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A generalization of the Roberts orthogonality: from normed linear spaces to C^*-algebras

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Abstract

We introduce and study an extension of the Roberts orthogonality, in the setting of C^*-algebras. More precisely, in a C^*-algebra $\mathscr{A}$, for $a, b \in \mathscr{A}$ and a nonempty subset of $\mathscr{A}$, say $\mathscr{B}$, a is called $\mathscr{B}$-Roberts orthogonal to b, denoted by $a \perp_R^{\mathscr{B}} b$, if $\|a + bc\| = \|a - bc\|$ for all $c \in \mathscr{B}$. For certain special C^*-algebras, including the C^*-algebra of all 2×2 complex matrices, we obtain a nontrivial subset $\mathscr{B}$ such that the $\mathscr{B}$-Roberts orthogonality coincides with the usual orthogonality. We also introduce a new concept of smoothness in normed linear spaces in terms of the additivity property of the usual Roberts orthogonality. A complete characterization of the same is obtained in the case of the classical ℓ_∞^n spaces.

Keywords Roberts orthogonality · Inner product orthogonality · Additivity · C^*-algebra

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1 Introduction

It is interesting to extend the usual orthogonality in inner product spaces to the other realms. Roberts [13] was one of the earliest mathematicians who generalized the geometric concept of orthogonality to the setting of real normed linear spaces. In a complex normed linear space $(\mathbb{X}, \|\cdot\|)$, a vector $x \in \mathbb{X}$ is called *Roberts orthogonal* to a vector $y \in \mathbb{X}$, written as $x \perp_R y$, if $\|x + \lambda y\| = \|x - \lambda y\|$ for all $\lambda \in \mathbb{C}$; see also the survey article [6]. It is elementary to note that the Roberts orthogonality is both symmetric (i.e., $x \perp_R y \Rightarrow y \perp_R x$) and homogeneous (i.e., $x \perp_R y \Rightarrow \alpha x \perp_R \beta y$ for all $\alpha, \beta \in \mathbb{C}$). The notion of approximate Roberts orthogonality was introduced and studied in [17, 18].

A vector $x \in \mathbb{X}$ is called *Birkhoff–James orthogonal* to a vector $y \in \mathbb{X}$, written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for each $\lambda \in \mathbb{C}$; see [7, 10]. The Birkhoff–James orthogonality is homogeneous, but in contrast to the Roberts orthogonality, it is not symmetric.

The Roberts orthogonality implies the Birkhoff–James orthogonality through the following chain of (in)equalities:

$$2\|x\| = \|2x + \lambda y - \lambda y\| \leq \|x + \lambda y\| + \|x - \lambda y\| = 2\|x + \lambda y\| \qquad (\lambda \in \mathbb{C}).$$

If the norm on $\mathbb{X}$ is induced by an inner product $\langle \cdot, \cdot \rangle$, then it is easy to verify that x and y are both Roberts and Birkhoff–James orthogonal if and only if $\langle x, y \rangle = 0$.

By virtue of the Hahn–Banach theorem, $x \perp_B y$ if and only if there is a norm one linear functional f on $\mathbb{X}$ such that $f(x) = \|x\|$ and $f(y) = 0$. By $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ we denote the unit sphere of a normed linear space $\mathbb{X}$.

Let us denote by $\mathbb{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators acting on a Hilbert space $\mathcal{H}$. The identity operator on a space is denoted by I, and an operator $A \in \mathbb{B}(\mathcal{H})$ is called a *similarity* when A is a scalar multiple of I, that is, $A = \lambda I$ for some $\lambda \in \mathbb{C}$. Also, we say that A is a *positive similarity* when $A = \lambda I$ for some $\lambda > 0$. For $A \in \mathbb{B}(\mathcal{H})$, the unique positive square root of A^*A is denoted by $|A|$. When $\mathcal{H}$ is of a finite dimension n, we identify $\mathbb{B}(\mathcal{H})$ with the algebra $\mathbb{M}_n(\mathbb{C})$ of all $n \times n$ matrices with complex entries. Let $\mathbb{K}(\mathcal{H})$ denote the collection of all compact operators on a Hilbert space $\mathcal{H}$.

Let $s_1(A), s_2(A), \ldots, s_n(A)$ be the singular values of $A \in \mathbb{M}_n(\mathbb{C})$, arranged in decreasing order and counted with their multiplicities. For $p > 0$, the p-Schatten norm is defined by

$$\|A\|_p := \left(\sum_{i=1}^{n} s_i(A)^p\right)^{\frac{1}{p}} = (\mathrm{tr}|A|^p)^{\frac{1}{p}}.$$

The norm $\|\cdot\|_1$ is the trace-class norm. Several authors have explored the orthogonality with respect to the norm $\|\cdot\|_p$ (for more details, we refer the interested reader to [8, 11, 16]).
It is known that if $A \in \mathbb{M}_2(\mathbb{C})$, then the eigenvalues of A^*A are as follows:

$$\lambda_1 = \frac{\operatorname{tr}(A^*A) + \sqrt{\operatorname{tr}^2(A^*A) - 4|\det(A)|^2}}{2},$$
$$\lambda_2 = \frac{\operatorname{tr}(A^*A) - \sqrt{\operatorname{tr}^2(A^*A) - 4|\det(A)|^2}}{2}.$$

Hence, $\|A\|_1 = \sqrt{\lambda_1} + \sqrt{\lambda_2}$, which ensures that

$$\|A\|_1^2 = \operatorname{tr}(A^*A) + 2|\det(A)|. \tag{1.1}$$

In the context of C^*-algebras, the authors of [3] presented a characterization of the Roberts orthogonality in terms of the notion of Davis-Wielandt shell. In the setting of Hilbert C^*-modules, [2] and [9] serve as useful references for understanding the Birkhoff–James and Roberts orthogonalities. In this paper, one of our aims is to present a meaningful extension of the Roberts orthogonality in C^*-algebras and to derive some interesting relations on the structure of the space, in light of a newly introduced generalized version.

Arambašić and Rajić [1] introduced a new version of the Birkhoff–James orthogonality in the setting of C^*-algebras in the following way. An element $a \in \mathscr{A}$ is called *strong Birkhoff–James orthogonal* to an element $b \in \mathscr{A}$, in short $a \perp_B^s b$, if $\|a\| \leq \|a+bc\|$ for each $c \in \mathscr{A}$. The strong Birkhoff–James orthogonality implies the Birkhoff–James orthogonality. Furthermore, $a \perp_B^s b$ if and only if there exists a positive linear functional φ of norm one on $\mathscr{A}$ such that $\varphi(a^*a) = \|a\|^2$ and $\varphi(|a^*b|^2) = 0$; see [1, Theorem 2.5].
We say that two elements a and b in a C^*-algebra $\mathscr{A}$ are *orthogonal* if $a^*b = 0$, and in that case, we write $a \perp b$. Applying [4, Theorem 2], one gets the following characterization of the orthogonality in a C^*-algebra.

Theorem 1.1 [4, Theorem 2] *Let $\mathscr{A}$ be a C^*-algebra and let $a, b \in \mathscr{A}$. Then the following conditions are equivalent:*

(i) $\|a + bc\| = \|a - bc\|$ *for all* $c \in \mathscr{A}$*;*
(ii) $a \perp b$.

For a comprehensive study of the theory of C^*-algebras, we refer the reader to [12].

In this paper, we introduce a nontrivial subset $\mathscr{B}$ in some C^*-algebra such that even after restricting the element c to be in $\mathscr{B}$, the above theorem remains valid. In Sect. 2, we first present an extension of the Roberts orthogonality, to the setting of C^*-algebras. Then we describe several basic properties of such an orthogonality relation. In Sect. 3, we obtain some characterizations of the usual orthogonality in certain C^*-algebras. In the final section, we introduce the concept of R-smoothness in a normed linear space, in order to essentially study the additivity property of the usual Roberts orthogonality. We completely characterize the R-smoothness in ℓ_∞^n spaces.

2 Roberts orthogonality with respect to a subset

Let us begin this section with a definition.

Definition 2.1 Let $\mathscr{B}$ be a nonempty subset of a C^*-algebra $\mathscr{A}$ and let $a, b \in \mathscr{A}$. Then a is said to be *$\mathscr{B}$-Roberts orthogonal* to b, denoted by $a \perp_R^{\mathscr{B}} b$, if

$$\|a + bc\| = \|a - bc\| \qquad \text{forall } c \in \mathscr{B}.$$

The following proposition presents some useful properties of this relationship.

