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# On the index of pseudo B-Fredholm operator

**Zakariae Aznay**[1] · **Abdelmalek Ouahab**[1] · **Hassan Zariouh**[2]

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**Abstract**
The index of a pseudo B-Fredholm operator will be defined and generalize the usual index of a B-Fredholm operator. This concept will be used to extend some known results in Fredholm's theory. Among other results, the nullity, the deficiency, the ascent and the descent will be extended and defined for a pseudo-Fredholm operator.

**Keywords** Pseudo semi-B-Fredholm · Index

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## 1 Introduction

Let $T \in L(X)$, where $L(X)$ is the Banach algebra of bounded linear operators acting on an infinite dimensional complex Banach space $X$. $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are respectively the kernel and the range of $T$. $T$ is said to be upper semi-Fredholm, if $\mathcal{R}(T)$ is closed and $\dim \mathcal{N}(T) < \infty$, while $T$ is called lower semi-Fredholm, if $\operatorname{codim} \mathcal{R}(T) < \infty$. If $T$ is an upper or a lower semi-Fredholm then it is called a semi-Fredholm operator and its index is defined by $\operatorname{ind}(T) = \dim \mathcal{N}(T) - \operatorname{codim} \mathcal{R}(T)$. $T$ is called a Fredholm operator if it is a semi-Fredholm with an integer index.

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✉ Zakariae Aznay
aznay.zakariae@ump.ac.ma

Abdelmalek Ouahab
ouahab05@yahoo.fr

Hassan Zariouh
h.zariouh@yahoo.fr

[1] Laboratory (L.A.N.O), Department of Mathematics, Faculty of Science, Mohammed I University, Oujda 60000, Morocco

[2] Department of Mathematics (CRMEFO), Laboratory (L.A.N.O), Faculty of Science, Mohammed I University, Oujda 60000, Morocco

A subspace $M$ of $X$ is $T$-invariant if $T(M) \subset M$ and in this case $T_M$ means the restriction of $T$ on $M$. We say that $T$ is completely reduced by a pair $(M, N)$ ($(M, N) \in Red(T)$ for brevity) if $M$ and $N$ are closed $T$-invariant subspaces of $X$ and $X = M \oplus N$; here $M \oplus N$ means that $M \cap N = \{0\}$. Let $(M, N) \in Red(T)$ and let the list of the following points:

(i) $T_M$ is semi-regular (i.e, $\mathcal{R}(T_M)$ is closed and $\mathcal{N}(T_M) \subset \bigcap_{n\in\mathbb{N}} \mathcal{R}(T_M^n)$).
(i') $T_M$ is semi-Fredholm.
(ii) $T_N$ is nilpotent of degree $d$ (i.e, $T_N^d = 0$ and $T_N^{d-1} \neq 0$).
(ii') $T_N$ is quasi-nilpotent (i.e, 0 is the only point of the spectrum $\sigma(T_N)$ of $T_N$).

In [13, Theorem 4], T. Kato proved that if $T$ is a semi-Fredholm operator, then there exists $(M, N) \in Red(T)$ satisfies the points (i) and (ii) listed above; this decomposition $(M, N)$ is called Kato's decomposition associated to $T$. In the case of $X$ is a Hilbert space, J. P. Labrousse [15] studied and characterized the operators which admit such a decomposition and he called them quasi-Fredholm operators of degree $d$. The class of quasi-Fredholm operators has been extended by Mbekhta and Muller in [18, p. 143], Poon in [20] to the general case of Banach space operators.
In 1999 and 2001, Berkani and Sarih [3, 6] generalized the concept of semi-Fredholm operators to a class called semi-B-Fredholm operators as follows: For $n \in \mathbb{N}$, let $T_{[n]} : \mathcal{R}(T^n) \to \mathcal{R}(T^n)$ be the restriction of $T$ to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular, $T_{[0]} = T$). $T$ is said to be semi-B-Fredholm if for some integer $n \geq 0$ the range $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is semi-Fredholm. And in this case, they showed in [6, Proposition 2.1] that $\mathcal{R}(T^m)$ is closed, $T_{[m]}$ is a semi-Fredholm operator. Moreover, we show in Proposition 2.12 below that $\text{ind}(T_{[m]}) = \text{ind}(T_{[n]})$, for each $m \geq n$. This defines the index of a semi-B-Fredholm operator $T$ as the index of the semi-Fredholm operator $T_{[n]}$. In particular, if $T$ is a semi-Fredholm operator we find the usual definition of the index of $T$. Furthermore, in the case of $X$ is a Hilbert space they showed that $T$ is semi-B-Fredholm if and only if there exists $(M, N) \in Red(T)$ such that $T$ satisfies the points (i') and (ii) defined above, see [6, Theorem 2.6]. Note that the B-Fredholm operators are also characterized in the general case of Banach spaces following this last decomposition, see [19, Theorem 7].
The class of quasi-Fredholm operators has been generalized by M. Mbekhta [17] to the class of pseudo-Fredholm operators and the decomposition $(M, N)$ associated to a pseudo-Fredholm $T$ called generalized Kato decomposition ($(M, N) \in GKD(T)$ for brevity). More precisely, $(M, N) \in GKD(T)$ if $(M, N)$ satisfies the points (i) and (ii') defined above. In the same way as the generalization of the notion of quasi-Fredholm operators to the notion of semi-B-Fredholm operators [3, 6], the notion of pseudo-Fredholm operators has been generalized to the notion called pseudo semi-B-Fredholm [2, 7, 8, 21, 23]. More precisely, $T$ is said to be pseudo semi-B-Fredholm if there is $(M, N) \in \text{Red}(T)$ which satisifies the points (i') and (ii'). Moreover, the authors [3, 6] have well defined the index of a semi-B-Fredholm operator as a natural extension of the useul index of a semi-Fredholm operator. In a natural way, we ask the following question: can we define the index of a pseudo semi-B-Fredholm operator?

