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Arnold stability and Misiołek curvature

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Abstract
Let M be a compact 2-dimensional Riemannian manifold with smooth boundary and consider the incompressible Euler equation on M. In the case that M is the straight periodic channel, the annulus or the disc with the Euclidean metric, it was proved by T. D. Drivas, G. Misiołek, B. Shi, and the second author that all Arnold stable solutions have no conjugate point on the volume-preserving diffeomorphism group $\mathcal{D}^s_\mu(M)$. They also proposed a question which asks whether this is true or not for any M. In this article, we give a partial positive answer. More precisely, we show that the Misiołek curvature of any Arnold stable solution is nonpositive. The positivity of the Misiołek curvature is a sufficient condition for the existence of a conjugate point.

Keywords Euler equation · Arnold stable flow · Diffeomorphism group · Conjugate point

Mathematics Subject Classification Primary 35Q35; Secondary 35Q31

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1 Introduction

Let (M, g) be a compact 2-dimensional Riemannian manifold possibly with smooth boundary ∂M and consider the incompressible Euler equation on M:

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla_u u &= -\operatorname{grad} p \qquad \text{on } M, \\ \operatorname{div} u &= 0 \qquad \text{on } M, \\ g(u, \nu) &= 0 \qquad \text{on } \partial M, \end{aligned} \tag{1.1}$$

where ν is a unit normal vector field on ∂M. For the case that M is the straight periodic channel, the annulus or the disc with the Euclidean metric, it was proved by T. D. Drivas, G. Misiołek, B. Shi, and the second author [6, Thm. 3] that all Arnold stable solutions (see Defition 2.5) contain no conjugate points when viewed as geodesics in the group $\mathcal{D}^s_\mu(M)$ of volume-preserving Sobolev H^s diffeomorphisms of M starting from the identity (fluid's initial configuration). They also proposed a question [6, Question 2] which asks whether this is true or not for any compact two-dimensional Riemannian manifold M with smooth boundary. In this article, we give a partial positive answer. For the precise statement, we recall the Misiołek curvature. Let μ be the volume form on M and set

$$\langle V, W \rangle := \int_M g(V, W)\mu, \tag{1.2}$$

$$|V|^2 := \langle V, V \rangle \tag{1.3}$$

for any vector fields V, W on M, which are tangent to ∂M.

Definition 1.1 (cf. [12, (1.3)], [13, Lems. B.6, B.7]) Let u be a stationary solution of (1.1) and Y a divergence-free vector field on M, which is tangent to ∂M. The Misiołek curvature defined as

$$\mathfrak{mc}_{u,Y} := -|[u, Y]|^2 - \langle [[u, Y], Y], u \rangle. \tag{1.4}$$

The importance of the Misiołek curvature is the following. We write $T_e\mathcal{D}^s_\mu(M)$ for the tangent space of $\mathcal{D}^s_\mu(M)$ at the identity element $e \in \mathcal{D}^s_\mu(M)$. We identify $T_e\mathcal{D}^s_\mu(M)$ with the space of all Sobolev H^s divergence-free vector fields on M, which are tangent to ∂M.

Fact 1.2 ([10] (see also [12])) *Let $s > 2 + \frac{n}{2}$ and M be a compact n-dimensional Riemannian manifold, possibly with smooth boundary. Suppose that $V \in T_e\mathcal{D}^s_\mu(M)$ is a stationary solution of the Euler Eq. (1.1) on M and take a geodesic η on $\mathcal{D}^s_\mu(M)$ satisfying $V = \dot{\eta} \circ \eta^{-1}$. Then if we have $\mathfrak{mc}_{V,W} > 0$ for some $W \in T_e D^s_\mu(M)$, there exists a point conjugate to $e \in \mathcal{D}^s_\mu(M)$ along $\eta(t)$ on $0 \leq t \leq t_0$ for some $t_0 > 0$.*

Remark 1.3 This was only proved for the case that M has no boundary in [10] (and [12]). Thus, we explain how to apply the proof in [10] to the case M has a boundary in the appendix.

This fact states that the positivity of the Misiołek curvature ensures the existence of a conjugate point. This criteria for the existence of a conjugate point by using $\mathfrak{mc}$ was first used in [10] by G. Misiołek and recently attracts attention again [6, 12, 13]. We note that this is only a sufficient condition. In fact, there is a stationary solution having a conjugate point, whose Misiołek curvature is all nonpositive (see [12, Rem. 3]). However, philosophically, the nonpositivity of the Misiołek curvature suggests the nonexistence of a conjugate point.

Our main theorem of this article is the following. See Sect. 2 for unexplained notions.

Theorem 1.4 *Let M be a two-dimensional Riemannian manifold possibly with smooth boundary, u an Arnold stable solution of* (1.1)*, and Y a divergence-free vector field on M, which is tangent to ∂M. Suppose that there exist stream functions of u and Y. Then, we have*

$$\mathfrak{mc}_{u,Y} \leq 0.$$

As a corollary, we have the following. Let S^1 be the one-dimensional sphere and $I := [-1, 1]$.

