Monatshefte fÃ¼r Mathematik (2022) 199:717–732
https://doi.org/10.1007/s00605-021-01655-0

Gouyon waves in water of finite depth

A. A. Abrashkin[1,2]

Received: 28 October 2021 / Accepted: 23 November 2021 / Published online: 21 January 2022

Abstract
Propagation of periodic stationary weakly vortical gravitational waves on the free water surface is considered. Similar wave motion was studied by Gouyon (Ann de la Fac des Sci de l'Université de Toulouse Sér 4(22):1–55, 1958) in linear and quadratic approximations in small parameter of the wave's steepness ε for the deep water conditions. In this paper this result is considered for the water of finite depth. Contrary to Gouyon who used the Euler approach a study of wave's motion is performed here basing on the method of the modified Lagrangian coordinates. The wave's vorticity Ω is specified as a series in the small parameter of steepness ε in the form: $\Omega = \sum_{n=1}^{\infty} \varepsilon^n \cdot \Omega_n(b)$, where Ω_n are arbitrary functions of the vertical Lagrangian coordinate b. Explicit expressions for the coordinates of the liquid particle trajectories and pressure distribution are obtained for the first two orders of perturbation theory. The nonlinear proportional to ε correction to the wave velocity is determined.

Keywords Water waves · Vorticity · Lagrangian variables

Mathematics Subject Classification 76B15 · 76 B47

1 Introduction

In the theory of water waves the assumption of the potentiality of the flow is used widely. But in a number of natural phenomena this approximation turns out to be unjustified. Rotational effects are significant in many circumstances. First of all, this is relevant to surface waves at the background of shear currents. At the same time,

Communicated by Adrian Constantin.

✉ A. A. Abrashkin
aabrashkin@hse.ru

1 National Research University Higher School of Economics, Nizhny Novgorod, Russia

2 Federal Research Center, Institute of Applied Physics of the Russian Academy of Sciences, Nizhny Novgorod, Russia

when waves are generated by the wind, which is one of the main mechanisms for the wave formation on water, a shear flow arises in the near-surface layer. It introduces vorticity into the liquid, so that the waves propagating along the free surface have a fundamentally vortex character.

It is noteworthy that historically the first analytical description of water waves was obtained by Gerstner just for vortex waves [17]. The Gerstner wave is a nonlinear elevation of a free surface of a trochoidal profile running in the horizontal direction [4, 6, 7, 19, 21, 22, 24]. The depth of the liquid is infinite. Liquid particles move in circles with a radius exponentially decreasing with depth.

The Gerstner wave has quite particular kind of vorticity. So, the description of waves with a more general vorticity distribution is required. In a stationary flow the vorticity along streamlines ψ is preserved and can be defined as an arbitrary function of this variable. For the case of vorticity the existence of large classes of small amplitude solutions was proven by Dubreil-Jacotin [15] in 1934 and large-amplitude solutions by Constantin and Strauss [8].

In 1958 Gouyon turned to the problem of description of periodic stationary weakly vortical waves in a fluid of infinite depth basing on the method of perturbation theory. In this case it is convenient to specify vorticities in each of the approximations. Gouyon suggested that the wave vorticity has the following form [18, 24]:

$$\Omega = \sum_{n=1}^{\infty} \varepsilon^n \cdot \Omega_n(\Psi). \tag{1}$$

where ε is the small parameter of the wave steepness This formula is the most general representation of the vorticity of a stationary plane flow in the absence of a shear flow (zero approximation vortex). Gouyon suggested a general scheme of the perturbation theory for periodic waves in the deep water with vorticity distribution (1), proved its convergence and found explicit solutions for the first two approximations by developing an original approach to describe flows in variables (x, Ψ) [18, 24] (x is the horizontal Cartesian coordinate).

The purpose of this work is to develop and supplement the results of [18] for fluid of finite depth. The vorticity in our description is given as follows:

$$\Omega = \sum_{n=1}^{\infty} \varepsilon^n \cdot \Omega_n(b), \tag{2}$$

here b is the vertical Lagrangian coordinate. The properties of Gouyon waves will be studied on the basis of the modified Lagrangian coordinates method [1, 2]. Instead of the usual Lagrangian coordinates a new pair of variables $q = a+\sigma(b)t$, b is introduced, where the function $\sigma(b)$ describes the inhomogeneous drift of liquid particles along the streamlines (isolines b and ψ coincide here). As a result, it is possible to construct a perturbation theory in the new variables without secular terms. The method of modified Lagrangian coordinates was used previously to describe nonlinear Guyon waves in deep water [3] (Guyon's results were generalized to the case of a cubic approximation).

