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Local in time solution to Kolmogorov's two-equation model of turbulence

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Abstract
We prove the existence of local in time solution to Kolmogorov's two-equation model of turbulence in three dimensional domain with periodic boundary conditions. We apply Galerkin method for appropriate truncated problem. Next, we obtain estimates for a limit of approximate solutions to ensure that it satisfies the original problem.

Keywords Kolmogorov's two-equation model of turbulence · Local in time solution · Galerkin method

Mathematics Subject Classification 35Q35 · 76F02

1 Introduction

Firstly, we will provide a short introduction to turbulence modeling. We introduce an idea behind RANS (Reynolds Averaged Navier Stokes, see [1–4]) and explain the necessity of incorporating additional equations to model turbulence. Next, we will introduce Kolmogorov's two equation model and its connection to currently used turbulence models.

Turbulent flow is a fluid motion characterized by rapid changes in velocity and pressure. These fluctuations cause difficulties mainly in finding solutions using numerical methods, which require dense mesh and very short time steps to properly reproduce

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the turbulent flow. Additionally, turbulences appear to be self-similar and display a chaotic behaviour. This bolster a need for precise simulations.

The simplest idea that would decrease the apparent fluctuations of solutions is to consider the average value of the velocity and of the pressure. This is the case in RANS, where the average is taken with respect to the time. Now, let us decompose the velocity v and pressure p:

$$v(x,t) = \overline{v}(x,t) + \widetilde{v}(x,t),\ p(x,t) = \overline{p}(x,t) + \widetilde{p}(x,t),$$

where $\overline{v}$, $\overline{p}$ are time-averaged values and $\widetilde{v}$, $\widetilde{p}$ are fluctuations. We substitute the decomposed functions into the Navier Stokes system and we get (for details see chapter 2 of [1]).

$$\partial_t \overline{v} + \overline{v} \cdot \nabla \overline{v} - \nu \operatorname{div} D\overline{v} + \nabla \overline{p} = -\operatorname{div}\left(\overline{\widetilde{v} \cdot \widetilde{v}}\right).$$

The last term on the right hand side can be approximated by Boussinesq approximation (see [1])

$$-\overline{\widetilde{v} \cdot \widetilde{v}} = \nu_T(\nabla \overline{v} + \nabla^T \overline{v}) - \frac{2}{3} k I,$$

where $\nu_T = \frac{k}{\omega}$, k is the tubulent kinetic energy and ω is the dissipation rate. Finaly, we obtain

$$\partial_t \overline{v} + \overline{v} \cdot \nabla \overline{v} - \nabla \cdot ((\nu + \nu_T) D\overline{v}) + \nabla \left(\overline{p} + \frac{2}{3} k\right) = 0. \tag{1}$$

We see that to close the system we need to introduce additional equations for ω and k. For further details see [1] and [3].

Nowadays, $k - \varepsilon$ and $k - \omega$ are two of the most commonly used models to calculate k and ω. They bear a strong resemblance to Kolmogorov's turbulence model in the way they deal with diffusive terms. In both models, the equation on k uses a squared matrix norm of the symmetric gradient as a source term.

In 1941 Kolmogorov introduced following system of equations describing turbulent flow ([5], English translation in Appendix A [6])

$$\partial_t v + \operatorname{div}(v \otimes v) - 2\nu_0 \operatorname{div}\left(\frac{b}{\omega} D(v)\right) = -\nabla p, \tag{2}$$

$$\partial_t \omega + \operatorname{div}(\omega v) - \kappa_1 \operatorname{div}\left(\frac{b}{\omega} \nabla \omega\right) = -\kappa_2 \omega^2, \tag{3}$$

$$\partial_t b + \operatorname{div}(b v) - \kappa_3 \operatorname{div}\left(\frac{b}{\omega} \nabla b\right) = -b\omega + \kappa_4 \frac{b}{\omega} |D(v)|^2, \tag{4}$$

$$\operatorname{div} v = 0, \tag{5}$$

where v is the mean velocity, ω is the dissipation rate, b represents 2/3 of the mean kinetic energy, p is the sum of the mean pressure and b. The novelty of Kolmogorov's formulation is that it no longer requires prior knowledge of the length scale (size of large eddies) - it can be calculated as $\frac{\sqrt{b}}{\omega}$. Let us notice that the proposed equation on velocity highly resembles the Eq. (1), which appeared in RANS. The $k-\varepsilon$ and $k-\omega$ systems provide similar equations for ω and b with the addition of a source term in the equation for ω.

The physical motivation of the proposed system can be found in [6] and [7]. A mathematical analysis of the difficulties that occur in proving the existence of solutions of such a system can also be found in [7].

Now, we would like to discus the known mathematical results related to Kolmogorov's two-equation model of turbulence. There are two recent results devoted to this problem: [7] and [8] (see the announcement [9]) and our result is inspired by them. In the first one, the Authors consider the system in a bounded $C^{1,1}$ domain with mixed boundary conditions for b and ω and a stick-slip boundary condition for the velocity v. In order to overcome the difficulties related with the last term on the right hand side of (4) the problem is reformulated and the quantity $E := \frac{1}{2}|v|^2 + \frac{2\nu_0}{\kappa_4}b$ is introduced. Then, the Eq. (4) is replaced by

$$\partial_t E + \operatorname{div}(v(E+p)) - 2\nu_0 \operatorname{div}\left(\frac{\kappa_3 b}{\kappa_4 \omega}\nabla b + \frac{b}{\omega}D(v)v\right) + \frac{2\nu_0}{\kappa_4}b\omega = 0.$$

The existence of global-in-time weak solution of the reformulated problem is established. It is also worth mentioning that in [7] the assumption related to the initial value of b tolerates the vanishing of b_0 in some points of the domain. More precisely, the existence of weak solution is proved under the conditions $b_0 \in L^1$, $b_0 > 0$ a.e. and $\ln b_0 \in L^1$.

In the article [8] the Authors consider the system (2–5) in a periodic domain. The existence of global-in-time weak solution is proved, but due to the presence of the strongly nonlinear term $\frac{b}{\omega}|D(v)|^2$, the weak form of equation (4) has to be corrected by a positive measure μ, which is zero, if the weak solution is sufficiently regular. There are also estimates for ω and b (see (4.2) in [8]). These observations are crucial in our reasoning presented below. Concerning to the initial value of b, the assumption is that b_0 is uniformly positive.

2 Notation and main result

Assume that $\Omega = \prod_{i=1}^{3}(0, L_i)$, $L_i, T > 0$ and $\Omega^T = \Omega \times (0, T)$. We shall consider the following problem

$$\partial_t v + \operatorname{div}(v \otimes v) - \nu_0 \operatorname{div}\left(\frac{b}{\omega}D(v)\right) = -\nabla p, \tag{6}$$

$$\partial_t \omega + \operatorname{div}(\omega v) - \kappa_1 \operatorname{div}\left(\frac{b}{\omega}\nabla\omega\right) = -\kappa_2\omega^2, \tag{7}$$

$$\partial_t b + \operatorname{div}(bv) - \kappa_3 \operatorname{div}\left(\frac{b}{\omega}\nabla b\right) = -b\omega + \kappa_4 \frac{b}{\omega}|D(v)|^2, \tag{8}$$

$$\operatorname{div} v = 0, \tag{9}$$

in Ω^T with periodic boundary condition on $\partial\Omega$ and initial condition

$$v_{|t=0} = v_0, \omega_{|t=0} = \omega_0, b_{|t=0} = b_0. \tag{10}$$

Here $\nu_0, \kappa_1, \dots, \kappa_4$ are positive constants. For simplicity, we assume further that all constants except κ_2 are equal to one. The reason is that the constant κ_2 plays an important role in the a priori estimates.
We shall show the local-in-time existence of regular solution of problem (6–10) under some assumption imposed on the initial data. Namely, suppose that there exists positive numbers $b_{\min}, \omega_{\min}, \omega_{\max}$ such that

$$0 < b_{\min} \le b_0(x), \tag{11}$$

$$0 < \omega_{\min} \le \omega_0(x) \le \omega_{\max} \tag{12}$$

on Ω and we set

$$\begin{aligned} b^t_{\min} &= \frac{b_{\min}}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}}, \omega^t_{\min} = \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}, \\ \omega^t_{\max} &= \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}, \mu^t_{\min} = \frac{1}{4}\frac{b^t_{\min}}{\omega^t_{\max}}. \end{aligned} \tag{13}$$

If $m \in \mathbb{N}$, then by $\mathcal{V}^m$ we denote the space of restrictions to Ω of the functions, which belong to the space

$$\{u \in H^m_{loc}(\mathbb{R}^3) : u(\cdot + kL_i e_i) = u(\cdot) \text{ for } k \in \mathbb{Z}, i = 1, 2, 3\}, \tag{14}$$

where $\{e_i\}_{i=1}^3$ form a standard basis in $\mathbb{R}^3$. Next, we define

$$\dot{\mathcal{V}}^m_{\operatorname{div}} = \{v \in \mathcal{V}^m : \operatorname{div} v = 0, \int_\Omega v dx = 0\}. \tag{15}$$

We shall find the solution of the system (6–8) such that $(v, \omega, b) \in \mathcal{X}(T)$, where

$$\mathcal{X}(T) = L^2(0, T; \dot{\mathcal{V}}^3_{\operatorname{div}}) \times L^2(0, T; \mathcal{V}^3)) \times (L^2(0, T; \mathcal{V}^3) \cap (H^1(0, T; H^1(\Omega)))^5. \tag{16}$$

We shall denote by $\|\cdot\|_{k,2}$ the norm in the Sobolev space, i.e.

$$\|f\|_{k,2} = (\|\nabla^k f\|_2^2 + \|f\|_2^2)^{\frac{1}{2}}, \tag{17}$$

where $\|\cdot\|_2$ is L^2 norm on Ω.