Proposition 2.2 *Let $\mathscr{A}$ be a C^*-algebra and let $\mathscr{B} \subseteq \mathscr{A}$. For every $a, b \in \mathscr{A}$, the following properties hold:*

(i) $a \perp_R^{\mathscr{B}} a$ if and only if $a = 0$, where $\mathscr{B}$ is a nonzero C^*-subalgebra of $\mathscr{A}$ and $a \in \mathscr{B}$;
(ii) The $\mathscr{B}$-Roberts orthogonality is homogeneous, where $\mathscr{B}$ is a subalgebra of $\mathscr{A}$;
(iii) $a \perp_R^{\mathscr{B}} b$ if and only if $a \perp_R^{\mathscr{B}} bc$ for each $c \in \mathscr{B}$, where $\mathscr{B}$ is a C^*-subalgebra of $\mathscr{A}$;
(iv) If $a \perp_R^{\mathscr{A}} b$, then $a \perp_B^{s} b$;
(v) If $\mathscr{B}$ and $\mathscr{C}$ are nonempty subsets of $\mathscr{A}$ such that $\mathscr{B} \subseteq \mathscr{C}$, then $a \perp_R^{\mathscr{C}} b$ implies $a \perp_R^{\mathscr{B}} b$.

Proof (i) Let $\mathscr{B}$ be a nonzero C^*-subalgebra of $\mathscr{A}$. Then it has an approximate unit (u_i). Hence,

$$2\|a\| = \lim_i \|a + au_i\| = \lim_i \|a - au_i\| = 0.$$

The other proofs are straightforward and so we omit them. □

It should be noted that the case (i) of Proposition 2.2 no longer holds true in general, for elements of $\mathscr{A}$. The following easy example illustrates this fact.

Example 2.3 Let p and q be nonzero projections in $\mathbb{B}(\mathcal{H})$ such that the range of q is a subset of the kernel of p. Let $\mathscr{B} = \mathbb{C}q$. Then it is immediate that $p \perp_R^{\mathscr{B}} p$ but $p \neq 0$.

The converse of the case (iv) of Proposition 2.2 also does not hold in general, as we will show in the next example.

Example 2.4 Consider the C^*-algebra $\mathbb{M}_2(\mathbb{C})$. If $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then for each $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$, we have

$$\begin{aligned} \|I + BC\| &= \sup_{|x|^2+|y|^2=1} \left\| \begin{bmatrix} x \\ c_3 x + (1 + c_4) y \end{bmatrix} \right\| \\ &= \sup_{|x|^2+|y|^2=1} \sqrt{|x|^2 + |c_3 x + (1 + c_4) y|^2} \geq \|I\|, \end{aligned}$$

where the last inequality is obtained by considering $x = 1$ and $y = 0$. Therefore, $I \perp_B^s B$. If we take $C = B$, then $\|I + BC\| = 2 \neq 1 = \|I - BC\|$, that is, $I \not\perp_R^{\mathbb{M}_2(\mathbb{C})} B$.

Remark 2.5 If $\mathscr{B}$ is a nonzero subset of a C^*-algebra $\mathscr{A}$, then clearly $\perp \subseteq \perp_R^{\mathscr{B}}$. Moreover, if $\mathscr{B}$ is a C^*-subalgebra of $\mathscr{A}$ and if $a, b \in \mathscr{B}$, then by employing the approximate unit (u_i) of $\mathscr{B}$, we obtain

$$\|a + \lambda b\| = \lim_i \|a + \lambda b u_i\| = \lim_i \|a - \lambda b u_i\| = \|a - \lambda b\|$$

for each $\lambda \in \mathbb{C}$, ensuring that $a \perp_R b$.

The next example shows that $a \perp_R b$ does not imply $a \perp_R^{\mathscr{B}} b$, in general. Let us fix our notation. Let $\mathbf{C}^b(\mathbb{X})$ be the normed linear space of all continuous bounded scalar-valued functions on a metric space $\mathbb{X}$, endowed with the usual supremum norm. For $f \in \mathbf{C}^b(\mathbb{X})$, the *support* $\operatorname{supp}(f)$ of f is the closure of the set $\{x \in \mathbb{X} : f(x) \neq 0\}$. The diameter of a set $\mathscr{S} \subseteq \mathbb{X}$ is denoted by $\operatorname{diam}(\mathscr{S})$.

Example 2.6 Let $\mathscr{C} = \{f \in \mathbf{C}([-1, 1]) : f = 0 \text{ on } [-1, 0]\}$. It is easy to see that $\mathscr{C}$ is an ideal of the C^*-algebra $\mathbf{C}([-1, 1])$.
Suppose $f, g \in \mathbf{C}([-1, 1])$ with $f(x) = x$ and $g(x) = |x|$. Then

$$\begin{aligned}\|f + \lambda g\|_\infty &= \sup\big(\{|x - \lambda x| : -1 \le x \le 0\} \cup \{|x + \lambda x| : 0 \le x \le 1\}\big)\\ &= \sup\big(\{|x + \lambda x| : -1 \le x \le 0\} \cup \{|x - \lambda x| : 0 \le x \le 1\}\big)\\ &= \|f - \lambda g\|_\infty.\end{aligned}$$

Hence, $f \perp_R g$.
Now, if $h(x) = \begin{cases} x, & 0 \le x \le 1, \\ 0, & -1 \le x \le 0, \end{cases}$ then $h \in \mathscr{C}$, and we have

$$\|f + gh\|_\infty = \sup\big(\{|x| : -1 \le x \le 0\} \cup \{|x + x^2| : 0 \le x \le 1\}\big) = 2$$

and

$$\|f - gh\|_\infty = \sup\big(\{|x| : -1 \le x \le 0\} \cup \{|x - x^2| : 0 \le x \le 1\}\big) = 1.$$

Thus, $f \not\perp_R^{\mathscr{C}} g$.

Furthermore, for some C^*-subalgebras $\mathscr{B}$, the inclusion $\perp_R^{\mathscr{B}} \subseteq \perp$ is generally not true. The next example illustrates this fact.

Example 2.7 Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Let $\mathscr{B}$ be the unital C^*-subalgebra of $\mathbb{M}_4(\mathbb{C})$ generated by A and B. Now, let $X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Obviously, $I \not\perp X$. However, for each $C \in \mathscr{B}$, we have

$$\begin{aligned} \|I + XC\| &= \left\| \begin{bmatrix} 1+\lambda & 0 & 0 & 0 \\ 0 & 1+\mu & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\mu \end{bmatrix} \right\| \\ &= \max\big\{|1+\lambda|, |1-\lambda|, |1+\mu|, |1-\mu|\big\} \\ &= \left\| \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\mu & 0 & 0 \\ 0 & 0 & 1+\lambda & 0 \\ 0 & 0 & 0 & 1+\mu \end{bmatrix} \right\| = \|I - XC\|. \end{aligned}$$

Therefore, $I \perp_R^{\mathscr{B}} X$.

3 Characterization of the usual orthogonality in some C^*-algebras

In this section, we present some characterizations of the usual orthogonality in certain special C^*-algebras. Let $\mathbb{X}$ be a metric space. For each positive integer n, we define the subset $\mathscr{B}_n$ of $\mathbf{C}^b(\mathbb{X})$ as follows:

$$\mathscr{B}_n = \left\{ f \in \mathbf{C}^b(\mathbb{X}) : \text{diam}\,(\text{supp}(f)) \leq \frac{1}{n} \right\}.$$

It is rather easy to observe that $\mathscr{B}_n$ is closed in $\mathbf{C}^b(\mathbb{X})$ for each n. Indeed, assume that $\{f_m\} \subset \mathscr{B}_n$ with $f_m \longrightarrow f_0 \in \mathbf{C}^b(\mathbb{X})$. We claim that $f_0 \in \mathscr{B}_n$. Consider any $x, y \in \mathbb{X}$ such that $f_0(x), f_0(y) \neq 0$. It is easy to deduce that there exists a sufficiently large $m_0 \in \mathbb{N}$ such that $f_m(x), f_m(y) \neq 0$ for all $m \geq m_0$. In particular, we obtain that $|f_m(x) - f_m(y)| \leq \frac{1}{n}$, whenever $m \geq m_0$. Since the diameter of a set equals the diameter of its closure, it is now clear that $f_0 \in \mathscr{B}_n$ establishes our claim.

It is also elementary to note that $\mathscr{B}_n$ is not dense in $\mathbf{C}^b(\mathbb{X})$, in general. For example, let $\mathbb{X} = \mathbb{R}$ with its usual metric. If $f \in \mathscr{B}_n$, then clearly $\mathbb{R} \setminus \text{supp}(f) \neq \emptyset$. We claim that the constant function 1 is not in $\overline{\mathscr{B}_n}$. Suppose on the contrary that there exists a sequence $\{f_m\}$ in $\mathscr{B}_n$ such that $f_m \longrightarrow 1$. Thus, for any $x \in \mathbb{R}$ and for sufficiently large m, we have

$$|f_m(x) - 1| \leq \|f_m - 1\|_\infty < \frac{1}{2}.$$

In particular, for $x \in \mathbb{R}\backslash\mathrm{supp}(f_m)$, we have $|f_m(x)-1| < \dfrac{1}{2}$, which is an obvious contradiction. Therefore, $\overline{\mathscr{B}_n} \subsetneq \mathbf{C}^b(\mathbb{R})$.