Vorticity can add qualitatively new properties into the structure and properties of a periodic stationary wave. This was vividly demonstrated in [12], where it was shown that uniformly vortical waves over a flat bottom could have internal stagnation points and critical layers. They can also have overturning profiles, that is, profiles that are not graphs. Such phenomena cannot occur for the much-studied irrotational flows. For an irrotational steady flow the wave profile is necessarily the graph of a single-valued function, and there are no interior stagnation points or critical layers (see references in [12, 13]).

In this sense, it is important to note the specific features of Guyon waves differing from potential waves (Stokes waves). For weakly nonlinear Guyon waves the proportional correction ε to the linear velocity of wave propagation is nonzero [3, 19]. It is determined by the vorticity of the first approximation Ω_1. For a potential wave $\Omega_1 = 0$, therefore this correction to the velocity of wave propagation is absent. Another interesting property of Guyon waves is related to the type of trajectories of liquid particles. They can be either closed or looped with different directions of the averaged drift velocity under a certain choice of the type of vorticity $\Omega_1(b)$ and depend on the depth of the fluid (value of b) [3]. In this paper these properties of Guyon waves are analyzed in detail.

The paper is structured as follows. Section 2 gives the formulation of the problem of the steady periodic Gouyon waves in modified Lagrangian variables. In Sects. 3, 4 linear and quadratic approximations in the small parameter of the wave steepness are considered respectively. The first correction to the linear velocity of wave propagation is found. It is shown that taking into account a weak shear flow can qualitatively change the trajectories of liquid particles in waves, transforming them from the closed (circular) into loop-like ones.In the Conclusion, the main results of the work are discussed.

2 The formulation of the problem

Let us consider a stationary plane wave propagating along the free surface of the liquid at a constant velocity in the positive direction of the OX axis and write the equations of two-dimensional hydrodynamics in Lagrangian variables in the following form [2, 4, 24]:

$$X_a Y_b - X_b Y_a = 1, \tag{3}$$

$$X_{tt} = -H_a Y_b + H_b Y_a; Y_{tt} = -H_b X_a + H_a X_b; H = \frac{p}{\rho} + gY, \tag{4}$$

here $X(a, b, t)$, $Y(a, b, t)$ are the coordinates of the trajectory of a liquid particle, a and b are the horizontal and vertical Lagrangian coordinates (the condition $b = 0$ specifies a free surface and the condition $b = -h$ corresponds to a bottom), p is the pressure, ρ is the density, g—acceleration due to gravity (Y axis is directed upwards). Equation (3) is the condition of fluid continuity, and the relations (4) are equivalent to the momentum equations.

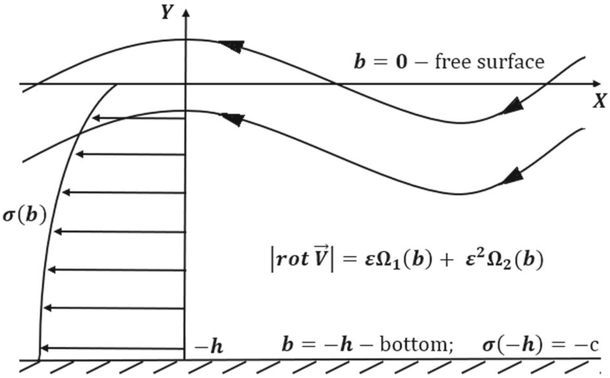

Fig. 1 The picture of a stationary flow in the frame of reference associated with the wave: periodic disturbances of the flow with a profile $\sigma(b)$

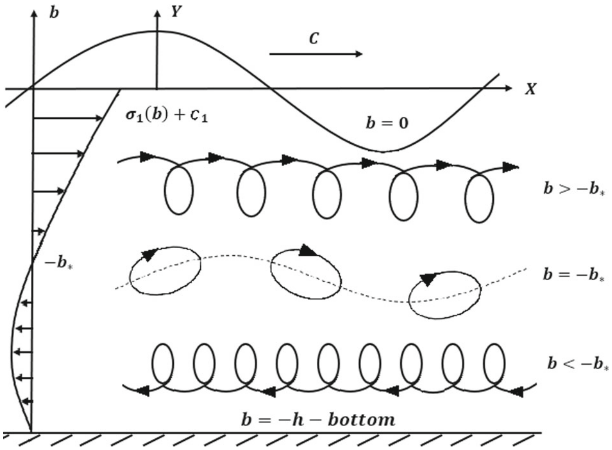

Fig. 2 Trajectories of liquid particles in the linear Gouyon wave in the laboratory frame for the model flow profile $\sigma_1(b) + c_1 = \gamma(b + b_*((b + h), b_* > 0$

When studying the wave motion, it is convenient to go over to the frame of reference moving with the wave propagation speed c, where the flow is stationary (see Fig. 1). In Lagrangian variables, the two-dimensional stationary flow has the form [1, 2, 5]

$$X = X(q, b), Y = Y(q, b), q = a + \sigma(b)t, \tag{5}$$

where $\sigma(b)$ is an arbitrary function. The easiest way to verify the validity of this statement is to write down the flow velocity field (5) in Eulerian coordinates. Since the Lagrangian velocity components $X_t = \sigma X_q, Y_t = \sigma Y_q$, like the functions X, Y, depend only on two variables q and b, the Euler velocity field $X_t(X, Y)$, $Y_t(X, Y)$ does not explicitly depend on time, which means that it describes a stationary flow. We note right away that the q coordinate is no longer the mark of an individual particle. Because of this, the suggested approach can no longer be called Lagrangian. The relationship between the function $\sigma(b)$ and the wave velocity is shown in Fig. 1 and will be explained below.

