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Universally fully and Krylov transitive torsion-free abelian groups

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Abstract
Extending results from our recent paper in Chekhlov et al. (J Algebra 566(2):187–204, 2021), we define and explore the classes of *universally fully transitive* and *universally Krylov transitive* torsion-free Abelian groups. A characterization theorem is proved in which numerous interesting properties of such groups are demonstrated. In addition, we prove the curious fact that these two classes do coincide as well as that in the reduced case these groups are just homogeneous separable and thus, in particular, they are both fully transitive and transitive. Some related results pertaining to *H-full transitivity* and *H-Krylov transitivity* for some special (fixed) groups H which, in particular, can be viewed as subgroups of a torsion-free Abelian group G are also obtained. Our achieved here results somewhat strengthen those established by Goldsmith and Strüngmann (Commun Algebra 33(4):1177–1191, 2005).

Keywords Torsion-free groups · Separable groups · Transitive groups · Fully (Krylov) transitive groups · Universally fully (Krylov) transitive groups

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1 Introduction and fundamentals

Throughout the rest of the paper, unless specified, all groups will be additively written torsion-free Abelian groups. We will primarily use the notation and terminology of [5–7], but we will follow somewhat those from [9,11] as well. We begin with a quick review of some of the most important.

If a is an element of the group G and p is a prime, we denote the p-height of a by $|a|_p$, or perhaps $|a|_{G,p}$ if we wish to specify the context in which group the height is calculated. By a *characteristic* we mean a sequence $\chi = (\chi_1, \chi_2, \ldots)$ of non-negative integers or the symbol ∞. In particular, if $\mathcal{P} := \{p_1, p_2, \ldots\}$ is the set of all primes, then the height sequence of $a \in G$, $\chi(a) = \chi_G(a) = (|a|_{p_1}, |a|_{p_2}, \ldots)$, is a characteristic.

For a characteristic χ there is clearly a unique subgroup $\mathbb{Q}_\chi \subseteq \mathbb{Q}$ such that $1 \in \mathbb{Q}_\chi$ and $\chi_{\mathbb{Q}_\chi}(1) = \chi$. And if $\mathbf{e}$ is an element of some rank-1 group A, and $\chi = \chi_A(\mathbf{e})$, then we often denote A by $\mathbb{Q}_\chi \mathbf{e} := \{q\mathbf{e} : q \in \mathbb{Q}_\chi\}$. For the group G and characteristic χ, we let $p^\chi G = \{a \in G : \chi_G(a) \geq \chi\}$. Clearly, $a \in p^\chi G$ if, and only if, $\mathbb{Q}_\chi a \subseteq G$.

If χ is a characteristic and $n \in \mathbb{Z}$, then we let $n\chi$ be the characteristic such that for all $i = 1, 2, \ldots$, $(n\chi)_i = \chi_i + |n|_{\mathbb{Z}, p_i}$. In particular, if $a \in G$, then $\chi_G(na) = n\chi_G(a)$. The characteristics χ, χ' are equivalent if there are non-zero $n, n' \in \mathbb{Z}$ such that $n\chi = n'\chi'$. It is easy to check this is an equivalence relation. A resulting equivalence class is called a *type*; we typically denote this by $\tau = \overline{\chi}$.

If $a \in G$, then we write $\tau(a) = \overline{\chi}(a)$. The *type set* of G, written $\tau(G) = \{\tau(a) \mid 0 \neq a \in G\}$, is the set of types of all non-zero elements of G. We say G is *homogeneous of type* τ, or simply τ*-homogeneous*, if $\tau(G) = \{\tau\}$.

The natural ordering of characteristics leads to a natural ordering of types. Clearly, for a type $\tau = \overline{\chi}$ and group G, $G(\tau) = \{a \in G \mid \tau(a) \geq \tau\}$ is a pure fully invariant subgroup of G. It is easy to see that $G(\tau) = G$ if, and only if, $G/(p^\chi G)$ is torsion.

Finally, the group G is *separable* if every finite subset of G is contained in a summand C which is a finite rank completely decomposable group. If G is separable, then $\Omega(G) \subseteq \tau(G)$ stands for the set of types of all rank-1 direct summands of G. Besides, if G is arbitrary, then we set $\pi(G) = \{p \in \mathcal{P} \mid pG \neq G\}$. And if τ is type, then we put $\pi(\tau) = \pi(G)$, where G is rank-1 of type τ.