Now, we introduce the notion of solution to the system (6–8). We shall show that for any $v_0 \in \dot{\mathcal{V}}^2_{\rm div}$ and strictly positive $\omega_0, b_0 \in \mathcal{V}^2$ there exist positive T and $(v, \omega, b) \in \mathcal{X}(T)$ such that

$$(\partial_t v, w) - (v \otimes v, \nabla w) + (\mu D(v), D(w)) = 0 \text{ for } w \in \dot{\mathcal{V}}^1_{\rm div}, \tag{18}$$

$$(\partial_t \omega, z) - (\omega v, \nabla z) + (\mu \nabla \omega, \nabla z) = -\kappa_2(\omega^2, z) \text{ for } z \in \mathcal{V}^1, \tag{19}$$

$$(\partial_t b, q) - (bv, \nabla q) + (\mu \nabla b, \nabla q) = -(b\omega, q) + (\mu |D(v)|^2, q) \text{ for } q \in \mathcal{V}^1, \tag{20}$$

for a.a. $t \in (0, T)$, where $\mu = \frac{b}{\omega}$ and (10) holds. Recall that $D(v)$ denotes the symmetric part of ∇v and $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$.

Our main result concerning the existence of local in time regular solutions is as follows.

Theorem 1 *Suppose that $\omega_0, b_0 \in \mathcal{V}^2$, $v_0 \in \dot{\mathcal{V}}^2_{\rm div}$ and (11), (12) are satisfied. Then there exist positive t^* and $(v, \omega, b) \in \mathcal{X}(t^*)$ such that (18–20) hold for a.a. $t \in (0, t^*)$ and (10) is satisfied. Furthermore, for each $(x, t) \in \Omega \times [0, t^*)$ the following estimates*

$$\frac{\omega_{\min}}{1 + \kappa_2 \omega_{\min} t} \le \omega(x, t) \le \frac{\omega_{\max}}{1 + \kappa_2 \omega_{\max} t}, \tag{21}$$

$$\frac{b_{\min}}{(1 + \kappa_2 \omega_{\max} t)^{\frac{1}{\kappa_2}}} \le b(x, t) \tag{22}$$

hold. The time of existence of the solution is estimated from below in the following sense: for each positive δ and compact $K \subseteq \{(a, b, c) : 0 < a \le b, 0 < c\}$ there exists positive $t^*_{K,\delta}$, which depends only on κ_2, Ω, δ and K such that if

$$\|v_0\|^2_{2,2} + \|\omega_0\|^2_{2,2} + \|b_0\|^2_{2,2} \le \delta \text{ and } (\omega_{\min}, \omega_{\max}, b_{\min}) \in K, \tag{23}$$

then $t^* \ge t^*_{K,\delta}$. The Sobolev norm is defined by (17).

We note that the last part of the theorem is needed for proving the existence of global in time solution for small data. We address this issue in another paper.

In the next section we prove the above theorem by applying Galerkin method for an appropriate truncated problem. We obtain a priori estimates for the sequence of approximate solutions and by a weak-compactness argument we get a solution of the truncated problem. Finally, after proving some bounds for ω and b we deduce that the obtained solution satisfies the original system of equations.

3 Proof of the main result

The proof of theorem 1 is based on Galerkin method. Hence, we need a basis of the spaces $\mathcal{V}^1$ and $\dot{\mathcal{V}}^1_{\rm div}$. Let $\{w_i\}_{i\in\mathbb{N}}$ be a system of eigenfunctions of Stokes operator in $\dot{\mathcal{V}}^1_{\rm div}$, which is complete and orthogonal in $\dot{\mathcal{V}}^1_{\rm div}$ and orthonormal in $L^2(\Omega)$ (see chap. II.6 in [10]). In particular, $\{w_i\}_{i\in\mathbb{N}}$ are smooth (see formula (6.17), chap. II in

[10]). By $\{\lambda_i\}_{i\in\mathbb{N}}$ we denote the corresponding system of eigenvalues. Similarly, let $\{z_i\}_{i\in\mathbb{N}}$ be an complete and orthogonal system in $\mathcal{V}^1$, which is orthonormal in $L^2(\Omega)$, which is obtained by taking eigenvectors of the minus Laplace operator. The system of corresponding eigenvalues is denoted by $\{\tilde{\lambda}_i\}_{i\in\mathbb{N}}$. We shall find approximate solutions of (18–20) in the following form

$$v^l(t,x) = \sum_{i=1}^{l} c_i^l(t) w_i(x),\ \omega^l(t,x) = \sum_{i=1}^{l} e_i^l(t) z_i(x),\ b^l(t,x) = \sum_{i=1}^{l} d_i^l(t) z_i(x). \tag{24}$$

We have to determine the coefficients $\{c_i^l\}_{i=1}^l$, $\{e_i^l\}_{i=1}^l$ and $\{d_i^l\}_{i=1}^l$. In order to define an approximate problem we have to introduce a few auxiliary functions. For fixed $t > 0$ we denote by $\Psi_t = \Psi_t(x)$ a smooth function such that

$$\Psi_t(x) = \begin{cases} \frac{1}{2} b^t_{\min} & \text{for } x < \frac{1}{2} b^t_{\min}, \\ x & \text{for } x \geq b^t_{\min}, \end{cases} \tag{25}$$

where $b^t_{\min}$ is defined by (13). We assume that the function Ψ_t also satisfies

$$0 \leq \Psi_t'(x) \leq c_0,\ |\Psi_t''(x)| \leq c_0 (b^t_{\min})^{-1}, \tag{26}$$

where, here and c_0 is a constant independent on x and t (see in the appendix for details (formula 107). We also need smooth functions Φ_t, ψ_t and ϕ_t such that

$$\Phi_t(x) = \begin{cases} \frac{1}{2}\omega^t_{\min} & \text{for } x < \frac{1}{2}\omega^t_{\min}, \\ x & \text{for } x \in [\omega^t_{\min}, \omega^t_{\max}], \\ 2\omega^t_{\max} & \text{for } x > 2\omega^t_{\max}, \end{cases} \tag{27}$$

$$\psi_t(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2} b^t_{\min}, \\ x & \text{for } x \geq b^t_{\min}, \end{cases} \tag{28}$$

$$\phi_t(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2}\omega^t_{\min}, \\ x & \text{for } x \geq \omega^t_{\min}. \end{cases} \tag{29}$$

We assume that these functions additionally satisfy

$$0 \leq \Phi_t'(x) \leq c_0,\ |\Phi_t''(x)| \leq c_0 (\omega^t_{\min})^{-1}, \tag{30}$$

$$\psi_t(x) \leq x \text{ for } x \geq 0,\ 0 \leq \psi_t'(x) \leq c_0 \text{ for } x \in \mathbb{R}, \tag{31}$$

$$\phi_t(x) \leq x \text{ for } x \geq 0,\ 0 \leq \phi_t'(x) \leq c_0 \text{ for } x \in \mathbb{R}, \tag{32}$$

for some constant c_0 (the construction of Φ_t, ψ_t and ϕ_t are similar to argument from the appendix).

An approximate solution will be found in the form (24), where the coefficients $\{c_i^l\}_{i=1}^l$, $\{e_i^l\}_{i=1}^l$ and $\{d_i^l\}_{i=1}^l$ are determined by the following truncated system

$$(\partial_t v^l, w_i) - (v^l \otimes v^l, \nabla w_i) + \Big(\mu^l D(v^l), D(w_i)\Big) = 0, \tag{33}$$

$$(\partial_t \omega^l, z_i) - (\omega^l v^l, \nabla z_i) + \Big(\mu^l \nabla \omega^l, \nabla z_i\Big) = -\kappa_2(\phi_t^2(\omega^l), z_i), \tag{34}$$

$$(\partial_t b^l, z_i) - (b^l v^l, \nabla z_i) + \Big(\mu^l \nabla b^l, \nabla z_i\Big) = -(\psi_t(b^l)\phi_t(\omega^l), z_i) + (\mu^l |D(v^l)|^2, z_i),$$

$$c_i^l(0) = (v_0, w_i), e_i^l(0) = (\omega_0, z_i), d_i^l(0) = (b_0, z_i), \tag{35}$$

where $i \in \{1, \ldots, l\}$ and we denote

$$\mu^l = \frac{\Psi_t(b^l)}{\Phi_t(\omega^l)}. \tag{36}$$

In the computations below, the exponent l systematically refers to this Galerkin approximation.

Remark 1 We emphasize that in order to control the second derivatives of approximated solutions we need the conditions (30–32). In particular, we can not apply piecewise linear functions.

Firstly, we note that μ^l is positive and then, by standard ODE theory the system (33–35) has a local-in-time solution. Now, we shall obtain an estimate independent on l.

Lemma 1 *The approximate solutions obtained above satisfies the following estimates*

$$\frac{d}{dt}\|v^l\|_2^2 + 2\mu_{\min}^t \|D(v^l)\|_2^2 \le 0, \tag{37}$$

$$\frac{d}{dt}\|\omega^l\|_2^2 + 2\mu_{\min}^t \|\nabla \omega^l\|_2^2 \le 0, \tag{38}$$

$$\frac{d}{dt}\|b^l\|_2^2 + 2\mu_{\min}^t \|\nabla b^l\|_2^2 \le 2\|b^l\|_\infty \|\mu^l\|_\infty \|\nabla v^l\|_2^2, \tag{39}$$

where $\mu_{\min}^t$ is defined by (13).

Proof We multiply (33) by c_i^l, sum over i and we obtain

$$\frac{1}{2}\frac{d}{dt}\|v^l\|_2^2 + (\mu^l D(v^l), D(v^l)) = 0,$$

where we used (24). Applying the properties of functions Ψ_t, Φ_t and (13) we get

$$\frac{1}{2}\frac{d}{dt}\|v^l\|_2^2 + \mu_{\min}^t \|D(v^l)\|_2^2 \le 0. \tag{40}$$

Similarly, we multiply (34) by e_i^l and we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega^l\|_2^2 + (\mu^l\nabla\omega^l, \nabla\omega^l) = -\kappa_2(\phi_t^2(\omega^l), \omega^l).$$

By the properties of ϕ_t the right-hand side is non-positive thus, we obtain (38). Finally, after multiplying (35) by d_i^l we get

$$\frac{1}{2}\frac{d}{dt}\|b^l\|_2^2 + (\mu^l\nabla b^l, \nabla b^l) = -(\psi_t(b^l)\phi_t(\omega^l), b^l) + (\mu^l|D(v^l)|^2, b^l).$$

We note that $\psi_t(b^l)\phi_t(\omega^l)b^l \geq 0$ hence, we obtain

$$\frac{1}{2}\frac{d}{dt}\|b^l\|_2^2 + \mu_{\min}^t\|\nabla b^l\|_2^2 \leq (\mu^l|D(v^l)|^2, b^l) \leq \|b^l\|_\infty\|\mu^l\|_\infty\|\nabla v^l\|_2^2$$

and the proof is finished. □

We also need the higher order estimates.