The coordinates q, b were introduced for the first time in [1] and called the modified Lagrangian variables [2]. In the variables q, b, Eqs. (3), (4) will be written as follows:

$$X_q Y_b - X_b Y_q = 1, \tag{6}$$

$$\sigma^2 X_{qq} = -H_q Y_b + H_b Y_q; \sigma^2 Y_{qq} = -H_b X_q + H_q X_b. \tag{7}$$

Let us assume that

$$X = q + \xi, Y = b + \eta,$$

where the functions ξ, η have the meaning of periodic wave perturbations of the trajectory of a liquid particle from the modified Lagrangian coordinates. Equations (6), (7) in this case can be rewritten as follows

$$\xi_q + \eta_b = -\frac{D(\xi, \eta)}{D(q, b)}, \tag{8}$$

$$\sigma^2 \xi_{qq} = -H_q + \frac{D(\eta, H)}{D(q, b)}, \tag{9}$$

$$\sigma^2 \eta_{qq} = -H_b - \frac{D(\xi, H)}{D(q, b)}. \tag{10}$$

These equations should be supplemented with boundary conditions. At the bottom, the vertical speed should drop to zero:

$$Y_t = \sigma Y_q = \sigma \eta_q = 0 \text{for} b = -h \tag{11}$$

and on the free surface ($b = 0$), the pressure should be constant, i.e.

$$H(q, 0) - g\eta(q, 0) = \frac{p_0}{\rho} = \text{const}. \tag{12}$$

Conditions (11), (12) should also be supplemented with the requirement that the OX axis corresponds to the average liquid level. It is written like this:

$$\int_0^\lambda Y dX \bigg|_{b=0} = \int \eta\left(1 + \xi_q\right) dq \bigg|_{b=0} = 0. \tag{13}$$

The horizontal velocity of liquid particles is $X_t = \sigma(b) + \sigma(b)\xi_q$. Here the first term describes a shear flow with a profile $\sigma(b)$. The second one is related to wave perturbations and is a periodic function q. The bottom coincides with the boundary streamline (see Fig. 1). Liquid particles move along the streamline, and their velocity consists of a homogeneous drift $\sigma(-h)$ and an oscillatory horizontal motion at the level $Y = -h$. The value of the function σ at the bottom is equal to the negative velocity of a stationary wave:

$$\sigma(-h) = -c. \tag{14}$$

The difference between the shear flow $\sigma(b)$ and the value $\sigma(-h)$ has the meaning of the drift velocity of particles

$$u(b) = \sigma(b) - \sigma(-h) = \sigma(b) + c. \tag{15}$$

Our goal is to describe the weakly nonlinear Gouyon waves. The expression for the vorticity of a plane flow, taking into account Eq. (6), can be represented as follows:

$$\begin{aligned}\Omega &= \frac{\partial Y_t}{\partial X} - \frac{\partial X_t}{\partial Y} = \frac{D(X_t, X)}{D(X, Y)} + \frac{D(Y_t, Y)}{D(X, Y)} \\ &= \frac{D(X_t, X)}{D(a, b)} + \frac{D(Y_t, Y)}{D(a, b)} = \frac{D\left(\sigma X_q, X\right)}{D(q, b)} + \frac{D\left(\sigma Y_q, Y\right)}{D(q, b)}.\end{aligned} \tag{16}$$

After passing over to new unknown functions, this relation will be rewritten as follows

$$\Omega = -\sigma'\left(1 + 2\xi_q\right) + \sigma\left(\eta_{qq} - \xi_{qb}\right) + \frac{D\left(\sigma\xi_q, \xi\right)}{D(q, b)} + \frac{D\left(\sigma\eta_q, \eta\right)}{D(q, b)}.$$

In the two-dimensional flow, the vorticity of a liquid particle is conserved. For the stationary flow (5), the vorticity depends only on the variable b. Since the functions ξ, η are periodic in the variable q, this formula can be simplified by applying the averaging operation. In this case, the expression for Ω takes the form

$$\Omega = -\sigma' + \frac{\overline{D\left(\sigma\xi_q, \xi\right)}}{D(q, b)} + \frac{\overline{D\left(\sigma\eta_q, \eta\right)}}{D(q, b)}, \tag{17}$$

where the bar is the sign of averaging over the variable q at the wavelength $\lambda = 2\pi/k$, k is the wavenumber.