In his famous book [11], Irving Kaplansky introduced two major properties of groups as follows:

Definition 1.1 A group G is called *transitive* if, for any two elements x, y with $\chi_G(x) = \chi_G(y)$, there exists an automorphism of G mapping x to y, and *fully transitive* if, for any two elements x, y with $\chi_G(x) \leq \chi_G(y)$, there exists an endomorphism ϕ of G with $\phi(x) = y$.

It is known that these two notions are independent for torsion-free groups, i.e., there is a group that is transitive, but not fully transitive, and a group that is fully transitive, but not transitive (see page 385 of [14] and the references listed there). Likewise, in [14, pp. 475–476] is announced that Kaplansky actually defined (full) transitivity in a more global setting, namely for modules over a complete domain of discrete normalization and thus, in particular, for p-primary groups over some prime p.

Both notions in Definition 1.1 were extended by Krylov [13] to the following concept:

Definition 1.2 A group G is called *Krylov transitive* if, for any elements $x, y \in G$ with $\chi_G(x) = \chi_G(y)$, there exists an endomorphism of G mapping x to y.

Clearly, a group that is transitive, but not fully transitive, will also be Krylov transitive, but not fully transition. Similarly, a group that is fully transitive, but not transitive, will also be Krylov transitive, but not transitive.

Turning to the contents of the paper, after this introduction, our results fall into two related areas. The second section of the paper is devoted to the following notion that is parallel to a definition in [4] for p-groups.

Definition 1.3 We shall say that the group G is *universally fully transitive* if, for every group M and every pair of elements $x \in G$ and $y \in M$, if $\chi_G(x) \leq \chi_M(y)$ then there is a homomorphism $\phi : G \to M$ such that $\phi(x) = y$. If this holds whenever $\chi_G(x) = \chi_M(y)$, the group G is said to be *universally Krylov transitive*.

Similarly to the p-torsion case considered in [4], we give a complete description of these two classes (Theorem 2.1). As in the torsion case, though there are groups that are Krylov transitive without being fully transitive, a group is universally Krylov transitive if, and only if, it is universally fully transitive. In fact, the groups in this class are quite familiar; they are precisely those groups that are a direct sum of a divisible group and a homogeneous separable group.

We pause to note one simple consequence of this characterization. Clearly, if G is universally fully transitive, then it must be fully transitive. On the other hand, there are many examples of reduced fully transitive groups that are not separable (for example, the p-adic integers for some prime p). In particular, such a fully transitive group will not be universally fully transitive.

The class of universally fully/Krylov transitive groups naturally breaks down into the study of those groups whose (maximal) reduced parts are τ-homogeneous for some fixed type τ. We next use some homological machinery to reduce this to the case where τ is *idempotent* (i.e., $\tau = \overline{\chi}$ where for all k, χ_k is 0 or ∞ - Theorem 2.10). When τ is idempotent we obtain a characterization of the class of τ-homogeneous universally fully transitive groups that parallels the characterization obtained in the torsion case in [4] (Corollary 2.11).

Continuing as processed in [4], in the third section we explore the notion of full transitivity and Krylov transitivity when we restrict the element x as well as the endomorphism ϕ in Definition 1.1 to a certain subgroup H of G. We, furthermore, also extend Definition 1.2 to the following (which slightly differs from the corresponding definition in the p-primary case stated in [4], where it was not assumed that H is contained in G).

Definition 1.4 Given $H \leq G$, we shall say that the group G is *H-fully transitive* if, for every pair of elements $x \in H$ and $y \in G$, if $\chi_H(x) \leq \chi_G(y)$ then there is a homomorphism $f : H \to G$ such that $f(x) = y$. If this holds when $\chi_H(x) = \chi_G(y)$, then the group G is said to be *H-Krylov transitive*.

Of particular interest is where H is actually a pure subgroup of G. When this is true, then for all $x \in H$, $\chi_H(x) = \chi_G(x)$. If $K \subseteq H$ are both pure subgroups of G, it readily follows that if G is H-fully or H-Krylov transitive, then G is also K-fully or K-Krylov transitive, respectively.

