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Monotonicity of horizontal fluid velocity and pressure gradient distribution beneath equatorial Stokes waves

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Abstract
In this paper, we study the irrotational periodic equatorial surface travelling waves in flows without neglecting Coriolis forces due to the Earth's rotation. The monotonicity of horizontal velocity and pressure gradient distribution are obtained by the physical structures for the problem itself and the maximum principles.

Keywords Equatorial flow · Velocity · Pressure gradient · Coriolis forces · Maximum principles

Mathematics Subject Classification 34B15 · 34D20 · 76B03

1 Introduction

In this paper, we consider a qualitative description of horizontal velocity and pressure gradient of irrotational equatorial flows without neglecting Coriolis forces due to the Earth's rotation, cf. the discussion in the papers [1, 2]. The change of sign of the Coriolis force across the Equator produces an effective waveguide, with the Equator acting as a

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fictitious wall that forces an azimuthal flow propagation. Relative to the monotonicity of velocity and to the pressure gradient distribution, refer to the theoretical papers [3, 4], the numerical simulations in [5], and the experiments reported in [6]. The particle trajectories [7–9] can be controlled by the velocity and pressure in the fluid domain, which is very important to discuss the monotonicity of velocity, and pressure gradient distribution. Considering the simple case where Coriolis forces are ignored, many very nice results for the monotonicity of the horizontal fluid velocities [7, 8, 10], pressure gradient distribution which refers to [7, 8, 10–12] and references therein, and wave heights can be estimated from pressure data, see the discussions in [13–15]. More work on ocean currents, such as constant vorticity flows and geophysical fluid flows, cf. [16–20] and so on.

As far as we know, without ignoring the Coriolis force, the monotonicity of the horizontal velocity is only discussed in [9], and the pressure gradient distribution has not yet been studied. Inspired by the above facts, in the present paper, we study the monotonicity of horizontal velocity, and pressure gradient distribution of the irrotational equatorial flows without neglecting the Earth's rotation. The existence can refer to [21]. We firstly use the modified height function in [22] to transform the free boundary value problem into a elliptic boundary value problem. Secondly, inspired by [7, 8, 10], we use the physical properties of the fluid itself and the strong maximum principle to obtain the monotonicity of horizontal velocity along every streamlines in the fluid domain. Our approach is different from [9]. Inspired by [7, 8, 10–12], we further study the pressure gradient distribution by using appropriate auxiliary function, the properties of the fluid and the maximum principles. Since we take Coriolis forces into account, our results complement the existing literature.

The paper is organized as follows. In Sect. 2, we give some preliminary results, mainly including governing equations and their two equivalent forms. In Sect. 3, we derive the monotonicity of horizontal fluid velocity, and pressure gradient distribution through the fluid.

2 Preliminaries

Throughout this section, we collect some preliminary results. Firstly, let $L > 0$ be wave period, t be time. We choose the Cartesian coordinate system $(x, y) \in \mathbb{R}^2$, where x-axis is horizontal to the east and y-axis points upwards. Let the free surface of the water flow denote by $y = \eta(t, x)$, which satisfies $\int_{-L/2}^{L/2} \eta(t, x)dx = 0$. Set $y = 0$ be the mean surface level for the water flow. Let $y = -d$ be the impermeable flat bed with $0 < d < \infty$, which below the free surface $y = \eta(x)$ of the flow. We assume that the Earth is a perfect sphere rotating with the speed $\Omega = 7.292 \times 10^{-5}$ rad/s. Here, the surface flow is westward due to prevailing westward winds, which implies that $c < 0$. The form of the velocity field is $(u(x - ct, y), v(x - ct, y))$ and the wave surface profile has the form $\eta(x - ct)$. Here, we use the transformation $(x - ct, y) \to (x, y)$ in which the origin moves in the direction of propagation of the wave with wave speed c. In stationary domain $G_\eta := \{(x, y) \in \mathbb{R}^2 : -d \le y \le \eta(x)\}$, we employ the f-plane approximation to the full governing equations satisfy the Euler equations

$$\begin{cases} (u-c)u_x + vu_y + 2\Omega v = -\frac{1}{\rho}P_x, \\ (u-c)v_x + vv_y - 2\Omega u = -\frac{1}{\rho}P_y - g \end{cases} \tag{1}$$

with equation of mass conservation

$$u_x + v_y = 0. \tag{2}$$

Here $P := P(x, y)$ is the pressure, ρ is the constant density, $g \approx 9.8$ m/s is the acceleration due to gravity. The kinematic boundary conditions are given by

$$v = (u-c)\eta_x \quad \text{on} \quad y = \eta(x); \qquad v = 0 \quad \text{on} \quad y = -d. \tag{3}$$