We represent the unknown functions ξ, η, u, H, c in the form of series in powers of the parameter ε:

$$\{\xi, \eta, u\} = \sum_{n=1}^{\infty} \varepsilon^n \{\xi_n, \eta_n, u_n\}; H = \frac{p_0}{\rho} + \sum_{n=1}^{\infty} \varepsilon^n H_n; c = \sum_{n=0}^{\infty} \varepsilon^n c_n. \tag{18}$$

Expression (15) can be rewritten as follows:

$$\sigma = -c_0 + \sum_{n=1}^{\infty} \varepsilon^n (u_n - c_n) = \sigma_0 + \sum_{n=1}^{\infty} \varepsilon^n \sigma_n. \tag{19}$$

We will be interested in the first two approximations. Within the framework of such a description, the vorticity is determined by the relation

$$\Omega = \varepsilon\Omega_1(b) + \varepsilon^2\Omega_2(b) + O\left(\varepsilon^3\right), \tag{20}$$

where $\Omega_1(b)$, $\Omega_2(b)$ are given arbitrary functions. They will determine the propagation specificity of the studied vortex waves.

3 Linear approximation

Let us write out, according to (8)–(10), the equations of the first approximation

$$\xi_{1q} + \eta_{1b} = 0, \tag{21}$$

$$\sigma_0^2\xi_{1qq} = -H_{1q}, \tag{22}$$

$$\sigma_0^2\eta_{1qq} = -H_{1b}. \tag{23}$$

Then we will differentiate Eq. (22) with respect to q, Eq. (23) with respect to b and add. Taking into account (21), we obtain

$$H_{1qq} + H_{1bb} = 0.$$

Let us choose a periodic solution of this equation with a period λ in the variable q:

$$H_1 = H_1^* f_1(b)\cos kq, \tag{24}$$

here H_1^* is a constant, $k = 2\pi/\lambda$, а функция $f_1(b)$ удовлетворяет уравнению

$$f_1'' - k^2 f_1 = 0. \tag{25}$$

Substituting this expression into Eqs. (22), (23), we obtain

$$\xi_{1qq} = \frac{kH_1^*}{\sigma_0^2} f_1\, sin\, kq; \quad \eta_{1qq} = -\frac{H_1^*}{\sigma_0^2} f_1' \cos kq. \tag{26}$$

For simplicity, it is convenient to put in them $H_1^* = \sigma_0^2$. Integrating relations (26) with consideration of the periodicity of the disturbances ξ_1, η_1 with respect to the variable q, we find

$$\xi_1 = -\frac{1}{k} f_1 \sin kq + l_1(b); \quad \eta_1 = \frac{1}{k^2} f_1' \cos kq + m_1(b). \tag{27}$$

Substituting these relations into the equation of continuity of this approximation (21), we obtain

$$\frac{\partial m_1(b)}{\partial b} = 0,$$

whence it follows that $m_1 = const$. For the average level to coincide with the horizontal $Y = 0$, that is, the condition is fulfilled (see (13))

$$\int_0^\lambda \eta_1 dq = 0,$$

constant m_1 should be set equal to zero. We will also choose the function $l_1(b)$ equal to zero, but already from completely different considerations given below. Our flow description can be viewed as a mapping of a certain region of variables q, b onto the plane of variables X, Y. If $l_1 = 0$, then the half-band $\{0 \le q \le \lambda, -h \le b \le 0\}$ is mapped to the region $\{0 \le X \le \lambda, -h \le Y \le \eta_1(q, 0)\}$. If $l_1 \neq 0$, the display area will be shifted by $l_1(b)$: at each of the Lagrangian horizons $b = const$, in the general case, by a different amount.

The form of the function l_1 does not affect the flow velocity, waveform and other characteristics, therefore, for the calculation simplicity, we will assume it to be zero. In the Lagrangian description, fluid particles can be relabeled in an infinite number of ways, but this, obviously, should not affect the mathematical representation of the flow in any way. Our choice of mapping between the Lagrangian and physical regions with $l_1 = 0$ was dictated by the convenience of description.

Conditions (11), (12) turn to the following form in the terms of the function f_1:

$$f_1{}'(-h) = 0; \quad \sigma_0^2 f_1(0) = \frac{g}{k^2} f_1'(0). \tag{28}$$

A solution of the boundary-value problem (25), (28) can be written:

$$f_1 = C chk(b+h)); \quad \sigma_0^2 = c_0^2 = \frac{g}{k} thkh, \tag{29}$$

here C is a constant. The second ratio (29) determines the square of the phase velocity of the wave in a linear approximation. It coincides with the velocity of linear potential waves in a fluid of finite depth [21, 22]. Taking into account (29), the expressions for wave disturbances are determined by the formulas

$$\xi_1 = -\frac{C}{k} chk\ (b+h) \sin kq; \quad \eta_1 = \frac{C}{k} shk(b+h) \cos kq; \tag{30}$$

$$H_1 = C\sigma_0^2 chk(b+h) \cos kq. \tag{31}$$

The amplitude of the wave in the linear approximation A_1 equals to $\varepsilon\eta_1(0)$ or

$$A_1 = \frac{\varepsilon C}{k} shkh.$$

Let us consider $C = 1/sh\ \ kh$, so the parameter $\varepsilon = kA_1$ means the wave's steepness.