The main purpose of the present paper is to answer affirmatively to this question. For a given pseudo semi-B-Fredholm operator $T$, we define its index as the index of the semi-Fredholm operator $T_M$; where $(M, N) \in Red(T)$ such that $T_M$ is semi-

Fredholm and $T_N$ is quasi-nilpotent. Furthermore, we prove that this definition of the index of $T$ is independent of the choice of the decomposition $(M, N)$ of $T$. In particular, in the case of $T$ is a semi-Fredholm or a B-Fredholm or a Hilbert space semi-B-Fredholm operator, we find the usual definition of the index of $T$ as semi-Fredholm or a B-Fredholm or a Hilbert space semi-B-Fredholm operator. Using this notion of the index for pseudo semi-B-Fredholm operator, we generalize some known properties in the index theory for Fredholm and B-Fredholm operators.
As an application of the results obtained in this paper, we prove that if $T \in L(X)$ is a B-Fredholm operator, then $\mathcal{R}(T^*) + \mathcal{N}(T^{*d})$ is closed in $\sigma(X^*, X)$ and $T^*$ is a B-Fredholm operator with $\text{ind}(T) = -\text{ind}(T^*)$; where $d = \text{dis}(T)$ is the degree of stable iteration of $T$ and $\sigma(X^*, X)$ is the weak*-topology on $X^*$.

## 2 Index of pseudo semi-B-Fredholm operator

We begin this section with the following lemma which will be useful in everything that follows. Hereafter, $\sigma_{sf}(T)$ means the semi-Fredholm spectrum of an operator $T$ and $B(0, \epsilon)$ means the open ball centered at 0 with radius $\epsilon > 0$.

**Lemma 2.1** *Let $T \in L(X)$. If there exist two pair of closed $T$-invariant subspaces $(M, N)$, $(M^{'}, N^{'})$ such that $M \oplus N = M^{'} \oplus N^{'}$ is closed, $T_M$ and $T_{M'}$ are semi-Fredholm operators, $T_N$ and $T_{N'}$ are quasi-nilpotent operators, then $ind(T_M) = ind(T_{M'})$.*

***Proof*** Since $T_M$ and $T_{M'}$ are semi-Fredholm operators, then from punctured neighborhood theorem for semi-Fredholm operators, there exists $\epsilon > 0$ such that $B(0, \epsilon) \subset \sigma_{sf}(T_M)^C \cap \sigma_{sf}(T_{M'})^C$, $\text{ind}(T_M - \lambda I) = \text{ind}(T_M)$ and $\text{ind}(T_{M'} - \lambda I) = \text{ind}(T_{M'})$ for every $\lambda \in B(0, \epsilon)$. As $T_N$ and $T_{N'}$ are quasi-nilpotent, then $B_0 := B(0, \epsilon) \setminus \{0\} \subset \sigma_{sf}(T_M)^C \cap \sigma_{sf}(T_{M'})^C \cap \sigma(T_N)^C \cap \sigma(T_{N'})^C \subset \sigma_{sf}(T_{M\oplus N})^C$. Let $\lambda \in B_0$, as $\text{ind}(T_M - \lambda I) + \text{ind}(T_N - \lambda I) = \text{ind}(T_{M'} - \lambda I) + \text{ind}(T_{N'} - \lambda I)$, then $\text{ind}(T_M) = \text{ind}(T_{M'})$. □

**Definition 2.2** [2, 7, 8, 21, 23] $T \in L(X)$ is said to be

(a) An upper pseudo semi-B-Fredholm (resp., a lower pseudo semi-B-Fredholm, a pseudo B-Fredholm) operator if there exists $(M, N) \in Red(T)$ such that $T_M$ is an upper semi-Fredholm (resp., a lower semi-Fredholm, a Fredholm) operator and $T_N$ is quasi-nilpotent.
(b) An upper pseudo semi-B-Weyl (resp., a lower pseudo semi-B-Weyl, a pseudo-B-Weyl) operator if there exists $(M, N) \in Red(T)$ such that $T_M$ is an upper semi-Weyl (resp., a lower semi-Weyl, a Weyl) operator and $T_N$ is quasi-nilpotent.
(c) A pseudo semi-B-Fredholm (resp., pseudo semi-B-Weyl) operator if it is an upper pseudo semi-B-Fredholm (resp., an upper pseudo semi-B-Weyl) or a lower pseudo semi-B-Fredholm (resp., a lower pseudo semi-B-Weyl) operator.

Let us denote by $\sigma_{upbf}(T)$, $\sigma_{lpbf}(T)$, $\sigma_{spbf}(T)$, $\sigma_{pbf}(T)$, $\sigma_{upbw}(T)$, $\sigma_{lpbw}(T)$, $\sigma_{spbw}(T)$ and $\sigma_{pbw}(T)$ respectively, the upper pseudo semi-B-Fredholm spectrum, the lower pseudo semi-B-Fredholm spectrum, the pseudo semi-B-Fredholm spectrum, the pseudo B-Fredholm spectrum, the upper pseudo semi-B-Weyl spectrum, the lower pseudo semi-B-Weyl spectrum, the pseudo semi-B-Weyl spectrum and the pseudo-B-Weyl spectrum of a given $T \in L(X)$.

**Definition 2.3** Let $T \in L(X)$ be a pseudo semi-B-Fredholm operator. We define the index $\text{ind}(T)$ of $T$ as the index of $T_M$; where $M$ is a closed $T$-invariant subspace which has a complementary closed $T$-invariant subspace $N$ with $T_M$ is semi-Fredholm and $T_N$ is quasi-nilpotent. From Lemma 2.1, it is clear that the index of $T$ is independent of the choice of the pair $(M, N)$ appearing in the definition of the pseudo semi-B-Fredholm $T$ (see Definition 2.2).

As a consequence of the notion of the index of a pseudo semi-B-Fredholm operator, we deduce the following remark.

***Remark 2.4***
(i) Every quasi-nilpotent operator is a pseudo B-Fredholm operator and its index equal to zero. And every semi-Fredholm operator is also a pseudo semi-B-Fredholm, and its usual index as a semi-Fredholm coincides with its index as a pseudo semi-B-Fredholm operator.
(ii) $T \in L(X)$ is an upper pseudo semi-B-Weyl (resp., a lower pseudo semi-B-Weyl, a pseudo B-Weyl) operator if and only if $T$ is an upper pseudo semi-B-Fredholm (resp., a lower pseudo semi-B-Fredholm, a pseudo B-Fredholm) with $\text{ind}(T) \leq 0$ (resp., $\text{ind}(T) \geq 0$, $\text{ind}(T) = 0$).
(iii) If $T \in L(X)$ and $S \in L(Y)$ are pseudo semi-B-Fredholm operators, then $T \oplus S$ is pseudo semi-B-Fredholm and $\text{ind}(T \oplus S) = \text{ind}(T) + \text{ind}(S)$.

