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ORIGINAL RESEARCH

A partition statistic for partitions with even parts distinct

Robert X. J. Hao[1]

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Abstract

Andrews, Hirschhorn, and Sellers studied the partition function $ped(n)$ which enumerates the number of partitions of n with even parts distinct, and obtained a number of interesting congruences. This paper aims to introduce a partition statistic to investigate the partition function $ped(n)$. We give combinatorial interpretations for some properties of $ped(n)$ including the infinite families of congruences given by Andrews et al.

Keywords Partition statistic · Crank · Bijection · Combinatorial interpretation · Refinement

Mathematics Subject Classification 05A17 · 11P83

1 Introduction

A partition λ of a positive integer n is a finite non-increasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_l)$ such that $|\lambda| = \sum_{i=1}^{l} \lambda_i = n$. The Ferrers graph of a partition λ is a set of coordinates in the bottom right quadrant of the plane where the i-th row contains λ_i dots. We denote by λ' the conjugate of λ, which is the partition whose graph is obtained by reflecting the Ferrers graph of λ about the main diagonal. For example, we give $\lambda = (4, 4, 2, 2, 1)$ and its conjugate partition $\lambda' = (5, 4, 2, 2)$ in Fig. 1.

Let $p(n)$ denote the ordinary partition function. The partition statistic crank defined by Andrews and Garvan [2, 6] can be used to provide combinatorial interpretations

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✉ Robert X. J. Hao
haoxj@njit.edu.cn

1 College of Science and Mathematics, Nanjing Institute of Technology, Nanjing 211167, People's Republic of China

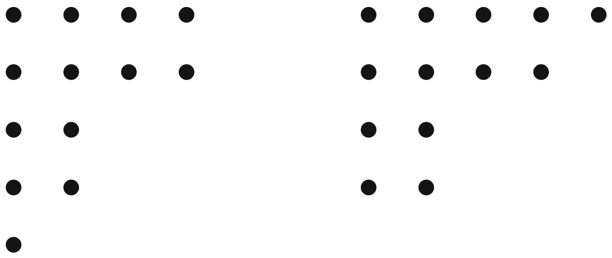

Fig. 1 Ferrers graph of partitions $\lambda = (4, 4, 2, 2, 1)$ and $\lambda' = (5, 4, 2, 2)$

for Ramanujan's famous congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{1.1}$$

The crank of a partition $\lambda \neq (1)$ is defined as follows:

$$c(\lambda) = \begin{cases} \lambda_1, & if \ \ n_1(\lambda) = 0, \\ \mu(\lambda) - n_1(\lambda), & if \ n_1(\lambda) > 0, \end{cases}$$

where $n_1(\lambda)$ denotes the number of parts equal to one in λ and $\mu(\lambda)$ denotes the number of parts in λ larger than $n_1(\lambda)$. Let $M(m, n)$ enumerate partitions of n with crank m. It should be pointed out that when $\lambda = (1)$,

$$M(0, 1) = -1, \quad M(-1, 1) = M(1, 1) = 1.$$

Andrews and Garvan [2, 6] established the generating function of $M(m, n)$ as given by

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m, n) z^m q^n = \frac{(q; q)_\infty}{(zq; q)_\infty (z^{-1}q; q)_\infty}. \tag{1.2}$$

Here and throughout this paper, $(a; q)_\infty$ stands for the q-shifted factorial

$$(a; q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}), \ \ |q| < 1,$$

and for any positive integer k,

$$f_k = (q^k; q^k)_\infty.$$

Let $M(m,t,n)$ denote the number of partitions of n with crank congruent to m modulo t. In 1990, Garvan [7] presented a graceful refinement of the congruence (1.1)

$$M(m,2,5n+4) \equiv 0 \pmod 5, \quad m=0,1,$$

together with the combinatorial interpretation

$$M(m+2k,10,5n+4) = \frac{M(m,2,5n+4)}{5}, \quad 0 \le k \le 4, m=0,1.$$

Let $ped(n)$ be the function that enumerates partitions of n with even parts distinct. Obviously,

$$\sum_{n=0}^{\infty} ped(n)q^n = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

The sequence $ped(n)_{n\ge 0}$ is well known and can be seen in [11, A001935], as well as other combinatorial interpretations. In 2010, Andrews, Hirschhorn, and Sellers [3] proved the following congruences.

Theorem 1.1 *For $\alpha, n \ge 0$,*

$$ped\left(3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right) \equiv 0 \pmod 2,$$
$$ped\left(3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right) \equiv 0 \pmod 2.$$

Theorem 1.2 *For $n \ge 0$,*

$$ped(9n+4) \equiv 0 \pmod 4, \tag{1.3}$$
$$ped(9n+7) \equiv 0 \pmod{12}. \tag{1.4}$$

In 2017, Merca [9] provided a simple criterion for deciding the parity of $ped(n)$.

Theorem 1.3 *The number of partitions of n with distinct even parts is odd if and only if n is a triangular number.*

In this paper, we aim at introducing a partition statistic which we call ped-crank to study the partition function $ped(n)$. Let $M_{ped}(m,n)$ denote the number of partitions of n with even parts distinct with ped-crank m, and let

$$M_{ped}(m,t,n) = \sum_{k\equiv m \pmod t} M_{ped}(k,n). \tag{1.5}$$

The main results of this paper are summarized below.