The expression for the linear Guyon wave in the Lagrangian variables a, b is written as

$$X = a - c_0 t + \varepsilon\sigma_1(b)t - \varepsilon\frac{chk(b+h)}{kshkh}\sin k[a - c_0 t + \varepsilon\sigma_1(b)t];$$

$$Y = b + \varepsilon\frac{shk(b+h)}{k\ sh\ kh}\cos k[a - c_0 t + \varepsilon\sigma_1(b)t]. \tag{32}$$

In addition to oscillatory motion, liquid particles participate in a non-uniform drift, which is associated with the function $\sigma_1(b)$. It follows from relation (17) that it is related to the first approximation vorticity:

$$\sigma_1' = -\Omega_1.$$

It is convenient to represent this function as follows

$$\sigma_1(b) = \sigma_1(-h) - \int_{-h}^{b}\Omega_1(b)db = -c_1 - \int_{-h}^{b}\Omega_1(b)db. \tag{33}$$

As can be seen, the vorticity distribution $\Omega_1(b)$ is insufficient for the complete determination of the function $\sigma_1(b)$. In the first approximation, it is found accurate to a constant—a linear correction to the propagation velocity c_0, taken with a minus sign. Its value should be calculated in the following approximation. This feature is inherent in all other approximations. The integral term in formula (33) corresponds to the drift velocity of liquid particles

$$u_1(b) = \sigma_1(b) + c_1 = -\int_{-h}^{b}\Omega_1(b)db.$$

In each of the approximations for the given vorticity, there will be an inhomogeneous drift flow, and the correction value to the propagation velocity should be calculated in the next approximation.

In the laboratory frame of reference, the solution of linear problem (27) has the form

$$X = a + \varepsilon[\sigma_1(b) + c_1]t - \varepsilon\cdot\frac{chk(b+h)}{kshkh}\sin k[a - c_0 t + \varepsilon\sigma_1(b)t];$$

$$Y = b + \varepsilon\cdot\frac{sh\ k(b+h)}{k\ sh\ kh}\cos k[a - c_0 t + \varepsilon\sigma_1(b)t]. \tag{34}$$

The wave propagates to the right with the speed $c_0 + \varepsilon c_1$. In the case $\sigma_1 = 0$, $c_1 = 0$, when vorticity is absent, expressions (34) describe the linear potential wave (Stokes wave). In this case, liquid particles rotate in ellipses [18, 19]

$$\frac{(X-a)^2}{\alpha^2} + \frac{(Y-b)^2}{\beta^2} = 1;$$
$$\alpha = \frac{chk(b+h)}{kshkh}; \beta = \frac{shk(b+h)}{kshkh},$$

here α, β– are the values of their horizontal and vertical semi-axes respectively. Both of these values decrease with depth. The value of β for $b = -h$ equals to zero, so that the liquid particles at the bottom oscillate along a straight lines.

For the Gouyon wave $\sigma_1(b) \neq 0$ (and is not constant), therefore, liquid particles, in addition to rotating in an ellipse, take part in a non-uniform drift in depth, so that their trajectory is a loop-like line in general. If $\sigma_1(b) + c_1 > 0$ the fluid particles move in the direction of the wave motion, and if the inequality sign is opposite they move backward. The function $\sigma_1(b)$ in solution (34) describes the shear flow and due to the arbitrariness of $\Omega_1(b)$ can also be arbitrary. Figure 2 shows the trajectories of fluid particles for the following distribution of the drift velocity

$$\sigma_1(b) + c_1 = \gamma(b + b_*)(b + h), b_* > 0,$$

where γ—is a constant with dimension $(\text{cm} \cdot \text{c})^{-1}$. It is worth noting that in the laboratory frame of reference, the particles, located on the Lagrangian horizon $b = -b_*$ move in ellipses, and at higher and lower horizons—along the loop trajectories, while the directions of the particle drift, relative to this horizon, are different.

A similar pattern of particle's paths may be realized in nonlinear potential waves in the presence of an adverse uniform underlying current [11] as well as in small-amplitude waves with large positive constant vorticity [16] (where the smallness of the vorticity is not imposed, in contrast to our case). Some examples of the analytical description of particle trajectories for these linear waves were obtained in [20].