**Proposition 2.5** *Let $T \in L(X)$. The following statements hold.*

(i) *$T$ is a pseudo B-Fredholm if and only if $T$ is an upper and lower pseudo semi-B-Fredholm.*
(ii) *$T$ is a pseudo B-Weyl if and only if $T$ is an upper and lower pseudo semi-B-Weyl.*
(iii) *$T$ is a pseudo B-Fredholm if and only if $T$ is a pseudo semi-B-Fredholm with $ind(T) \in \mathbb{Z}$.*

***Proof***
(i) Suppose that $T$ is an upper and lower pseudo semi-B-Fredholm. Then there exist $(M, N)$, $(M^{'}, N^{'}) \in Red(T)$ such that $T_M$ is an upper semi-Fredholm, $T_{M'}$ is a lower semi-Fredholm, $T_N$ and $T_{N'}$ are quasi-nilpotent. From Lemma 2.1 we have $\text{ind}(T) = \text{ind}(T_M) = \text{ind}(T_{M'})$, and so $\dim \mathcal{N}(T_M) + \operatorname{codim} \mathcal{R}(T_{M'}) - \dim \mathcal{N}(T_{M'}) = \operatorname{codim} \mathcal{R}(T_M) \geq 0$. Thus $T_M$ and $T_{M'}$ are Fredholm operators. The converse is obvious.
(ii) Is a consequence of the first point. The point (iii) is obvious.

□

From Proposition 2.5 we obtain the following corollary.

**Corollary 2.6** *For every $T \in L(X)$, we have $\sigma_{pbf}(T) = \sigma_{upbf}(T) \cup \sigma_{lpbf}(T)$ and $\sigma_{pbw}(T) = \sigma_{upbw}(T) \cup \sigma_{lpbw}(T)$.*

The following proposition extends [16, Proposition 3.7.1] to pseudo B-Fredholm operators.

**Proposition 2.7** *Let $T \in L(X)$, and let $A \subset X$ be a closed $T$-invariant subspace of finite codimension. If $T$ is a pseudo B-Fredholm operator and $(M, N) \in GKD(T)$ such that $T_M$ is Fredholm and $M \cap A + N \cap A = A$, then $T_A$ is also a pseudo B-Fredholm operator and in this case, $ind(T) = ind(T_A)$. The converse is true if $A$ has a complementary $T$-invariant subspace.*

***Proof*** As $A$ is a closed $T$-invariant subspace, $(M, N) \in \text{GKD}(T)$ and $M \cap A + N \cap A = A$ then $(M \cap A, N \cap A) \in Red(T_A)$. Moreover, from [12, Lemma 2.2] we have $\text{codim}_M(M \cap A) := \dim \frac{M}{M \cap A} = \dim \frac{A+M}{A} \leq \text{codim}\, A < \infty$. Then we get from [16, Proposition 3.7.1] that $T_{M \cap A}$ is a Fredholm operator and $\text{ind}(T_M) = \text{ind}(T_{M \cap A})$. As $T_N$ is quasi-nilpotent then $T_{N \cap A}$ is also a quasi-nilpotent operator. Consequently, $T_A$ is a pseudo B-Fredholm operator and $\text{ind}(T) = \text{ind}(T_A)$. Conversely, suppose that $T_A$ is a pseudo B-Fredholm operator. Then there exists $(M, N) \in Red(T_A)$ such that $T_M$ is a Fredholm operator and $T_N$ is quasi-nilpotent. Since by hypotheses, $A$ is a closed $T$-invariant subspace of finite codimension and has a complementary $T$-invariant subspace $F$, then $(M \oplus F, N) \in Red(T)$. As $F$ is of finite dimension, then $T_F$ is Weyl and so $T_{M \oplus F}$ is Fredholm. Hence, $T$ is pseudo B-Fredholm and $\text{ind}(T) = \text{ind}(T_{M \oplus F}) = \text{ind}(T_M) + \text{ind}(T_F) = \text{ind}(T_M) = \text{ind}(T_A)$. □

**Proposition 2.8** *Let $T \in L(X)$ be a pseudo semi-B-Fredholm operator. Then for every strictly positive integer $n$, the operator $T^n$ is pseudo semi-B-Fredholm and $ind(T^n) = n.ind(T)$.*

***Proof*** Since $T$ is a pseudo semi-B-Fredholm, then there exists $(M, N) \in Red(T)$ such that $T_M$ is a semi-Fredholm operator and $T_N$ is quasi-nilpotent. So $(M, N) \in Red(T^n)$, $T_M^n$ is a semi-Fredholm operator and $T_N^n$ is quasi-nilpotent. As it is well known that $\text{ind}(T_M^n) = n.\text{ind}(T_M)$ then $\text{ind}(T^n) = n.\text{ind}(T)$. □

Let $T \in L(X)$ and let

$$\Delta(T) := \{m \in \mathbb{N} : \mathcal{R}(T^m) \cap \mathcal{N}(T) = \mathcal{R}(T^r) \cap \mathcal{N}(T), \forall r \in \mathbb{N}\ r \geq m\}.$$

The degree of stable iteration $\text{dis}(T)$ of $T$ is defined as $\text{dis}(T) = \inf\Delta(T)$; with the infimum taken $\infty$ in the case of empty set, see [1, 15]. We say that $T$ is semi-regular if $\mathcal{R}(T)$ is closed and $\text{dis}(T) = 0$.
Let $r \in \mathbb{N}$. It is easily seen that $r \geq \text{dis}(T)$ if and only if $\mathcal{R}(T) + \mathcal{N}(T^m) = \mathcal{R}(T) + \mathcal{N}(T^r)$, $\forall m \in \mathbb{N}$ such that $m \geq r$.