The next theorem shows that $\perp_R^{\mathscr{B}_n}$ and $\perp$ may coincide in a commutative C^*-algebra.

Theorem 3.1 *Let $\mathbb{X}$ be a metric space and let $f, g \in \mathbf{C}^b(\mathbb{X})$. Then $f \perp_R^{\mathscr{B}_n} g$ for some positive integer n if and only if $f \perp g$.*

Proof Let $f \perp g$. Combining Theorem 1.1 and the case (v) of Proposition 2.2, we conclude that $f \perp_R^{\mathscr{B}_n} g$ for all n.
Conversely, suppose that $f \perp_R^{\mathscr{B}_{n_0}} g$ for some positive integer n_0. We show that $f(x)g(x)=0$ for every $x \in \mathbb{X}$.
Let $x_0 \in \mathbb{X}$. Without loss of generality, we can assume that $g(x_0) \neq 0$. We prove that $f(x_0)=0$. Suppose that $\mathrm{Re}\,(f(x_0)) \geq 0$ (if $\mathrm{Re}\,(f(x_0)) \leq 0$, then we replace f by $-f$).
Let $\mathcal{N}_{x_0}$ be the set of all open neighborhoods U of x_0. Let $0 < \varepsilon < \frac{|g(x_0)|}{2}$. Then for each $0 < \varepsilon_0 \leq \min\left\{\varepsilon, \dfrac{4\|f\|_\infty}{\left|f(x_0)+4\|f\|_\infty\right|}\varepsilon\right\}$, there exist $n \geq n_0$ and $U \in \mathcal{N}_{x_0}$ of the radius $\dfrac{1}{2n}$ such that $|f(x)-f(x_0)| < \varepsilon_0$ and $|g(x)-g(x_0)| < \varepsilon_0$, for every $x \in U$.
Selection $0 < \varepsilon < \frac{|g(x_0)|}{2}$ requires that $g(x) \neq 0$ for each $x \in U$.
By the Uryshon's lemma, there exists $0 \leq h'_U \leq 1$ such that $h'_U(x_0) = 1$ and $h'_U(x) = 0$ for all $x \in \mathbb{X} \setminus U$. Define h_U as follows:

$$h_U(x) = \begin{cases} \dfrac{4\|f\|_\infty}{g(x)} h'_U(x), & x \in U, \\ 0, & x \in \mathbb{X} \setminus U. \end{cases}$$

Then

$$\mathrm{supp}(h_U) = \overline{\left\{x : h'_U(x) \neq 0\right\}} \subseteq \overline{U}.$$

Hence, $\mathrm{diam}(\mathrm{supp}(h_U)) \leq \dfrac{1}{n}$, that is, $h_U \in \mathscr{B}_n$.
Moreover, for every $x \in \mathbb{X} \setminus U$, we have

$$\begin{aligned} |f(x)+g(x)h_U(x)| &= |f(x)| \\ &\leq \|f\|_\infty \\ &\leq \left|f(x_0)+4\|f\|_\infty\right| \\ &= \left|f(x_0)+4\|f\|_\infty h'_U(x_0)\right| \\ &= \left|f(x_0)+g(x_0)h_U(x_0)\right|, \end{aligned} \tag{3.1}$$

and for every $x \in U$, we get

$$\begin{aligned}
\left|f(x)+g(x)h_U(x)\right| &\leq \left|f(x_0)+g(x)h_U(x)\right|+\left|f(x)-f(x_0)\right| \\
&\leq \left|f(x_0)+4\|f\|_\infty h'_U(x)\right|+\varepsilon_0 \\
&= \sqrt{\left(\mathrm{Re}\,(f(x_0))+4\|f\|_\infty h'_U(x)\right)^2+(\mathrm{Im}\,(f(x_0)))^2}+\varepsilon_0 \\
&\leq \sqrt{(\mathrm{Re}\,(f(x_0))+4\|f\|_\infty)^2+(\mathrm{Im}\,(f(x_0)))^2}+\varepsilon_0 \\
&= \left|f(x_0)+g(x_0)h_U(x_0)\right|+\varepsilon_0. \qquad (3.2)
\end{aligned}$$

Using (3.1) and (3.2), we obtain $\|f+gh_U\|_\infty \leq \left|f(x_0)+g(x_0)h_U(x_0)\right|+\varepsilon_0$. Therefore,

$$\left|\|f+gh_U\|_\infty-\left|f(x_0)+g(x_0)h_U(x_0)\right|\right| \leq \varepsilon_0. \qquad (3.3)$$

In the next step, we have

$$\begin{aligned}
|f(x)-g(x)h_U(x)| &= |f(x)| \\
&\leq \|f\|_\infty \\
&< \sqrt{(3\|f\|_\infty)^2+(\mathrm{Im}\,(f(x_0)))^2} \\
&\leq \sqrt{(4\|f\|_\infty-\mathrm{Re}\,(f(x_0)))^2+(\mathrm{Im}\,(f(x_0)))^2} \\
&= \left|f(x_0)-4\|f\|_\infty\right| \\
&= \left|f(x_0)-4\|f\|_\infty h'_U(x_0)\right| \\
&= \left|f(x_0)-g(x_0)h_U(x_0)\right| \qquad (3.4)
\end{aligned}$$

for every $x \in \mathbb{X} \setminus U$. On the other hand, if $x \in U$, then

$$\begin{aligned}
\left|f(x)-g(x)h_U(x)\right| &\leq \left|f(x_0)-g(x)h_U(x)\right|+\left|f(x)-f(x_0)\right| \\
&\leq \left|f(x_0)-4\|f\|_\infty h'_U(x)\right|+\varepsilon_0 \\
&= \sqrt{\left(\mathrm{Re}\,(f(x_0))-4\|f\|_\infty h'_U(x)\right)^2+(\mathrm{Im}\,(f(x_0)))^2}+\varepsilon_0 \\
&\leq \sqrt{(\mathrm{Re}\,(f(x_0))-4\|f\|_\infty)^2+(\mathrm{Im}\,(f(x_0)))^2}+\varepsilon_0 \\
&= \left|f(x_0)-g(x_0)h_U(x_0)\right|+\varepsilon_0. \qquad (3.5)
\end{aligned}$$

Thus, using (3.4) and (3.5), we conclude that $\|f-gh_U\|_\infty \leq \left|f(x_0)-g(x_0)h_U(x_0)\right|+\varepsilon_0$. Hence,

$$\left|\|f-gh_U\|_\infty-\left|f(x_0)-g(x_0)h_U(x_0)\right|\right| \leq \varepsilon_0. \qquad (3.6)$$

Now, (3.3), (3.6), and the assumption $\|f + gh_U\|_\infty = \|f - gh_U\|_\infty$ ensure that

$$
\begin{aligned}
&\frac{8\|f\|_\infty}{\big|f(x_0)+4\|f\|_\infty\big|}\big|\mathrm{Re}\,(f(x_0))\big| \\
&= \frac{\Big|\big|f(x_0)+4\|f\|_\infty\big|^2 - \big|f(x_0)-4\|f\|_\infty\big|^2\Big|}{2\big|f(x_0)+4\|f\|_\infty\big|} \\
&\leq \frac{\Big|\big|f(x_0)+g(x_0)h_U(x_0)\big|^2 - \big|f(x_0)-g(x_0)h_U(x_0)\big|^2\Big|}{\big|f(x_0)+g(x_0)h_U(x_0)\big| + \big|f(x_0)-g(x_0)h_U(x_0)\big|} \qquad (\text{since } \mathrm{Re}\,(f(x_0)) \geq 0) \\
&= \Big|\big|f(x_0)+g(x_0)h_U(x_0)\big| - \big|f(x_0)-g(x_0)h_U(x_0)\big|\Big| \\
&= \Big|\big|f(x_0)+g(x_0)h_U(x_0)\big| - \big|f(x_0)-g(x_0)h_U(x_0)\big| - \|f+gh_U\|_\infty + \|f-gh_U\|_\infty\Big| \\
&\leq \Big|\big|f(x_0)+g(x_0)h_U(x_0)\big| - \|f+gh_U\|_\infty\Big| + \Big|\big|f(x_0)-g(x_0)h_U(x_0)\big| - \|f-gh_U\|_\infty\Big| \\
&\leq 2\varepsilon_0 \\
&\leq \frac{8\|f\|_\infty}{\big|f(x_0)+4\|f\|_\infty\big|}\varepsilon.
\end{aligned}
$$

Letting ε tend to zero, we obtain that $\mathrm{Re}\,(f(x_0)) = 0$. Using analogous techniques and using the function ih_U, we can prove that $\mathrm{Im}\,(f(x_0)) = 0$. □

In the following, we obtain similar results for 2×2 complex matrices. More precisely, we introduce a nontrivial subset $\mathscr{C}$ of $\mathbb{M}_2(\mathbb{C})$ such that $\perp_R^{\mathscr{C}}$ and $\perp$ coincide with respect to both the operator norm and the trace-class norm.
The following relation is a famous equality for an element A of the C^*-algebra $\mathbb{M}_2(\mathbb{C})$:

$$\|A^*A\| = \frac{1}{2}\left(\mathrm{tr}(A^*A) + \sqrt{\mathrm{tr}^2(A^*A) - 4\mathrm{det}(A^*A)}\right). \tag{3.7}$$

Recently, Arambašić and Rajić [5] proved that for any $A, B \in \mathbb{M}_2(\mathbb{C})$,

$$A \perp_R B \Longrightarrow \mathrm{tr(A^*B)} = 0. \tag{3.8}$$

Moreover, the following characterization of the Roberts orthogonality for matrices A and B in $\mathbb{M}_2(\mathbb{C})$ was established in [5].