As it is known, linear waves on the surface of the flow $\sigma_0(b)$ with an inflection point on the profile are unstable [14]. In our case $\sigma_0 = const$, the flow profile $\sigma_1(b)$ can be any. In particular, the flow profile $\sigma_1(b)$ may have an inflection point.

It is necessary to stress one more important fact. In the Lagrangian description of flows the trajectories of liquid particles are found explicitly. It is an important advantage of the Lagrangian approach. Let us explain this statement by an example of the linear Stokes wave. To determine coordinatesX, Y of a trajectory of liquid particles in the Euler variables a system of two equations of the first order should be solved. In the left parts of this system there are terms $\frac{dX}{dt}$ and $\frac{dY}{dt}$, in the right—the expressions for the horizontal and vertical velocity determined by the potential for the linear wave (see, for example, [21]). This system is non-linear and non-integrable. For the linear wave, however, the values ofX, Y on the right-hand sides can be replaced by their mean values. In this case the system can be easily integrated and, as a consequence, the classical result is obtained: liquid particles move over elliptical trajectories. But quite recently Constantin and Villari paid attention to a very interesting property of solutions of a complete nonlinear system forX, Y: none of these solutions describes the motions with closed trajectories of liquid particles [9, 10]. Thus, in the Euler description the liquid particle drift is already potentially present in the linear approximation, but definitely it is the effect of the quadratic approximation. In this case the Lagrangian description looks simpler and more convenient.

4 Quadratic approximation

Let us write the equations of the second order of the perturbation theory:

$$\xi_{2q} + \eta_{2b} = -\frac{D(\xi_1, \eta_1)}{D(q, b)}, \tag{35}$$

$$\sigma_0^2 \xi_{2qq} + 2\sigma_0\sigma_1\xi_{1qq} = -H_{2q} + \frac{D(\eta_1, H_1)}{D(q, b)}, \tag{36}$$

$$\sigma_0^2 \eta_{2qq} + 2\sigma_0\sigma_1\eta_{1qq} = -H_{2b} - \frac{D(\xi_1, H_1)}{D(q, b)}. \tag{37}$$

The unknown functions included in them should satisfy the boundary conditions

$$\left(\sigma_0\eta_{2q} + \sigma_1\eta_{1q}\right)\big|_{b=-h} = 0, \tag{38}$$

$$\int_0^\lambda \left(\eta_2 + \eta_1\xi_{1q}\right)\big|_{b=0} = 0. \tag{39}$$

$$H_2(q, 0) = g\eta_2(q, 0), \tag{40}$$

The quadratic terms in (35)–(37) are easily calculated and equal, respectively,

$$\frac{D(\xi_1, \eta_1)}{D(q, b)} = -\frac{C^2}{2}[\cos 2kq + ch\ 2k(b+h)]; \tag{41}$$

$$\frac{D(\eta_1, H_1)}{D(q, b)} = \frac{k\sigma_0^2 C^2}{2}\sin 2kq; \tag{42}$$

$$\frac{D(\xi_1, H_1)}{D(q, b)} = -\frac{k\sigma_0^2 C^2}{2} sh\ 2k(b+h). \tag{43}$$

Let us differentiate Eq. (36) with respect to q, Eq. (37) with respect to the variable b and add them. Taking into account relations (21), (35) and (41)–(43), we obtain

$$\Delta H_2 = 2kC\sigma_0\sigma_1'\ sh\ k(b+h)\cos kq + k^2C^2\sigma_0^2[3\ \cos 2\ kq + ch\ 2k(b+h)]. \tag{44}$$

We shall find the solution of this equation in the form:

$$H_2 = \sigma_0^2[f_{20}(b) + f_{21}(b)\cos kq + f_{22}(b)\cos 2kq], \tag{45}$$

then the proposed functions of b satisfy the following equations

$$f_{20}'' = k^2C^2\ ch2k(b+h); \tag{46}$$

$$f_{21}'' - k^2 f_{21} = 2kC\frac{\sigma_1'}{\sigma_0} shk(b+h); \tag{47}$$

$$f_{22}'' - 4k^2 f_{22} = 3k^2C^2. \tag{48}$$

The choice of functions f_{20}, f_{21}, f_{22} depends on conditions (11)–(13) including expressions for wave perturbations of the coordinates of the trajectories of liquid particles. The vertical deviation η_2 according to Eq. (37) and equality (43) is determined by the ratio

$$\eta_{2qq} = \frac{kC^2}{2} sh\ 2k(b+h) - f_{20}' + \left[\frac{2\sigma_1}{\sigma_0} kC\ sh\ k(b+h) - f_{21}'\right] \cos kq - f_{22}' \cos 2kq. \tag{49}$$