And dynamic boundary condition is expressed as

$$P = P_{atm} \quad \text{on} \quad y = \eta(x), \tag{4}$$

where P_{atm} is the constant atmospheric pressure. Assumed that the flow is irrotational, which implies

$$u_y - v_x = 0. \tag{5}$$

Governing Eqs. (1)–(5) see [9, 21, 23] for more details. Furthermore, we assume the absence of stagnation points, that is $u \neq c$ in G_η. Here, we only have to talk about the case where u is greater than c due to the case where u is less than c is symmetric. To avoid new notations, we describe the velocity field of the steady motion with the same symbols

$$(u-c, v) \to (u, v).$$

Under the above assumptions, we know $u > 0$ and rewrite (1)–(5) in the new reference frame as the form

$$\begin{cases} uu_x + vu_y + 2\Omega v = -\frac{1}{\rho}P_x, \\ uv_x + vv_y - 2\Omega u = -\frac{1}{\rho}P_y - (g - 2\Omega c), \\ u_x + v_y = 0, \\ v = u\eta_x, \quad y = \eta(x), \\ v = 0, \quad y = -d. \\ P = P_{atm}, \quad y = \eta(x), \\ u_y - v_x = 0 \end{cases} \tag{6}$$

in G_η. Since Ω is small enough, we assume that $g - 2\Omega c > 0$. Equatorial waves satisfying the Eq. (6) conform to the following (See [9, 21, 23]):

(1) u and v only one crest and one trough in per period;
(2) η is strictly monotonous between per successive crest and trough;

(3) η, u and P are symmetric with respect to the crest line, and v is antisymmetric about the crest line.

For more details, see [7].

By the third relation in (6), we can define the stream function Ψ up to a constant by

$$\Psi_x = -v, \qquad \Psi_y = u, \qquad (x, y) \in G_\eta. \tag{7}$$

We see that Ψ is a constant on $y = \eta(x)$ and $y = -d$, respectively. Thus, without any loss of generality, we can choose $\Psi = 0$ on $y = \eta(x)$ and $\Psi = m$ on $y = -d$, where

$$m = -\int_{-d}^{\eta(x)} u(x, y)dy < 0, \qquad (x, y) \in G_\eta$$

satisfies $\Psi(x, y) = m + \int_{-d}^{y} u(x, r)dr$ in G_η.

It is can be seen that $\Psi(x, y)$ is a L-periodic function related x. Then the problem (6) is equivalent to the free boundary problem

$$\begin{cases} \Delta\Psi = 0, \qquad -d < y < \eta(x), \\ \frac{|\nabla\Psi|^2}{2} + (g - 2\Omega c)(y + d) + \frac{1}{\rho} P_{atm} = Q, \qquad y = \eta(x), \\ \Psi = 0, \qquad y = \eta(x), \\ \Psi = m, \qquad y = -d, \end{cases} \tag{8}$$

where the constant $Q > 0$ is called total head. It is learned that Clamond [24] deals with waves without Coriolis effects, a derivation of Bernoulli's equation (the second in (8)) in the equatorial context being provided in [25].