**Definition 2.9** [17] An operator $T \in L(X)$ is said pseudo-Fredholm if there exists $(M, N) \in Red(T)$ such that $T_M$ is a semi-regular operator and $T_N$ is quasi-nilpotent. In this case, we say that the pair $(M, N)$ is a generalized Kato decomposition associated to $T$, and we write $(M, N) \in GKD(T)$ for brevity.

The next proposition gives a characterization of pseudo semi-B-Fredholm operators.

**Proposition 2.10** *Let $T \in L(X)$. $T$ is pseudo semi-B-Fredholm if and only if $T = T_1 \oplus T_2$; where $T_1$ is a semi-Fredholm and semi-regular operator and $T_2$ is quasi-nilpotent. In particular, a pseudo semi-B-Fredholm is pseudo-Fredholm.*

***Proof*** Let $T$ be a pseudo semi-B-Fredholm operator. Then there is $(M, N) \in GKD(T)$ such that $T_M$ is semi-Fredholm and $T_N$ is quasi-nilpotent. From [13, Theorem 4], there exists $(A, B) \in GKD(T_M)$ such that $\dim B < \infty$. Since $T_M$ is semi-Fredholm, then $T_A$ is semi-Fredholm. On the other hand, it is easy to get $(A, B \oplus N) \in GKD(T)$. The converse is obvious. □

Let $T \in L(X)$ be a pseudo semi-B-Fredholm operator. According to Proposition 2.10, we focus in the sequel only on the pairs $(M, N) \in GKD(T)$ such that $T_M$ is semi-Fredholm. We denote in the sequel [17] by: the analytic core and the quasi-nilpotent part of $T$ defined respectively, by

$$\begin{aligned}\mathcal{K}(T) &= \{x \in X : \exists \epsilon > 0 \text{ and } \exists (u_n)_n \subset X \text{ such that } x = u_0, Tu_{n+1} \\ &= u_n \text{ and } \|u_n\| \leq \epsilon^n \|x\| \, \forall n \in \mathbb{N}\} \\ \text{and } \mathcal{H}_0(T) &= \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.\end{aligned}$$

For the sake of completeness, and to give to the reader a good overview of the subject, we include here the following proposition.

**Proposition 2.11** [6, Proposition 2.1] *Let $T \in L(X)$. If there exists an integer $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and such that the operator $T_{[n]}$ is an upper semi-Fredholm (resp. a lower semi-Fredholm) operator, then $\mathcal{R}(T^m)$ is closed, $T_{[m]}$ is an upper semi-Fredholm (resp. a lower semi-Fredholm) operator, for each $m \geq n$. Moreover, if $T_{[n]}$ is a Fredholm operator, then $T_{[m]}$ is a Fredholm operator and ind$(T_{[m]}) =$ ind$(T_{[n]})$, for each $m \geq n$.*

Now, we prove in the following proposition that if $T$ is a semi-B-Fredholm operator then $\text{ind}(T_{[m]}) = \text{ind}(T_{[n]})$, for each $m \geq n$; where $n$ is any integer such that $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is semi-Fredholm.

**Proposition 2.12** *Let $T \in L(X)$. If there exists an integer $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is semi-Fredholm then $\mathcal{R}(T^m)$ is closed, $T_{[m]}$ is semi-Fredholm and ind$(T_{[m]}) =$ ind$(T_{[n]})$, for each $m \geq n$.*

***Proof*** Suppose that there exists an integer $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and such that the operator $T_{[n]}$ is semi-Fredholm. The first part of this proposition is proved in Proposition 2.11. Let us to show that $\text{ind}(T_{[m]}) = \text{ind}(T_{[n]})$, for each $m \geq n$. From [12, Lemma 3.2] we have $\text{codim}\, \mathcal{R}(T_{[n]}) = \dim \frac{X}{\mathcal{R}(T)+\mathcal{N}(T^n)}$. Moreover, from [12, Lemma 2.2] we have $\frac{\mathcal{R}(T)+\mathcal{N}(T^{n+1})}{\mathcal{R}(T)+\mathcal{N}(T^n)} \cong \frac{\mathcal{N}(T_{[n]})}{\mathcal{N}(T_{[n+1]})}$. We then obtain from [22, Lemma 2.1] that $\text{codim}\, \mathcal{R}(T_{[n]}) = \text{codim}\, \mathcal{R}(T_{[n+1]}) + k_n(T)$ and $\dim \mathcal{N}(T_{[n]}) = \dim \mathcal{N}(T_{[n+1]}) + k_n(T)$; where $k_n(T) \leq \min\{\dim \mathcal{N}(T_{[n]}), \text{codim}\, \mathcal{R}(T_{[n]})\}$. It is easily seen that $k_n(T) \leq \min\{\dim \mathcal{N}(T_{[n]}) - \text{codim}\, \mathcal{R}(T_{[n]})\}$. Since $T_{[n]}$ is semi-Fredholm, then $k_n(T)$ is finite. Hence $\text{ind}(T_{[n]}) = \alpha(T_{[n]}) - \beta(T_{[n]}) = \dim \mathcal{N}(T_{[n]}) - k_n(T) - (\text{codim}\, \mathcal{R}(T_{[n]}) - k_n(T)) = \dim \mathcal{N}(T_{[n+1]}) - \text{codim}\, \mathcal{R}(T_{[n+1]}) = \text{ind}(T_{[n+1]})$. It then follows by induction that $\text{ind}(T_{[m]}) = \text{ind}(T_{[n]})$, for each $m \geq n$. □

**Definition 2.13** Let $T \in L(X)$ be a semi-B-Fredholm operator. The index of $T$ is defined as the index of $T_{[n]}$; where $n$ is any integer such that $\mathcal{R}(T^n)$ is closed and $T_{[n]}$

is semi-Fredholm. From Proposition 2.12, this definition is independent of the choice of the integer $n$ (see also the first Remark given in [6, p. 459]). Furthermore, if $T$ is a semi-Fredholm operator this reduces to the usual definition of the index.