Theorem 1.4 *For* $\alpha, n \geq 0$,

$$M_{ped}\left(0, 6, 3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right) = M_{ped}\left(3, 6, 3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right),$$
$$M_{ped}\left(1, 6, 3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right) = M_{ped}\left(2, 6, 3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right),$$
$$M_{ped}\left(0, 6, 3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right) = M_{ped}\left(3, 6, 3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right),$$
$$M_{ped}\left(1, 6, 3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right) = M_{ped}\left(2, 6, 3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right).$$

Any of the following three corollaries deduced from Theorem 1.4 provides a combinatorial interpretation or a refinement of Theorem 1.1. When $\alpha = 0$, combining Corollary 1.5 and Corollary 1.6 refines (1.3). Meanwhile, Corollary 1.7 combinatorially interprets (1.3).

Corollary 1.5 *For* $m = 0, 1, 2$ *and* $\alpha, n \geq 0$,

$$M_{ped}\left(m, 6, 3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right) = \frac{M_{ped}\left(m, 3, 3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right)}{2},$$
$$M_{ped}\left(m, 6, 3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right) = \frac{M_{ped}\left(m, 3, 3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right)}{2}.$$

Corollary 1.6 *For* $\alpha, n \geq 0$,

$$M_{ped}\left(3, 12, 3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right) = \frac{M_{ped}\left(0, 3, 3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right)}{4},$$
$$M_{ped}\left(3, 12, 3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right) = \frac{M_{ped}\left(0, 3, 3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right)}{4}.$$

Corollary 1.7 *For* $\alpha, n \geq 0$,

$$M_{ped}\left(1, 4, 3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right) = \frac{ped\left(3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right)}{4},$$
$$M_{ped}\left(1, 4, 3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right) = \frac{ped\left(3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right)}{4}.$$

Theorem 1.8 *If* n *cannot be written as a sum of a triangular number and a square of even integer, we have*

$$M_{ped}(0, 4, n) = M_{ped}(2, 4, n).$$

Moreover, $M_{ped}(0, n)$ *is odd if and only if* n *is a triangular number.*

Theorem 1.9 *For* $m = 0, 1, 2, 3, 4, 5$ *and* $n \geq 0$*,*

$$M_{ped}(m, 6, 9n+7) = \frac{ped(9n+7)}{6},$$
$$M_{ped}(3, 12, 9n+7) = \frac{ped(9n+7)}{12}. \tag{1.6}$$

It is worth mentioning that Theorem 1.8 not only combinatorially interprets but also refines Theorem 1.3, and (1.6) provides a combinatorial interpretation for (1.4).

2 Definition of the *ped*-crank

In this section, we shall define the *ped*-crank of partitions with even parts distinct based on Glaisher's bijection and a modified version φ of the Wright map established by Seo and Yee [10].

We first give a quick overview of Glaisher's bijection and the Frobenius symbol. Let D_n denote the set of distinct partitions, and let O_n denote the set of odd partitions of n respectively. Glaisher's bijection ϕ: $O_n \to D_n$ is defined as follows. Let $\lambda = (1^{m_1}3^{m_3}\ldots) \in O_n$ be an odd partition. For every odd i, let $\phi(\lambda)$ contain part $i \cdot 2^r$, if and only if the integer m_i written in binary has 1 at the r-th position. In the other direction, let ψ: $D_n \to O_n$ be defined by an iterative procedure. Start with $\lambda = (\lambda_1, \lambda_2, \ldots) \in D_n$. Substitute every even part λ_i with two parts $\lambda_i/2$. Repeat until the resulting partition has no even parts.

The Frobenius symbol of n is a two-rowed array [1, 14]

$$F = \begin{pmatrix} \alpha_1 & \alpha_2 \cdots \alpha_\ell \\ \beta_1 & \beta_2 \cdots \beta_\ell \end{pmatrix},$$

where $\alpha_1 > \alpha_2 > \ldots > \alpha_\ell \geq 0$, $\beta_1 > \beta_2 > \ldots > \beta_\ell \geq 0$ and $n = |\alpha| + |\beta| + \ell$. If we express an ordinary partition by Ferrers graph, it is easy to see that α_i form rows to the right of the diagonal and β_i form columns below the diagonal. Thus the Frobenius symbol is another representation of an ordinary partition. For instance, the Frobenius symbol for (8, 7, 4, 3, 1) is

$$\begin{pmatrix} 7 & 5 & 1 \\ 4 & 2 & 1 \end{pmatrix}.$$

Giving a real number c, we define $c\lambda$ as the partition whose parts are c times each part of λ. For example, let $\lambda = (4, 2, 2)$. We have $4\lambda = (16, 8, 8)$ and $\frac{1}{2}\lambda = (2, 1, 1)$. Suppose μ and ν are two partitions. Let $\mu \cup \nu$ denote the partition consisting of all the parts of μ and ν. The definition of *ped*-crank is given based on the following theorem.

Theorem 2.1 *For integer* $k_1 \geq -1$*,* $k_2 \geq 1$*, there is a bijection* Δ *between the set of partitions of* n *with even parts distinct and the set of vector partitions* (α, β, γ) *with*

$|\alpha|+|\beta|+|\gamma|$ *equal to* n. *Here* α *is an even partition,* β *is a partition of the form* $(4k_1+1,\ldots,9,5,1)$ *or* $(4k_2-1,\ldots,11,7,3)$ *and* γ *is a distinct even partition.*

Proof The bijection Δ can be decomposed into six weight preserving steps.

Step 1. $\lambda \to (\omega,\gamma)$: Start with a partition λ of n with even parts distinct. Split λ into a pair of partitions (ω,γ) according to the parts odd or even. It is clear that ω is an odd partition and γ is a distinct even partition.

Step 2. $(\omega,\gamma) \to (\xi,\gamma)$: By Glaisher's bijection, let $\phi(\omega)=\xi$. One can see that ξ is a distinct partition.