Lemma 2 *There exist positive t^* and C_*, which depend on $b_{\min}$, $\omega_{\min}$, $\omega_{\max}$, Ω, κ_2, c_0, $\|v_0\|_{2,2}$, $\|\omega_0\|_{2,2}$ and $\|b_0\|_{2,2}$ such that for each $l \in \mathbb{N}$ the following estimate*

$$\begin{aligned}&\|v^l, \omega^l, b^l\|_{L^\infty(0,t^*;H^2(\Omega))} + \|v^l, \omega^l, b^l\|_{L^2(0,t^*;H^3(\Omega))}\\&+\|\partial_t v^l, \partial_t\omega^l, \partial_t b^l\|_{L^2(0,t^*;H^1(\Omega))} \leq C_*\end{aligned} \tag{41}$$

holds.

Furthermore, for each positive δ and compact $K \subseteq \{(a, b, c) : 0 < a \leq b, 0 < c\}$ there exists positive $t^*_{K,\delta}$, which depends only on κ_2, Ω, δ and K such that if

$$\|v_0\|_{2,2}^2 + \|\omega_0\|_{2,2}^2 + \|b_0\|_{2,2}^2 \leq \delta \text{ and } (\omega_{\min}, \omega_{\max}, b_{\min}) \in K, \tag{42}$$

then $t^* \geq t^*_{K,\delta}$.

Before we go to the proof of Lemma 2 we present its idea. First, we test the equation for approximate solution by its bi-Laplacian. Next, after integration by parts we obtain (43), (45) and (46). Further, we apply the lower bound for the "diffusive coefficient" μ^l (see 48) and use the Hölder and Gagliardo-Nirenberg inequalities which leads to (60). To estimate the H^2-norm of μ^l we use the properties of Ψ_t and Φ_t. After applying the energy estimates from Lemma 1 we obtain (71), which leads to a uniform bound of the H^2-norm of the sequence of approximate solution on the interval $(0, t^*)$ for some positive t^* (see 75). Immediately it gives a bound in L^2H^3. The last step is the l-independent estimate of the time derivative of the approximate solution.

Proof We multiply the equality (33) by $\lambda_i^2 c_i^l$ and sum over i

$$(\partial_t v^l, \Delta^2 v^l) - (v^l \otimes v^l, \nabla\Delta^2 v^l) + (\mu^l D(v^l), D(\Delta^2 v^l)) = 0.$$

After integrating by parts we obtain

$$(\partial_t v^l, \Delta^2 v^l) = \frac{1}{2}\frac{d}{dt}\|\Delta v^l\|_2^2,$$
$$(v^l \otimes v^l, \nabla\Delta^2 v^l) = (\Delta(v^l \otimes v^l), \nabla\Delta v^l),$$

$$\begin{aligned}(\mu^l D(v^l), D(\Delta^2 v^l)) &= (\Delta\mu^l D(v^l), \Delta D(v^l))\\ &\quad +2(\nabla\mu^l \cdot \nabla D(v^l), \Delta D(v^l)) + (\mu^l \Delta D(v^l), \Delta D(v^l)).\end{aligned}$$

Thus, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta v^l\|_2^2 + \int_\Omega \mu^l |\Delta D(v^l)|^2 dx &= -(\Delta(v^l \otimes v^l), \nabla\Delta v^l) - (\Delta\mu^l D(v^l), \Delta D(v^l))\\ &\quad -2(\nabla\mu^l \cdot \nabla D(v^l), \Delta D(v^l)).\end{aligned}$$

We estimate the right-hand side

$$|(\Delta(v^l \otimes v^l), \nabla\Delta v^l)| \leq \|v^l\|_\infty \|\nabla^2 v^l\|_2 \|\nabla^3 v^l\|_2 + \|\nabla v^l\|_4^2 \|\nabla^3 v^l\|_2.$$

Proceeding analogously we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \quad \|\Delta v^l\|_2^2 + \int_\Omega \mu^l |\Delta D(v^l)|^2 dx\\ \leq \|v^l\|_\infty \|\nabla^2 v^l\|_2 \|\nabla^3 v^l\|_2 + \|\nabla v^l\|_4^2 \|\nabla^3 v^l\|_2\\ +\Big(\|\Delta\mu^l D(v^l)\|_2 + 2\|\nabla\mu^l \cdot \nabla D(v^l)\|_2\Big)\|\Delta D(v^l)\|_2.\end{aligned} \tag{43}$$

Now, we multiply the Eq. (34) by $\tilde{\lambda}_i^2 e_i^l$ and we obtain

$$(\partial_t \omega^l, \Delta^2 \omega^l) - (\omega^l v^l, \nabla\Delta^2 \omega^l) + \Big(\mu^l \nabla\omega^l, \nabla\Delta^2\omega^l\Big) = -\kappa_2(\phi_t^2(\omega^l), \Delta^2\omega^l).$$

After integrating by parts we get

$$\begin{aligned}(\partial_t \omega^l, \Delta^2 \omega^l) &= \frac{1}{2}\frac{d}{dt}\|\Delta\omega^l\|_2^2,\\ (\omega^l v^l, \nabla\Delta^2\omega^l) &= (\Delta\omega^l v^l, \nabla\Delta\omega^l) + 2(\nabla v^l \nabla\omega^l, \nabla\Delta\omega^l) + (\omega^l \Delta v^l, \nabla\Delta\omega^l),\\ \Big(\mu^l \nabla\omega^l, \nabla\Delta^2\omega^l\Big) &= \Big(\Delta\mu^l \nabla\omega^l, \nabla\Delta\omega^l\Big) + 2\Big(\nabla^2\omega^l \nabla\mu^l, \nabla\Delta\omega^l\Big)\\ &\quad + \Big(\mu^l \nabla\Delta\omega^l, \nabla\Delta\omega^l\Big), -(\phi_t^2(\omega^l), \Delta^2\omega^l)\\ &= 2\Big(\phi_t(\omega^l)\phi_t'(\omega^l)\nabla\omega^l, \nabla\Delta\omega^l\Big)\end{aligned} \tag{44}$$

Thus, we may write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\ \|\Delta\omega^l\|_2^2 + \int_\Omega \mu^l \left|\nabla\Delta\omega^l\right|^2 dx \\ \le \Big(\|\Delta\omega^l v^l\|_2 + \|\nabla v^l \nabla\omega^l\|_2 + \|\omega^l \Delta v^l\|_2 + \|\Delta\mu^l \nabla\omega^l\|_2 \\ +2\|\nabla^2\omega^l \nabla\mu^l\|_2 + 2\kappa_2\|\phi_t(\omega^l)\phi_t'(\omega^l)\nabla\omega^l\|_2\Big) \|\nabla\Delta\omega^l\|_2. \end{aligned} \tag{45}$$

Finally, after multiplying (35) by $\tilde{\lambda}_i^2 d_i^l$ we obtain

$$\begin{aligned}(\partial_t b^l, \Delta^2 b^l) - (b^l v^l, \Delta^2\nabla b^l) + \left(\mu^l \nabla b^l, \nabla\Delta^2 b^l\right) \\ = -(\psi_t(b^l)\phi_t(\omega^l), \Delta^2 b^l) + (\mu^l |D(v^l)|^2, \Delta^2 b^l).\end{aligned}$$

We deal with the terms on the left hand-side as earlier and for the right-hand side terms we get

$$\begin{aligned}-(\psi_t(b^l)\phi_t(\omega^l), \Delta^2 b^l) &= \left(\psi_t'(b^l)\phi_t(\omega^l)\nabla b^l, \nabla\Delta b^l\right) + \left(\psi_t(b^l)\phi_t'(\omega^l)\nabla\omega^l, \nabla\Delta b^l\right), \\ (\mu^l |D(v^l)|^2, \Delta^2 b^l) &= -(|D(v^l)|^2 \nabla\mu^l, \nabla\Delta b^l) - (\mu^l \nabla(|D(v^l)|^2), \nabla\Delta b^l).\end{aligned}$$

Therefore, we obtain the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta b^l\|_2^2 + \int_\Omega \mu^l \left|\nabla\Delta b^l\right|^2 dx \le \Big(\|\Delta b^l v^l\|_2 + 2\|\nabla v^l \nabla b^l\|_2 + \|b^l \Delta v^l\|_2 \\ +\|\Delta\mu^l \nabla b^l\|_2 + 2\|\nabla^2 b^l \nabla\mu^l\|_2 + \|\phi_t(\omega^l)\psi_t'(b^l)\nabla b^l\|_2 \\ +\|\psi_t(b^l)\phi_t'(\omega^l)\nabla\omega^l\|_2 + \|\nabla\mu^l \left|D(v^l)\right|^2 \|_2 \\ +\|\mu^l |D(v^l)||\nabla D(v^l)|\|_2\Big)\|\nabla\Delta b^l\|_2.\end{aligned} \tag{46}$$

We note that

$$\int_\Omega \left|\Delta D(v^l)\right|^2 dx = \frac{1}{2}\int_\Omega \left|\nabla^3 v^l\right|^2 dx. \tag{47}$$

Indeed, integrating by parts yield

$$\begin{aligned}2\int_\Omega \left|\Delta D(v^l)\right|^2 dx = \sum_{k,m}\int_\Omega \left|\Delta v^l_{k,x_m}\right|^2 dx + \int_\Omega \Delta v^l_{k,x_m}\cdot\Delta v^l_{m,x_k} dx \\ = \sum_{k,m,p,q}\int_\Omega v^l_{k,x_m x_p x_p}\cdot v^l_{k,x_m x_q x_q} dx + \sum_{k,m,p,q}\int_\Omega \Delta v^l_{k,x_k}\cdot\Delta v^l_{m,x_m} dx\end{aligned}$$

$$= \sum_{k,m,p,q} \int_{\Omega} \left| v^l_{k,x_m x_p x_q} \right|^2 dx,$$

where we applied the condition div $v^l = 0$ and used the tensor notation for components and derivatives. After applying (13), (25), (27) and (36) we get

$$\mu^t_{\min} \le \mu^l \tag{48}$$

for each l thus, (43) together with (47) and (48) give

$$\begin{aligned} &\frac{d}{dt}\|\Delta v^l\|_2^2 + \mu^t_{\min}\|\Delta D(v^l)\|_2^2 \\ &\le \frac{32}{\mu^t_{\min}}\Big(\|v^l\|_\infty^2\|\nabla^2 v^l\|_2^2 + \|\nabla v^l\|_4^4 + \|\Delta\mu^l D(v^l)\|_2^2 + \|\nabla\mu^l \cdot \nabla D(v^l)\|_2^2\Big). \end{aligned} \tag{49}$$