On the other hand, from the seventh relation of (6), we define the velocity potential

$$\Phi(x, y) = \int_0^x u(l, -d)dl + \int_{-d}^{y} v(x, r)dr, \quad (x, y) \in G_\eta$$

by

$$\Phi_x = u, \qquad \Phi_y = v, \qquad (x, y) \in G_\eta. \tag{9}$$

Let $\zeta = \int_{-L/2}^{L/2} u(x, -d)dx > 0$, then $\Phi - \zeta x/L$ is L-periodic in x, and Φ is an odd function in the x, while

$$\Phi(Ln, y_0) = \int_0^{Ln} u(l, y_0)dl = \zeta n, \quad n = 1, 2, \cdots$$

for $y_0 \in [-d, \min \eta(x))$. From Ψ and Φ we can perform the conformal hodograph transformation $\Xi : G_\eta \to \widetilde{G}_\eta$ by

$$\Xi(x, y) := (q, s)(x, y) := \left(\Phi(x, y), -\frac{\Psi(x, y)}{m}\right), \quad (x, y) \in G_\eta. \tag{10}$$

Here $\widetilde{G}_\eta = \{(q,s) \in \mathbb{R}^2 : -1 \leq s \leq 0\}$. The mapping Ξ is a diffeomorphism like such [8].

Because of the periodicity of u, v, P and η, we only need to consider one period. Thereby, we're talking about the free region between $x = -L/2$ and $x = L/2$. Here, we represent the interior regions

$$G_- = \left\{(x, y) \in \mathbb{R}^2 : -L/2 < x < 0, -d < y < \eta(x)\right\},$$
$$G_+ = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < L/2, -d < y < \eta(x)\right\},$$

and their lateral edges

$$\left\{(x, y) \in \mathbb{R}^2 : x = 0, -d \leq y \leq \eta(0)\right\},$$
$$\left\{(x, y) \in \mathbb{R}^2 : x = \pm L/2, -d \leq y \leq \eta(\pm L/2)\right\}.$$

It also gives the free surface

$$S_- = \left\{(x, y) \in \mathbb{R}^2 : -L/2 < x < 0, y = \eta(x)\right\},$$
$$S_+ = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < L/2, y = \eta(x)\right\},$$

and the lower boundaries are also given

$$B_- = \left\{(x, y) \in \mathbb{R}^2 : -L/2 < x < 0, y = -d\right\},$$
$$B_+ = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < L/2, y = -d\right\}.$$

After conformal transformation, the regions G_- and G_+ become

$$\widetilde{G}_- = \left\{(q, s) \in \mathbb{R}^2 : -\zeta/2 < q < 0, -1 < s < 0\right\},$$
$$\widetilde{G}_+ = \left\{(q, s) \in \mathbb{R}^2 : 0 < q < \zeta/2, -1 < s < 0\right\},$$

respectively. Meanwhile lateral edges are given by

$$\{(q, s) \in \mathbb{R}^2 : q = 0, -1 \leq s \leq 0\}, \quad \{(q, s) \in \mathbb{R}^2 : q = \pm\zeta/2, -1 \leq s \leq 0\}.$$

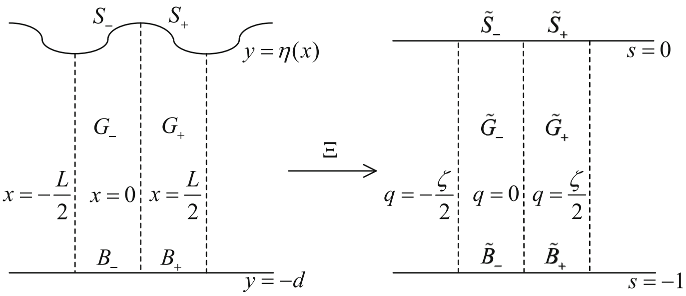

Fig. 1 The hodograph transform between the free boundary domain G_η and the fixed domain $\widetilde{G}_\eta$

The free surface and lower boundaries are shown

$$
\begin{aligned}
\widetilde{S}_- &= \left\{(q,s) \in \mathbb{R}^2 : -\zeta/2 < q < 0, s = 0\right\},\\
\widetilde{S}_+ &= \left\{(q,s) \in \mathbb{R}^2 : 0 < q < \zeta/2, s = 0\right\},\\
\widetilde{B}_- &= \left\{(q,s) \in \mathbb{R}^2 : -\zeta/2 < q < 0, s = -1\right\},\\
\widetilde{B}_+ &= \left\{(q,s) \in \mathbb{R}^2 : 0 < q < \zeta/2, s = -1\right\}.
\end{aligned}
$$

See Fig. 1.