Step 3. $(\xi,\gamma) \to (\mu^1,\mu^2,\pi,\gamma)$: Divide ξ into a triple of partitions (μ^1,μ^2,π) according to the remainder of the parts mod 4. Here μ^1 (μ^2) consist of all the parts congruent to $1(3)$ mod 4 and π consists of all the even parts of ξ.

Step 4. $(\mu^1,\mu^2,\pi,\gamma) \to (\mu^1,\mu^2,\zeta,\gamma)$: Let $\zeta = 2\psi(\frac{1}{2}\pi)$ by applying Glaisher's bijection. Since $\frac{1}{2}\pi$ is a distinct partition, we can say that ζ is a partition with all parts congruent to 2 mod 4.

Step 5. $(\mu^1,\mu^2,\zeta,\gamma) \to (\eta,\beta,\zeta,\gamma)$: Write μ^1 and μ^2 as

$$\mu^1 = (4a_1+1, 4a_2+1, \ldots, 4a_{s+m}+1),$$
$$\mu^2 = (4b_1+3, 4b_2+3, \ldots, 4b_s+3),$$

where $a_1 > a_2 > \cdots > a_{s+m} \geq 0$ and $b_1 > b_2 > \cdots > b_s \geq 0$.

Case 1. $m \geq 0$. Using the bijection φ established by Seo and Yee [10], a Frobenius symbol

$$\mu = \begin{pmatrix} a_{1+m} & a_{2+m} & \cdots & a_{s+m} \\ b_1 & b_2 & \cdots & b_s \end{pmatrix}$$

and a partition $\nu = (a_1-m+1, a_2-m+2, \ldots, a_m)$ can be constructed. Let $\varphi(\mu^1,\mu^2)=(\eta,\beta)$, where $\eta = 4(\mu\cup\nu)$ and $\beta = (4(m-1)+1, 4(m-2)+1, \ldots, 5, 1)$.

Case 2. $m < 0$. Correspondingly, a Frobenius symbol

$$\mu = \begin{pmatrix} b_{1-m} & b_{2-m} & \cdots & b_s \\ a_1 & a_2 & \cdots & a_{s+m} \end{pmatrix}$$

and a partition $\nu = (b_1+m+1, b_2+m+2, \ldots, b_{-m})$ can be constructed. Let $\varphi(\mu^1,\mu^2)=(\eta,\beta)$, where $\eta = 4(\mu\cup\nu)'$ and $\beta = (4(-m-1)+3, 4(-m-2)+3, \ldots, 7, 3)$.

Step 6. $(\eta,\beta,\zeta,\gamma) \to (\alpha,\beta,\gamma)$: Ultimately, let $\alpha = \eta\cup\zeta$ and define $\Delta(\lambda) = (\alpha,\beta,\gamma)$.

Furthermore, one sees that the above construction can be reversed. This completes the proof. □

An example of the bijection Δ is given below.

Example 2.2

$$
\begin{gathered}
\lambda = (30, 25, 18, 13, 11, 9, 6, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3, 3, 1, 1, 1) \\
\Updownarrow \text{Step 1.} \\
(\omega, \gamma) = \big((25, 13, 11, 9, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3, 3, 1, 1, 1), (30, 18, 6)\big) \\
\Updownarrow \text{Step 2.} \\
(\xi, \gamma) = \big((25, 20, 13, 12, 11, 9, 6, 3, 2, 1), (30, 18, 6)\big) \\
\Updownarrow \text{Step 3.} \\
(\mu^1, \mu^2, \pi, \gamma) = \big((25, 13, 9, 1), (11, 3), (20, 12, 6, 2), (30, 18, 6)\big) \\
\Updownarrow \text{Step 4.} \\
(\mu^1, \mu^2, \zeta, \gamma) = \big((25, 13, 9, 1), (11, 3), (10, 10, 6, 6, 6, 2), (30, 18, 6)\big) \\
\Updownarrow \text{Step 5.} \\
(\eta, \beta, \zeta, \gamma) = \big((20, 12, 12, 8, 4), (5, 1), (10, 10, 6, 6, 6, 2), (30, 18, 6)\big) \\
\Updownarrow \text{Step 6.} \\
(\alpha, \beta, \gamma) = \big((20, 12, 12, 10, 10, 8, 6, 6, 6, 4, 2), (5, 1), (30, 18, 6)\big)
\end{gathered}
$$

Now we are ready to give the definition of the ped-crank of a partition with even parts distinct under the bijection Δ.

Definition 2.3 Let λ be a partition with even parts distinct and $\Delta(\lambda) = (\alpha, \beta, \gamma)$. The ped-crank of λ, denoted by $c_{ped}(\lambda)$, is defined as the crank of $\frac{1}{2}\alpha$.

3 Generating function of $M_{ped}(m, n)$

This section focuses on the generating function of $M_{ped}(m, n)$.