Theorem 1.5 *Let M be a two-dimensional Riemannian manifold possibly with smooth boundary. Suppose that either $H^1_{dR}(M) = 0$ or M is diffeomorphic to $I \times S^1$. Then, for any Arnold stable solution u of* (1.1) *and any divergence-free vector field Y on M, which is tangent to ∂M, we have*

$$\mathfrak{mc}_{u,Y} \leq 0.$$

Remark 1.6 Note that if M is the disc, then we have $H^1_{dR}(M) = 0$. Moreover, if M is either the straight periodic channel or the annulus, then M is diffeomorphic to $I \times S^1$.

Remark 1.7 It looks like that Theorem 1.5 agrees with the intuitive argument in [6] before Question 2.

Remark 1.8 Let $a > 1$ and

$$M_a := \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2(1 - z^2)\right\}$$

be a two-dimensional ellipsoid with the Riemannian metric induced by that of $\mathbb{R}^3$. Note that we have $H^1_{dR}(M_a) = 0$ because M_a is diffeomorphic to S^2 for any $a > 1$. Thus, Theorem 1.5 implies $\mathfrak{mc}_{u,Y} \leq 0$ for any Arnold stable solution u of (1.1) and any divergence-free vector field Y on M_a.

On the other hand, Fact 1.10, which is given below, implies that for any zonal flow u (see Definition 1.9 given below for the definition) on M_a whose support is contained in $M_a \setminus \{(0, 0, 1), (0, 0, -1)\}$, there exists a divergence-free vector field Y on M_a satisfying $\mathfrak{mc}_{u,Y} > 0$. This implies that any zonal flow u on M_a whose support is contained in $M_a \setminus \{(0, 0, 1), (0, 0, -1)\}$ never be Arnold stable by the assertion of the previous paragraph.

Definition 1.9 ([12, (1.4)]) We say that a vector field Z on M_a is a *zonal flow* if Z has the following form

$$Z = F(z)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

for some function $F(z) : [-1, 1] \to \mathbb{R}$.

Note that a zonal flow is always a stationary solution of the incompressible Euler equation (1.1) on M_a.

Fact 1.10 ([12, Thm. 1.2]) *Let $a > 1$. Then, for any zonal flow u on M_a whose support is contained in $M_a \backslash \{(0, 0, 1), (0, 0, -1)\}$, there exists a divergence-free vector field Y on M_a satisfying $\mathfrak{mc}_{u,Y} > 0$.*

By V. I. Arnold [1], geodesics on $\mathcal{D}^s_\mu(M)$ correspond to solutions of (1.1). One can thus speculate that existence of a conjugate point is indicative of Lagrangian stability of the corresponding solution.

This article is organized as follows. In Sect. 2, we recall the definition and properties of Arnold stability. In Sects. 3 and 4, we prove Theorems 1.4 and 1.5, respectively. In Appendix A, we explain how to apply the proof in [10] to the case M has a boundary. In Appendix B, we state the basic results, which are used in the proof of Theorem 1.4.

2 Arnold stable flow

In this section, we recall that the definition of an Arnold stable flow and its basic property. Although almost all the materials in this section are well known, we prove some results for the convenience. Main references are [2, Sect. II.4.A], [5] and [6, Sect. 5].

Let (M, g) be a compact 2-dimensional Riemannian manifold possibly with smooth boundary ∂M and consider the incompressible Euler Eq. (1.1) on M.

Definition 2.1 Let u be a divergence-free vector field on M, which is tangent to ∂M. A function ψ on M is called a stream function of u if ψ satisfies

$$\star \operatorname{grad} \psi = u, \tag{2.1}$$

where $\star$ is the Hodge star. We write

$$\Delta := \operatorname{div} \circ \operatorname{grad}$$

for the Laplace-Beltrami operator. In the case (2.1), we set

$$\omega := -\operatorname{div} \star u = \Delta\psi. \tag{2.2}$$

Lemma 2.2 *Let u be a stationary solution of (1.1) on a two-dimensional Riemannian manifold M possibly with smooth boundary ∂M. Suppose that there exists a function ψ on M such that $u = \star \operatorname{grad} \psi$. Then $\star \operatorname{grad} \psi$ and $\operatorname{grad} \omega$ are orthogonal. In particular, $\operatorname{grad} \psi$ and $\operatorname{grad} \omega$ are collinear.*

Proof Because u is a time independent solution of (1.1), we have

$$\nabla_u u = -\operatorname{grad} p, \qquad \operatorname{div}(u) = 0. \tag{2.3}$$

Recall that $\operatorname{div}(\cdot) = \star d \star (\cdot)^\flat$, where d is the exterior derivative and $\flat$ is the musical isomorphism. We note that the Hodge star $\star$ commutes with $\flat$ and $\star^2 = -1$ as an operator on the space of vector fields. Thus, applying the operator $\star \circ \operatorname{div} \circ \star = d(\cdot)^\flat$ to the first equation of (2.3), we have

$$d(\nabla_u u)^\flat = 0 \tag{2.4}$$

by $(\operatorname{grad} p)^\flat = dp$ and $d^2 = 0$. Recall (cf. [2, Thm. 1.17 in Sect. IV.1.D])

$$(\nabla_u u)^\flat = L_u(u^\flat) - \frac{1}{2} d\left(g(u, u)\right),$$

where L_u is the Lie derivative. Thus, (2.4) implies

$$L_u(d(u^\flat)) = 0 \tag{2.5}$$

by $[L_u, d] = 0$. On the other hand, the assumption $u = \star \operatorname{grad} \psi$ implies