***Remark 2.14*** Let $T \in L(X)$ be a pseudo-Fredholm and $(M, N) \in GKD(T)$. Let $n \in \mathbb{N}^*$, then $(M, N) \in GKD(T^n)$. Hence $\mathcal{N}(T_M^n) = \mathcal{K}(T) \cap \mathcal{N}(T^n)$ and $\mathcal{R}(T_M^n) \oplus N = \mathcal{R}(T^n) + \mathcal{H}_0(T)$. From [1, Theorem 1.44] it follows that

$$\mathcal{K}(T^n) = \mathcal{K}(T_M^n) = \mathcal{R}^\infty(T_M^n) = \mathcal{R}^\infty(T_M) = \mathcal{K}(T_M) = \mathcal{K}(T),$$

which with [1, Theorem 1.63] implies that

$$\mathcal{N}(T_M^n) = \mathcal{N}(T^n) \cap \mathcal{K}(T^n) = \mathcal{N}(T^n) \cap \mathcal{K}(T).$$

On the other hand, as $T_M$ is semi-regular then [1, Corollary 2.38] entails that $\mathcal{H}_0(T_M) = T^n(\mathcal{H}_0(T_M)) \subset \mathcal{R}(T_M^n)$. Thus $\mathcal{R}(T^n) + \mathcal{H}_0(T) = \mathcal{R}(T_M^n) + \mathcal{R}(T_N^n) + \mathcal{H}_0(T_M) + N = \mathcal{R}(T_M^n) \oplus N$. It is easily seen that $\mathcal{R}(T_M^n) = \mathcal{R}(T_M^m)$ if and only if $\mathcal{R}(T^n) + \mathcal{H}_0(T) = \mathcal{R}(T^m) + \mathcal{H}_0(T)$, for every integers $m, n \in \mathbb{N}^*$. Moreover, $\operatorname{codim}_M \mathcal{R}(T_M) := \dim \frac{M}{\mathcal{R}(T_M)} = \dim \frac{X}{\mathcal{R}(T_M)\oplus N} = \dim \frac{X}{\mathcal{R}(T)+\mathcal{H}_0(T)}$.

The previous remark allows us to introduce the following definition.

**Definition 2.15** Let $T \in L(X)$ be a pseudo-Fredholm operator, and let $(M, N) \in GKD(T)$. We define the nullity, the deficiency, the ascent and the descent of $T$ respectively, by $\alpha(T) := \dim \mathcal{N}(T_M)$, $\beta(T) := \operatorname{codim}_M \mathcal{R}(T_M)$, $p(T) := \inf\{n \in \mathbb{N} : \mathcal{N}(T_M^n) = \mathcal{N}(T_M^m)$ for all integer $m \geq n\}$ and $q(T) := \inf\{n \in \mathbb{N} : \mathcal{R}(T_M^n) = \mathcal{R}(T_M^m)$ for all integer $m \geq n\}$. From the previous remark, the nullity, the deficiency, the ascent and the descent of $T$ are independent of the choice of the generalized Kato decomposition $(M, N)$ of $T$.

In particular, if $T$ is semi-regular then $\alpha(T) = \dim \mathcal{N}(T)$ and $\beta(T) = \operatorname{codim} \mathcal{R}(T)$. And if $T$ is a B-Fredholm, then $\mathcal{R}(T^d)$ is closed, $T_{[d]}$ is semi-regular, $\alpha(T) = \alpha(T_{[d]})$ and $\beta(T) = \beta(T_{[d]})$; where $d = \operatorname{dis}(T)$, see Theorem 2.21 below and [1, Theorem 1.64]. And if $T$ is semi-Fredholm, then there exists $n \in \mathbb{N}$ such that $\alpha(T) = \dim \mathcal{N}(T) - n$ and $\beta(T) = \operatorname{codim} \mathcal{R}(T) - n$.

From [1, Theorem 1.22] and Definition 2.15, we deduce the relationships between the quantities $\alpha(T)$, $\beta(T)$, $p(T)$ and $q(T)$.

***Remark 2.16*** Let $T$ be a pseudo-Fredholm operator. We have the following statements.

(i) If $p(T) < \infty$ then $\alpha(T) \leq \beta(T)$.
(ii) If $q(T) < \infty$ then $\alpha(T) \geq \beta(T)$.
(iii) If $\max\{p(T), q(T)\} < \infty$ then $p(T) = q(T)$ and $\alpha(T) = \beta(T)$.

**Lemma 2.17** *Let $T \in L(X)$ be an operator with stable iteration. The following statements hold.*

(i) *If $\alpha(T) < \infty$ then $T$ is one-to-one if and only if $p(T) < \infty$.*
(ii) *If $\beta(T) < \infty$ then $T$ is onto if and only if $q(T) < \infty$.*

(iii) *If max* $\{\alpha(T), \beta(T)\} < \infty$ *then* $T$ *is bijective if and only if* $p(T) = q(T) < \infty$.

***Proof*** (i) Let $n \in \mathbb{N}$. Since $\text{dis}(T) = 0$ then $T(\mathcal{N}(T^{n+1})) = \mathcal{N}(T^n) \cap \mathcal{R}(T) = \mathcal{N}(T^n)$, and so the operator $T : \mathcal{N}(T^{n+1}) \longrightarrow \mathcal{N}(T^n)$ is onto. As $\alpha(T) < \infty$, then $\alpha(T^{n+1}) = \alpha(T) + \alpha(T^n)$. By induction we obtain $\alpha(T^n) = n.\alpha(T)$. Consequently, $T$ is one-to-one if and only if $p(T) < \infty$.

(ii) Suppose that $q(T) < \infty$. Since $T$ is semi-regular then from [1, Theorem 1.43], $T^*$ is semi-regular. As $\beta(T) < \infty$ then $\mathcal{R}(T)$ is closed and $\alpha(T^*) < \infty$. From [1, Lemma 1.26] $q(T) = p(T^*) < \infty$ and this implies by the first point that $T$ is onto.

(iii) Is a direct consequence of the first and the second points.

□

The next proposition gives a characterization of pseudo B-Fredholm and generalized Drazin invertible operators. We recall [11, 14] that $T \in L(X)$ is said to be left generalized Drazin invertible (resp., right generalized Drazin invertible, generalized Drazin invertible) if $T = T_1 \oplus T_2$; where $T_1$ is bounded below (resp., onto, invertible) and $T_2$ is quasi-nilpotent.