Next, we will use similar arguments in [8] to show that (8) is equivalent to the equation which related to the modified height function. Now we introduce the modified height function $h : \widetilde{G}_\eta \to \mathbb{R}$ by

$$
h(q,s) = \frac{y}{d} - s, \quad (q,s) \in \widetilde{G}_\eta \tag{11}
$$

in [22]. By the chain rule and (7), (9), (10), (11), it follows that

$$
h_q = \frac{v}{d(u^2+v^2)}, \qquad h_s = \frac{-mu}{d(u^2+v^2)} - 1. \tag{12}
$$

By (7), (9) and (10), we obtain that

$$
\partial_x = u\partial_q + \frac{v}{m}\partial_s, \qquad \partial_y = v\partial_q - \frac{u}{m}\partial_s. \tag{13}
$$

Directly calculating (13) and combining with (12), we have

$$
\partial_q = -\frac{d}{m}(h_s+1)\partial_x + dh_q\partial_y, \qquad \partial_s = dmh_q\partial_x + d(h_s+1)\partial_y. \tag{14}
$$

By (7) and the seventh relation in (6), it follows that

$$0 = mdh_{sq}[m^2h_q^2 + (h_s+1)^2][(m^2h_{qq} - h_{ss})(h_s+1) - 2m^2h_q] \\ - 2[(h_s+1) + h_q][m^2h_q(h_{qq} + h_{qs}) + (h_s+1) + (h_{sq} + h_s)] \\ \text{in} \quad -1 < s < 0.$$

By straightforward calculation, the second Eq. in (8) is equal to

$$m^2[m^2h_q^2 + (h_s+1)^2] + 2d^2\left[d(g - 2\Omega c)(h+s+1) - Q + \frac{1}{\rho}P_{atm}\right] \\ (m^2h_q^2 + (h_s+1)^2)^2 = 0, \quad s = 0.$$

On the flat bed, it is easy to see that

$$h = 0, \quad s = -1.$$

On the other hand, h satisfies

$$\int_{-L/2}^{L/2} h(q,0)dq = 0.$$

Conesquently, the problem (8) is rewritten to

$$\begin{cases} mdh_{sq}[m^2h_q^2 + (h_s+1)^2][(m^2h_{qq} - h_{ss})(h_s+1) - 2m^2h_q] \\ = 2[(h_s+1) + h_q][m^2h_q(h_{qq} + h_{qs}) + (h_s+1) + (h_{sq} + h_s)], \\ -1 < s < 0, \\ m^2[m^2h_q^2 + (h_s+1)^2] + 2d^2\left[d(g - 2\Omega c)(h+s+1) - Q + \frac{1}{\rho}P_{atm}\right] \\ (m^2h_q^2 + (h_s+1)^2)^2 = 0, \quad s = 0, \\ h = 0, \quad s = -1, \\ \int_{-L/2}^{L/2} h(q,0)dq = 0. \end{cases}$$

Next, we give two lemmas, which are important in our discussion.

Lemma 2.1 (See [10]) *For $u > 0$ in G_η, the following inequalities hold:*
(a) $u_x(x, -d) > 0, \quad x \in (0, L/2)$; (b) $u_y(L/2, y) > 0, \quad y \in (-d, \eta(L/2)$;
(c) $v(x, y) < 0, \quad (x, y) \in G_+$; (d) $v(x, y) > 0, \quad (x, y) \in G_-$.

Since the realistic values of u are less than $\frac{g-2\Omega c}{2\Omega}$, there is a following lemma.