The function η_2 ought to be periodic with respect to the variable q, so it follows from formula (49) that

$$f_{20}(b) = \frac{C^2}{4} ch\ 2k(b+h) + C_{20},$$

where C_{20} is a constant. Its value could be found from the boundary condition for pressure (40). In this case it reduces (see (45)) to the ratio $f_{20}(0) = 0$, so that

$$C_{20} = -\frac{C^2}{4} ch\ 2kh.$$

Equation (47) should be added with the boundary condition:

$$f_{21}'(-h) = 0; \tag{50}$$

which follows from no leakage condition across the bottom (38). The solution satisfying this condition will be written like this:

$$f_{21}(b) = \frac{C}{\sigma_0}\left[e^{kb}\int_{-h}^{b} \sigma_1' e^{-kb} sh\ k(b+h) db - e^{-kb}\int_{-h}^{b} \sigma_1' e^{kb} sh\ k(b+h) db\right]. \tag{51}$$

The expression for f_{21} does not include the general solution of the homogeneous Eq. (47), since it has the structure of the function H_1, and taking it into account would lead to a change in the amplitude of the first harmonic only.

The pressure constancy condition for oscillations with a wave number k (see (40)) is written as follows

$$\sigma_0^2 f_{21}(0) = \frac{g}{k^2}\left[f'_{21}(0) - \frac{2\sigma_1(0)}{\sigma_0}k\right]. \tag{52}$$

The value $\sigma_1(0)$ follows from this expression as

$$\sigma_1(0) = \int_{-h}^{0} \sigma'_1(b)\frac{sh\ k(b+h)}{sh2kh}db. \tag{53}$$

When taking into account the following relation

$$\sigma_1(b) = \sigma_1(-h) + \int_{-h}^{b} \sigma'_1(b)db = -c_1 - \int_{-h}^{b} \Omega_1(b)db,$$

we find the value of the correction to the linear wave velocity

$$c_1 = \int_{-h}^{0} \Omega_1(b)\left[\frac{sh2k(b+h)}{sh2kh} - 1\right]db. \tag{54}$$

In the case of deep water $kh \gg 1$ and this expression takes the form

$$c_1 = \int_{-h}^{0} \Omega_1(b)\left[e^{2kb} - 1\right]db,$$

which is equivalent to the result obtained by Gouyon (see [3]).

It remains to define expressions for second harmonic perturbations in k. As it follows from conditions (38), (40), (45), (49), the function $f_{22}(b)$ should satisfy the following boundary conditions

$$f'_{22}(-h) = 0;$$

$$\sigma_0^2 f_{22}(0) = \frac{g}{4k^2} f'_{22}(0).$$

When solving Eq. (48) with their consideration we obtain

$$f_{22}(b) = \frac{3}{4sh^2kh}\left[1 - \frac{ch\ 2k(b+h)}{sh^2kh}\right]. \tag{55}$$

Expressions for second-order wave perturbations ξ_2, η_2 could be found from relations (36), (37) and in the final form will be written as follows:

$$\xi_2 = -\frac{2\sigma_1}{\sigma_0}\xi_1 - \frac{f_{21}}{k}\sin kq - \frac{1}{2k}\left(f_{22} + \frac{1}{4sh^2\ kh}\right)\sin 2\ kq; \tag{56}$$

$$\eta_2 = \frac{1}{2k\ th\ kh} + \frac{1}{k}\left[\frac{f_{21}'}{k} - \frac{2\sigma_1 sh\ k(b+h)}{\sigma_0 sh\ kh}\right]\cos kq + \frac{f_{22}'}{4k^2}\cos 2kq. \tag{57}$$

The first term in Eq. (57) is caused by the requirement of zero average liquid level (condition (39)).

In the Lagrangian coordinates, the second approximation solution is written as follows:

$$X = a - c_0 t + \varepsilon(\sigma_1 t + \xi_1) + \varepsilon^2(\sigma_2 t + \xi_2) + O\left(\varepsilon^3\right); \tag{58}$$

$$Y = b + \varepsilon\eta_1 + \varepsilon^2\eta_2 + O\left(\varepsilon^3\right). \tag{59}$$

The variable q in the expressions for ξ_1, η_1 should be taken equal to $a-(c_0 - \varepsilon\sigma_1)t$, and in the expressions for ξ_2, η_2 equal to $a - c_0 t$. In this approximation, a term containing the function $\sigma_2(b)$ is added to the shear flow. It is related to the quadratic approximation vorticity (see (17)):

$$\sigma_2' = -\Omega_2(b) + kc_0\frac{sh\ 2k(b+h)}{sh^2\ kh}, \tag{60}$$

let us recall that the form of Ω_2 is assumed to be given. The function $\sigma_2(b)$ is determined up to a constant, the value of which, and hence the value of the correction c_2 to the wave phase velocity, is calculated in the following approximation. The case for $\Omega_2 = 0$ relate to a potential wave [23].