**Proposition 2.18** *Let* $T \in L(X)$. *The following assertions hold.*

(i) $T$ *is pseudo B-Fredholm if and only if* $T$ *is pseudo-Fredholm and sup* $\{\alpha(T), \beta(T)\} < \infty$.

(ii) $T$ *is semi pseudo B-Fredholm if and only if* $T$ *is pseudo-Fredholm and inf* $\{\alpha(T), \beta(T)\} < \infty$.

(iii) $T$ *is left generalized Drazin invertible if and only if* $T$ *is upper pseudo semi-B-Fredholm and* $p(T) < \infty$ *if and only if* $T$ *is pseudo-Fredholm and* $p(T) = 0$.

(iv) $T$ *is right generalized Drazin invertible if and only if* $T$ *is lower pseudo semi-B-Fredholm and* $q(T) < \infty$ *if and only if* $T$ *is pseudo-Fredholm and* $q(T) = 0$.

(v) $T$ *is generalized Drazin invertible if and only if* $T$ *is pseudo-Fredholm and* $p(T) = q(T) < \infty$.

***Proof*** We left its proof as an exercise to the reader. □

**Conjecture.** $T \in L(X)$ is pseudo B-Fredholm if and only if $\dim \mathcal{K}(T) \cap \mathcal{N}(T) < \infty$ and $\dim \frac{X}{\mathcal{R}(T)+\mathcal{H}_0(T)} < \infty$.

Our next theorem gives a punctured neighborhood theorem for pseudo semi-B-Fredholm operators. Which in turn it extends [1, Theorem 1.117] and [21, Theorem 3.1] by using the notions of nullity, the deficiency and the index of pseudo semi-B-Fredholm operator. Hereafter $\sigma_{se}(T)$ means the semi-regular spectrum of $T$.

**Theorem 2.19** *Let* $T \in L(X)$ *be a pseudo semi-B-Fredholm operator, then there exists* $\epsilon > 0$ *such that* $B(0,\epsilon)\backslash\{0\} \subset (\sigma_{sf}(T))^C \cap (\sigma_{se}(T))^C$. *Moreover,* $\alpha(T) = \alpha(T - \lambda I)$, $\beta(T) = \beta(T - \lambda I)$ *and* $ind(T) = ind(T - \lambda I)$ *for every* $\lambda \in B_0$.

***Proof*** Is a direct consequence of Proposition 2.10 and [25, Lemma 2.4]. □

The next corollary is a consequence of Theorem 2.19.

**Corollary 2.20** *Let* $T \in L(X)$. *Then*

(i) $\sigma_{upbf}(T)$, $\sigma_{lpbf}(T)$, $\sigma_{spbf}(T)$, $\sigma_{pbf}(T)$, $\sigma_{upbw}(T)$, $\sigma_{lpbw}(T)$, $\sigma_{spbw}(T)$ *and* $\sigma_{pbw}(T)$ *are a compact subsets of* $\mathbb{C}$.
(ii) *If* $\Omega$ *is a component of* $(\sigma_{upbf}(T))^C$ *or of* $(\sigma_{lpbf}(T))^C$, *then the index* $ind(T-\lambda I)$ *is constant as* $\lambda$ *ranges over* $\Omega$.

For proving [4, Theorem 2.4], the authors used the characterization of B-Fredholm operators in the case of Hilbert spaces based on the Kato's decomposition of quasi-Fredholm operators [15]. As an application of Lemma 2.1, we extend [4, Theorem 2.4] to the general case of Banach space. Precisely, we shows that if $T \in L(X)$ is B-Fredholm, then the index of $T$ as a B-Fredholm operator coincides with its index as a pseudo B-Fredholm.

In the sequel, for a semi-B-Fredholm operator $T \in L(X)$, we take $n$ an integer such that $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is semi-Fredholm.

**Theorem 2.21** $T \in L(X)$ *is a B-Fredholm operator if and only if* $T = T_1 \oplus T_2$ *such that* $T_1$ *is Fredholm and semi-regular and* $T_2$ *is nilpotent. Moreover, in this case* $T$ *is pseudo B-Fredholm and* $\alpha(T) = \alpha(T_1)$, $\beta(T) = \beta(T_1)$ *and* $ind(T) = ind(T_1) = ind(T_{[n]})$.

***Proof*** Follows directly from [19, Theorem 5] and the proof of [19, Theorem 7]. □

We don't know if Theorem 2.21 can be extended to the case of semi-B-Fredholm operators. Whereas the following proposition gives a version of Theorem 2.21 for semi-B-Fredholm operators and gives also an improvement of [8, Proposition 4.5]. Note that in the case of $X$ is a Hilbert space, it is proved in [6, Theorem 2.6] that $T$ is a semi-B-Fredholm operator if and only if $T = T_1 \oplus T_2$ such that $T_1$ is semi-Fredholm and $T_2$ is nilpotent. In this case and if $T$ is an upper semi-B-Fredholm operator, then $\text{ind}(T_1) = \text{ind}(T)$, see [5, Proposition 2.9].

**Proposition 2.22** $T \in L(X)$ *is a semi-B-Fredholm and pseudo-Fredholm if and only if* $T$ *is a direct sum of a semi-Fredholm operator and a nilpotent operator. Moreover, the index of* $T$ *as a semi-B-Fredholm coincides with its index as a pseudo semi-B-Fredholm.*

***Proof*** Let $(M, N) \in GKD(T)$, then $T_M$ is semi-regular, $\alpha(T_M) = \alpha((T_M)_{[m]})$ and $\beta(T_M) = \beta((T_M)_{[m]})$ for every $m \in \mathbb{N}$. Since $T$ is semi-B-Fredholm then $T_M$ is semi-B-Fredholm, and so $T_M$ is a semi-Fredholm operator. On the other hand, as $T_N$ is semi-B-Fredholm and quasi-nilpotent then its semi-B-Fredholm spectrum is empty, which implies by [24, Corollary 2.10] that its Drazin spectrum is empty and thus $T_N$ is Drazin invertible. Hence $T_N$ is nilpotent.