According to the bijection Δ, the generating function of β can be derived by using Jacobi's triple product identity.

$$
\sum_{n=0}^{\infty} q^{4\binom{n}{2}+n} + \sum_{n=-\infty}^{-1} q^{4\binom{n}{2}+n} = (-q; q^4)_\infty (-q^3; q^4)_\infty (q^4; q^4)_\infty.
$$

Moreover, it is trivial that the generating function of γ is

$$
(-q^2; q^2)_\infty.
$$

Since the ped-crank only relies on the even partition α, by (1.2), the generating function of $M_{ped}(m, n)$ can be given as

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M_{ped}(m,n)z^m q^n$$
$$= \frac{(q^2;q^2)_\infty}{(zq^2;q^2)_\infty(z^{-1}q^2;q^2)_\infty} \cdot (-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty \cdot (-q^2;q^2)_\infty.$$
$$= \frac{(q^2;q^2)_\infty}{(zq^2;q^2)_\infty(z^{-1}q^2;q^2)_\infty}\frac{(q^4;q^4)_\infty}{(q;q^2)_\infty} \tag{3.1}$$

By considering the transformation that interchanges z and z^{-1} in (3.1), we have

$$M_{ped}(m,n) = M_{ped}(-m,n).$$

Thus, for any positive integer t,

$$M_{ped}(m,t,n) = M_{ped}(-m,t,n).$$

In other words,

$$M_{ped}(m,t,n) = M_{ped}(t-m,t,n). \tag{3.2}$$

4 Preliminaries

In this section we present some results that will be used in Section 5.

Lemma 4.1

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}.$$

Proof Replacing q by $-q$ in $(q;q)_\infty$, we have

$$\begin{aligned}(-q;-q)_\infty &= (-q;q^2)_\infty(q^2;q^2)_\infty\\ &= \frac{(-q;q)_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty}\\ &= \frac{(q;q)_\infty(-q;q)_\infty(q^2;q^2)_\infty^2}{(q;q)_\infty(-q^2;q^2)_\infty(q^2;q^2)_\infty}\\ &= \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty(q^4;q^4)_\infty}.\end{aligned}$$

□

Lemma 4.2 ([4, Entry 22, p. 36])

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}.$$

Replacing q by $-q$ in the above equation, we get

$$\phi(-q) = 2\sum_{n=0}^{\infty}(-1)^n q^{n^2} - 1 = \frac{f_1^2}{f_2}.$$

Lemma 4.3 ([4, p. 49])

$$\psi(q) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}.$$

Theorem 4.4 ([5, Lemma 2.2])

$$\frac{1}{\varphi(-q)} = \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)}\left(1 + 2q\omega(q^3) + 4q^2\omega^2(q^3)\right),$$
$$\frac{1}{\psi(q)} = \frac{\psi^3(q^9)}{\psi^4(q^3)}\left(\frac{1}{\omega^2(q^3)} - q\frac{1}{\omega(q^3)} + q^2\right),$$

where

$$\omega(q) = \frac{f_1 f_6^3}{f_2 f_3^3}.$$

Lemma 4.5 ([13, Lemma 2.5])

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}.$$

Lemma 4.6 ([8, p. 5])

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}.$$

Lemma 4.7 ([3, Theorem 3.1])

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}.$$

The following two theorems are crucial for establishing combinatorial interpretations.

Theorem 4.8 *For any fixed n, if*

$$M_{ped}(0, 2, n) = M_{ped}(1, 2, n), \tag{4.1}$$
$$M_{ped}(0, 3, n) = M_{ped}(1, 3, n), \tag{4.2}$$
$$M_{ped}(0, 6, n) + M_{ped}(1, 6, n) = M_{ped}(2, 6, n) + M_{ped}(3, 6, n), \tag{4.3}$$

then

$$M_{ped}(m, 6, n) = \frac{ped(n)}{6}, \quad m = 0, 1, 2, 3, 4, 5, \quad \text{and} \quad M_{ped}(3, 12, n) = \frac{ped(n)}{12}.$$

Proof By (3.2), we have

$$M_{ped}(0, 2, n) = M_{ped}(0, 6, n) + 2M_{ped}(2, 6, n), \tag{4.4}$$
$$M_{ped}(1, 2, n) = 2M_{ped}(1, 6, n) + M_{ped}(3, 6, n), \tag{4.5}$$
$$M_{ped}(0, 3, n) = M_{ped}(0, 6, n) + M_{ped}(3, 6, n), \tag{4.6}$$
$$M_{ped}(1, 3, n) = M_{ped}(1, 6, n) + M_{ped}(2, 6, n). \tag{4.7}$$

Substituting (4.4)–(4.7) into (4.1)–(4.2), we have

$$M_{ped}(0, 6, n) - 2M_{ped}(1, 6, n)+2M_{ped}(2, 6, n) - M_{ped}(3, 6, n) = 0, \tag{4.8}$$
$$M_{ped}(0, 6, n) - M_{ped}(1, 6, n)-M_{ped}(2, 6, n) + M_{ped}(3, 6, n) = 0. \tag{4.9}$$

Solving system of linear homogeneous equations (4.3), (4.8) and (4.9), we get

$$M_{ped}(m, 6, n) = \frac{ped(n)}{6}, \quad m = 0, 1, 2, 3, 4, 5.$$

Moreover,

$$M_{ped}(3, 6, n) = M_{ped}(3, 12, n) + M_{ped}(9, 12, n) = 2M_{ped}(3, 12, n), \tag{4.10}$$

which implies

$$M_{ped}(3, 12, n) = \frac{M_{ped}(3, 6, n)}{2} = \frac{ped(n)}{12}.$$

This completes the proof. □

The following theorem can be checked similarly.

Theorem 4.9 *For any fixed n, if*

$$M_{ped}(0, 2, n) = M_{ped}(1, 2, n),$$
$$M_{ped}(0, 6, n) + M_{ped}(1, 6, n) = M_{ped}(2, 6, n) + M_{ped}(3, 6, n),$$

then

$$M_{ped}(0,6,n) = M_{ped}(3,6,n),$$
$$M_{ped}(1,6,n) = M_{ped}(2,6,n).$$

5 Proofs of main results

In this section, we give proofs of our main results. Hereafter we always assume $\alpha, n \geq 0$ unless specified otherwise.