5 Conclusion

For a long time, researchers did not pay due attention to the Gouyon waves. Of the numerous monographs on the theory of water waves, they are described only in the book [24]. The present paper calls for more status to be given to the Gouyon model. We generalized Gouyon's result for the fluid of finite depth. Gouyon used the Euler approach, and performed calculations in the variables "x-coordinate—stream function ψ". Our approach is based on calculations in the modified Lagrangian variables and can be called "quasi-Lagrangian".

The main result of this work is the formulation of the first nonlinear correction to the phase velocity of wave propagation. Its value is determined by the vorticity distribution $\Omega_1(b)$. This function may be arbitrary, and therefore the solution obtained in this work defines a wide class of nonlinear wave oscillations with their properties differing from the Stokes wave qualitatively.

We have shown the effectiveness of the modified Lagrangian coordinate method. This method allows one to solve equation systems of higher approximations effectively and to find the terms of series (18), (19) in higher orders of the perturbation theory. The only difficulty on this path is the calculation cumbersomeness. The numerical analysis of the systems (6), (7) should become an important addition to the suggested perturbation theory method. Only on its basis it will be possible to study the structure of highly nonlinear vortex waves.

Acknowledgements This work is supported by the state assignment of the IAP RAS, topic №0030-2021-0009.

References

1. Abrashkin, A.A., Zen'kovich, D.A.: Vortex stationary waves on a shear flow. Izv. Akad. Nauk SSSR. Fiz. Atmos. Okeana **26**, 35–46 (1990). ((**in Russian**))
2. Abrashkin, A.A., Yakubovich, E.I.: Vortex Dynamics in Lagrangian Description. Fizmatlit, Moscow (2006).. ((**in Russian**))
3. Abrashkin, A.A., Pelinovsky, E.N.: Nonlinear Gouyon waves. J. Phys. A Math. Theor. **54**(39), 395701–395719 (2021)
4. Bennett, A.: Lagrangian Fluid Dynamics. Cambridge University Press, Cambridge (2006)
5. Clamond, D.: On the Lagrangian description of steady surface gravity waves. J. Fluid Mech. **589**, 434–454 (2007)
6. Constantin, A.: On the deep water wave motion. J. Phys. A Math. Gen. **34**(7), 1405–1422 (2001)
7. Constantin, A.: Nonlinear water waves with applications to wave–current interactions and tsunamis. CBMS-NSF Conference Series in Applied Mathematics, vol. 81. PA: SIAM, Philadelphia (2011)
8. Constantin, A., Strauss, W.: Exact steady periodic water waves with vorticity. Comm. Pure Appl. Math. **54**(4), 481–527 (2004)
9. Constantin, A.: The trajectories of particles in Stokes waves. Invent. Math. **166**, 523–535 (2006)
10. Constantin, A., Villari, G.: Particle trajectories in linear water waves. J. Math. Fluid Mech. **10**, 1–18 (2008)
11. Constantin, A., Strauss, W.: Pressure beneath a Stokes wave. Comm. Pure Appl. Math. **63**, 533–557 (2010)
12. Constantin, A., Strauss, W., Vărvărucă, E.: Global bifurcation of steady gravity water waves with critical layers. Acta Math. **217**, 195–262 (2016)
13. Constantin, A., Strauss, W., Vărvărucă, E.: Large-amplitude steady downstream water waves. Commun. Math. Phys. **387**, 237–266 (2021)
14. Drazin, P.G.: Introduction to Hydrodynamic Stability. Cambridge University Press, Cambridge (2002)
15. Dubreil-Jacotin, M.-L.: Sur la détermination rigoureuse des ondes permanentes périodiques d'ampleur finie. J. Math. Pures Appl. **13**, 217–291 (1934)
16. Ehrnström, M., Villari, G.: Linear water waves with vorticity: rotational features and particle paths. J. Differ. Equ. **244**, 1888–1909 (2008)
17. Gerstner, F.: Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile. Ann. Phys. **2**, 412–445 (1809)
18. Gouyon, R.: Contribution à la théorie des houles. Ann. de la Fac des Sci. de l'Université de Toulouse Sér. **4**(22), 1–55 (1958)
19. Henry, D.: On Gerstner's water wave. J. Nonlinear Math. Phys. **15**(2), 87–95 (2008)
20. Ionescu-Kruse, D.: Elliptic and hyperelliptic functions, describing the particle motion beneath small-amplitude water waves with constant vorticity. Commun. Pure Appl. Anal. **11**, 1475–1496 (2012)
21. Kochin, N.E., Kibel, I.A., Roze, N.V.: Theoretical Hydromechanics. Interscience Publ., New York (1964)
22. Lamb, G.: Hydrodynamics, 6th edn. Cambridge University Press, Cambridge (1932)

23. Phillips, O.M.: Dynamics of the Upper Ocean, 2nd edn. Cambridge University Press, Cambridge (1980)
24. Sretensky, L.N.: Theory of the Fluid Wave Motions. Nauka, Moscow (1977).. (**(in Russian)**)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.