$$\begin{aligned} d(u^\flat) &= d(\star(\operatorname{grad} \psi)^\flat) \\ &= \operatorname{div}(\operatorname{grad} \psi)\mu \\ &= \omega\mu \end{aligned}$$

by $\star\mu = 1$ and (2.2). Thus, (2.5) implies

$$\begin{aligned} 0 &= L_u(d(u^\flat)) \\ &= L_u(\omega\mu). \end{aligned}$$

By $L_u(\mu) = \operatorname{div}(u)\mu = 0$ and the Leibniz rule of L_u, this is equal to

$$\begin{aligned} &= L_u(\omega)\mu \\ &= g(u, \operatorname{grad} \omega)\mu, \end{aligned}$$

which completes the proof by $u = \star \operatorname{grad} \psi$. □

Lemma 2.3 *Let M be a two-dimensional Riemannian manifold possibly with smooth boundary ∂M and u a stationary solution of (1.1) on M having ψ as its stream function. Set $\omega := \Delta\psi$. Then, there exits a (possibly multivalued) function F on $\mathbb{R}$ satisfying*

$$\omega(x) = F(\psi(x)) \quad \textit{for any } x \in M.$$

Proof By Lemma 2.2, grad ψ and grad ω are collinear. Thus, there exits a (possibly multivalued) function f on $\mathbb{R}$ satisfying

$$\operatorname{grad}\omega(x) = f(\psi(x))\operatorname{grad}\psi(x) \quad \text{for any } x \in M.$$

Take a primitive function F of f (as a function on $\mathbb{R}$). By the chain rule, we have

$$\operatorname{grad}(F(\psi)) = F'(\psi)\operatorname{grad}\psi = f(\psi)\operatorname{grad}\psi = \operatorname{grad}\omega. \tag{2.6}$$

Note that the difference of functions which have the same gradient must be a constant function. Thus, adding a suitable constant to F (as a function on $\mathbb{R}$) if necessary, we have the lemma. □

Corollary 2.4 *Let M be a two-dimensional Riemannian manifold possibly with smooth boundary ∂M and u be a stationary solution of* (1.1) *on M having ψ as its stream function. Set $\omega := \Delta\psi$. Then, the function F in Lemma 2.3 satisfies*

$$F'(\psi) = \frac{\operatorname{grad}\omega}{\operatorname{grad}\psi} = \frac{\operatorname{grad}\Delta\psi}{\operatorname{grad}\psi}. \tag{2.7}$$

Proof This is a consequence of (2.6). Note that by the collinearity of grad ω and grad ψ (see Lemma 2.2), the fraction of (2.7) makes sense. □

Write $\lambda_1 > 0$ for the first eigenvalue of $-\Delta$. Therefore, we have

$$\Delta f \leq -\lambda_1 f \tag{2.8}$$

for any function f on M satisfying $\int_M f\mu = 0$ (resp. $f|_{\partial M} = 0$) if ∂M is empty (resp. nonempty), where μ is the volume form on M.

Definition 2.5 ([1, Sect. 10], or [2, Thm. 4.3 in Sect. II.4.A].) Let M be a two-dimensional Riemannian manifold possibly with smooth boundary ∂M. We say that a stationary solution u of (1.1) is *Arnold stable* if the corresponding function F in Lemma 2.3 satisfies

$$-\lambda_1 < F'(\psi) < 0, \qquad \text{or} \qquad 0 < F'(\psi) < \infty. \tag{2.9}$$

Lemma 2.6 ([5, Prop. 1.1]) *Let M be a two-dimensional Riemannian manifold possibly with smooth boundary ∂M and u an Arnold stable stationary solution of* (1.1) *with stream function ψ. Suppose that there exits a Killing vector field X on M, which is tangent to ∂M. Then we have $X\psi = 0$.*

Proof Note that $\Delta L_X = L_X\Delta$ as an operator on the space of functions because X is Killing, where L_X is the Lie derivative. By the definition (see (2.2) and Lemma 2.3), we have

$$\Delta\psi = F(\psi).$$

The chain rule and $L_X \Delta = \Delta L_X$ imply

$$(\Delta - F'(\psi))X\psi = 0.$$

Thus (2.8) and (2.9) imply the lemma in the case $\partial M \neq \emptyset$ because $X\psi|_{\partial M} = 0$ by the assumption that ψ is the stream function of u. In the case $\partial M = \emptyset$, we note that $\int_M X\psi\mu = \int_M L_X(\psi)\mu = 0$ by $L_X(\mu) = \mathrm{div}(X)\mu = 0$, the Leibniz rule of the Lie derivative, and the Stokes thoerem. Thus, (2.8) and (2.9) also imply the lemma in this case. □

Remark 2.7 The equation $\Delta L_X = L_X \Delta$ is also true as an operator on the space of p-forms if we interpret that Δ is the Laplace-de Rham operator $\Delta := (-1)^{n(p+1)+1}(d\star d\star + \star d\star d)$, where $n := \dim M$. This is because L_X commutes the Hodge star operator if X is Killing (see [14, (14)], for example).

3 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. In the proof, we use freely lemmas in Appendix B.