Theorem 3.2 *[5, Theorem 2.2] Let $A, B \in \mathbb{M}_2(\mathbb{C})$, $A, B \neq 0$. Then $A \perp_R B$ if and only if one of the following conditions holds:*

(i) $A = \alpha U_1 \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} U_2$ *and* $B = \beta U_1 \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} U_2$ *for some unitary matrices* $U_1, U_2 \in \mathbb{M}_2(\mathbb{C})$, $0 \leq a \leq 1$, *and* $\alpha, \beta, b, c \in \mathbb{C}$.

(ii) $A = \alpha U_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U_2$ *and* $B = \beta U_1 \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix} U_2$ *for some unitary matrices* $U_1, U_2 \in \mathbb{M}_2(\mathbb{C})$ *and* $\alpha, \beta, b \in \mathbb{C}$.

(iii) $A = \alpha U_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U_2$ *and* $B = \beta U_1 \begin{bmatrix} 0 & 0 \\ b & 1 \end{bmatrix} U_2$ *for some unitary matrices* $U_1, U_2 \in \mathbb{M}_2(\mathbb{C})$ *and* $\alpha, \beta, b \in \mathbb{C}$.

Let $\mathscr{C} \subseteq \mathbb{M}_2(\mathbb{C})$ be the set of all those elements $T \in \mathbb{M}_2(\mathbb{C})$ whose anticommutator $TT^* + T^*T$ is a positive similarity, that is,

$$\mathscr{C} = \left\{T \in \mathbb{M}_2(\mathbb{C}) : TT^* + T^*T = \lambda I, \text{forsome } \lambda \geq 0\right\}.$$

In the next theorem, we show that $\perp_R^{\mathscr{C}}$ and $\perp$ coincide.

Theorem 3.3 *Let $A, B \in \mathbb{M}_2(\mathbb{C})$, let $\mathscr{B}$ be the subalgebra of $\mathbb{M}_2(\mathbb{C})$ generated by B^*A, and let $\mathscr{C} \subseteq \mathbb{M}_2(\mathbb{C})$ be as above. Then the following properties are equivalent:*

(i) $A^*B = 0$,
(ii) $A \perp_R^{\mathbb{M}_2(\mathbb{C})} B$,
(iii) $A \perp_R^{\mathscr{B}} B$,
(iv) $A \perp_R^{\mathscr{C}} B$.
(v) $\|A + BC\|_1 = \|A - BC\|_1$ *for all* $C \in \mathbb{M}_2(\mathbb{C})$.

Proof (i) $\Leftrightarrow$ (ii) follows from Theorem 1.1. The implications (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) can be deduced directly from the case (v) of Proposition 2.2.
To prove the implication (iii) $\Rightarrow$ (i), suppose that $A \perp_R^{\mathscr{B}} B$. Then for all $\lambda \in \mathbb{C}$, we have

$$\|A + \lambda BC\| = \|A - \lambda BC\|,$$

where $C \in \mathscr{B}$. Therefore, $\|A + \lambda BB^*A\| = \|A - \lambda BB^*A\|$, that is, $A \perp_R BB^*A$. Using (3.8), we obtain that

$$\left\|\left|B^*A\right|^2\right\|_1 = \operatorname{tr}\left(\left|B^*A\right|^2\right) = \operatorname{tr}\left(A^*BB^*A\right) = 0.$$

Thus, $|B^*A|^2 = 0$, whence $A^*B = 0$. Hence, A is orthogonal to B.

Next, we show that (iv) $\Rightarrow$ (i). Suppose that $A \perp_R^{\mathscr{C}} B$. Since $\lambda I \in \mathscr{C}$ for each $\lambda \in \mathbb{C}$, thus, $A \perp_R B$. Therefore, one of the conditions in Theorem 3.2 holds. We show that $A^*B = 0$, for each of the conditions mentioned in Theorem 3.2.
Let us assume that the case (i) of Theorem 3.2 holds true. Then, $A = \alpha U_1 \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} U_2$ and $B = \beta U_1 \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} U_2$, for some unitary matrices $U_1, U_2 \in \mathbb{M}_2(\mathbb{C})$, $0 \leq a \leq 1$, and $\alpha, \beta, b, c \in \mathbb{C}$. Thus $A^*B = \overline{\alpha}\beta U_2^* \begin{bmatrix} 0 & b \\ ac & 0 \end{bmatrix} U_2$. If $\alpha = 0$ or $\beta = 0$, then obviously $A^*B = 0$. Hence, assume that $\alpha, \beta \neq 0$. We will now show that $b = ac = 0$.
If $b \neq 0$, then put $T = \dfrac{\alpha}{\beta} U_2^* \begin{bmatrix} 0 & 0 \\ \frac{1}{b} & 0 \end{bmatrix} U_2$. Thus

$$TT^* = \left|\frac{\alpha}{\beta}\right|^2 U_2^* \begin{bmatrix} 0 & 0 \\ \frac{1}{b} & 0 \end{bmatrix} U_2 U_2^* \begin{bmatrix} 0 & \frac{1}{\overline{b}} \\ 0 & 0 \end{bmatrix} U_2 = \left|\frac{\alpha}{\beta}\right|^2 U_2^* \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{|b|^2} \end{bmatrix} U_2,$$

and similarly, $T^*T = \left|\frac{\alpha}{\beta}\right|^2 U_2^* \begin{bmatrix} \frac{1}{|b|^2} & 0 \\ 0 & 0 \end{bmatrix} U_2$. Therefore, $TT^* + T^*T = \left|\frac{\alpha}{\beta}\right|^2 \frac{1}{|b|^2} I$, whence $T \in \mathscr{C}$. Furthermore,

$$A + BT = \alpha U_1 \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} U_2 + \left(\beta U_1 \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} U_2\right)\left(\frac{\alpha}{\beta} U_2^* \begin{bmatrix} 0 & 0 \\ \frac{1}{b} & 0 \end{bmatrix} U_2\right) = \alpha U_1 \begin{bmatrix} 2 & 0 \\ 0 & a \end{bmatrix} U_2,$$

and similarly, $A - BT = \alpha U_1 \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} U_2$. By the assumption $A \perp_R^{\mathscr{C}} B$ and in light of the relation (3.7), we have

$$\begin{aligned}
&\|A + BT\|^2 = \|A - BT\|^2 \\
&\quad \Rightarrow \operatorname{tr}\big((A + BT)^*(A + BT)\big) \\
&\qquad + \sqrt{\operatorname{tr}^2\big((A + BT)^*(A + BT)\big) - 4\det\big((A + BT)^*(A + BT)\big)} \\
&\quad = \operatorname{tr}\big((A - BT)^*(A - BT)\big) \\
&\qquad + \sqrt{\operatorname{tr}^2\big((A - BT)^*(A - BT)\big) - 4\det\big((A - BT)^*(A - BT)\big)} \\
&\quad \Rightarrow |\alpha|^2 \left(4 + a^2\right) + \sqrt{|\alpha|^4 \left(4 + a^2\right)^2 - 16|\alpha|^4 a^2} = |\alpha|^2 a^2 + \sqrt{|\alpha|^4 a^4}.
\end{aligned}$$

Hence, $a^2 = 4$, which contradicts with the structure of the matrix A, as $0 \le a \le 1$. Therefore, $b = 0$. Now, we are going to prove that $ac = 0$. Suppose on the contrary that $ac \neq 0$. Then by choosing $T = \frac{\alpha}{\beta} U_2^* \begin{bmatrix} 0 & \frac{1}{c} \\ 0 & 0 \end{bmatrix} U_2$, we have $T \in \mathscr{C}$, $A + BT = \alpha U_1 \begin{bmatrix} 1 & 0 \\ 0 & a+1 \end{bmatrix} U_2$, and $A - BT = \alpha U_1 \begin{bmatrix} 1 & 0 \\ 0 & a-1 \end{bmatrix} U_2$. Again, from $A \perp_R^{\mathscr{C}} B$ and (3.7), a straightforward computation shows that

$$\begin{aligned}
&1 + (a+1)^2 + \sqrt{\left(1 + (a+1)^2\right)^2 - 4(a+1)^2} \\
&\quad = 1 + (a-1)^2 + \sqrt{\left(1 + (a-1)^2\right)^2 - 4(a-1)^2} \\
&\quad \Rightarrow 1 + (a+1)^2 + \sqrt{((a+1)^2 - 1)^2} = 1 + (a-1)^2 + \sqrt{(1 - (a-1)^2)^2} \\
&\quad \Rightarrow \quad (a+1)^2 = 1 \\
&\quad \Rightarrow \quad a = 0, -2.
\end{aligned}$$

Since $0 \le a \le 1$, it follows that $a = 0$, and therefore, $A^*B = 0$.