Lemma 2.2 (See [23]) *Let function $f(x) = u(x, \eta(x)) + 2\Omega\eta(x)$ be monotonically increasing with respect to $x \in (0, L/2)$, that is $\partial_x u(x, \eta(x)) + 2\Omega\eta'(x) > 0$ for $x \in (0, L/2)$.*

3 Main results

Theorem 3.1 *The horizontal fluid velocity u is a strictly increasing function of x in $\overline{G}_+$ and a strictly decreasing function of x in $\overline{G}_-$ along any streamline.*

Proof If $(x, y(x))$ is a parametric equation for a streamline, then $\partial_x(\Psi(x, y(x))) = 0$ by definition. From this, we see $\Psi_x + y'\Psi_y = 0$. On the other hand, we notice that $u > 0$ in G_η, $m < 0$ and the second relation in (12) ensure the positive of $h_s + 1$. Hence, by (7) and (12), we get

$$\frac{dy}{dx} = y' = -\frac{\Psi_x}{\Psi_y} = \frac{v}{u} = -\frac{mh_q}{h_s + 1}.$$

From this and the first relation in (14), we have

$$\partial_x(u(x, y(x))) = u_x + y'u_y = u_x - \frac{mh_q}{h_s + 1}u_y = -\frac{m}{d(h_s + 1)}u_q. \tag{15}$$

From the third and seventh relations in (6), we can see that $\triangle_{(x,y)}u = 0$ and $\triangle_{(x,y)}v = 0$. They are also harmonic in $(q, s) \in \mathbb{R}^2$ for (10) is conformal change of variables. Thus, we get $\triangle u_q = 0$. Considering the restriction of u_q to $\widetilde{G}_+$, by (14) and (12), we obtain that

$$u_q = \frac{1}{u^2 + v^2}(uu_x + vu_y). \tag{16}$$

Note that v is odd and periodic in x, we can see $v = 0$ on the crest line $\{(x, y) : x = 0, -d < y < \eta(0)\}$ and trough line $\{(x, y) : x = \pm L/2, -d < y < \eta(0)\}$. Due to the images under (10) of the crest and trough lines, then we obtain

$$u_q = \frac{u_x}{u} = -\frac{v_y}{u} = 0 \quad \text{for} \quad q = 0, \pm\zeta/2 \tag{17}$$

by (16) and $u > 0$ in G_η. Note that $v = 0$ in $\widetilde{B}_+$, by (16) and Lemma 2.1(a), we have

$$u_q = \frac{u_x}{u} > 0 \quad \text{on} \quad \widetilde{B}_+. \tag{18}$$

From Lemma 2.2, we can see that

$$\partial_x u(x, \eta(x)) > -2\Omega\eta'(x) > 0, \quad x \in (0, L/2). \tag{19}$$

Thus, by (16), the fourth relation of (6) and (19), we obtain

$$u_q = \frac{u}{u^2 + v^2}\partial_x(u(x, \eta(x))) > 0 \quad \text{on} \quad \widetilde{S}_+. \tag{20}$$

By (17), (18), (20), and applying the strong maximum principle to the harmonic function u_q, we obtain that $u_q > 0$ in $\widetilde{G}_+$. But together with $h_s + 1 > 0$ and $m < 0$, by (15), it follows that $\partial_x(u(x, y(x))) > 0$ in G_+. Hence, u increases strictly along every streamline with respect to x in $\overline{G}_+$. Being even in the x-variable, u is a strictly decreasing function of x along every streamline in $\overline{G}_-$. This completes the proof of Theorem 3.1. □

Remark 3.2 We find the results in Theorem 3.1 is same as [10,Claim 1], which indicates that the Coriolis force does not affect the monotonicity of the horizontal fluid velocity. It makes sense mathematically.

The pressure gradient distribution results are given below. From a physical point of view, the realistic values of v are bounded, so we can define $M^- := \min_{(x,y)\in G_+} v(x, y)$ and $M^+ := \max_{(x,y)\in G_-} v(x, y)$ by (c), (d) of Lemma 2.1.

Theorem 3.3 *Assume $\eta_x(x)$ is bound for $\mathbb{R}$. The pressure gradient P_x at a point in fluid depends on the position of the point with respect to the crest line: $P_x = 0$ below the crest, $P_x < 0$ on $S_+ \cup B_+$, $P_x > 0$ on $S_- \cup B_-$, $P_x < -2\Omega\rho M^-$ in G_+, $P_x > -2\Omega\rho M^+$ in G_-; the component $P_y < 0$ on $B_+ \cup S_+ \cup \{x = \pm L/2\} \cup B_- \cup S_-$, $P_y < \rho(g - 2\Omega c)$ in $G_+ \cup \{x = 0\} \cup G_-$.*