Proof of Theorem 1.4. Setting $z = e^{\pi i} = -1$ in (3.1), we get

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M_{ped}(m,n)(-1)^m q^n = \sum_{n=0}^{\infty}\left(M_{ped}(0,2,n) - M_{ped}(1,2,n)\right) q^n = \frac{f_2^4}{f_1 f_4}. \tag{5.1}$$

Let $g(q)$ be a polynomial of q, observing that the coefficient of q^n in $g(q)$ is zero implies the coefficient of q^n in $g(-q)$ is zero and vice versa. Hence we consider the following equation. By Lemma 4.1, replacing q by $-q$ in (5.1), we have

$$\sum_{n=0}^{\infty}\left(M_{ped}(0,2,n) - M_{ped}(1,2,n)\right)(-q)^n = f_1 f_2. \tag{5.2}$$

According to Lemma 4.6, we find that

$$\sum_{n=0}^{\infty}\left(M_{ped}(0,2,n) - M_{ped}(1,2,n)\right)(-q)^n = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{5.3}$$

Extracting those terms associated with powers q^{3n+1} on both sides of (5.3), then dividing by q and replacing q^3 by q, we arrive at

$$\sum_{n=0}^{\infty}\left(M_{ped}(0,2,3n+1) - M_{ped}(1,2,3n+1)\right)(-1)^{3n+1} q^n = -f_3 f_6. \tag{5.4}$$

Since the coefficients of q^{3n+1} and q^{3n+2} in (5.4) are both zero, we can conclude that the coefficients of q^{9n+4} and q^{9n+7} in (5.2) are both zero. This yields

$$M_{ped}(0,2,9n+4) = M_{ped}(1,2,9n+4), \tag{5.5}$$
$$M_{ped}(0,2,9n+7) = M_{ped}(1,2,9n+7). \tag{5.6}$$

Extracting the terms involving q^{3n} in (5.4) and substituting q^3 by q gives

$$\sum_{n=0}^{\infty}\big(M_{ped}(0,2,9n+1)-M_{ped}(1,2,9n+1)\big)(-1)^{9n+1}q^n=-f_1f_2. \tag{5.7}$$

Since $9n$ has the same parity as n, (5.7) becomes

$$\sum_{n=0}^{\infty}\big(M_{ped}(0,2,9n+1)-M_{ped}(1,2,9n+1)\big)(-q)^n=f_1f_2. \tag{5.8}$$

From (5.2), (5.8) and mathematical induction, it follows that

$$\sum_{n=0}^{\infty}\left(M_{ped}\left(0,2,3^{2\alpha}n+\frac{3^{2\alpha}-1}{8}\right)-M_{ped}\left(1,2,3^{2\alpha}n+\frac{3^{2\alpha}-1}{8}\right)\right)(-q)^n=f_1f_2. \tag{5.9}$$

Comparing (5.9) with (5.2), the following equations can be proved by similar arguments for (5.5)–(5.6), and hence the proof is omitted.

$$M_{ped}\left(0,2,3^{2\alpha+2}n+\frac{11\cdot 3^{2\alpha+1}-1}{8}\right)=M_{ped}\left(1,2,3^{2\alpha+2}n+\frac{11\cdot 3^{2\alpha+1}-1}{8}\right), \tag{5.10}$$

$$M_{ped}\left(0,2,3^{2\alpha+2}n+\frac{19\cdot 3^{2\alpha+1}-1}{8}\right)=M_{ped}\left(1,2,3^{2\alpha+2}n+\frac{19\cdot 3^{2\alpha+1}-1}{8}\right). \tag{5.11}$$

Substituting $z=e^{\frac{\pi i}{3}}$ into (3.1), by (3.2) and $e^{\frac{\pi i}{3}}+e^{\frac{5\pi i}{3}}=-(e^{\frac{2\pi i}{3}}+e^{\frac{4\pi i}{3}})=-e^{\pi i}=1$, we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{5}M_{ped}(m,6,n)e^{\frac{m\pi i}{3}}q^n\\
&=\sum_{n=0}^{\infty}\big(M_{ped}(0,6,n)n+(e^{\frac{\pi i}{3}}+e^{\frac{5\pi i}{3}})M_{ped}(1,6,n)n\\
&\quad+(e^{\frac{2\pi i}{3}}+e^{\frac{4\pi i}{3}})M_{ped}(2,6,n)+e^{\pi i}M_{ped}(3,6,n)\big)q^nn\\
&=\sum_{n=0}^{\infty}\big(M_{ped}(0,6,n)+M_{ped}(1,6,n)n-M_{ped}(2,6,n)-M_{ped}(3,6,n)\big)q^nn\\
&=\frac{f_2^2f_4}{f_1}\prod_{n=0}^{\infty}\frac{1}{1-q^{2n}+q^{4n}}n
\end{aligned}$$

$$= \frac{f_2^2 f_4}{f_1} \prod_{n=0}^{\infty} \frac{1+q^{2n}}{1+q^{6n}} n$$
$$= \frac{f_2^2}{f_1} \frac{f_4^2}{f_2} \frac{f_6}{f_{12}}.$$

By Lemma 4.3, we have

$$\sum_{n=0}^{\infty} \big(M_{ped}(0,6,n) + M_{ped}(1,6,n) - M_{ped}(2,6,n) - M_{ped}(3,6,n)\big) q^n$$
$$= \left(\frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9}\right)\left(\frac{f_{12} f_{18}^2}{f_6 f_{36}} + q^2 \frac{f_{36}^2}{f_{18}}\right) \frac{f_6}{f_{12}}. \tag{5.12}$$