Now, assume that the case (ii) of Theorem 3.2 occurs. Then $A = \alpha U_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U_2$ and $B = \beta U_1 \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix} U_2$, for some unitary matrices $U_1, U_2 \in \mathbb{M}_2(\mathbb{C})$ and $\alpha, \beta, b \in$

$\mathbb{C}$. Thus, $A^*B = \overline{\alpha}\beta U_2 \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} U_2$. When $\alpha = 0$ or $\beta = 0$, obviously, $A^*B = 0$. Hence, assume that $\alpha, \beta \neq 0$. We show that $b = 0$.
Consider $T = \frac{\alpha}{\beta} U_2^* \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} U_2$. Then $T \in \mathscr{C}$, $A + BT = \alpha U_1 \begin{bmatrix} 1+b & 0 \\ 1 & 0 \end{bmatrix} U_2$ and $A - BT = \alpha U_1 \begin{bmatrix} 1-b & 0 \\ -1 & 0 \end{bmatrix} U_2$. From the assumption $\|A + BT\|^2 = \|A - BT\|^2$, we conclude that $|1+b|^2 = |1-b|^2$, which ensures that $\mathrm{Re}(b) = 0$. Similarly, by choosing $T = \frac{\alpha}{\beta} U_2^* \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} U_2$, we arrive at $\mathrm{Im}(b) = 0$. Hence, $b = 0$ and so $A^*B = 0$.
As a result, we get $A^*B = 0$ in a straightforward way from the case (iii) of Theorem 3.2.
Next, for the implication (i) $\Rightarrow$ (v), assume that $A^*B = 0$ and $C \in \mathbb{M}_2(\mathbb{C})$. Then

$$\begin{aligned}
\|A + BC\|_1^2 &= \mathrm{tr}\left(A^*A + C^*B^*BC + 2\mathrm{Re}(A^*BC)\right) \\
&\quad + 2\sqrt{\det\left(A^*A + C^*B^*BC + 2\mathrm{Re}(A^*BC)\right)} \\
&= \mathrm{tr}\left(A^*A + C^*B^*BC - 2\mathrm{Re}(A^*BC)\right) \\
&\quad + 2\sqrt{\det\left(A^*A + C^*B^*BC - 2\mathrm{Re}(A^*BC)\right)} \\
&= \|A - BC\|_1^2.
\end{aligned}$$

Finally, we prove the implication (v) $\Rightarrow$ (i). Let $\|A + BC\|_1 = \|A - BC\|_1$ for all $C \in \mathbb{M}_2(\mathbb{C})$. In view of the continuity of the determinant, we can choose $r \in \mathbb{R}^+$ such that $\det\left(I - \frac{1}{r}BB^*\right)$ and $\det\left(I + \frac{1}{r}BB^*\right)$ are positive.
Suppose that $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, and consider $C = B^*A$. Then assumption (v) entails that

$$\begin{aligned}
&\mathrm{tr}\left(A^*A + \left|BB^*A\right|^2 + 2\mathrm{Re}\left|B^*A\right|^2\right) + 2\left|\det(A + BB^*A)\right| \\
&\quad = \mathrm{tr}\left(A^*A + \left|BB^*A\right|^2 - 2\mathrm{Re}\left|B^*A\right|^2\right) + 2\left|\det(A - BB^*A)\right|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 \le 2\mathrm{tr}\left((B^*A)^*(B^*A)\right) &= 2\mathrm{tr}\left(\mathrm{Re}\left|B^*A\right|^2\right) \\
&= \left|\det(A - BB^*A)\right| - \left|\det(A + BB^*A)\right| \\
&= \left|\det(A)\right|\left(\det(I - BB^*) - \det(I + BB^*)\right) \\
&= -2\left|\det(A)\right|\left(|x|^2 + |y|^2 + |z|^2 + |w|^2\right) \le 0,
\end{aligned}$$

which establishes the theorem completely. □

The following result gives a sufficient condition for the $\mathbb{B}(\mathcal{H})$-Roberts orthogonality of $A, B \in \mathbb{B}(\mathcal{H})$. For $A \in \mathbb{B}(\mathcal{H})$, the image of A is denoted by $A(\mathcal{H})$, that is, $A(\mathcal{H}) = \{A(x) : x \in \mathcal{H}\}$. Let $\mathbb{A}$ and $\mathbb{B}$ be two subsets of $\mathcal{H}$. We write $\mathbb{A} \perp_R \mathbb{B}$ if $x \perp_R y$ for all $x \in \mathbb{A}$ and $y \in \mathbb{B}$.

Lemma 3.4 *Let $\mathcal{H}$ be a Hilbert space and let $A, B \in \mathbb{B}(\mathcal{H})$. Then the following are equivalent:*

(i) $A(\mathcal{H}) \perp_R B(\mathcal{H})$,
(ii) $A^*B = 0$,
(iii) $A \perp_R^{\mathbb{B}(\mathcal{H})} B$.

Proof (i) $\Longleftrightarrow$ (ii) follows by using the fact that $x \perp_R y$ if and only if $\langle x, y\rangle = 0$ for $x, y \in \mathcal{H}$. (ii) $\Longleftrightarrow$ (iii) follows from Theorem 1.1. □

For rank one operators in $\mathbb{B}(\mathcal{H})$, $\mathbb{B}(\mathcal{H})$-Roberts orthogonality can be characterized in the following way.

Theorem 3.5 *Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathbb{B}(\mathcal{H})$ be rank one. Suppose $B \in \mathbb{B}(\mathcal{H})$. Then the following properties are equivalent:*

(i) $A(\mathcal{H}) \perp_R B(\mathcal{H})$,
(ii) $A^*B = 0$,
(iii) $A \perp_R^{\mathbb{B}(\mathcal{H})} B$,
(iv) $A \perp_R^{\mathbb{K}(\mathcal{H})} B$.

Proof Equivalence of (i), (ii) and (iii) follows from Lemma 3.4. The implication (iii) $\Rightarrow$ (iv) is obvious. To complete the proof, we now prove (iv) $\Rightarrow$ (i). Suppose that (iv) holds. Let $x_0 \in \mathcal{H}$ and $y_0 \in S_{\mathcal{H}}$ such that $A(x) = \langle x, x_0\rangle y_0$ for $x \in \mathcal{H}$. Let $z \in \mathcal{H}$, and define $T : \mathcal{H} \longrightarrow \mathcal{H}$ by $T(x) = \langle x, x_0\rangle z$ for $x \in \mathcal{H}$. Then, (ii) implies $\|A + \lambda BT\| = \|A - \lambda BT\|$ for all $\lambda \in \mathbb{C}$. Let $\lambda \in \mathbb{C}$ such that $\|A + \lambda BT\| = \|A - \lambda BT\| \neq 0$. Also let $(x_n) \subset S_{\mathcal{H}}$ such that $\|A + \lambda BT\| = \lim_n \|(A + \lambda BT)(x_n)\|$. Then,

$$\begin{aligned}\lim_n |\langle x_n, x_0\rangle| \|y_0 - \lambda B(z)\| &= \lim_n \|(A - \lambda BT)(x_n)\| \\ &\leq \|A - \lambda BT\| \\ &= \|A + \lambda BT\| \\ &= \lim_n \|(A + \lambda BT)(x_n)\| \\ &= \lim_n |\langle x_n, x_0\rangle| \|y_0 + \lambda B(z)\|.\end{aligned}$$

Our assumption on λ implies that $\|y_0 - \lambda B(z)\| \leq \|y_0 + \lambda B(z)\|$. Using similar arguments, we can also show that $\|y_0 + \lambda B(z)\| \leq \|y_0 - \lambda B(z)\|$. Thus, $y_0 \perp_R B(z)$ for all $z \in \mathcal{H}$. Now, the homogeneity property of Roberts orthogonality implies that $A(\mathcal{H}) \perp_R B(\mathcal{H})$, and thus the implication (i) follows. □

4 Additivity of the Roberts orthogonality in ℓ_∞^n

Having generalized the notion of the Roberts orthogonality to the realm of C^*-algebras, we would like to end the present paper with a study of the additivity property of the original Roberts orthogonality in the real Banach spaces ℓ_∞^n. To put things into perspective, we start with the following general discussion.