Clearly, G is fully transitive if, and only if, it is G-fully transitive, and similarly for Krylov transitivity. In particular, G is fully or Krylov transitive if, and only if, it is H-fully or H-Krylov transitive, respectively, over all possible pure subgroups H of G.

So, if G is fully transitive and H is a pure subgroup of G, then G is H-fully transitive. On the other hand, such a pure subgroup H may not, itself, be fully transitive. For example, again suppose G is the p-adic integers, $a \in G \setminus pG$ is transcendental over $\mathbb{Z}_{(p)} \subseteq G$ and H is the smallest pure subgroup of G containing 1 and a. It follows that $\chi(1) = \chi(a)$ and multiplication by a is the only homomorphism $\phi : H \to G$ such that $\phi(1) = a$. But since $\phi(a) = a^2 \notin H$, it follows that ϕ is not an endomorphism of H.

In this third section we also discuss the question of how H-fully transitive groups and H-Krylov transitive groups behave under direct sums and summands (Proposition 3.5). In addition, if H is a fully transitive separable pure subgroup of G, we characterize when G is H-fully transitive (Proposition 3.12). It is worthwhile noticing that, in proving these two assertions, we do *not* require H to be necessarily a subgroup of G and so, analogously to the approach illustrated for p-torsion groups in [4], the letter H is used here to be any existing group.

We finish off the paper with a list of some open problems which naturally arise for further work.

2 Universally fully and Krylov transitive groups

We turn to our characterization of universal full and Krylov transitivity in the torsion-free case (again, the p-primary case was completely characterized in [4]). It can be treated as a generalization of the well-known results due to Baer (see, e.g., [7, Lemma 4.5]) which say that a homogeneous torsion-free group is separable if, and only if, every finite rank pure subgroup is a summand and that pure subgroups of homogeneous separable groups are again separable. Moreover, point (h) listed below is a non-trivial extension of [7, Proposition 4.8] as well.

It is also worth to notice that our used constructions in the present section (in particular, in the given below proof of the main result in this section) develop those utilized in a series of papers/books due to Goldsmith, Strüngmann, Salce and Kaplansky (see the papers cited in the bibliography list as well as the references therein).

Theorem 2.1 *Suppose G is a group and $G = R \oplus D$, where R is reduced and D is divisible. Then the following are equivalent:*

(a) G is universally fully transitive;
(b) G is universally Krylov transitive;
(c) Every pure rank-1 subgroup of G is a summand;
(d) Every pure finite-rank subgroup of G is a summand;
(e) Every group M containing G as a subgroup is G-fully transitive;
(f) For every rank-1 group Q, if $N = G \oplus Q \supseteq G$, then N is G-fully transitive.

(g) R is a homogeneous separable group;
(h) There is a type $\tau = \overline{\chi}$ such that R is τ-homogeneous and isomorphic to a pure subgroup of the vector group $\prod_{i\in I} \mathbb{Q}_\chi \mathbf{e}_i$.

Proof (a) $\Rightarrow$ (b): This is trivial.

(b) $\Rightarrow$ (c): Suppose J is a rank-1 pure subgroup of G and $0 \neq x \in J$. It follows that $\chi_G(x) = \chi_J(x)$, so there a homomorphism $\phi : G \to J$ which must be the identity on J. If K is the kernel of ϕ, it follows that $G = J \oplus K$, as required.

(c) $\Rightarrow$ (a): Suppose $x \in G$, $y \in M$ and $\chi_G(x) \leq \chi_M(y)$. By hypothesis, if J is the corresponding rank-1 pure subgroup of G containing x, then there is a splitting $G = J \oplus K$. Clearly, the assignment $x \mapsto y$ extends to a homomorphism $\phi : J \to M$. And setting ϕ equal to 0 on K, we have established (a).

(c) $\Rightarrow$ (d): Follows from an obvious induction, while (d) $\Rightarrow$ (c) is trivial.

(a) $\Rightarrow$ (e) $\Rightarrow$ (f): These are automatic.