Proof We firstly show the distribution case of P_x. Using the first relation in (6) and (16), it follows that

$$P_x = -\rho((u^2 + v^2)u_q + 2\Omega v), \quad x \in \mathbb{R}. \tag{21}$$

(18) and the fifth relation in (6) imply that $P_x < 0$ on B_+. Similarly, we obtain that $P_x > 0$ on B_-. On the free surface $S_+ \cup S_-$, P_x is rewritten

$$P_x = -\rho u(u_x + u_y\eta_x + 2\Omega\eta_x) = -\rho u(\partial_x u(x, \eta(x)) + 2\Omega\eta'(x)),$$
$$x \in (-L/2, 0) \cup (0, L/2).$$

By Lemma 2.2 we see that $P_x < 0$ on S_+. Similarly, we can figure out $P_x > 0$ on S_-.

Note that v is odd and periodic in x, we can see $v = 0$ on the crest line $\{(x, y) : x = 0, -d < y < \eta(0)\}$ and trough line $\{(x, y) : x = \pm L/2, -d < y < \eta(0)\}$, by (17) and (21) we have $P_x = 0$ on the crest and trough line.
By (21) and Lemma 2.1(c), and we also know $u_q > 0$ in G_+ from proof of Theorem 3.1, it follows that $P_x < -2\Omega\rho M^-$ in G_+. Similarly, we have $P_x > -2\Omega\rho M^+$ in G_-.

Next, we give the distribution case of P_y. Now we show that

$$\frac{d}{dx}(-uv|_{y=\eta(x)}) < g - 2\Omega c, \quad x \in (-L/2, 0) \cup (0, L/2). \tag{22}$$

As a matter of fact, by the third, fourth and seventh relations in (6), we obtain

$$\frac{d}{dx}(-u(x, \eta(x))v(x, \eta(x)) = -uv_x(1 + \eta_x^2), \quad x \in (-L/2, 0) \cup (0, L/2). \tag{23}$$

On the other hand, by differentiating the second expression in (8), and using the seventh, third and fourth relations in (6), we have

$$\begin{aligned} 0 &= u(u_x + u_y\eta_x) + v(v_x + v_y\eta_x) + (g - 2\Omega c)\eta_x \\ &= -uv_y(1 - \eta_x^2) + 2vv_x + (g - 2\Omega c)\eta_x, \quad y = \eta(x). \end{aligned}$$

Note that $\eta'(x) < 0$ in $(0, L/2)$ and $\eta'(x) > 0$ in $(-L/2, 0)$, so from the above and the four relation in (6), we get

$$u\left[\frac{v_y}{\eta_x}(1 - \eta_x^2) - 2v_x\right] = g - 2\Omega c \quad \text{on} \quad S_+. \tag{24}$$

By Lemma 2.2 and the third, fourth and seventh relations in (6), we can obtain $v_y < v_x\eta_x$ on S_+, which implies

$$\frac{v_y}{\eta_x}(1 - \eta_x^2) - 2v_x > -v_x(1 + \eta_x^2) \quad \text{on} \quad S_+. \tag{25}$$

Here, $1 - \eta_x^2(x) > 0$ in [7].
Combine (23), (24) and (25), we see (22) in $(0, L/2)$ holds. Further, we notice that u is symmetric and v is antisymmetric with respect to $x = 0$, then (22) in $(-L/2, 0)$ holds. In order to talk about P_y in G_+, analogous to the considerations in [7], we define

$$F(x, y) = -u(x, y)v(x, y) - (g - 2\Omega c)x, \quad (x, y) \in G_\eta. \tag{26}$$

Using the third and seventh relations in (6), it is can be seen that $\triangle_{(x,y)}F = 0$. By (10), we see $\triangle_{(q,s)}F = 0$, then we have $\triangle F_q = 0$ in $\widetilde{G}_+$. By the first relation in (14), the third and seventh relations in (6), we have

$$F_q = -v_x - \frac{(g - 2\Omega c)u}{u^2 + v^2}, \quad (q, s) \in \widetilde{G}_\eta. \tag{27}$$