Proof of Theorem 1.4 By Lemma B.16, $(M, g, \omega, \star)$ is an almost Kähler manifold, where $\star$ is the Hodge star operator. We write H_f for the Hamiltonian vector field of a function f on M (Definition B.1). By the assumption, there exist functions ψ and ϕ satisfying

$$u = \star\,\mathrm{grad}\,\psi, \quad Y = \star\,\mathrm{grad}\,\phi \qquad \in \mathfrak{X}^t(M),$$

where $\mathfrak{X}^t(M)$ is the space of vector fields on M, which are tangent to ∂M. Then, Lemma B.10 implies

$$u = H_\psi, \quad Y = H_\phi \qquad \in \mathfrak{X}^t(M).$$

Thus, we have

$$\begin{aligned} |[u, Y]|^2 &= \langle [H_\psi, H_\phi], [H_\psi, H_\phi]\rangle \\ &= \langle H_{\{\psi,\phi\}}, H_{\{\psi,\phi\}}\rangle \\ &= -\int_M \{\psi, \phi\}\Delta\{\psi, \phi\}\mu \end{aligned} \tag{3.1}$$

by Lemmas B.8 and B.19, where $\langle , \rangle$ is given by (1.2) and $\{,\}$ is the Poisson bracket.

On the other hand, we have

$$\begin{aligned} \langle [[u, Y], Y], u\rangle &= \langle [[H_\psi, H_\phi], H_\phi], H_\psi\rangle \\ &= \langle H_{\{\{\psi,\phi\},\phi\}}, H_\psi\rangle \\ &= \int_M -\{\{\psi, \phi\}, \phi\}\Delta\psi\mu \end{aligned}$$

by Lemmas B.8 and B.19. By Lemmas B.12 and B.17, this is equal to

$$
\begin{aligned}
&= -\int_M \{\psi, \phi\}\{\phi, \Delta\psi\} \\
&= \int_M \{\psi, \phi\}\{\Delta\psi, \phi\}\mu
\end{aligned}
\tag{3.2}
$$

by Lemmas B.5.

The definition (1.4) of $\mathfrak{mc}$ and Eqs. (3.1), (3.2) imply

$$
\begin{aligned}
\mathfrak{mc}_{u,Y} &= \int_M \{\psi, \phi\} \left(\Delta\{\psi, \phi\} - \{\Delta\psi, \phi\}\right) \\
&= \int_M H_\psi(\phi) \left(\Delta H_\psi - H_\omega\right)(\phi)\mu
\end{aligned}
\tag{3.3}
$$

by Lemma B.6 and (2.2). On the other hand, there exists a function F satisfying

$$
F'(\psi)\,\mathrm{grad}\,\psi = \mathrm{grad}\,\omega
$$

by the Arnold stable assumption (Lemma 2.3). Applying the Hodge star, we have

$$
F'(\psi)H_\psi = H_\omega \tag{3.4}
$$

by Lemma B.10. Thus, (3.3) and (3.4) imply

$$
\begin{aligned}
\mathfrak{mc}_{u,Y} &= \int_M H_\psi(\phi) \left(\Delta H_\psi - F'(\psi)H_\psi\right)(\phi)\mu \\
&= \int_M H_\psi(\phi) \left(\Delta - F'(\psi)\right) H_\psi(\phi)\mu.
\end{aligned}
$$

Note that $H_\psi(\phi)|_{\partial M} = \{\psi, \phi\}|_{\partial M} = 0$ by Lemma B.17. Therefore, the theorem is a consequence of (2.8) and (2.9) in the case $\partial M \neq \emptyset$. Moreover, if $\partial M = \emptyset$, we have

$$
\begin{aligned}
\int_M H_\psi(\phi)\mu &= \int_M L_{H_\psi}(\phi\mu) \\
&= \int_M d(\iota_{H_\psi}(\phi\mu)) \\
&= 0
\end{aligned}
$$

by $\mathrm{div}(H_\mu) = 0$ (Lemma B.3) and the Stokes theorem. Thus, (2.8) and (2.9) also imply the theorem in this case. □

4 Proof of Theorem 1.5

In this section, we prove Theorem 1.5. Let M be a two-dimensional Riemannian manifold possibly with smooth boundary ∂M. Recall that $\mathfrak{X}^t(M)$ is the space of vector fields on M, which are tangent to ∂M. For the notational simplicity, we set

$$\begin{aligned}\mathfrak{X}^t_\mu(M) &:= \{Y \in \mathfrak{X}^t(M) \mid \operatorname{div}(Y) = 0\},\\ \mathfrak{X}^t_\mu(M)^{str} &:= \{Y \in \mathfrak{X}^t_\mu(M) \mid Y \text{ has a stream function}\},\\ \mathfrak{X}^t_\mu(M)^{no} &:= \mathfrak{X}^t_\mu(M)/\mathfrak{X}^t_\mu(M)^{str}.\end{aligned}$$

Moreover, we write

$$H^1_{dR}(M) := \{\alpha \in \mathcal{E}^1(M) \mid d\alpha = 0\}/d(C^\infty(M)). \tag{4.1}$$

for the 1st de Rham cohomology, where $\mathcal{E}^1(M)$ is the space of one-forms on M. Before proving Theorem 1.5, we need a lemma.