For a normed linear space $\mathbb{X}$, let $\mathbb{X}^*$ denote the topological dual of X. Given a nonzero $x \in \mathbb{X}$, let $J(x) = \{x^* \in S_{\mathbb{X}^*} : x^*(x) = \|x\|\}$ denote the collection of all supporting functionals at x. We say that $\mathbb{X}$ is *smooth* (in the classical sense) at x if $J(x)$ is singleton. It is well known that the local smoothness at $x \in \mathbb{X}$ is equivalent to the Gateaux differentiability of the norm at x. The study of smoothness in normed linear spaces is intimately connected to the concept of Birkhoff–James orthogonality [7, 10]. We recall that smoothness in a normed linear space is equivalent to the right-additivity of the Birkhoff–James orthogonality. In other words, $x \in \mathbb{X}$ is smooth if and only if, given any $y, z \in \mathbb{X}$, $x \perp_B y$ and $x \perp_B z$ imply that $x \perp_B (y + z)$. We further note that unlike the Birkhoff–James orthogonality, the existence of (nontrivial) Roberts orthogonal elements to a given vector is not a priory guaranteed, in a general normed linear space. Motivated by these basic observations, we would like to introduce the following notion of smoothness in normed linear spaces, induced by the Roberts orthogonality. In this context, let us also mention that the study of smoothness in normed linear spaces is a vast area of research in the normed geometry, and we refer to the recent works [14, 15] for two different notions of smoothness in normed linear spaces and in spaces of operators.

Definition 4.1 Let $\mathbb{X}$ be a normed linear space and let $0 \neq x \in \mathbb{X}$. We say that x is *R-smooth* if the following conditions are satisfied:

(a) There exists $0 \neq y \in \mathbb{X}$ such that $x \perp_R y$.
(b) If $0 \neq y_1, y_2 \in \mathbb{X}$ with $x \perp_R y_1$ and $x \perp_R y_2$, then $x \perp_R (y_1 + y_2)$.

We completely characterize the R-smooth points in $\ell_\infty^n(\mathbb{R})$, which we simply write as ℓ_∞^n. For our purpose, let us first fix a few notations. For $x = (x_1, x_2, \ldots, x_n) \in \ell_\infty^n$, we follow the usual convention to write $\text{supp}(x) = \{i : x_i \neq 0\}$, and $|\text{supp}(x)|$ denotes the cardinality of $\text{supp}(x)$. We recall the sign function on $\mathbb{R}$ by

$$\text{sgn}(t) = \frac{t}{|t|} \text{ and } \text{sgn}(0) = 0.$$

We now prove the following result, which will be used throughout this section. We also recall that a unit vector $x = (x_1, x_2, \ldots, x_n) \in \ell_\infty^n$ is a smooth point if and only if x has precisely one unimodular coordinate.

Lemma 4.2 *(i) If* $x, y \in \ell_\infty^n$ *and* $\text{supp}(x) \cap \text{supp}(y) = \emptyset$*, then* $x \perp_R y$.
(ii) Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in S_{\ell_\infty^n}$ *such that* x *is a smooth point with* $|x_i| = 1$ *and* $x \perp_R y$*. Then* $y_i = 0$.

Proof (i) The proof follows directly from the definition of R-orthogonality.

(ii) Let $x = (x_1, x_2, \ldots, x_n) \in S_{\ell_\infty^n}$ be a smooth point. Then $|x_i| = 1$ for some $1 \leq i \leq n$ and $|x_j| < 1$ for all $j \neq i$. In this case, the unique norming functional for x is $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in S_{\ell_1^n}$, where $x_i^* = \text{sgn}(x_i)$ and $x_j^* = 0$ for all $j \neq i$. Now, if $y = (y_1, y_2, \ldots, y_n) \in S_{\ell_\infty^n}$ with $x \perp_R y$, then $x \perp_B y$ and thus $x^*(y) = 0$. This shows that $y_i = 0$. □

The following result provides a characterization of R-smoothness among the smooth points of ℓ_∞^n.

Theorem 4.3 *Let $x = (x_1, x_2, \dots, x_n) \in S_{\ell_\infty^n}$ be a smooth point.*

(a) If $|\mathrm{supp}(x)| = 1$, then x is R-smooth.
(b) If $1 < |\mathrm{supp}(x)| < n$, then x is not R-smooth.
(c) If $|\mathrm{supp}(x)| = n$ and $|x_i| \neq |x_j|$ for all $1 \leq i \neq j \leq n$, then there is no $0 \neq y \in \ell_\infty^n$ such that $x \perp_R y$.
(d) If $n = 3$, $|\mathrm{supp}(x)| = n$, and $|x_i| = |x_j|$ for some $1 \leq i \neq j \leq n$, then x is R-smooth.
(e) If $n \geq 4$, $|\mathrm{supp}(x)| = n$, and $|x_i| = |x_j|$ for some $1 \leq i \neq j \leq n$, then x is not R-smooth.

Proof (a) Let $1 \leq i \leq n$ such that $|x_i| = 1$ and $x_j = 0$ for all $j \neq i$. If we consider $y = (y_1, y_2, \dots, y_n) \in \ell_\infty^n$, where $y_i = 0$, then the case (i) of Lemma 4.2 implies that $x \perp_R y$. Now, if we assume that $y \in \ell_\infty^n$ with $x \perp_R y$, then the case (ii) of Lemma 4.2 implies that $y_i = 0$. This shows that $\{y \in \ell_\infty^n : x \perp_R y\} = \{y = (y_1, y_2, \dots, y_n) \in \ell_\infty^n : y_i = 0\}$. Thus it follows that x is R-smooth.

(b) Let $1 \leq i, j, k \leq n$ such that $|x_i| = 1$, $x_j \neq 0$, and $x_k = 0$. Let $y_1 = (y_1^1, y_2^1, \dots, y_n^1)$, $y_2 = (y_1^2, y_2^2, \dots, y_n^2) \in \ell_\infty^n$ be defined by

$$y_\ell^1 = \begin{cases} 1 - |x_j| & \text{if } \ell = j, \\ 1 & \text{if } \ell = k, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$y_\ell^2 = \begin{cases} 1 - |x_j| & \text{if } \ell = j, \\ -1 & \text{if } \ell = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} \|x + \lambda y_1\| &= \max\{1, |x_j + \lambda(1 - |x_j|)|, |x_\ell|, |\lambda| : \ell \neq i, j, k\} \\ &= \max\{1, |x_j + \lambda(1 - |x_j|)|, |\lambda|\} \end{aligned}$$

and

$$\|x - \lambda y_1\| = \max\{1, |x_j - \lambda(1 - |x_j|)|, |\lambda|\}.$$

If $|\lambda| \leq 1$, then

$$|x_j \pm \lambda(1 - |x_j|)| \leq |x_j| + |\lambda|(1 - |x_j|) \leq 1$$

and if $|\lambda| > 1$, then

$$|x_j \pm \lambda(1 - |x_j|)| \leq |x_j| + |\lambda|(1 - |x_j|) < |x_j||\lambda| + |\lambda|(1 - |x_j|) \leq |\lambda|.$$

Thus,

$$\|x + \lambda y_1\| = \max\{1, |\lambda|\} = \|x - \lambda y_1\|.$$

By using similar arguments, we can show that $\|x+\lambda y_2\| = \|x-\lambda y_2\|$ for all $\lambda \in \mathbb{R}$. Thus $x \perp_R y_1$ and $x \perp_R y_2$. If we assume that x is R-smooth, then $x \perp_R (y_1 + y_2)$. Now, using $\mathrm{supp}(y_1 + y_2) = \{j\}$ and the case (ii) of Lemma 4.2, we get $x_j = 0$. This leads to a contradiction, and thus x is not R-smooth.

(c) Suppose on the contrary that there exists $0 \neq y = (y_1, y_2, \dots, y_n) \in \ell_\infty^n$ such that $x \perp_R y$. Let $1 \leq i \leq n$ such that $|x_i| = 1$. Now, the case (ii) of Lemma 4.2 implies that $y_i = 0$.

If $n = 2$, then using $|\mathrm{supp}(x)| = n$, $y \perp_R x$, and the case (ii) of Lemma 4.2, we get $y = 0$. This leads to a contradiction.

If $n > 2$, then the case (ii) of Lemma 4.2 implies that there exist $1 \leq i_0 \neq j_0 \leq n$ such that $|y_{i_0}| = |y_{j_0}| = \|y\|$. By our assumption, $\|x+\lambda y\| = \|x-\lambda y\|$ for all $\lambda \in \mathbb{R}$. Thus,

$$\max\{1, |x_j + \lambda y_j| : j \neq i\} = \max\{1, |x_j - \lambda y_j| : j \neq i\}$$

for all $\lambda \in \mathbb{R}$. Using $x_j \neq 0$ for all $1 \leq j \leq n$ and $|y_{i_0}| = |y_{j_0}| \neq 0$, we can choose infinitely many values of λ (sufficiently large) such that $\|x + \lambda y\| > 1$.