Applying Gagliardo-Nirenberg interpolation inequality

$$\|\nabla v^l\|_\infty \le C\|\nabla^3 v^l\|_2^{\frac{1}{2}}\|\nabla v^l\|_6^{\frac{1}{2}} \tag{50}$$

and Sobolev embedding inequality we get

$$\|\Delta\mu^l D(v^l)\|_2^2 \le \|\Delta\mu^l\|_2^2\|D(v^l)\|_\infty^2 \le C\|\nabla^3 v^l\|_2\|v^l\|_{2,2}\|\mu^l\|_{2,2}^2,$$

where C depends only on Ω. Again, by Gagliardo-Nirenberg inequality

$$\|\nabla^2 v^l\|_3 \le C\|\nabla^3 v^l\|_2^{\frac{1}{2}}\|\nabla^2 v^l\|_2^{\frac{1}{2}} \tag{51}$$

and Hölder inequality we have

$$\|\nabla\mu^l \cdot \nabla D(v^l)\|_2^2 \le \|\nabla\mu\|_6^2\|\nabla^2 v^l\|_3^2 \le C\|\nabla^3 v^l\|_2\|v^l\|_{2,2}\|\mu^l\|_{2,2}^2.$$

Thus, applying after the Young inequality with exponents (2, 6, 3) we get

$$\|\Delta\mu^l D(v^l)\|_2^2 + \|\nabla\mu^l \cdot \nabla D(v^l)\|_2^2 \le \varepsilon\|\nabla^3 v^l\|_2^2 + \frac{C}{\varepsilon}(\|v^l\|_{2,2}^6 + \|\mu^l\|_{2,2}^6), \tag{52}$$

where $\varepsilon > 0$ and C depends only on Ω. Applying the above inequality and (47) in (49) we obtain

$$\frac{d}{dt}\|\nabla^2 v^l\|_2^2 + \mu^t_{\min}\|\nabla^3 v^l\|_2^2 \le \frac{C}{\mu^t_{\min}}\Big(\|v^l\|_{2,2}^4 + (\mu^t_{\min})^{-2}(\|v^l\|_{2,2}^6 + \|\mu^l\|_{2,2}^6)\Big), \tag{53}$$

where $C = C(\Omega)$. Now, we proceed similarly with (45) and we obtain

$$\frac{d}{dt}\|\Delta\omega^l\|_2^2 + \mu_{\min}^t\|\nabla\Delta\omega^l\|_2^2 \le \frac{C}{\mu_{\min}^t}\Big(\|v^l\|_\infty^2\|\nabla^2\omega^l\|_2^2 + \|\nabla v^l\|_4^2\|\nabla\omega^l\|_4^2 + \|\omega^l\|_\infty^2\|\nabla^2 v^l\|_2^2 + \|\Delta\mu^l\nabla\omega^l\|_2^2 + \|\nabla^2\omega^l\nabla\mu^l\|_2^2 + \kappa_2^2 c_0^2\|\omega^l\|_\infty^2\|\nabla\omega^l\|_2^2\Big), \tag{54}$$

where we applied (32). We repeat the reasoning leading to (52) and we obtain

$$\|\Delta\mu^l\nabla\omega^l\|_2^2 + \|\nabla^2\omega^l\nabla\mu^l\|_2^2 \le \varepsilon\|\nabla^3\omega^l\|_2^2 + \frac{C}{\varepsilon}(\|\omega^l\|_{2,2}^6 + \|\mu^l\|_{2,2}^6).$$

Thus, the above inequality and (54) give

$$\frac{d}{dt}\|\nabla^2\omega^l\|_2^2 + \mu_{\min}^t\|\nabla^3\omega^l\|_2^2 \le \frac{C}{\mu_{\min}^t}\Big(\|v^l\|_{2,2}^4 + (1+\kappa_2^4 c_0^4)\|\omega^l\|_{2,2}^4 + (\mu_{\min}^t)^{-2}(\|\omega^l\|_{2,2}^6 + \|\mu^l\|_{2,2}^6)\Big), \tag{55}$$

where $C = C(\Omega)$. Further, from (46) we get

$$\frac{d}{dt}\|\Delta b^l\|_2^2 + \mu_{\min}^t\|\nabla\Delta b^l\|_2^2 \le \frac{C}{\mu_{\min}^t}\Big(\|v^l\|_\infty^2\|\nabla^2 b^l\|_2^2 + \|\nabla v^l\|_4^2\|\nabla b^l\|_4^2 + \|b^l\|_\infty^2\|\nabla^2 v^l\|_2^2 + \|\nabla^2\mu^l\nabla b^l\|_2^2 + \|\nabla^2 b^l\nabla\mu^l\|_2^2 + c_0^2\|\omega^l\|_\infty^2\|\nabla b^l\|_2^2 + c_0^2\|b^l\|_\infty^2\|\nabla\omega^l\|_2^2 + \|\nabla\mu^l|D(v^l)|^2\|_2^2 + \|\mu^l\nabla(|D(v^l)|^2)\|_2^2\Big),$$

where we applied (31) and (32). Applying integrating by parts and Sobolev embedding theorem we get

$$\frac{d}{dt}\|\nabla^2 b^l\|_2^2 + \mu_{\min}^t\|\nabla^3 b^l\|_2^2 \le \frac{C}{\mu_{\min}^t}\Big(\|v^l\|_{2,2}^4 + \|b^l\|_{2,2}^4 + \|\nabla^2\mu^l\nabla b^l\|_2^2 + \|\nabla^2 b^l\nabla\mu^l\|_2^2 + c_0^4\|\omega^l\|_{2,2}^4 + \|\mu^l\|_{2,2}^6 + \|v^l\|_{2,2}^6 + \|\nabla^2 v^l\|_3^2\|\mu^l\|_{2,2}^2\|v^l\|_{2,2}^2\Big), \tag{56}$$

Applying again the Gagliardo-Nirenberg inequality and Young inequality we get

$$\|\nabla^2\mu^l\nabla b^l\|_2^2 + \|\nabla^2 b^l\nabla\mu^l\|_2^2 \le \varepsilon\|\nabla^3 b^l\|_2^2 + \frac{C}{\varepsilon}(\|b^l\|_{2,2}^6 + \|\mu^l\|_{2,2}^6).$$

From (51) we get

$$\|\nabla^2 v^l\|_3^2\|v^l\|_{2,2}^2\|\mu^l\|_{2,2}^2 \le C\|\nabla^3 v^l\|_2\|v^l\|_{2,2}^3\|\mu^l\|_{2,2}^2 \le \varepsilon\|\nabla^3 v^l\|_2^2 + \frac{C}{\varepsilon}(\|v^l\|_{2,2}^{10} + \|\mu^l\|_{2,2}^{10}).$$

hence, from (56) we obtain the following estimate

$$\frac{d}{dt}\|\nabla^2 b^l\|_2^2 + \mu_{\min}^t\|\nabla^3 b^l\|_2^2 \le \frac{C}{\mu_{\min}^t}\Big(\|v^l\|_{2,2}^4 + \|b^l\|_{2,2}^4 + c_0^4\|\omega^l\|_{2,2}^4 + \|\mu^l\|_{2,2}^6 + \|v^l\|_{2,2}^6\Big) + \frac{C}{(\mu_{\min}^t)^3}\Big(\|b^l\|_{2,2}^6 + \|\mu^l\|_{2,2}^6 + \|v^l\|_{2,2}^{10} + \|\mu^l\|_{2,2}^{10}\Big) + \frac{\mu_{\min}^t}{2}\|\nabla^3 v^l\|_2^2, \tag{57}$$

where $C = C(\Omega)$. We sum the inequalities (53), (55), (57) and we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla^2 v^l\|_2^2 + \|\nabla^2\omega^l\|_2^2 + \|\nabla^2 b^l\|_2^2\Big) + \mu_{\min}^t\Big(\|\nabla^3 v^l\|_2^2 + \|\nabla^3\omega^l\|_2^2 + \|\nabla^3 b^l\|_2^2\Big)\\
&\le \frac{C}{\mu_{\min}^t}\Big(\|v^l\|_{2,2}^4 + \|b^l\|_{2,2}^4 + (1 + c_0^4 + c_0^4\kappa_2^4)\|\omega^l\|_{2,2}^4 + \|\mu^l\|_{2,2}^6 + \|v^l\|_{2,2}^6\Big)\\
&\quad + \frac{C}{(\mu_{\min}^t)^3}\Big(\|v^l\|_{2,2}^6 + \|b^l\|_{2,2}^6 + \|\omega^l\|_{2,2}^6 + \|\mu^l\|_{2,2}^6 + \|v^l\|_{2,2}^{10} + \|\mu^l\|_{2,2}^{10}\Big)
\end{aligned} \tag{58}$$

for some C, which depends only on Ω. We note that

$$\mu_{\min}^t = \frac{1}{4}\frac{b_{\min}}{\omega_{\max}}(1+\kappa_2\omega_{\max}t)^{1-\frac{1}{\kappa_2}} \tag{59}$$

hence, we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla^2 v^l\|_2^2 + \|\nabla^2\omega^l\|_2^2 + \|\nabla^2 b^l\|_2^2\Big) + \mu_{\min}^t\Big(\|\nabla^3 v^l\|_2^2 + \|\nabla^3\omega^l\|_2^2 + \|\nabla^3 b^l\|_2^2\Big)\\
&\le C\left(\frac{\omega_{\max}}{b_{\min}} + \left(\frac{\omega_{\max}}{b_{\min}}\right)^3\right)(1+\kappa_2\omega_{\max}t)^{\beta}\\
&\Big(1 + \|b^l\|_{2,2}^6 + \|\omega^l\|_{2,2}^6 + \|\mu^l\|_{2,2}^{10} + \|v^l\|_{2,2}^{10}\Big),
\end{aligned} \tag{60}$$

where $\beta = \max\{\frac{1}{\kappa_2} - 1, \frac{3}{\kappa_2} - 3\}$ and C depends only on Ω, c_0 and κ_2.