Lemma 4.1 *Let M be a two-dimensional Riemannian manifold possibly with smooth boundary ∂M and $j : \partial M \hookrightarrow M$ the inclusion. Then, $\mathfrak{X}^t_\mu(M)^{no}$ is isomorphic to the kernel of $j^* : H^1_{dR}(M) \to H^1_{dR}(\partial M)$, where j^* is the pull back. (We set $H^1_{dR}(\partial M) := 0$ if $\partial M = \emptyset$.)*

Remark 4.2 The kernel $j^* : H^1_{dR}(M) \to H^1_{dR}(\partial M)$ is isomorphic to the relative de Rham cohomology $H^1(j)$, see [4, Sect. 6 of Ch. 1] or [15, Sect. 8.2], for example.

Proof of Lemma 4.1 Let Y be a vector field on M (which is not necessarily tangent to ∂M). Note that

$$\operatorname{div}(Y) = \star d(\star Y^\flat).$$

Thus, Y is divergence-free if and only if the one-form $\star Y^\flat$ is closed. Therefore, we have

$$\begin{aligned}\mathfrak{X}_\mu(M) &\simeq \{\alpha \in \mathcal{E}^1(M) \mid d\alpha = 0\}\\ Y &\mapsto \star Y^\flat,\end{aligned} \tag{4.2}$$

where $\mathfrak{X}_\mu(M)$ is the space of divergence-free vector fields (which are not necessarily tangent to ∂M). Moreover, by definition, Y has a stream function if and only if

$$Y = \star \operatorname{grad} \phi$$

for some function ϕ on M. Applying the musical isomorphism $\flat$ and the Hodge operator $\star$, we have

$$\star Y^\flat = -d\phi.$$

Thus, Y has a stream function if and only if the one-form $\star Y^\flat$ is exact. Therefore, we have an isomorphism

$$\begin{aligned}\{Y \in \mathfrak{X}_\mu(M) \mid Y \text{ has a stream function}\} &\simeq d(C^\infty(M)) \\ Y &\mapsto \star Y^\flat. \end{aligned} \tag{4.3}$$

Moreover, Y is tangent to ∂M if and only if

$$g(\star Y, W)|_{\partial M} = 0$$

for any vector fields W on ∂M because $\star$ is the $\frac{\pi}{2}$ rotation operator. This equation is equivalent to

$$\star Y^\flat(W)|_{\partial M} = 0$$

for any vector fields W on ∂M. Thus, we have an isomorphism

$$\begin{aligned}\mathfrak{X}^t(M) &\simeq \{\alpha \in \mathcal{E}^1(M) \mid j^*(\alpha) = 0\} \\ Y &\mapsto \star Y^\flat. \end{aligned} \tag{4.4}$$

Then, the lemma is a consequence of (4.2), (4.3), and (4.4) by the definition (4.1) of $H^1_{dR}(M)$. □

We prove Theorem 1.5 by using this lemma.

Proof of Theorem 1.5 By Theorem 1.4, it is enough to show $\mathfrak{X}^t_\mu(M)^{no} = 0$. Moreover, by Lemma 4.1, it is enough to show $j^* : H^1_{dR}(M) \to H^1_{dR}(\partial M)$ is injective. In the case $H^1_{dR}(M) = 0$, this is obvious. Therefore, we only consider the case that M is diffeomorphic to $I \times S^1$. Then, the de Rham cohomology only depends on the differentiable structure of M, it is enough to prove the theorem in the case $M = I \times S^1$. Thus, we have to show that if $\alpha \in \mathcal{E}^1(I \times S^1)$ satisfy $d\alpha = 0$ and $j^*\alpha = 0$, then, there exists a function ϕ on $I \times S^1$ such that $d\phi = \alpha$. For this end, we take a coordinate $(r, \theta) \in I \times S^1$ and $\alpha \in \mathcal{E}^1(I \times S^1)$ satisfying $d\alpha = 0$ and $j^*\alpha = 0$. Write

$$\alpha = f(r, \theta)dr + h(r, \theta)d\theta. \tag{4.5}$$

Then, $d\alpha = 0$ implies

$$(-\partial_\theta f + \partial_r h)dr \wedge d\theta = 0.$$

Thus, by considering the Fourier series

$$f(r, \theta) = \sum_{n \in \mathbb{Z}} f_n(r)e^{in\theta}, \qquad h(r, \theta) = \sum_{n \in \mathbb{Z}} h_n(r)e^{in\theta},$$

we have

$$inf_n(r) = \partial_r h_n(r) \tag{4.6}$$

for all $n \in \mathbb{Z}$. In particular, we have

$$\partial_r h_0(r) = 0. \tag{4.7}$$

On the other hand, $j^*(\alpha) = 0$ implies

$$h(\pm 1, \theta) = \sum_{n \in \mathbb{Z}} h_n(\pm 1) e^{in\theta} = 0$$

for any $\theta \in S^1$ because j is the inclusion $\partial(I \times S^1) = \{\pm 1\} \times S^1 \hookrightarrow I \times S^1$. In particular, we have

$$h_0(\pm 1) = 0. \tag{4.8}$$

Thus, (4.7) and (4.8) imply

$$h_0 = 0. \tag{4.9}$$

Take a primitive function $F_0(r)$ of $f_0(r)$ and define a function ϕ on $I \times M$ by

$$\phi(r, \theta) := F_0(r) + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{h_n(r)}{in} e^{in\theta}.$$

Then, (4.5) and (4.6) imply

$$d\phi = \alpha.$$

This completes the proof. □

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Appendix: A sufficient criterion of Misiołek

In this appendix, we explain how to apply the proof of Fact 1.2 in [10] to the case M has a boundary.

