 (2009)
28. Liu, M.S.: Landau's theorems for planar harmonic mappings. Comput. Math. Appl. **57**(7), 1142–1146 (2009)
29. Liu, M.S., Ponnusamy, S.: Landau-type theorems for certain bounded bianalytic functions and biharmonic mappings. Can. Math. Bull. (2023). https://doi.org/10.4153/S0008439523000577

30. Liu, M.S., Liu, Z.X.: Landau-type theorems for p-harmonic mappings or log-p-harmonic mappings. Appl. Anal. **11**, 2462–2477 (2014)
31. Liu, M.S., Luo, L.F.: Landau-type theorems for certain bounded biharmonic mappings. Results Math. **74**, 170 (2019)
32. Liu, M.S., Luo, L.F.: Precise values of the Bloch constants of certain log-p-harmonic mappings. Acta Math. Sci. **41B**(1), 297–310 (2021)
33. Mao, Zh., Ponnusamy, S., Wang, X.: Schwarzian derivative and Landau's theorem for logharmonic mappings. Complex Var. Elliptic Eq. **58**(8), 1093–1107 (2013)
34. Zhu, Y.C., Liu, M.S.: Landau-type theorems for certain planar harmonic mappings or biharmonic mappings. Complex Var. Elliptic Eq. **58**(12), 1667–1676 (2013)