Since the fourth relation in (6), $v = u\eta_x$ on $S_+ \cup S_- \cup \{(0, \eta(0))\}$ and (27), we see that

$$F_q = -v_x - \frac{g - 2\Omega c}{u(1 + \eta_x^2)} \quad \text{along} \quad s = 0.$$

Invoking again (22) and (23), we obtain

$$F_q < 0 \quad \text{for} \quad s = 0. \tag{28}$$

Note that $v = 0$ on $\{(\zeta/2, s) : -1 < s < 0\}$, and by Lemma 2.1(b), (27), we obtain that

$$F_q = -v_x - \frac{g - 2\Omega c}{u} < 0 \quad \text{for} \quad q = \zeta/2.$$

Since $v = 0$, $v_x = 0$ and $u > 0$ on the flat bed, by (27), it follows that

$$F_q = -\frac{g - 2\Omega c}{u} < 0 \quad \text{for} \quad s = -1. \tag{29}$$

Since F_q is even in the q-variable and satisfies $\triangle F_q = 0$ in $[-\zeta/2, \zeta/2] \times [-1, 0]$, by (28), (29) and the maximum principle, we can get

$$F_q = -v_x - \frac{(g - 2\Omega c)u}{u^2 + v^2} < 0 \quad \text{in} \quad [-\zeta/2, \zeta/2] \times [-1, 0]. \tag{30}$$

On the other hand, from the second relation in (6), we obtain

$$P_y = \rho[2\Omega u - (g - 2\Omega c) - uv_x - vv_y], \quad (x, y) \in G_\eta. \tag{31}$$

We know $u_q > 0$ in $\widetilde{S}_+ \cup \widetilde{B}_+ \cup \widetilde{G}_+$ by Theorem 3.1. Combing (16), the third and seventh relations of (6), we get $uv_y < vv_x$ in $S_+ \cup B_+ \cup G_+$. Hence, from this and (31) we obtain

$$P_y < \rho(2\Omega u - (g - 2\Omega c)) - \rho v_x \frac{u^2 + v^2}{u}, \quad (x, y) \in G_+. \tag{32}$$

Due to $u < \frac{g-2\Omega c}{2\Omega}$, and combine (32), (30) we can see that $P_y < \rho(g - 2\Omega c)$ in G_+. Similarly, $P_y < \rho(g - 2\Omega c)$ in G_-.
By the fifth relation in (6), (31) and $2\Omega u < g - 2\Omega c$, we have

$$P_y = \rho(2\Omega u - (g - 2\Omega c)) < 0 \quad \text{on} \quad y = -d.$$

From (6) we can see that $\Delta P \leq 0$, then the minimum value of P is obtained on the free surface $y = \eta(x)$. Hence, by (31) and Hopf's maximum principle, $P_y < 0$ on $S_+ \cup S_- \cup \{(0, \eta(0))\}$. Because of $0 < u < \frac{g-2\Omega c}{2\Omega}$, (31) and Lemma 2.1(b), it follows that

$$P_y = \rho(2\Omega u - (g - 2\Omega c) - uv_x) < 0 \quad \text{on} \quad x = L/2.$$

Similarly, we can get $P_y < 0$ on $x = -L/2$. On the crest line $x = 0$, by (27), we can see that

$$uF_q(0, s) = -(uv_x + g - 2\Omega c). \tag{33}$$

Recalling $v = 0$, and combine (31), (33), it follows that

$$P_y = \rho u[2\Omega + F_q(0, s)] \quad \text{on} \quad x = 0.$$

Invoking (30) and $2\Omega u < g - 2\Omega c$, we obtain that

$$P_y < \rho(g - 2\Omega c) \quad \text{on} \quad x = 0.$$

The proof is completed. □

Remark 3.4 We find the results in Theorem 3.3 is different from [10,Claim 3], which indicates that the Coriolis force affect the distribution of the pressure gradient. It is easily obtainable that P_x and P_y in (6) depend on u, v and their partial derivatives. On the other hand, Coriolis force also depends on u and v, so the Coriolis force causes changes in P_x and P_y, which is mathematically reasonable.

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