Conversely, let $(M, N) \in Red(T)$ such that $T_M$ is semi-Fredholm and $T_N$ is nilpotent. Then there exists $(A, B) \in Red(T_M)$ such that $T_A$ is semi-Fredholm and semi-regular and $T_B$ is nilpotent. So $(A, B \oplus N) \in Red(T)$ and $T_{B\oplus N}$ is nilpotent of degree $d$. Hence $\mathcal{R}(T^d) = \mathcal{R}(T_A^d)$ and $\mathcal{N}(T_A) = \mathcal{N}(T_{[d]})$ and $T(A) \oplus (B \oplus N) = \mathcal{N}(T^d) + \mathcal{R}(T)$. Therefore $\alpha(T_A) = \alpha(T_{[d]})$ and $\beta(T_A) = \beta(T_{[d]})$. So $\mathcal{R}(T^d)$ is closed and $T_{[d]}$ is semi-Fredholm, and then $T$ is semi-B-Fredholm. Moreover, $\text{ind}(T) = \text{ind}(T_{[d]}) = \text{ind}(T_{[n]})$, where $n$ is any integer such that $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is semi-Fredholm. □

The following Corollary 2.24 shows that Theorem 2.21 can be extended to semi-B-Fredholm operators in the case of $X$ is a Hilbert space, since every closed subspace of a Hilbert space is complemented. Before that we recall some basic definitions which will be needed later.

**Definition 2.23** [13, 18] Let $T \in L(X)$.

(i) $T$ is called a quasi-Fredholm operator of degree $d$ if $d = \text{dis}(T) \in \mathbb{N}$ and $R(T^{d+1})$ is closed.
(ii) We say that $T$ is decomposable in the Kato's sense of degree $d$ if there exists $(M, N) \in Red(T)$ such that $T_M$ is semi-regular and $T_N$ is nilpotent of degree $d$.

It is well known [15] that the degree $d$ of a decomposable operator $T \in L(X)$ in the Kato's sense is well defined.

**Corollary 2.24** *Let $T \in L(X)$ be an upper semi-B-Fredholm (resp., a lower semi-B-Fredholm) operator such that $\mathcal{R}(T) + \mathcal{N}(T^d)$ (resp., $\mathcal{R}(T^d) \cap \mathcal{N}(T)$) has a complementary in $X$; where $d = dis(T)$. Then $T$ is pseudo semi-B-Fredholm and $ind(T) = ind(T_{[n]})$.*

***Proof*** If $T \in L(X)$ is an upper semi-B-Fredholm then from [6, Proposition 2.5], $T$ is quasi-Fredholm operator of degree $d$ and the subspace $\mathcal{N}(T_{[d]}) = \mathcal{R}(T^d) \cap \mathcal{N}(T)$ is of finite dimension. If $d = 0$ then $T$ is an upper semi-Fredholm, since $\mathcal{R}(T^d)$ is closed and $T_{[d]}$ is upper semi-Fredholm. Thus, $T$ is a pseudo semi-B-Fredholm operator. Suppose that $d > 0$, by assumption we have $\mathcal{R}(T) + \mathcal{N}(T^d)$ is complemented. So $T$ is decomposable in the Kato's sense of degree $d$ (see [15, Remark p. 206]), that's there exists $(M, N) \in Red(T)$ such that $T_M$ is semi-regular and $T_N^d = 0$. Thus $\mathcal{R}(T_M)$ is closed, $\alpha(T_{[d]}) = \alpha((T_M)_{[d]}) = \alpha(T_M) < \infty$ and $\beta(T_{[d]}) = \beta((T_M)_{[d]}) = \beta(T_M)$. Hence $T_M$ is an upper semi-Fredholm operator. By Proposition 2.22 we deduce the desired result. The case of $T$ is a lower semi-B-Fredholm operator with $\mathcal{R}(T^d) \cap \mathcal{N}(T)$ has a complementary goes similarly. □

Let $M$ be a subset of $X$ and $N$ a subset of $X^*$. The annihilator of $M$ and the pre-annihilator of $N$ are the closed subspaces defined respectively, by

$$M^{\perp} := \{f \in X^* : f(x) = 0 \text{ for every } x \in M\},$$

and

$$^{\perp}N := \{x \in X : f(x) = 0 \text{ for every } f \in N\}.$$

Let $T \in L(X)$ and let $(M, N) \in Red(T)$, we denote by $P_M$ the projection on $M$ according to the decomposition of $X = M \oplus N$.

**Lemma 2.25** *Let $T \in L(X)$ and let $(M, N) \in Red(T)$ such that $\mathcal{R}(T_M) \oplus N$ is closed, then $\mathcal{R}(T^*_{N^{\perp}}) \oplus M^{\perp}$ is closed in $\sigma(X^*, X)$; where $\sigma(X^*, X)$ is the weak*-topology on $X^*$.*

***Proof*** Suppose that $\mathcal{R}(T_M) \oplus N$ is closed, and let $\overline{T} \in L(\frac{X}{\mathcal{N}(T_M)}, \mathcal{R}(T_M) \oplus N)$ the operator defined by $\overline{T}(\overline{x}) = T(P_M(x)) + P_N(x)$. It is easily seen that $\overline{T}$ is well defined and it is an isomorphism. On the other hand, as $\mathcal{N}(T_M)^{\perp} = \overline{\mathcal{R}(T^*_{N^{\perp}}) \oplus M^{\perp}}^{\sigma(X^*,X)}$ then it suffices to show that $\mathcal{N}(T_M)^{\perp} \subset \mathcal{R}(T^*_{N^{\perp}}) \oplus M^{\perp}$. Let $f \in \mathcal{N}(T_M)^{\perp}$ and let $\overline{f} \in L(\frac{X}{\mathcal{N}(T_M)}, \mathbb{C})$ the linear form defined by $\overline{f}(\overline{x}) = f(x)$. Let $g \in X^*$ be the extension of $\overline{f}(\overline{T})^{-1}$ given by the Hahn-Banach theorem. Hence $f = T^*(g) + f(I - T)P_M \in \mathcal{R}(T^*) + M^{\perp} = \mathcal{R}(T^*_{N^{\perp}}) \oplus M^{\perp}$. □

Let $X$ be a Banach space, it is well known that $\dim X \leq \dim X^*$; where $X^*$ is the topological dual. In the next theorem we do not distinguish between $\dim X$ and $\dim X^*$, that is if $\dim X = \infty$ then we write $\dim X = \dim X^* = \infty$.
The proof of the next theorem is based on the classical theorems [16, Theorem A.1.8, Theorem A. 1.9].