Now, we shall estimate μ^l in terms of ω^l and b^l. Firstly, we note that from (25) and (27) we have

$$\Psi_t(b^l) \le \max\left\{\frac{1}{2}b_{\min}^t, b^l\right\}, \; \Phi_t(\omega^l) \ge \frac{1}{2}\omega_{\min}^t. \tag{61}$$

Hence, by definition (36) we get

$$0 < \mu^l \le 2(\omega_{\min}^t)^{-1}\max\{b_{\min}^t, b^l\} \le c_1(\Omega)\frac{1}{\omega_{\min}}(1+\kappa_2\omega_{\min}t)\Big(b_{\min} + |b^l|\Big), \tag{62}$$

where c_1 depends only on Ω. Thus, we obtain

$$\|\mu^l\|_2 \leq c_1 \frac{1}{\omega_{\min}} (1+\kappa_2 \omega_{\min} t) (b_{\min} + \|b^l\|_2). \tag{63}$$

Now, we have to estimate the derivatives of μ^l. Direct calculation gives

$$\begin{aligned} |\nabla^2 \mu^l| &= \left|\nabla^2 \left(\Psi_t(b^l)\cdot(\Phi_t(\omega^l))^{-1}\right)\right| \leq (\Phi_t(\omega^l))^{-1}\left|\nabla^2(\Psi_t(b^l))\right| \\ &+2(\Phi_t(\omega^l))^{-2}\left|\nabla(\Psi_t(b^l))\right|\left|\nabla(\Phi_t(\omega^l))\right| \\ &+2\Psi_t(b^l)(\Phi_t(\omega^l))^{-3}\left|\nabla(\Phi_t(\omega^l))\right|^2 + \Psi_t(b^l)(\Phi_t(\omega^l))^{-2}\left|\nabla^2(\Phi_t(\omega^l))\right|. \end{aligned} \tag{64}$$

Using (26) and (30) we may estimate the derivatives

$$\left|\nabla(\Psi_t(b^l))\right| \leq c_0\left|\nabla b^l\right|, \left|\nabla(\Phi_t(\omega^l))\right| \leq c_0\left|\nabla \omega^l\right|, \tag{65}$$

$$\begin{aligned} \left|\nabla^2(\Psi_t(b^l))\right| &\leq c_0(b^t_{\min})^{-1}\left|\nabla b^l\right|^2 + c_0\left|\nabla^2 b^l\right|, \\ \left|\nabla^2(\Phi_t(\omega^l))\right| &\leq c_0(\omega^t_{\min})^{-1}\left|\nabla \omega^l\right|^2 + c_0\left|\nabla^2 \omega^l\right|. \end{aligned} \tag{66}$$

If we apply estimates (61), (65) and (66) in (64) then we obtain

$$\begin{aligned} \left|\nabla^2\mu^l\right| \leq c_2 Q_1 (1+\kappa_2\omega_{\max} t)^{\max\{3,1+\frac{1}{\kappa_2}\}} \Big[\left|\nabla b^l\right|^2 + \left|\nabla^2 b^l\right| + |b^l|\left|\nabla\omega^l\right|^2 \\ + \left|\nabla b^l\right| + \left|\nabla \omega^l\right| + \left|\nabla\omega^l\right|^2 + \left|b^l\nabla^2\omega^l\right| + \left|\nabla^2\omega^l\right|\Big] \end{aligned} \tag{67}$$

where c_2 depends only on c_0 and $Q_1 = \frac{b_{\min}}{\omega_{\min}}\left(1 + b_{\min}^{-3} + \omega_{\min}^{-3}\right)$. Thus, we obtain

$$\begin{aligned} \|\nabla^2\mu^l\|_2 \leq c_2 Q_1 (1+\kappa_2\omega_{\max} t)^{\max\{3,1+\frac{1}{\kappa_2}\}} \Big[\|\nabla b^l\|_4^2 + \|\nabla^2 b^l\|_2 \\ +\|b^l\|_\infty\|\nabla\omega^l\|_4^2 + \|\nabla\omega^l\|_4^2 + \|\nabla^2\omega^l\|_2 + \|b^l\|_\infty\|\nabla^2\omega^l\|_2\Big]. \end{aligned} \tag{68}$$

If we take into account (63) then we get

$$\|\mu^l\|_{2,2} \leq c_3 Q_1 (1+\kappa_2\omega_{\max} t)^{\max\{3,1+\frac{1}{\kappa_2}\}} \left(\|b^l\|^3_{2,2} + \|\omega^l\|^3_{2,2} + 1\right), \tag{69}$$

where $c_3 = c_3(c_0, \Omega)$. Applying the above estimate in (60) we obtain

$$\frac{d}{dt}\left(\|\nabla^2 v^l\|_2^2 + \|\nabla^2\omega^l\|_2^2 + \|\nabla^2 b^l\|_2^2\right) + \mu^t_{\min}\left(\|\nabla^3 v^l\|_2^2 + \|\nabla^3\omega^l\|_2^2 + \|\nabla^3 b^l\|_2^2\right)$$

$$\leq C Q_2 \,(1+\kappa_2\omega_{\max} t)^{\bar{\beta}} \left(1+\|v^l\|^2_{2,2}+\|b^l\|^2_{2,2}+\|\omega^l\|^2_{2,2}\right)^{15}, \tag{70}$$

where

$$\begin{aligned} Q_2 &= \left[1+\left(\frac{\omega_{\max}}{b_{\min}}\right)^3\right]\left[\frac{b_{\min}}{\omega_{\min}}(1+b_{\min}^{-3}+\omega_{\min}^{-3})^{10}+1\right], \bar{\beta} \\ &= 10\max\left\{1+\frac{1}{\kappa_2}, 3\right\}+\beta \end{aligned}$$

and C depends only on Ω, c_0 and κ_2. If we take into account the estimates (37–39) then we have

$$\begin{aligned} &\frac{d}{dt}\left(\|v^l\|^2_{2,2}+\|\omega^l\|^2_{2,2}+\|b^l\|^2_{2,2}\right)+\mu^t_{\min}\left(\|v^l\|^2_{3,2}+\|\omega^l\|^2_{3,2}+\|b^l\|^2_{3,2}\right) \\ &\quad\leq C Q_3 \,(1+\kappa_2\omega_{\max} t)^{\bar{\beta}} \left(1+\|v^l\|^2_{2,2}+\|b^l\|^2_{2,2}+\|\omega^l\|^2_{2,2}\right)^{15}, \end{aligned} \tag{71}$$

where $C = C(c_0, \Omega, \kappa_2)$ and $Q_3 = Q_1^2 + Q_2 + 1$. If we divide both sides by the last term and next integrate with respect time variable then we get

$$\begin{aligned} &\left(1+\|v^l(t)\|^2_{2,2}+\|b^l(t)\|^2_{2,2}+\|\omega^l(t)\|^2_{2,2}\right)^{-14} \geq \Big(1+\|v^l(0)\|^2_{2,2} \\ &\quad+\|b^l(0)\|^2_{2,2}+\|\omega^l(0)\|^2_{2,2}\Big)^{-14} - \frac{14 C Q_3}{(\bar{\beta}+1)\kappa_2\omega_{\max}}\left((1+\kappa_2\omega_{\max}t)^{\bar{\beta}+1}-1\right) \\ &\geq \left(1+\|v_0\|^2_{2,2}+\|b_0\|^2_{2,2}+\|\omega_0\|^2_{2,2}\right)^{-14} - \frac{14 C Q_3}{(\bar{\beta}+1)\kappa_2\omega_{\max}}\left((1+\kappa_2\omega_{\max}t)^{\bar{\beta}+1}-1\right), \end{aligned} \tag{72}$$

where the last estimate is a consequence of Bessel inequality. Now, we define time t^* as the unique solution of the equality

$$\left(1+\|v_0\|^2_{2,2}+\|b_0\|^2_{2,2}+\|\omega_0\|^2_{2,2}\right)^{-14} = \frac{15 C Q_3}{(\bar{\beta}+1)\kappa_2\omega_{\max}}\left((1+\kappa_2\omega_{\max}t^*)^{\bar{\beta}+1}-1\right). \tag{73}$$

We note that t^* is positive and depends on $\|v_0\|^2_{2,2}+\|b_0\|^2_{2,2}+\|\omega_0\|^2_{2,2}$, κ_2, Ω, c_0, $\omega_{\min}$, $\omega_{\max}$ and $b_{\min}$. It is evident that t^* is decreasing function of $\|v_0\|^2_{2,2}+\|b_0\|^2_{2,2}+\|\omega_0\|^2_{2,2}$. Moreover, for any $\delta > 0$ and compact $K \subseteq \{(a,b,c) : 0 < a \leq b, 0 < c\}$ there exists $t^*_{K,\delta} > 0$ such that $t^* \geq t^*_{K,\delta}$ for any initial data satisfying $\|v_0\|^2_{2,2}+\|b_0\|^2_{2,2}+\|\omega_0\|^2_{2,2} \leq \delta$ and $(\omega_{\min}, \omega_{\max}, b_{\min}) \in K$. From (73) we deduce that $t^*_{K,\delta}$ depends only on δ, K, Ω κ_2 and c_0.