Extracting the terms involving q^{3n+1} in (5.12), then dividing by q and replacing q^3 by q, we find that

$$\sum_{n=0}^{\infty} \big(M_{ped}(0,6,3n+1) + M_{ped}(1,6,3n+1) - M_{ped}(2,6,3n+1)$$
$$-M_{ped}(3,6,3n+1)\big) q^n = \frac{f_6^4}{f_3 f_{12}}. \tag{5.13}$$

Obviously, the coefficients of q^{3n+1} and q^{3n+2} in (5.13) are both zero, which gives

$$M_{ped}(0,6,9n+4) + M_{ped}(1,6,9n+4) = M_{ped}(2,6,9n+4) + M_{ped}(3,6,9n+4), \tag{5.14}$$
$$M_{ped}(0,6,9n+7) + M_{ped}(1,6,9n+7) = M_{ped}(2,6,9n+7) + M_{ped}(3,6,9n+7). \tag{5.15}$$

Considering the terms involving q^{3n} in (5.13), after simplification, we get

$$\sum_{n=0}^{\infty} \big(M_{ped}(0,6,9n+1) + M_{ped}(1,6,9n+1) - M_{ped}(2,6,9n+1)$$
$$-M_{ped}(3,6,9n+1)\big) q^n = \frac{f_2^4}{f_1 f_4}. \tag{5.16}$$

Comparing (5.16) with (5.1), according to (5.14)–(5.15) and the proofs of (5.10)–(5.11), a simple deduction shows that

$$M_{ped}\left(0,6,3^{2\alpha+2}n + \frac{11 \cdot 3^{2\alpha+1} - 1}{8}\right) + M_{ped}\left(1,6,3^{2\alpha+2}n + \frac{11 \cdot 3^{2\alpha+1} - 1}{8}\right)$$
$$= M_{ped}\left(2,6,3^{2\alpha+2}n + \frac{11 \cdot 3^{2\alpha+1} - 1}{8}\right) + M_{ped}\left(3,6,3^{2\alpha+2}n + \frac{11 \cdot 3^{2\alpha+1} - 1}{8}\right), \tag{5.17}$$

$$M_{ped}\left(0,6,3^{2\alpha+2}n+\frac{19\cdot 3^{2\alpha+1}-1}{8}\right)+M_{ped}\left(1,6,3^{2\alpha+2}n+\frac{19\cdot 3^{2\alpha+1}-1}{8}\right)$$
$$=M_{ped}\left(2,6,3^{2\alpha+2}n+\frac{19\cdot 3^{2\alpha+1}-1}{8}\right)+M_{ped}\left(3,6,3^{2\alpha+2}n+\frac{19\cdot 3^{2\alpha+1}-1}{8}\right). \tag{5.18}$$

Combining (5.10)–(5.11), (5.17)–(5.18) and Theorem 4.9, Theorem 1.4 follows immediately. □

Proof of Corollary 1.5. Corollary 1.5 can be checked easily by (4.6)–(4.7) and Theorem 1.4, hence we omitted the details. □

Proof of Corollary 1.6 By (4.6), (4.10) and Theorem 1.4, one can see that

$$M_{ped}\left(3,12,3^{2\alpha+2}n+\frac{11\cdot 3^{2\alpha+1}-1}{8}\right)=\frac{M_{ped}\left(3,6,3^{2\alpha+2}n+\frac{11\cdot 3^{2\alpha+1}-1}{8}\right)}{2}$$
$$=\frac{M_{ped}\left(0,3,3^{2\alpha+2}n+\frac{11\cdot 3^{2\alpha+1}-1}{8}\right)}{4},$$
$$M_{ped}\left(3,12,3^{2\alpha+2}n+\frac{19\cdot 3^{2\alpha+1}-1}{8}\right)=\frac{M_{ped}\left(3,6,3^{2\alpha+2}n+\frac{19\cdot 3^{2\alpha+1}-1}{8}\right)}{2}$$
$$=\frac{M_{ped}\left(0,3,3^{2\alpha+2}n+\frac{19\cdot 3^{2\alpha+1}-1}{8}\right)}{4}.$$

Hence Corollary 1.6 holds.

Proof of Corollary 1.7 By (3.2), we have

$$M_{ped}(1,2,n)=M_{ped}(1,4,n)+M_{ped}(3,4,n)=2M_{ped}(1,4,n). \tag{5.19}$$

Then Corollary 1.7 follows immediately according to (5.10)–(5.11) and the fact that $ped(n)=M_{ped}(0,2,n)+M_{ped}(1,2,n)$. □

Proof of Theorem 1.8 Substituting $z=e^{\frac{\pi i}{2}}=i$ into (3.1), by (3.2) and Lemmas 4.2–4.3, we find that

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}M_{ped}(m,n)i^m q^n=\sum_{n=0}^{\infty}\sum_{m=0}^{3}M_{ped}(m,4,n)i^m q^n$$
$$=\sum_{n=0}^{\infty}\Big(M_{ped}(0,4,n)+(i+i^3)M_{ped}(1,4,n)$$
$$+i^2M_{ped}(2,4,n)\Big)q^n$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \left(M_{ped}(0,4,n) - M_{ped}(2,4,n) \right) q^n \\
&= \frac{f_2^2}{f_1} \frac{f_4^2}{f_8} \\
&= \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} \left(2 \sum_{n=0}^{\infty} (-1)^n q^{4n^2} - 1 \right). \qquad (5.20)
\end{aligned}
$$

By (3.2), one can see that

$$
ped(n) = \sum_{m=0}^{3} M_{ped}(m,4,n) = M_{ped}(0,4,n) + 2M_{ped}(1,4,n) + 2M_{ped}(2,8,n).
$$

In light of (5.20) and the fact that $M_{ped}(0,n)$ has the same parity as $M_{ped}(0,4,n)$, Theorem 1.8 holds. □

We next aim to prove Theorem 1.9.