A.1 $\mathcal{D}^s_\mu(M)$ in the case M has a boundary

In this subsection, we recall briefly the theory of volume-preserving diffeomorphism group $\mathcal{D}^s_\mu(M)$ in the case that M has a boundary. Main reference is [7].

Let M be a compact n-dimensional Riemannian manifold with smooth boundary, $\mathcal{D}^s_\mu(M)$ the group of all diffeomorphisms of Sobolev class H^s preserving the volume form on M. Then, the tangent space $T_e\mathcal{D}^s_\mu(M)$ of $\mathcal{D}^s_\mu(M)$ at the identity element $e \in \mathcal{D}^s_\mu(M)$ is identified with the space of divergence-free vector fields on M which are tangent to ∂M. If $s > \frac{n}{2}+1$, $\mathcal{D}^s_\mu(M)$ has an infinite-dimensional Hilbert manifold structure with the right-invariant L^2 Riemannian metric given by

$$\langle X, Y\rangle := \int_M g(X, Y)\mu,$$

where $X, Y \in T_e\mathcal{D}^s_\mu(M)$.

By V. I. Arnold [1], a solution u of the incompressible Euler Eq. (1.1) on M corresponds to a geodesic η on $\mathcal{D}^s_\mu(M)$ starting at $e \in \mathcal{D}^s_\mu(M)$ via $u = \dot{\eta}\circ\eta^{-1}$. Thus, it is important to study of the geometry of $\mathcal{D}^s_\mu(M)$. In particular, the existence of a conjugate point on a geodesic has attractive considerable attention because it is related to the Lagrangian stability of the corresponding solution.

A.2 Sketch of the proof of Fact 1.2

In this subsection, we explain how to apply the proof of Fact 1.2 in [10] to the case that M has a boundary. For the convenience, we rewrite Fact 1.2.

Fact 1.2 *Let M be a compact n-dimensional Riemannian manifold with smooth boundary and $s > 2 + \frac{n}{2}$. Suppose that $V \in T_e\mathcal{D}^s_\mu(M)$ is a stationary solution of the Euler Eq. (1.1) on M and take a geodesic η on $\mathcal{D}^s_\mu(M)$ satisfying $V = \dot{\eta}\circ\eta^{-1}$. Then if we have $\mathfrak{mc}_{V,W} > 0$ for some $W \in T_e D^s_\mu(M)$, there exists a point conjugate to $e \in \mathcal{D}^s_\mu(M)$ along $\eta(t)$ on $0 \le t \le t_0$ for some $t_0 > 0$.*

Sketch of the proof of Fact 1.2 Because the Riemannian metric of $\mathcal{D}^s_\mu(M)$ is right invariant, Theorem B.5 in [13] shows that there exist $t_0 > 0$ and a vector field $\widetilde{W}$ on η satisfying $\widetilde{W}(0) = \widetilde{W}(t_0) = 0$ and

$$E''(\eta)_0^{t_0}(\widetilde{W}, \widetilde{W}) < 0 \tag{A.1}$$

by the assumption $\mathfrak{mc}_{V,W} > 0$. Here $E''(\eta)_0^{t_0}(\widetilde{W}, \widetilde{W})$ is the second variation of the energy function $E_0^{t_0}(\eta)$ of η:

$$E_0^{t_0}(\eta) := \frac{1}{2}\int_0^{t_0}\langle\dot{\eta}, \dot{\eta}\rangle dt.$$

On the other hand, the same argument of [10, Lem. 3] gives

$$E''(\eta)_0^{t_0}(Z, Z) \geq 0 \tag{A.2}$$

for any vector field $Z(t)$ on η with $Z(0) = Z(t_0) = 0$ if there exists no conjugate point on $\eta(t)$ $(0 \leq t \leq t_0)$. The essential point of the argument of [10, Lem. 3] is that the differential of the exponential map is bounded operator, which is deduced by the boundedness of the curvature of $\mathcal{D}^s_\mu(M)$ in [10, Lem. 3]. This boundedness of the curvature is also guaranteed for the case that M has a boundary by [9, Prop. 3.6]. Thus, the same argument is valid in the case that M has a boundary and the contradiction of (A.1) to (A.2) gives the desired result. □

B Some basic results

In this section, we recall basic results on symplectic and almost Kähler manifolds. Although almost all the materials in this section are well known, we prove some results for the convenience. Main references are [3, Sect. 4], [8, Sect. 22] and [11, Sect. 2].

B.1 Symplectic manifold with boundary

Let (M, ω) be a compact symplectic manifold possibly with smooth boundary ∂M. We write $\mathfrak{X}(M)$ (resp. $\mathfrak{X}^t(M)$) for the space of vector fields on M (resp. which are tangent to ∂M).

Definition B.1 Let $f \in C^\infty(M)$. Then, the Hamilton vector field $H_f \in \mathfrak{X}(M)$ of f is defined by the equation

$$\iota_{H_f}\omega = df \tag{B.1}$$

where d is the exterior derivative and ι_{H_f} is the interior derivative.

We always take

$$\mu := \frac{1}{n!}\omega^n := \frac{1}{n!}\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}. \tag{B.2}$$

as the volume form on M, where $n := \frac{\dim M}{2}$.