From (72) and (73) we have

$$\left(1+\|v^l(t)\|^2_{2,2}+\|b^l(t)\|^2_{2,2}+\|\omega^l(t)\|^2_{2,2}\right)^{-14} \geq \frac{C Q_3}{(\bar{\beta}+1)\kappa_2\omega_{\max}}\left((1+\kappa_2\omega_{\max}t)^{\bar{\beta}+1}-1\right)$$

for $t \in [0, t^*]$ hence,

$$\|v^l(t)\|_{2,2}^2 + \|b^l(t)\|_{2,2}^2 + \|\omega^l(t)\|_{2,2}^2 \leq \left[\frac{CQ_3}{(\bar{\beta}+1)\kappa_2\omega_{\max}} \left((1+\kappa_2\omega_{\max}t^*)^{\bar{\beta}+1} - 1 \right) \right]^{-\frac{1}{14}} \tag{74}$$

for $t \in [0, t^*]$. In particular, there exists $C^* = C^*(t^*)$ such that

$$\|v^l\|_{L^\infty(0,t^*;\dot{\mathcal{V}}^2_{\mathrm{div}})} + \|\omega^l\|_{L^\infty(0,t^*;\mathcal{V}^2)} + \|b^l\|_{L^\infty(0,t^*;\mathcal{V}^2)} \leq C^* \tag{75}$$

uniformly with respect to $l \in \mathbb{N}$. Next, from (59), (71) and (75) we get the bound

$$\|v^l\|_{L^2(0,t^*;\dot{\mathcal{V}}^3_{\mathrm{div}})} + \|\omega^l\|_{L^2(0,t^*;\mathcal{V}^3)} + \|b^l\|_{L^2(0,t^*;\mathcal{V}^3)} \leq C_*, \tag{76}$$

where C_* depends on t^*, κ_2, $b_{\min}$, $\omega_{\max}$ and C^*. It remains to show the estimate of time derivative of solution. We do this by multiplying the equality (33) by $\frac{d}{dt}c_i^l$ and after summing it over i we get

$$(\partial_t v^l, \partial_t v^l) - (v^l \otimes v^l, \nabla\partial_t v^l) + (\mu^l D(v^l), D(\partial_t v^l)) = 0.$$

Thus, by after integration by parts and applying Hölder inequality we have

$$\|\partial_t v^l\|_2^2 \leq \|\operatorname{div}(v^l \otimes v^l)\|_2\|\partial_t v^l\|_2 + \|\nabla\left(\mu^l D(v^l)\right)\|_2\|\partial_t v^l\|_2.$$

By applying Young inequality we get

$$\|\partial_t v^l\|_2^2 \leq 2\|\operatorname{div}(v^l \otimes v^l)\|_2^2 + 2\|\nabla\left(\mu^l D(v^l)\right)\|_2^2.$$

Next, Hölder inequality gives us

$$\|\partial_t v^l\|_2^2 \leq C\Big(\|\nabla v^l\|_4^2\|v^l\|_4^2 + \|\nabla\mu^l\|_4^2\|D(v^l)\|_4^2 + \|\mu^l\|_\infty^2\|\nabla D(v^l)\|_2^2\Big).$$

Finally, Sobolev embedding theorem leads us to the following inequality

$$\|\partial_t v^l\|_2^2 \leq C\Big(\|v^l\|_{2,2}^4 + \|\mu^l\|_{2,2}^2\|v^l\|_{2,2}^2\Big),$$

where C depends only on Ω. If we apply (69) and (75) then we get

$$\|\partial_t v^l\|_{L^\infty(0,t^*;L^2(\Omega))} \leq C_*, \tag{77}$$

where C_* depends on Ω, c_0, t^*, κ_2, $b_{\min}$, $\omega_{\max}$ and C^*.

Now, we shall consider (34). Proceeding as earlier we get

$$\|\partial_t\omega^l\|_2^2 \leq 4\|\nabla\omega^l \cdot v^l\|_2^2 + 4\|\nabla(\mu^l\nabla\omega^l)\|_2^2 + 4\kappa_2\|\phi_t^2(\omega^l)\|_2^2$$

$$\leq 4\|v^l\|_\infty^2\|\nabla\omega^l\|_2^2 + 8\|\nabla\mu^l\|_4^2\|\nabla\omega^l\|_4^2 + 8\|\mu^l\|_\infty^2\|\nabla^2\omega^l\|_2^2 + 4\kappa_2\|\omega^l\|_4^4,$$

where we applied (32). Thus, using (69) and (75) we get

$$\|\partial_t\omega^l\|_{L^\infty(0,t^*;L^2(\Omega))} \leq C_*, \tag{78}$$

where C_* is as earlier. It remains to deal with (35). In similar way we obtain

$$\begin{aligned}\|\partial_t b^l\|_2^2 &\leq 4\|\nabla b^l v^l\|_2^2 + 4\|\nabla(\mu^l\nabla b^l)\|_2^2 + 4\|\psi_t(b^l)\phi_t(\omega^l)\|_2^2 + 4\|\mu^l|D(v^l)|^2\|_2^2\\ &\leq 4\|\nabla b^l\|_2^2\|v^l\|_\infty^2 + 8\|\nabla\mu^l\|_4^2\|\nabla b^l\|_4^2 + 8\|\mu^l\|_\infty^2\|\nabla^2 b^l\|_2^2\\ &\quad +4\|b^l\|_\infty^2\|\omega^l\|_2^2 + 4\|\mu^l\|_\infty^2\|\nabla v^l\|_4^4.\end{aligned}$$

Applying again (69) and (75) we obtain

$$\|\partial_t b^l\|_{L^\infty(0,t^*;L^2(\Omega))} \leq C_*, \tag{79}$$

where C_* depends on Ω, c_0, t^*, κ_2, $b_{\min}$, $\omega_{\max}$ and C^*.

Now, we prove the higher order estimates for time derivative of approximate solution. Firstly, we multiply the equality (33) by $-\lambda_i\frac{d}{dt}c_i^l$ and sum over i

$$(\partial_t v^l, -\Delta\partial_t v^l) + (v^l\otimes v^l, \nabla\Delta\partial_t v^l) - (\mu^l D(v^l), D(\Delta\partial_t v^l)) = 0.$$

After integration by parts we get

$$\|\nabla\partial_t v^l\|_2^2 = -\left(\Delta\left(v^l\otimes v^l\right), \nabla\partial_t v^l\right) + \left(\Delta\left(\mu^l D(v^l)\right), D(\partial_t v^l)\right).$$

If we apply Hölder and Young inequalities, then we get

$$\|\nabla\partial_t v^l\|_2^2 \leq 2\|\Delta\left(v^l\otimes v^l\right)\|_2^2 + \|\Delta\left(\mu^l D(v^l)\right)\|_2^2,$$

where we used the equality $2\|D(\partial_t v^l)\|_2^2 = \|\nabla\partial_t v^l\|_2^2$. We estimate further

$$\begin{aligned}\|\nabla\partial_t v^l\|_2^2 \leq\ & 8\|v^l\|_\infty^2\|\nabla^2 v^l\|_2^2 + 8\|\nabla v^l\|_4^4 + 4\|\mu^l\|_\infty^2\|\Delta D(v^l)\|_2^2\\ &+16\|\nabla\mu^l\|_3^2\|\nabla D(v^l)\|_6^2 + 4\|\Delta\mu^l\|_2^2\|D(v^l)\|_\infty^2.\end{aligned}$$

Using Sobolev embedding we obtain

$$\|\nabla\partial_t v^l\|_2^2 \leq C\left(\|v^l\|_{2,2}^4 + \|\mu^l\|_{2,2}^2\|v^l\|_{2,2}^2 + \|\mu^l\|_{2,2}^2\|v^l\|_{3,2}^2.\right),$$

where C depends only on Ω. Applying (69), (75) and (76) we get

$$\|\nabla\partial_t v^l\|_{L^2(0,t^*;L^2(\Omega)} \leq C_*, \tag{80}$$

where C_* depends on c_0, Ω, t^*, κ_2, $b_{\min}$, $\omega_{\max}$ and C^*. Proceeding analogously we get

$$\|\nabla \partial_t \omega^l\|_{L^2(0,t^*;L^2(\Omega))} \le C_*. \tag{81}$$

It remains to estimate $\nabla\partial_t b^l$. If we multiply the equality (35) by $-\tilde{\lambda}_i \frac{d}{dt} d_i^l$ and sum over i, then we get

$$\begin{aligned}(\partial_t b^l, -\Delta\partial_t b^l) + (b^l v^l, \nabla\Delta\partial_t b^l) - \left(\mu^l \nabla b^l, \nabla\Delta\partial_t b^l\right)\\ = (\psi_t(b^l)\phi_t(\omega^l), \Delta\partial_t b^l) - (\mu^l |D(v^l)|^2, \Delta\partial_t b^l).\end{aligned}$$

Integrating by parts and Hölder inequality lead to

$$\begin{aligned}\|\nabla\partial_t b^l\|_2^2 \le \|\Delta\left(b^l v^l\right)\|_2 \|\nabla\partial_t b^l\|_2 + \|\Delta\left(\mu^l \nabla b^l\right)\|_2 \|\nabla\partial_t b^l\|_2\\ + \|\nabla\left(\psi_t(b^l)\phi_t(\omega^l)\right)\|_2 \|\nabla\partial_t b^l\|_2 + \|\nabla\left(\mu^l |D(v^l)|^2\right)\|_2 \|\nabla\partial_t b^l\|_2.\end{aligned}$$

After applying Young inequality we get

$$\begin{aligned}\|\nabla\partial_t b^l\|_2^2 \le 4\|\Delta\left(b^l v^l\right)\|_2^2 + 4\|\Delta\left(\mu^l \nabla b^l\right)\|_2^2\\ + 4\|\nabla\left(\psi_t(b^l)\phi_t(\omega^l)\right)\|_2^2 + 4\|\nabla\left(\mu^l |D(v^l)|^2\right)\|_2^2.\end{aligned}$$

Using Hölder inequality we obtain

$$\begin{aligned}\|\nabla\partial_t b^l\|_2^2 \le {} & 16\|\Delta b^l\|_2^2 \|v^l\|_\infty^2 + 32\|\nabla b^l\|_4^2 \|\nabla v^l\|_4^2 + 16\|b^l\|_\infty^2 \|\nabla^2 v^l\|_2^2\\ & + 16\|\Delta\mu^l\|_2^2 \|\nabla b^l\|_\infty^2 + 32\|\nabla\mu^l\|_4^2 \|\nabla^2 b^l\|_4^2 + 16\|\mu^l\|_\infty^2 \|\nabla\Delta b^l\|_2^2\\ & + 8\|\nabla(\psi_t(b^l))\|_2^2 \|\phi_t(\omega^l)\|_\infty^2 + 8\|\psi_t(b^l)\|_\infty^2 \|\nabla(\phi_t(\omega^l))\|_2^2\\ & + 8\|\nabla\mu^l\|_6^2 \|D(v^l)\|_6^4 + 16\|\mu^l\|_\infty^2 \|D(v^l)\|_3^2 \|\nabla D(v^l)\|_6^2.\end{aligned} \tag{82}$$

After applying (31) and (32) we get $\|\psi_t(b^l)\|_\infty \le \|b^l\|_\infty$, $\|\psi_t(\omega^l)\|_\infty \le \|\omega^l\|_\infty$ and

$$\begin{aligned}\|\nabla(\phi_t(\omega^l))\|_2 = \|\phi_t'(\omega^l)\nabla\omega^l\|_2 \le c_0\|\nabla\omega^l\|_2,\\ \|\nabla(\psi_t(\omega^l))\|_2 = \|\psi_t'(b^l)\nabla\omega^l\|_2 \le c_0\|\nabla b^l\|_2.\end{aligned}$$