Proof of Theorem 1.9 Substituting $z = e^{\frac{2\pi i}{3}}$ into (3.1), we obtain

$$
\sum_{n=0}^{\infty} \sum_{m=0}^{2} M_{ped}(m,3,n) e^{\frac{2m\pi i}{3}} q^n = \frac{(q^2;q^2)_\infty}{(\zeta q^2;q^2)_\infty (\zeta^{-1} q^2;q^2)_\infty} \frac{(q^4;q^4)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1} \frac{f_2 f_4}{f_6}. \qquad (5.21)
$$

Using Lemmas 4.3, 4.6, by (3.2) and the fact that $1 + e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}} = 0$, we get

$$
\begin{aligned}
\sum_{n=0}^{\infty} (M_{ped}(0,3,n) - M_{ped}(1,3,n)) q^n &= \left(\frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9} \right) \\
&\left(\frac{f_{12} f_{18}^4}{f_6 f_{36}^2} - q^2 f_{18} f_{36} - 2q^4 \frac{f_6 f_{36}^4}{f_{12} f_{18}^2} \right) \frac{1}{f_6}.
\end{aligned}
$$

Extracting those terms associated with powers q^{3n+1} on both sides of the above equation, then dividing by q and replacing q^3 by q, one can see that

$$
\sum_{n=0}^{\infty} (M_{ped}(0,3,3n+1) - M_{ped}(1,3,3n+1)) q^n = \frac{f_4}{f_2^2} \frac{f_6^6}{f_3 f_{12}^2} - 2q \frac{f_2}{f_1 f_4} \frac{f_3^2 f_{12}^4}{f_6^3}. \qquad (5.22)
$$

Since

$$
\frac{f_2}{f_1 f_4} = \frac{1}{\psi(-q)},
$$

using Lemma 4.4,

$$\sum_{n=0}^{\infty}(M_{ped}(0,3,3n+1)-M_{ped}(1,3,3n+1))q^n$$
$$=\frac{f_{12}^2 f_{18}^6}{f_3 f_6^2 f_{36}^3}-2q^3\frac{f_6 f_9^3 f_{36}^3}{f_3^2 f_{18}^3}-2q\frac{f_{12}^2 f_{18}^9}{f_6^3 f_9^3 f_{36}^3}+4q^4\frac{f_{36}^3}{f_3} \tag{5.23}$$

after simplification.

Clearly, the coefficient of q^{3n+2} in (5.23) is zero. We can conclude that

$$M_{ped}(0,3,9n+7)=M_{ped}(1,3,9n+7). \tag{5.24}$$

Combining (5.11), (5.18), (5.24) and Theorem 4.8, we complete the proof of Theorem 1.9. □

Remark 1 Andrews, Hirschhorn, and Sellers [3] presented an interesting infinite family of congruences modulo 3 as given by

$$ped\left(3^{2\alpha+1}n+\frac{17\cdot 3^{2\alpha}-1}{8}\right)\equiv 0 \pmod 3,\quad \alpha\geq 1, \tag{5.25}$$

and deduced that

$$ped\left(3^{2\alpha+1}n+\frac{17\cdot 3^{2\alpha}-1}{8}\right)\equiv 0 \pmod 6,\quad \alpha\geq 1. \tag{5.26}$$

Actually, based on a substantial amount of numerical evidence, we conjecture that the *ped*-crank can be used to provide a combinatorial interpretation of (5.25), namely

$$M_{ped}\left(0,3,3^{2\alpha+1}n+\frac{17\cdot 3^{2\alpha}-1}{8}\right)=M_{ped}\left(1,3,3^{2\alpha+1}n+\frac{17\cdot 3^{2\alpha}-1}{8}\right),$$
$$\alpha\geq 1.$$

Here, we only prove the case for $\alpha=1$, and for any $\alpha>1$, we are not able to provide an elementary proof of this conjecture.

Proof Extracting the terms involving q^{3n} in (5.23) and substituting q^3 by q, we obtain

$$\sum_{n=0}^{\infty}(M_{ped}(0,3,9n+1)-M_{ped}(1,3,9n+1))q^n=\frac{f_4}{f_1}\frac{f_4}{f_2^2}\frac{f_6^6}{f_{12}^3}-2q\frac{f_2 f_3^3 f_{12}^3}{f_1^2 f_6^3}. \tag{5.27}$$

From Lemmas 4.4, 4.5, 4.7, considering the terms involving q^{3n+2} in (5.27) leads to

$$\sum_{n=0}^{\infty}(M_{ped}(0,3,27n+19)-M_{ped}(1,3,27n+19))q^n$$

$$\begin{aligned}
&= \frac{f_4 f_6^7}{f_1^3 f_2 f_{12}^2} - \frac{f_3^3 f_4^3}{f_1^4} + q\frac{f_2^2 f_3^3 f_{12}^4}{f_1^4 f_4 f_6^2} \\
&= \frac{f_3^3}{f_1^4}\left(\frac{f_1}{f_3^3}\frac{f_4 f_6^7}{f_2 f_{12}^2} - f_4^3 + q\frac{f_2^2 f_{12}^4}{f_4 f_6^2}\right) \\
&= \frac{f_3^3}{f_1^4}\left(\left(\frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}\right)\frac{f_4 f_6^7}{f_2 f_{12}^2} - f_4^3 + q\frac{f_2^2 f_{12}^4}{f_4 f_6^2}\right) \\
&= 0.
\end{aligned}$$

That means

$$M_{ped}(0, 3, 27n + 19) = M_{ped}(1, 3, 27n + 19).$$

□

Unfortunately, the *ped*-crank cannot be employed to interpret (5.26) even for $\alpha = 1$. Hence, it will be interesting to introduce another partition statistic that could combinatorially interpret (5.26).