Definition B.2 Let $V \in \mathfrak{X}(M)$. The divergence of V is defined by

$$\operatorname{div}(V)\mu = L_V(\mu)$$

where L_V is the Lie derivative.

Lemma B.3 *Let $f \in C^\infty(M)$. Then, we have*

$$\operatorname{div}(H_f) = 0.$$

Proof By (B.2) and the Cartan magic formula $L_{H_f} = d \circ \iota_{H_f} + \iota_{H_f} \circ d$, we have

$$n! L_{H_f}(\mu) = d(\iota_{H_f}(\omega^n))$$

because $d\omega = 0$. By the graded Leibniz rule of the interior derivative and (B.1), this is equal to

$$= nd(df \wedge \omega^{n-1}).$$

By the Leibniz rule of d, and $d^2 = 0$, this is equal to

$$\begin{aligned} &= n\left(ddf \wedge \omega^{n-1} - df \wedge (d\omega^{n-1})\right) \\ &= 0, \end{aligned}$$

which completes the proof. □

Definition B.4 Let $f, g \in C^\infty(M)$. The Poisson bracket of f and g is defined by

$$\{f, g\} := -\omega(H_f, H_g) = \omega(H_g, H_f). \tag{B.3}$$

Lemma B.5 *For $f, g \in C^\infty(M)$, we have*

$$\{f, g\} = -\{g, f\}.$$

Proof By the skew-symmetry of ω and the definition (B.3), this lemma is obvious. □

Lemma B.6 *For $f, g \in C^\infty(M)$, we have*

$$\{f, g\} = -df(H_g) = -H_g(f) = dg(H_f) = H_f(g).$$

Proof This is obvious from (B.1), (B.4) and the definition of the exterior derivative d. □

Lemma B.7 *For $f, g \in C^\infty(M)$, we have*

$$\{f, gh\} = \{f, g\}h + \{f, h\}g. \tag{B.4}$$

Proof Lemma B.6 implies

$$\begin{aligned} \{f, gh\} &= d(gh)(H_f) \\ &= hdg(H_f) + gdh(H_f) \end{aligned}$$

by the Leibniz rule of d. This completes the proof by Lemma B.6. □

Lemma B.8 *For $f, g \in C^\infty(M)$, we have*

$$[H_f, H_g] = H_{\{f,g\}}.$$

Proof Recall that the Lie derivative and the interior derivative satisfy

$$\iota_{[V,W]} = L_V \circ \iota_W - \iota_W \circ L_V, \tag{B.5}$$
$$L_V = \iota_V \circ d + d \circ \iota_V \tag{B.6}$$

for any $V, W \in \mathfrak{X}(M)$. Thus, we have

$$\begin{aligned}\iota_{[H_f,H_g]}(\omega) &= (L_{H_f} \circ \iota_{H_g} - \iota_{H_g} \circ L_{H_f})(\omega)\\ &= (\iota_{H_f} \circ d + d \circ \iota_{H_f})(\iota_{H_g}(\omega)) - \iota_{H_g} \circ L_{H_f}(\omega)\end{aligned}$$

Moreover, we have

$$\begin{aligned}d\iota_{H_g}(\omega) &= ddg\\ &= 0,\\ L_{H_f}(\omega) &= (\iota_{H_f} \circ d + d \circ \iota_{H_f})(\omega)\\ &= 0\end{aligned}$$

by $d\omega = 0$. These impliy

$$\begin{aligned}\iota_{[H_f,H_g]}(\omega) &= d \circ \iota_{H_f} \circ \iota_{H_g}(\omega)\\ &= d(dg(H_f)).\end{aligned}$$

This completes the proof by Definition B.1 and Lemma B.6. □

B.2 Almost Kähler manifold

Let (M, g, ω, J) be a almost Kähler manifold possibly with smooth boundary ∂M. Namely, g is a Riemannian metric on M, ω is a symplectic form on M, and J is an operator on the tangent bundle TM on M satisfying

$$J^2 = -1, \tag{B.7}$$
$$g(V, W) = \omega(JV, W) \tag{B.8}$$

for any $V, W \in \mathfrak{X}(M)$.

Lemma B.9 *Let $V, W \in \mathfrak{X}(M)$. Then, we have*

$$g(JV, JW) = g(V, W)$$

for any $V, W \in \mathfrak{X}(M)$.

Proof By (B.7), (B.8), and the skew-symmetry of ω, we have

$$g(JV, JW) = -\omega(V, JW) = \omega(JW, V) = g(W, V) = g(V, W).$$

This completes the proof. □

Lemma B.10 *Let* $f \in C^\infty(M)$*. Then, we have*

$$H_f = J \operatorname{grad} f.$$

Proof By the definition of the gradient, we have

$$df(\cdot) = g(\operatorname{grad} f, \cdot).$$

This implies the lemma by Definition B.1 and (B.8). □

Lemma B.11 *Let* $f, g \in C^\infty(M)$*. Then, we have*

$$\int_M \{f, g\}\mu = -\int_{\partial M} f\iota_{H_g}(\mu).$$

Proof By Lemma B.6, we have

$$\begin{aligned}\int_M \{f, g\}\mu &= -\int_M H_g(f)\mu \\ &= -\int_M L_{H_g}(f)\mu.\end{aligned}$$