**Theorem 2.26** *If $T \in L(X)$ is a pseudo-Fredholm then $T^*$ is pseudo-Fredholm, $\alpha(T) = \beta(T^*)$, $\beta(T) = \alpha(T^*)$, $p(T) = q(T^*)$ and $q(T) = p(T^*)$. In particular, if $T$ is pseudo semi-B-Fredholm then $T^*$ is pseudo semi-B-Fredholm and $ind(T) = -ind(T^*)$.*

***Proof*** Suppose that $T$ is a pseudo-Fredholm. Then there exists $(M, N) \in GKD(T)$. From (which is also true in the Banach spaces) [17, Theorem 3.3] that $(N^{\perp}, M^{\perp}) \in GKD(T^*)$. Let $n \in \mathbb{N}$, then $\mathcal{N}((T^*_{N^{\perp}})^n) = (N + \mathcal{R}(T^n))^{\perp} = (N \oplus \mathcal{R}(T^n_M))^{\perp}$. As $N \oplus \mathcal{R}(T_M)$ is closed then $\alpha(T^*) = \alpha(T^*_{N^{\perp}}) = \dim(\frac{X}{N \oplus \mathcal{R}(T_M)})^* = \dim(\frac{M}{\mathcal{R}(T_M)})^* = \beta(T_M) = \beta(T)$, and from Remark 2.14 we then obtain $p(T^*) = p(T^*_{N^{\perp}}) = q(T_M) = q(T)$. Since $\mathcal{N}(T^n) = {}^{\perp}\mathcal{R}((T^*)^n)$ then $\mathcal{N}(T^n_M) = {}^{\perp}(\mathcal{R}((T^*)^n) + M^{\perp}) = {}^{\perp}(\mathcal{R}((T^*_{N^{\perp}})^n) \oplus M^{\perp})$. Using again Remark 2.14 we deduce $p(T) = q(T^*)$. On the other hand, the previous lemma shows that $\mathcal{R}(T^*_{N^{\perp}}) \oplus M^{\perp}$ is closed in $\sigma(X^*, X)$ and then $(\mathcal{N}(T_M)^*) \cong \frac{X^*}{M^{\perp} \oplus \mathcal{R}(T^*_{N^{\perp}})} \cong \frac{N^{\perp}}{\mathcal{R}(T^*_{N^{\perp}})}$. Thus $\alpha(T) = \alpha(T_M) = \dim(\mathcal{N}(T_M)^*) = \beta(T^*_{N^{\perp}}) = \beta(T^*)$. Consequently, if $T_M$ is semi-Fredholm then $T^*_{N^{\perp}}$ is semi-Fredholm and $\mathrm{ind}(T) = \mathrm{ind}(T_M) = \alpha(T_M) - \beta(T_M) = \beta(T^*_{N^{\perp}}) - \alpha(T^*_{N^{\perp}}) = -\mathrm{ind}(T^*_{N^{\perp}}) = -\mathrm{ind}(T^*)$. □

From the previous results, we obtain the next corollary.

**Corollary 2.27** *If $T \in L(X)$ is a B-Fredholm operator, then $\mathcal{R}(T^*) + \mathcal{N}(T^{*d})$ is closed in $\sigma(X^*, X)$ and $T^*$ is a B-Fredholm operator with $\alpha(T_{[d]}) = \beta(T^*_{[d]})$, $\beta(T_{[d]}) = \alpha(T^*_{[d]})$ and $ind(T) = ind(T_{[d]}) = -ind(T^*_{[d]}) = -ind(T^*)$; where $d = dis(T)$.*

***Proof*** We know from [19, Theorem 7] that $T$ is decomposable in the Kato's sense of degree $d'$. More precisely, there exists $(M, N) \in Red(T)$ such that $T_M$ is semi-regular, $T_N$ is nilpotent of degree $d'$ and $N \subset \mathcal{N}(T^d)$. Thus $N = \mathcal{N}(T^d_N) = \mathcal{N}(T^{d'}_N)$ and then $d \geq d'$. Let us to show that $d = d'$. Let $m \geq d'$, since $T_M$ is semi-regular then $\mathcal{R}(T) + \mathcal{N}(T^m) = \mathcal{R}(T_N) + \mathcal{N}(T^m_N) + \mathcal{R}(T_M) + \mathcal{N}(T^m_M) = N \oplus \mathcal{R}(T_M) = \mathcal{R}(T_N) + \mathcal{N}(T^d_N) + \mathcal{R}(T_M) + \mathcal{N}(T^d_M) = \mathcal{R}(T) + \mathcal{N}(T^d)$. Hence $d = d'$ and $\mathcal{R}(T_M) \oplus N = \mathcal{R}(T) + \mathcal{N}(T^d)$ is closed. On the other hand, it is well known that

$T^*$ is decomposable in the Kato's sense of degree $\mathrm{dis}(T^*) = d$, and from Theorem 2.26 we have $\alpha(T_{[d]}) = \beta(T^*)$ and $\beta(T_{[d]}) = \alpha(T^*)$. Hence $T^*$ is a B-Fredholm, and from Lemma 2.25 and what precedes we have $\mathcal{R}(T^*_{N^\perp}) \oplus M^\perp = \mathcal{R}(T^*) + \mathcal{N}(T^{*d})$ is closed in $\sigma(X^*, X)$. Consequently, $\alpha(T_{[d]}) = \beta(T^*_{[d]})$, $\beta(T_{[d]}) = \alpha(T^*_{[d]})$ and $\mathrm{ind}(T) = \mathrm{ind}(T_{[d]}) = -\mathrm{ind}(T^*_{[d]}) = -\mathrm{ind}(T^*)$. □

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