Using these inequalities in (82) we obtain

$$\begin{aligned}\|\nabla\partial_t b^l\|_2^2 \le C\Big(&\|b^l\|_{2,2}^2 \|v^l\|_{2,2}^2 + \|\mu^l\|_{2,2}^2 \|b^l\|_{3,2}^2 + \|\nabla b^l\|_2^2 \|\omega^l\|_{2,2}^2 + \|\nabla\omega^l\|_2^2 \|b^l\|_{2,2}^2\\ & + \|\mu^l\|_{2,2}^2 \|v^l\|_{2,2}^4 + \|\mu^l\|_{2,2}^2 \|v^l\|_{2,2}^2 \|v^l\|_{3,2}^2\Big),\end{aligned}$$

where $C = C(\Omega, c_0)$. Finally, from (69), (75) and (76) we obtain

$$\|\nabla\partial_t b^l\|_{L^2(0,t^*;L^2(\Omega))} \le C_*, \tag{83}$$

where C_* depends on c_0, Ω, t^*, κ_2, $b_{\min}$, $\omega_{\max}$ and C^*. The estimates (75–79), (80), (81) and (83) give (41) and the proof of lemma 2 is finished. □

Now, we draw the idea of the remain part of the proof of theorem 1. From the l-independent estimate (41) we deduce the existence of a subsequence, which converges weakly in some spaces (see 84–85). Next, by applying Aubin-Lions lemma we get strong convergence of the approximate solution, see (87), (88). Further, we prove the convergence of "diffusive coefficient" μ^l (89), which allows us to take the limit in the approximate problem. As a result, we obtain (91–93). In the last step we prove a series of inequalities (94–96), (98), (101), which show that the truncated problem is in fact the original one.

Having the estimate (41) from lemma 2 we may apply weak-compactness argument to the sequence of approximate solutions and we obtain a subsequence (still numerated by superscript l) weakly convergent in appropriate spaces. To be more precise, there exist v, ω and b such that

$$\begin{aligned} v \in L^2(0,t^*;\dot{\mathcal{V}}^3_{\text{div}}) \cap L^\infty(0,t^*;\dot{\mathcal{V}}^2_{\text{div}}),\ \partial_t v \in L^2(0,t^*;H^1(\Omega)) \\ \omega, b \in L^2(0,t^*;\mathcal{V}^3) \cap L^\infty(0,t^*;\mathcal{V}^2),\ \partial_t\omega, \partial_t b \in L^2(0,t^*;H^1(\Omega)) \end{aligned}$$

and

$$v^l \rightharpoonup v \text{ in } L^2(0,t^*;\dot{\mathcal{V}}^3_{\text{div}}),\ v^l \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,t^*;\dot{\mathcal{V}}^2_{\text{div}}),\ \partial_t v^l \rightharpoonup \partial_t v \text{ in } L^2(0,t^*;H^1(\Omega)), \tag{84}$$

$$(\omega^l, b^l) \rightharpoonup (\omega, b) \text{ in } L^2(0,t^*;\mathcal{V}^3),\ (\omega^l, b^l) \overset{*}{\rightharpoonup} (\omega, b) \text{ in } L^\infty(0,t^*;\mathcal{V}^2), \tag{85}$$

$$(\partial_t\omega^l, \partial_t b^l) \rightharpoonup (\partial_t\omega, \partial_t b) \text{ in } L^2(0,t^*;H^1(\Omega)). \tag{86}$$

Thus, by the Aubin-Lions lemma there exists a subsequence (again denoted by l) such that

$$(v^l, \omega^l, b^l) \longrightarrow (v, \omega, b) \text{ in } L^2(0,t^*;H^s(\Omega)) \text{ for } s < 3, \tag{87}$$

and

$$(v^l, \omega^l, b^l) \longrightarrow (v, \omega, b) \text{ in } C([0,t^*];H^q(\Omega)) \text{ for } q < 2. \tag{88}$$

Now, we characterize the limits of nonlinear terms. Firstly, we note that for fixed (x,t) we may write

$$\begin{aligned} \Psi_t(b^l(x,t)) - \Psi_t(b(x,t)) &= \int_0^1 \frac{d}{ds}\left[\Psi_t\left(sb^l(x,t) + (1-s)b(x,t)\right)\right] ds \\ &= \int_0^1 \Psi_t'(sb^l(x,t) + (1-s)b(x,t))ds \cdot [b^l(x,t) - b(x,t)]. \end{aligned}$$

Taking into account (26) we get

$$|\Psi_t(b^l(x,t)) - \Psi_t(b(x,t))| \le c_0|b^l(x,t) - b(x,t)|.$$

Similarly we obtain

$$|\Phi_t(\omega^l(x,t)) - \Phi_t(\omega(x,t))| \le c_0|\omega^l(x,t) - \omega(x,t)|.$$

and

$$|\Phi_t(b(x,t))| \le c_0(|b(x,t)| + b^t_{\min}).$$

Therefore, applying (27) we obtain

$$\begin{aligned}\left|\frac{\Psi_t(b^l)}{\Phi_t(\omega^l)} - \frac{\Psi_t(b)}{\Phi_t(\omega)}\right| &\le 4(\omega^t_{\min})^{-2}\Big[|\Phi_t(\omega)||\Psi_t(b^l) - \Psi_t(b)| + |\Psi_t(b)||\Phi_t(\omega) - \Phi_t(\omega^l)|\Big] \\ &\le 4(\omega^t_{\min})^{-2}\Big[2\omega_{\max}|b^l - b| + c_0(|b| + b^t_{\min})|\omega - \omega^l|\Big].\end{aligned}$$

From (88) and the above estimate we have

$$\mu^l \longrightarrow \mu_{\Psi_t\Phi_t} \equiv \frac{\Psi_t(b)}{\Phi_t(\omega)} \text{ uniformly on } \overline{\Omega} \times [0, t^*]. \tag{89}$$

Now, we shall take the limit $l \to \infty$ in the system (33–35). First, we multiply (33) by a_i and sum over $i \in \{1, \dots, l\}$ and after integrating with respect time variable we get

$$\int_0^t (\partial_t v^l, w)dt - \int_0^t (v^l \otimes v^l, \nabla w)dt + \int_0^t \Big(\mu^l D(v^l), D(w)\Big)\, dt = 0,$$

where $w = \sum_{i=1}^{l} a_i w_i$ and $t \in (0, t^*)$. We note that from (88) we have for some $\lambda > 0$

$$(v^l, \omega^l, b^l) \longrightarrow (v, \omega, b) \text{ in } C([0, t^*]; C^{0,\lambda}(\overline{\Omega})) \tag{90}$$

hence, (85), (88) and (89) imply that

$$\int_0^t (\partial_t v, w)dt - \int_0^t (v \otimes v, \nabla w)dt + \int_0^t \big(\mu_{\Psi_t\Phi_t} D(v), D(w)\big)\, dt = 0$$

for $t \in (0, t^*)$ and $w = \sum_{i=1}^{l} a_i w_i$. By density, the above identity holds for $w \in \dot{\mathcal{V}}^1_{\mathrm{div}}$. As a consequence, we obtain

$$\int_{t_1}^{t_2} (\partial_t v, w)dt - \int_{t_1}^{t_2} (v \otimes v, \nabla w)dt + \int_{t_1}^{t_2} \big(\mu_{\Psi_t\Phi_t} D(v), D(w)\big)\, dt = 0$$

for $0 < t_1 < t_2 < t^*$ and $w \in \dot{\mathcal{V}}^1_{\mathrm{div}}$. After dividing both sides by $|t_2 - t_1|$ and taking the limit $t_2 \to t_1$ we get

$$(\partial_t v, w) - (v \otimes v, \nabla w) + \big(\mu_{\Psi_t\Phi_t} D(v), D(w)\big) = 0 \text{ for } w \in \dot{\mathcal{V}}^1_{\mathrm{div}} \tag{91}$$

for a.a. $t \in (0, t^*)$. Further, we have

$$\psi_t(b^l) \longrightarrow \psi_t(b), \phi_t(\omega^l) \longrightarrow \phi_t(\omega^l) \text{ uniformly on } \overline{\Omega} \times [0, t^*]$$

thus, using (34) and (35) and arguing as earlier we obtain

$$(\partial_t \omega, z) - (\omega v, \nabla z) + \left(\mu_{\Psi_t, \Phi_t} \nabla \omega, \nabla z\right) = -\kappa_2(\phi_t^2(\omega), z) \text{ for } z \in \mathcal{V}^1, \tag{92}$$
$$(\partial_t b, q) - (bv, \nabla q) + \left(\mu_{\Psi_t, \Phi_t} \nabla b, \nabla q\right) = -(\psi_t(b)\phi_t(\omega), q) + (\mu_{\Psi_t, \Phi_t} |D(v)|^2, q)$$
$$\text{for } q \in \mathcal{V}^1 \tag{93}$$

for a.a. $t \in (0, t^*)$.

Now, we shall prove the bounds for b and ω. The proof is similar to one found in [8]. We denote by b_+ (b_-) the positive (negative resp.) part of b. Then $b = b_+ + b_-$. We shall show that

$$b \geq 0 \text{ in } \overline{\Omega} \times [0, t^*]. \tag{94}$$

For this purpose we test the Eq. (93) by b_- and we obtain

$$\begin{aligned}&(\partial_t b, b_-) - (bv, \nabla b_-) + \left(\mu_{\Psi_t, \Phi_t} \nabla b, \nabla b_-\right)\\ &\quad = -(\psi_t(b)\phi_t(\omega), b_-) + (\mu_{\Psi_t, \Phi_t} |D(v)|^2, b_-).\end{aligned}$$

We note that from (89) we have $0 \leq \mu_{\Psi_t, \Phi_t}$ and by (28) we obtain $\psi_t(b)b_- \equiv 0$ thus, we get

$$(\partial_t b_-, b_-) - (b_- v, \nabla b_-) + \left(\mu_{\Psi_t, \Phi_t} \nabla b_-, \nabla b_-\right) \leq 0$$

and then

$$\frac{d}{dt}\|b_-\|_2^2 \leq 0.$$

By the assumption (11) the negative part of initial value of b is zero hence, $b_- \equiv 0$ and we obtained (94).