Note $\operatorname{div}(H_f) = 0$ by Lemma B.3. Thus, this is equal to

$$\begin{aligned}&= -\int_M L_{H_g}(f\mu) \\ &= -\int_M d\left(\iota_{H_g}(f\mu)\right)\end{aligned}$$

by the Leibniz rule of the Lie derivative and the Cartan magic formula (B.6). Thus, the Stokes theorem implies the lemma. □

Lemma B.12 *For any* $f, g, h \in C^\infty(M)$*, we have*

$$\int_{\partial M} fh\iota_{H_g}(\mu) = \int_M \left(-\{f, g\}h + f\{g, h\}\right)\mu.$$

In particular, if $fh|_{\partial M} = 0$*, we have*

$$\int_M \{f, g\}h\mu = \int_M f\{g, h\}\mu.$$

Proof By Lemma B.7, we have

$$\int_M \{g, fh\}\mu = \int_M \left(\{g, f\}h + f\{g, h\}\right)\mu.$$

By Lemmas B.5 and B.11, we have the lemma. □

Lemma B.13 *For $f, g \in C^\infty(M)$, we have*

$$\int_M g(H_f, H_g)\mu = \int_M g\,(\mathrm{grad}\, f, \mathrm{grad}\, g)\,\mu.$$

Proof This is obvious by Lemmas B.9 and B.10. □

B.3 L^2 inner product on almost Kähler manifold

Let (M, g, ω, J) be an almost Kähler manifold possibly with smooth boundary ∂M. Set

$$\langle V, W\rangle := \int_M g(V, W)\mu, \tag{B.9}$$

$$|V|^2 := \langle V, V\rangle \tag{B.10}$$

for any $V, W \in \mathfrak{X}(M)$.

Definition B.14 The Laplace-Beltrami operator is defined by

$$\Delta := \mathrm{div} \circ \mathrm{grad}\,.$$

Lemma B.15 *Let $f, g \in C^\infty(M)$. Then, we have*

$$\langle H_f, H_g\rangle = \int_{\partial M} f\,\iota_{\mathrm{grad}\, g}(\mu) - \int_M f\,\Delta(g)\mu.$$

In particular, if $f|_{\partial M} = 0$, we have

$$\langle H_f, H_g\rangle = -\int_M f\,\Delta(g)\mu$$

Proof We have

$$\begin{aligned}\langle H_f, H_g\rangle &= \int_M g(H_f, H_g)\mu\\ &= \int_M g(\mathrm{grad}\, f, \mathrm{grad}\, g)\mu\\ &= \int_M L_{\mathrm{grad}\, g}(f)\mu\end{aligned}$$

by Lemma B.13 and the definition of the gradient. By the Leibniz rule of the Lie derivative, this is equal to

$$= \int_M L_{\mathrm{grad}\, g}(f\mu) - f L_{\mathrm{grad}\, g}(\mu)$$
$$= \int_M d \circ \iota_{\mathrm{grad}\, g}(f\mu) - f\Delta(g)\mu.$$

This completes the proof by the Stokes theorem. □

B.4 2D Riemannian manifold

Let M be an orientable two-dimensional Riemannian manifold possibly with smooth boundary ∂M. Note that $\dim M = 2$ implies that the Hodge star operator $\star$ satisfies

$$\star^2 = -1$$

as an operator on $\mathfrak{X}(M)$.

Lemma B.16 *Define a two-form ω on M by*

$$\omega(V, W) := g(\star V, W),$$

where $V, W \in \mathfrak{X}(M)$. Then, $(M, g, \omega, \star)$ is an almost Kähler manifold.

Proof This follows from the definition. □

Lemma B.17 *Let $f, g \in C^\infty(M)$ with $H_f, H_g \in \mathfrak{X}^t(M)$. Then, we have*

$$\{f, g\}|_{\partial M} = 0.$$

Proof Note that H_f and H_g are tangent to ∂M by the assumption. Therefore, we have

$$g(\star H_f, H_g)|_{\partial M} = 0$$

because $\star$ is the $\frac{\pi}{2}$ rotation operator. On the other hand, we have

$$\{f, g\} = -\omega(H_f, H_g)$$
$$= g(\star H_f, H_g)$$

by Definition B.4 and (B.8). This completes the proof. □

Lemma B.18 *Let $f, g, h \in C^\infty(M)$ with $H_f, H_g, H_h \in \mathfrak{X}^t(M)$. Then, we have*

$$\{\{f, g\}, h\}|_{\partial M} = 0.$$

Proof By Lemma B.17, $\{f, g\}$ is constant on ∂M. Thus, we have the lemma because H_h is tangent to ∂M and $\{\{f, g\}, h\} = -H_h(\{f, g\})$ by Lemma B.6. □

Lemma B.19 *Let $f, g, h \in C^\infty(M)$ with $H_f, H_g \in \mathfrak{X}^t(M)$. Then, we have*

$$\langle H_{\{f,g\}}, H_h\rangle = -\int_M \{f, g\}\Delta(h)\mu,$$
$$\langle H_{\{\{f,g\},g\}}, H_h\rangle = -\int_M \{\{f, g\}, g\}\Delta(h)\mu.$$

Proof This follows from Lemmas B.15, B.17, and B.18. □

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