For each such λ, we can find $1 \leq i_\lambda \neq j_\lambda \leq n$ such that

$$|x_{i_\lambda} + \lambda y_{i_\lambda}| = |x_{j_\lambda} - \lambda y_{j_\lambda}|.$$

Thus, one of the following equalities is true:

$$x_{i_\lambda} + \lambda y_{i_\lambda} = x_{j_\lambda} - \lambda y_{j_\lambda} \tag{4.1}$$

or

$$x_{i_\lambda} + \lambda y_{i_\lambda} = -x_{j_\lambda} + \lambda y_{j_\lambda}. \tag{4.2}$$

Now, using $x_j \neq 0$ and $x_j \neq x_k$ for all $1 \leq j \neq k \leq n$, from (4.1) and (4.2), we get that either $\lambda = \frac{x_{j_\lambda} - x_{i_\lambda}}{y_{j_\lambda} + y_{i_\lambda}}$ or $\lambda = \frac{x_{i_\lambda} + x_{j_\lambda}}{y_{j_\lambda} - y_{i_\lambda}}$.

This shows that $\|x + \lambda y\| > 1$ is possible only for finitely many values of λ, which is a contradiction. Therefore, there is no $0 \neq y \in \ell_\infty^n$ such that $x \perp_R y$.

(d) Without loss of generality, we assume that $|x_1| = 1$ and that $|x_2| = |x_3| = \alpha > 0$. If $y = (y_1, y_2, y_3) \in S_{\ell_\infty^3}$ with $x \perp_R y$, then the case (ii) of Lemma 4.2 implies that $y_1 = 0$ and that $|y_2| = |y_3| = 1$.

We now claim that either (i) $\mathrm{sgn}(y_2) = \mathrm{sgn}(x_2)$ and $\mathrm{sgn}(y_3) = -\mathrm{sgn}(x_3)$ or (ii) $\mathrm{sgn}(y_2) = -\mathrm{sgn}(x_2)$ and $\mathrm{sgn}(y_3) = \mathrm{sgn}(x_3)$.

If $\mathrm{sgn}(y_2) = \mathrm{sgn}(x_2)$ and $\mathrm{sgn}(y_3) = \mathrm{sgn}(x_3)$, then $x \perp_R y$ implies that

$$\max\{1, |\alpha + \lambda|\} = \max\{1, |\alpha - \lambda|\}$$

for all $\lambda \in \mathbb{R}$. This leads to a contradiction.

Similar arguments show that $\operatorname{sgn}(y_2) = -\operatorname{sgn}(x_2)$ and $\operatorname{sgn}(y_3) = -\operatorname{sgn}(x_3)$ are also impossible.

Thus,

$$\begin{aligned}\{y \in \ell_\infty^3 : x \perp_R y\} = \Big\{&y = (y_1, y_2, y_3) \in \ell_\infty^3 : y_1 = 0, |y_2| = |y_3|, \operatorname{sgn}(y_2)\\ &= \operatorname{sgn}(x_2) \text{and } \operatorname{sgn}(y_3) = -\operatorname{sgn}(x_3)\Big\}\\ \cup \Big\{&y = (y_1, y_2, y_3) \in \ell_\infty^3 : y_1 = 0, |y_2| = |y_3|, \operatorname{sgn}(y_2) = -\operatorname{sgn}(x_2)\\ &\text{and } \operatorname{sgn}(y_3) = \operatorname{sgn}(x_3)\Big\}.\end{aligned}$$

This shows that x is R-smooth.

(e) We break the proof in the following three subcases.

(i) Let $|x_i| = 1$ for $1 \le i \le n$, and suppose that there exist $1 \le i_1 \neq i_2 \neq i_3 \le n$ such that $|x_{i_1}| = |x_{i_2}| = |x_{i_3}| = \alpha$. Let $y_1 = (y_1^1, y_2^1, \dots, y_n^1)$, $y_2 = (y_1^2, y_2^2, \dots, y_n^2) \in \ell_\infty^n$ be defined by

$$y_\ell^1 = \begin{cases} \operatorname{sgn}(x_{i_1}) & \text{if } \ell = i_1,\\ -\operatorname{sgn}(x_{i_2}) & \text{if } \ell = i_2,\\ \operatorname{sgn}(x_{i_3}) & \text{if } \ell = i_3,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$y_\ell^2 = \begin{cases} \operatorname{sgn}(x_{i_1}) & \text{if } \ell = i_1,\\ \operatorname{sgn}(x_{i_2}) & \text{if } \ell = i_2,\\ -\operatorname{sgn}(x_{i_3}) & \text{if } \ell = i_3,\\ 0 & \text{otherwise.}\end{cases}$$

Then,
$\|x + \lambda y_1\| = \max\{1, |\alpha + \lambda|, |\alpha - \lambda|, |x_k| : k \neq i, i_1, i_2, i_3\} = \|x - \lambda y_1\|$
and
$\|x + \lambda y_2\| = \max\{1, |\alpha + \lambda|, |\alpha - \lambda|, |x_k| : k \neq i, i_1, i_2, i_3\} = \|x - \lambda y_2\|$
for all $\lambda \in \mathbb{R}$.

Thus $x \perp_R y_1$ and $x \perp_R y_2$. Now, if we assume that x is R-smooth, then we get $x \perp_R z$, where $z = (z_1, z_2, \dots, z_n)$ is given by

$$z_k = \begin{cases} 2\operatorname{sgn}(x_{i_1}) & \text{if } k = i_1,\\ 0 & \text{otherwise.}\end{cases}$$

However, the case (ii) of Lemma 4.2 implies that this is impossible, and therefore, x is not R-smooth.

(ii) Let $1 \leq i, j, k \leq n$ such that $|x_i| = |x_j| = \alpha$, $|x_k| = 1$, and $\alpha \neq \min_{1\leq \ell \leq n} |x_\ell|$. We now choose $1 \leq m \leq n$ such that $|x_m| = \min_{1\leq \ell \leq n} |x_\ell|$ and define $y_1 = (y_1^1, y_2^1, \ldots, y_n^1)$, $y_2 = (y_1^2, y_2^2, \ldots, y_n^2)$ in the following way:

$$y_\ell^1 = \begin{cases} \operatorname{sgn}(x_i) & \text{if } \ell = i, \\ -\operatorname{sgn}(x_j) & \text{if } \ell = j, \\ 1 & \text{if } \ell = m, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_\ell^2 = \begin{cases} -\operatorname{sgn}(x_i) & \text{if } \ell = i, \\ \operatorname{sgn}(x_j) & \text{if } \ell = j, \\ 1 & \text{if } \ell = m, \\ 0 & \text{otherwise.} \end{cases}$$

From our choice of m, we have $|x_m \pm \lambda| \leq |x_m| + |\lambda| < \alpha + |\lambda|$ for any $\lambda \in \mathbb{R}$. This shows that

$$\begin{aligned} \|x + \lambda y_1\| &= \max\{1, |x_\ell|, |\alpha + \lambda|, |\alpha - \lambda|, |x_m + \lambda| : \ell \neq i, j, k, m\} \\ &= \max\{1, \alpha + |\lambda|\} \\ &= \|x - \lambda y_1\|. \end{aligned}$$

Thus, $x \perp_R y_1$. Similar arguments can be applied to show that $x \perp_R y_2$. Now, if x is R-smooth, then $x \perp_R (y_1 + y_2)$. However, the case (ii) of Lemma 4.2 implies that this is impossible, and therefore, x is not R-smooth.

(iii) We are left with the case where $1 \leq i, j, k \leq n$ with $|x_i| = |x_j| = \alpha$, $|x_k| = 1$ and $\alpha = \min_{1\leq \ell \leq n} |x_\ell|$. We now choose $1 \leq m \leq n$ such that $m \neq i, j, k$. Without loss of generality, we assume that $x_m > 0$. We define

$$\beta = \max\left\{2, \frac{x_m - \alpha}{1 - x_m} + 1\right\}$$

and $y_1 = (y_1^1, y_2^1, \ldots, y_n^1) \in \ell_\infty^n$, where

$$y_\ell^1 = \begin{cases} \beta \operatorname{sgn}(x_i) & \text{if } \ell = i, \\ -\beta \operatorname{sgn}(x_j) & \text{if } \ell = j, \\ 1 & \text{if } \ell = m, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $x \perp_R y_1$. For any scalar λ, we have

$$\begin{aligned} \|x + \lambda y_1\| &= \max\{1, |x_\ell|, |\alpha + \lambda\beta|, |\alpha - \lambda\beta|, |x_m + \lambda| : \ell \neq i, j, k, m\} \\ &= \max\{1, |\alpha + \lambda\beta|, |\alpha - \lambda\beta|, |x_m + \lambda|\}. \end{aligned}$$

Let $\lambda \geq 0$.