Proceeding similarly we introduce the decomposition $\omega = \omega_+ + \omega_-$ and test the Eq. (92) by ω_-

$$(\partial_t \omega, \omega_-) - (\omega v, \nabla \omega_-) + \left(\mu_{\Psi_t, \Phi_t} \nabla \omega, \nabla \omega_-\right) = -(\phi_t^2(\omega), \omega_-).$$

We note that by (29) the right-hand side of the above equality vanishes thus, we get $\frac{d}{dt}\|\omega_-\|_2^2 \leq 0$ and by assumption (12)

$$\omega \geq 0 \text{ in } \overline{\Omega} \times [0, t^*]. \tag{95}$$

Now, we shall prove that

$$\omega(x,t) \geq \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t} \text{ for } (x,t) \in \overline{\Omega}\times[0,t^*]. \tag{96}$$

We test the equation (92) by $(\omega-\omega_{\min}^t)_-$ and we obtain

$$\begin{aligned}&(\partial_t\omega, (\omega-\omega_{\min}^t)_-) - (\omega v, \nabla(\omega-\omega_{\min}^t)_-) + \left(\mu_{\Psi_t\Phi_t}\nabla\omega, \nabla\left(\omega-\omega_{\min}^t\right)_-\right)\\ &\quad = -\kappa_2(\phi_t^2(\omega), (\omega-\omega_{\min}^t)_-).\end{aligned} \tag{97}$$

Using (13) we get

$$(\partial_t\omega, (\omega-\omega_{\min}^t)_-) = \frac{1}{2}\frac{d}{dt}\|(\omega-\omega_{\min}^t)_-\|_2^2 - \kappa_2\left((\omega_{\min}^t)^2, (\omega-\omega_{\min}^t)_-\right)$$

hence, using inequality $0 \leq \mu_{\Psi_t\Phi_t}$ and $\operatorname{div} v = 0$ in (97) we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\omega-\omega_{\min}^t)_-\|_2^2 - \kappa_2\left((\omega_{\min}^t)^2, (\omega-\omega_{\min}^t)_-\right) \leq -\kappa_2(\phi_t^2(\omega), (\omega-\omega_{\min}^t)_-).$$

We write the above inequality the form

$$\frac{1}{2}\frac{d}{dt}\|(\omega-\omega_{\min}^t)_-\|_2^2 \leq -\kappa_2((\phi_t(\omega)-\omega_{\min}^t)(\phi_t(\omega)+\omega_{\min}^t), (\omega-\omega_{\min}^t)_-).$$

We note that $-\kappa_2((\phi_t(\omega)+\omega_{\min}^t), (\omega-\omega_{\min}^t)_-)$ is nonnegative thus, using (32) we get $\phi_t(\omega) \leq \omega$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\omega-\omega_{\min}^t)_-\|_2^2 &\leq -\kappa_2((\omega-\omega_{\min}^t)(\phi_t(\omega)+\omega_{\min}^t), (\omega-\omega_{\min}^t)_-)\\ &= -\kappa_2\big((\phi_t(\omega)+\omega_{\min}^t), \left|(\omega-\omega_{\min}^t)_-\right|^2\big) \leq 0.\end{aligned}$$

Therefore, we obtain $\frac{d}{dt}\|(\omega-\omega_{\min}^t)_-\|_2^2 \leq 0$ and by (12) we get (96). Now, we shall prove that

$$\omega(x,t) \leq \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t} \text{ for } (x,t) \in \overline{\Omega}\times[0,t^*]. \tag{98}$$

Indeed, firstly we note that from (13), (29) and (96) we have

$$\phi_t(\omega) = \omega \tag{99}$$

hence, if we test the equation (92) by $(\omega-\omega_{\max}^t)_+$ then we obtain

$$(\partial_t\omega, (\omega-\omega_{\max}^t)_+) - (\omega v, \nabla(\omega-\omega_{\max}^t)_+) + \left(\mu_{\Psi_t\Phi_t}\nabla\omega, \nabla\left(\omega-\omega_{\max}^t\right)_+\right)$$

$$= -\kappa_2(\omega^2, (\omega - \omega^t_{\max})_+).$$

Proceeding as earlier, we get

$$\frac{1}{2}\frac{d}{dt}\|(\omega - \omega^t_{\max})_+\|_2^2 - \kappa_2\left((\omega^t_{\max})^2, (\omega - \omega^t_{\max})_+\right) \leq -\kappa_2(\omega^2, (\omega - \omega^t_{\max})_+).$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\omega - \omega^t_{\max})_+\|_2^2 &\leq -\kappa_2((\omega - \omega^t_{\max})(\omega + \omega^t_{\max}), (\omega - \omega^t_{\max})_+) \\ &= -\kappa_2((\omega + \omega^t_{\max}), |(\omega - \omega^t_{\max})_+|^2)\end{aligned}$$

hence, we obtain

$$\frac{1}{2}\ \tfrac{d}{dt}\|(\omega - \omega^t_{\max})_+\|_2^2 \leq 0. \tag{100}$$

By (12) we get (98). We shall prove that

$$b(x,t) \geq b^t_{\min} \text{ for } (x,t) \in \overline{\Omega} \times [0, t^*]. \tag{101}$$

For this purpose we test the equation (93) by $(b - b^t_{\min})_-$. Then we get

$$\begin{aligned}&(\partial_t b, (b - b^t_{\min})_-) - (bv, \nabla((b - b^t_{\min})_-)) + \left(\mu_{\Psi_t\Phi_t}\nabla b, \nabla((b - b^t_{\min})_-)\right) \\ &\quad = -(\psi_t(b)\omega, (b - b^t_{\min})_-) + (\mu_{\Psi_t\Phi_t}\,|D(v)|^2\,, (b - b^t_{\min})_-).\end{aligned}$$

The first term on the left-hand side is equal to

$$\frac{1}{2}\frac{d}{dt}\|(b - b^t_{\min})_-\|_2^2 - \left(\frac{\omega_{\max} b_{\min}}{(1 + \omega_{\max}\kappa_2 t)^{\frac{1}{\kappa_2}+1}}, (b - b^t_{\min})_-\right).$$

The second term of the left-hand side vanishes and the third is nonnegative. Thus, we get

$$\frac{1}{2}\frac{d}{dt}\|(b - b^t_{\min})_-\|_2^2 - \left(\frac{\omega_{\max} b_{\min}}{(1 + \omega_{\max}\kappa_2 t)^{\frac{1}{\kappa_2}+1}}, (b - b^t_{\min})_-\right) \leq -(\psi_t(b)\omega, (b - b^t_{\min})_-).$$

Using (98) we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(b - b^t_{\min})_-\|_2^2 - \left(\frac{\omega_{\max} b_{\min}}{(1 + \omega_{\max}\kappa_2 t)^{\frac{1}{\kappa_2}+1}}, (b - b^t_{\min})_-\right) \\ &\quad \leq -\frac{\omega_{\max}}{1 + \omega_{\max}\kappa_2 t}(\psi_t(b), (b - b^t_{\min})_-)\end{aligned}$$

and by definition (13) we obtain

$$\frac{1}{2}\frac{d}{dt}\|(b-b^t_{\min})_-\|_2^2 \leq -\frac{\omega_{\max}}{1+\omega_{\max}\kappa_2 t}(\psi_t(b)-b^t_{\min},(b-b^t_{\min})_-).$$

From (94) and (31) we have $\psi_t(b) \leq b$ so, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(b-b^t_{\min})_-\|_2^2 &\leq -\frac{\omega_{\max}}{1+\omega_{\max}\kappa_2 t}(b-b^t_{\min},(b-b^t_{\min})_-)\\ &= -\frac{\omega_{\max}}{1+\omega_{\max}\kappa_2 t}\|(b-b^t_{\min})_-\|_2^2\end{aligned}$$

and then $\frac{d}{dt}\|(b-b^t_{\min})_-\|_2^2 \leq 0$. Using (11) and (13) we get (101).

Note that from (28) and (101) we get

$$\psi_t(b) = b. \tag{102}$$

Further, (25) and (101) give $\Psi_t(b) = b$. Finally, (13), (27), (96) and (98) yield $\Phi_t(\omega) = \omega$. Thus,

$$\mu_{\Psi_t\Phi_t} = \frac{\Psi_t(b)}{\Phi_t(\omega)} = \frac{b}{\omega}. \tag{103}$$

Applying (99), (102) and (103) we deduce that system (91)-(93) has the following form

$$(\partial_t v, w) - (v\otimes v, \nabla w) + \left(\frac{b}{\omega}D(v), D(w)\right) = 0 \text{ for } w \in \dot{\mathcal{V}}^1_{\mathrm{div}}, \tag{104}$$

$$(\partial_t \omega, z) - (\omega v, \nabla z) + \left(\frac{b}{\omega}\nabla\omega, \nabla z\right) = -\kappa_2(\omega^2, z) \text{ for } z \in \mathcal{V}^1, \tag{105}$$

$$(\partial_t b, q) - (bv, \nabla q) + \left(\frac{b}{\omega}\nabla b, \nabla q\right) = -(b\omega, q) + \left(\frac{b}{\omega}|D(v)|^2, q\right) \text{ for } q \in \mathcal{V}^1 \tag{106}$$

for a.a. $t \in (0, t^*)$.

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Appendix

The function Ψ_t may be defined as follows. We set $f(x) = e^{-1/x}$ for $x > 0$ and zero elsewhere. We put $g(x) = x - e^{-1/x}$ for $x < 0$ and $g(x) = x$ for $x > 0$. Then we set

$$\tilde{\eta}(x) = \frac{1}{c}\int_0^x f(y)f(-y+1)dy,$$

where $c = \int_0^1 f(y)f(-y+1)dy$. Function $\tilde{\eta}$ is smooth function, which vanishes for negative x and is equal to one for $x > 1$. Next, we put

$$\eta(x) = \tilde{\eta}(2(x - \frac{1}{4})), h(x) = (1 - \eta(x))f(x) + \eta(x)g(x).$$

Finally, we define

$$\Psi_t(x) = \frac{b^t_{\min}}{2} + \frac{b^t_{\min}}{2} h\left(\frac{2}{b^t_{\min}}\left(x - \frac{b^t_{\min}}{2}\right)\right). \tag{107}$$

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