Remark 2 For ordinary partitions, recall that we define $M(0, 1) = -1$, $M(-1, 1) = M(1, 1) = 1$. From the definition of *ped*-crank, one can see that a similar problem will arise when $\Delta(\lambda) = ((2), \beta, \gamma)$. So we make the following adjustment to the definition of *ped*-crank. Let

$$\begin{aligned}
\gamma &= (\gamma_1, \gamma_2, \cdots, \gamma_k), \\
A_n &= \{\lambda | \Delta(\lambda) = ((2), \beta, \gamma),\ \ |\lambda| = n\}, \\
B_n &= \{\lambda | \Delta(\lambda) = ((0), \beta, \gamma),\ \ |\lambda| = n,\ \ \gamma = (2) \text{ or } \gamma_1 - \gamma_2 \geq 4\}.
\end{aligned}$$

Definition 5.1 Let λ be a partition of n with even parts distinct. The $c_{mped}(\lambda)$ is given by

$$c_{mped}(\lambda) = \begin{cases} 1 & if\ \ \lambda \in A_n, \\ -1 & if\ \ \lambda \in B_n, \\ c_{ped}(\lambda) & otherwise, \end{cases}$$

where $c_{ped}(\lambda)$ is the *ped*-crank of λ.

When $\lambda \in A_n$, an injection from A_n to B_n can be constructed by changing α to $\emptyset$ and adding 2 to the largest part of γ. Another direction is obvious. Hence for any non-negative integer n, there is a bijection between A_n and B_n. Let $M_{mped}(m, n)$ denote the number of partitions of n with even parts distinct with $c_{mped}(\lambda) = m$. By the definition of $c_{mped}(\lambda)$, one can check $M_{mped}(m, n) = M_{ped}(m, n)$ for any integer m and non-negative integer n.

For example, if $\lambda = (6, 5, 4, 1, 1, 1)$, $\Delta(\lambda) = ((2), (5, 1), (6, 4))$, then $c_{mped}(\lambda) = 1$ and if $\lambda = (8, 5, 4, 1)$, $\Delta(\lambda) = ((0), (5, 1), (8, 4))$, then $c_{mped}(\lambda) = -1$.

Table 1 The case for $n = 7$

λ	$\Delta(\lambda) = (\alpha, \beta, \gamma)$	$\frac{1}{2}\alpha$	$c_{mped}(\lambda)$
(7)	$((4),\ (3),\ \emptyset)$	(2)	2
$(6, 1) \in B$	$(\emptyset,\ (1),\ (6))$	$\emptyset$	-1
$(5, 2)$	$((4),\ (1),\ (2))$	(2)	2
$(5, 1, 1)$	$((4, 2),\ (1),\ \emptyset)$	$(2,1)$	0
$(4, 3) \in B$	$(\emptyset,\ (3),\ (4))$	$\emptyset$	-1
$(4, 2, 1)$	$(\emptyset,\ (1),\ (4, 2))$	$\emptyset$	0
$(4, 1, 1, 1) \in A$	$((2),\ (1),\ (4))$	(1)	1
$(3, 3, 1)$	$((6),\ (1),\ \emptyset)$	(3)	3
$(3, 2, 1, 1) \in A$	$((2),\ (3),\ (2))$	(1)	1
$(3, 1, 1, 1, 1)$	$((2, 2),\ (3),\ \emptyset)$	$(1, 1)$	-2
$(2, 1, 1, 1, 1, 1)$	$((2, 2),\ (1),\ (2))$	$(1, 1)$	-2
$(1, 1, 1, 1, 1, 1, 1)$	$((2, 2, 2),\ (1),\ \emptyset)$	$(1, 1, 1)$	-3

Table 1 gives the 12 partitions of 7 with even parts distinct. It is easy to check that these partitions are divided into six equinumerous subsets by *ped*-crank. Moreover,

$$M_{ped}(1, 4, 7) = 3 = \frac{ped(7)}{4},$$
$$M_{ped}(3, 12, 7) = 1 = \frac{ped(7)}{12}.$$

6 Closing remarks

In 2014, Xia [12] proved the following congruence modulo 4 for $ped(n)$.

Theorem 6.1 [12, Equation (9), Theorem 1] *For* $\alpha, n \geq 0$,

$$ped\left(3^{2\alpha}n + \frac{3^{2\alpha} - 1}{8}\right) \equiv ped(n) \pmod 4.$$

Note that comparing (5.4) with (5.13), a simple deduction gives

$$M_{ped}(1, 6, 3n + 1) = M_{ped}(2, 6, 3n + 1).$$

Thus

$$ped(3n + 1) = M_{ped}(0, 3, 3n + 1) + 4M_{ped}(1, 6, 3n + 1).$$

Since for all $\alpha > 0$, $n \geq 0$, $3^{2\alpha}n + \frac{3^{2\alpha}-1}{8} \equiv 1 \pmod 3$, a refinement of Theorem 6.1 can be given as

$$M_{ped}\left(0, 3, 3^{2\alpha}n + \frac{3^{2\alpha}-1}{8}\right) \equiv ped(n) \pmod 4.$$

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