(i) If $0 \leq \lambda \leq 1 - x_m$, then $x_m + \lambda \leq 1$.

(ii) If $\lambda > 1 - x_m$, then we claim that $x_m + \lambda \leq \alpha + \lambda\beta$. If we assume that $x_m + \lambda > \alpha + \lambda\beta$, then

$$\beta < 1 + \frac{x_m - \alpha}{\lambda} < 1 + \frac{x_m - \alpha}{1 - x_m}.$$

Thus, if $\lambda > 1 - x_m$, then $x_m + \lambda \leq \alpha + \lambda\beta$.

Let $\lambda < 0$.

(i) If $-1 - x_m \leq \lambda < 0$, then $|x_m + \lambda| \leq 1$.

(ii) If $-1 - x_m > \lambda$, then we claim that $-x_m - \lambda \leq \alpha - \lambda\beta$. If we assume that $-x_m - \lambda > \alpha - \lambda\beta$, then

$$\beta < 1 + \frac{x_m + \alpha}{\lambda} < 1.$$

Thus, if $-1 - x_m > \lambda$, then $-x_m - \lambda \leq \alpha - \lambda\beta$. This shows that

$$\|x + \lambda y_1\| = \max\{1, |\alpha + \lambda\beta|, |\alpha - \lambda\beta|\}.$$

Using similar arguments, it can be shown that $\|x - \lambda y_1\| = \max\{1, |\alpha + \lambda\beta|, |\alpha - \lambda\beta|\}$. Thus, $x \perp_R y_1$.

If we define $y_2 = (y_1^2, y_2^2, \ldots, y_n^2) \in \ell_\infty^n$, where

$$y_\ell^2 = \begin{cases} -\beta \operatorname{sgn}(x_i) & \text{if } \ell = i, \\ \beta \operatorname{sgn}(x_j) & \text{if } \ell = j, \\ 1 & \text{if } \ell = m, \\ 0 & \text{otherwise,} \end{cases}$$

then similar arguments as above can show that $x \perp_R y_2$. Now, if x is R-smooth, then $x \perp_R (y_1 + y_2)$. However, the case (ii) of Lemma 4.2 implies that this is impossible, and therefore, x is not R-smooth. □

Our next result shows that among the nonsmooth points of ℓ_∞^n, the only R-smooth points are the elements with exactly two nonzero entries.

Theorem 4.4 *Let $x \in S_{\ell_\infty^n}$ be a nonsmooth point.*

(a) If $|\mathrm{supp}(x)| = 2$, then x is R-smooth.

(b) If $|\mathrm{supp}(x)| > 2$, then x is not R-smooth.

Proof (a) Let $x = (x_1, x_2, \ldots, x_n) \in S_{\ell_\infty^n}$ be a nonsmooth point with $|\mathrm{supp}(x)| = 2$. Let $1 \leq i, j \leq n$ such that $|x_i| = |x_j| = 1$ and $x_k = 0$ for all $k \neq i, j$. Let $y = (y_1, y_2, \ldots, y_n) \in S_{\ell_\infty^n}$ such that $x \perp_R y$. Then arguments similar to the proof of the case (d) of Theorem 4.3 entail that one of the following conditions is true.

(i) $y_i = y_j = 0$,

(ii) $|y_i| = |y_j| \neq 0$, $\operatorname{sgn}(x_i) = \operatorname{sgn}(y_i)$, and $\operatorname{sgn}(x_j) = -\operatorname{sgn}(y_j)$,
(iii) $|y_i| = |y_j| \neq 0$, $\operatorname{sgn}(x_i) = -\operatorname{sgn}(y_i)$, and $\operatorname{sgn}(x_j) = \operatorname{sgn}(y_j)$.

This shows that x is R-smooth.

(b) Let $x = (x_1, x_2, \ldots, x_n) \in S_{\ell_\infty^n}$ be a nonsmooth point with $|\operatorname{supp}(x)| > 2$. Let $1 \leq i, j, k \leq n$ such that $|x_i| = |x_j| = 1$ and $0 < |x_k| \leq 1$.

We now define $y_1 = (y_1^1, y_2^1, \ldots, y_n^1)$, $y_2 = (y_1^2, y_2^2, \ldots, y_n^2) \in S_{\ell_\infty^n}$ in the following way:

$$y_\ell^1 = \begin{cases} \operatorname{sgn}(x_i) & \text{if } \ell = i, \\ -\operatorname{sgn}(x_j) & \text{if } \ell = j, \\ 1 & \text{if } \ell = k, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_\ell^2 = \begin{cases} -\operatorname{sgn}(x_i) & \text{if } \ell = i, \\ \operatorname{sgn}(x_j) & \text{if } \ell = j, \\ 1 & \text{if } \ell = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} \|x + \lambda y_1\| &= \max\{|1 + \lambda|, |1 - \lambda|, |x_k + \lambda|, |x_\ell| : \ell \neq i, j, k\} \\ &= \max\{1 + |\lambda|, |x_k + \lambda|\} \end{aligned}$$

and

$$\|x - \lambda y_1\| = \max\{1 + |\lambda|, |x_k - \lambda|\}.$$

Indeed $|x_k \pm \lambda| \leq |x_k| + |\lambda| \leq 1 + |\lambda|$. Thus, $x \perp_R y_1$. Similar arguments will show that $x \perp_R y_2$.

Now, if we assume that x is R-smooth, then $x \perp_R (y_1 + y_2)$. Using the case (ii) of Lemma 4.2, we can conclude that this is impossible, and hence, x is not R-smooth. □

We end this section with the following remarks.

Remark 4.5 (i) The cases (b) and (e) of Theorem 4.3 show that, in general, a smooth point in a normed linear space need not be R-smooth.

(ii) The case (a) of Theorem 4.4 shows that a nonsmooth point in a normed linear space can be R-smooth.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Arambašić, Lj., Rajić, R.: A strong version of the Birkhoff-James orthogonality in Hilbert C^*-modules. Ann. Funct. Anal. **5**(1), 109–120 (2014)
2. Arambašić, Lj., Rajić, R.: On Birkhoff-James and Roberts orthogonality. Spec. Matrices **6**, 229–236 (2018)
3. Arambašić, Lj., Berić, T., Rajić, R.: Roberts orthogonality and Davis-Wielandt shell. Linear Algebra Appl. **539**, 1–13 (2018)
4. Arambašić, Lj., Rajić, R.: Another characterization of orthogonality in Hilbert C^*-modules. Math. Inequal. Appl. **22**(4), 142–1426 (2019)
5. Arambašić, Lj., Rajić, R.: Roberts orthogonality for 2×2 complex matrices. Acta Math. Hung. **157**, 220–228 (2019)
6. Balestro, V., Horváth, Á.G., Martini, H., Teixeira, R.: Angles in normed spaces. Aequ. Math. **91**(2), 201–236 (2017)
7. Birkhoff, G.: Orthogonality in linear metric spaces. Duke Math. J. **1**, 169–172 (1935)
8. Bottazzi, T., Conde, C., Moslehian, M.S., Wojcik, P., Zamani, A.: Orthogonality and parallelism of operators on various Banach spaces. J. Aust. Math. Soc. **106**(2), 160–183 (2019)
9. Eskandari, R., Moslehian, M.S., Popovici, D.: Operator equalities and characterizations of orthogonality in pre-Hilbert C^*-modules. Proc. Edinb. Math. Soc. (2) **64**(3), 594–614 (2021)
10. James, R.C.: Orthogonality and linear functionals in normed linear spaces. Trans. Am. Math. Soc. **61**, 265–292 (1947)
11. Kittaneh, F.: On zero-trace matrices. Linear Algebra Appl. **151**, 119–124 (1991)
12. Murphy, G.J.: C^*-Algebras and Operator Theory. Academic Press, New York (1990)
13. Roberts, D.B.: On the geometry of abstract vector spaces. Tohoku Math. J. **39**, 42–59 (1934)
14. Roy, S., Sain, D.: Numerical radius and a notion of smoothness in the space of bounded linear operators. Bull. Sci. Math. **173**, 15 (2021)
15. Sain, D.: Orthogonality and smoothness induced by the norm derivatives. Rev. R. Acad. Cienc. Exactas Fís Nat. Ser. A Mat. RACSAM **115**(3), 10 (2021)
16. Turnšk, A.: Orthogonality in $\mathscr{C}_p$ classes. Monatsh Math. **132**, 349–354 (2001)
17. Zamani, A., Moslehian, M.S.: Approximate Roberts orthogonality. Aequ. Math. **89**(3), 529–541 (2015)
18. Zamani, A., Moslehian, M.S.: Approximate Roberts orthogonality sets and (δ, ε)-(a, b)-isosceles-orthogonality preserving mappings. Aequ. Math. **90**(3), 647–659 (2016)