(f) $\Rightarrow$ (a): Suppose M is a group, $x \in G$, $y \in M$ and $\chi := \chi_G(x) \leq \chi_M(y)$. By hypothesis, if $Q = \mathbb{Q}_\chi \mathbf{e}$ and $N = G \oplus Q$, then there is a homomorphism $\theta : G \to N$ such that $\theta(x) = \mathbf{e}$. There is clearly a homomorphism $\nu : Q \to M$ such that $\nu(\mathbf{e}) = y$; extend ν to all of N by setting it equal to 0 on G. It follows that $\phi := \theta \circ \nu : G \to H$ satisfies $\phi(x) = y$.

(d) $\Rightarrow$ (g): Our hypothesis immediately implies that G is separable. If R were not homogeneous, then we could find characteristics λ and γ with $\overline{\lambda} \neq \overline{\gamma}$ and a summand L of R with $L \cong \mathbb{Q}_\lambda \mathbf{e} \oplus \mathbb{Q}_\gamma \mathbf{f}$.

Case 1: Suppose $\overline{\lambda}$ and $\overline{\gamma}$ are not comparable. If we let $\xi = \lambda \wedge \gamma$, then $\overline{\xi} < \overline{\lambda}$ and $\overline{\xi} < \overline{\gamma}$. Consider $x := \mathbf{e} + \mathbf{f} \in G$. If $J = \mathbb{Q}_\lambda x$ is the pure rank-1 subgroup determined by x, then it follows that J is a summand of G, and hence of L. Let $\pi : L \to J$ be the corresponding projection. Since $\overline{\xi} < \overline{\lambda}$, it follows that $\pi(\mathbb{Q}_\lambda \mathbf{e}) = 0$, and since $\overline{\xi} < \overline{\gamma}$, it follows that $\pi(\mathbb{Q}_\gamma \mathbf{f}) = 0$. But this would imply that $P = \pi(L) = 0$, which is a contradiction.

Case 2: Otherwise, we may suppose, without loss of generality, that $\overline{\lambda} < \overline{\gamma}$. Let p_k be some prime such that $\gamma_k \neq \infty$; so we must also have $\lambda_k \neq \infty$. We may clearly amend $\mathbf{e}$ and $\mathbf{f}$ so that $\lambda' := \chi(\mathbf{e}) < \chi(\mathbf{f}) := \gamma'$ and $\lambda'_k = 0 = \gamma'_k$. Replace λ and γ by λ' and γ' so that $\lambda_k = 0 = \gamma_k$.

Consider $x := p_k \mathbf{e} + \mathbf{f}$. It is straightforward to check that $J := \mathbb{Q}_\lambda x$ is pure in L and G. Again, let $\pi : L \to J$ be a projection, so that if K is the kernel of π, then $L = J \oplus K$.

Since $\lambda < \gamma$, it follows that $\pi(\mathbb{Q}_\gamma \mathbf{f}) = 0$; i.e., $\mathbb{Q}_\gamma \mathbf{f} \subseteq K$. And since $\mathbb{Q}_\gamma \mathbf{f}$ and K are both pure rank-1 subgroups of L, this implies that $\mathbb{Q}_\gamma \mathbf{f} = K$.

However, it is readily observed that

$$\mathbf{e} \notin p\,(\mathbb{Q}_\lambda \mathbf{e}) \oplus \mathbb{Q}_\gamma \mathbf{f} = J + \mathbb{Q}_\gamma \mathbf{f} = J + K = L.$$

This contradiction completes the proof.

(g) $\Rightarrow$ (h): Suppose R is $\tau = \overline{\chi}$-homogeneous. Let $\{\mathbb{Q}_\chi \mathbf{e}_i\}_{i\in I}$ be the collection of all rank-1 pure subgroups of R.

For each $i \in I$, there is a projection function $\pi_i : R \to \mathbb{Q}_\chi \mathbf{e_i}$. If $V := \prod_{i\in I} \mathbb{Q}_\chi \mathbf{e}_i$, then the diagonal map $\phi : R \to V$ given by $\phi(x) = (\pi_i(x))_{i\in I}$ is clearly a homomorphism.

For each $i \in I$, let $\nu_i : V \to \mathbb{Q}_\chi \mathbf{e}_i$ be the natural projection. It follows that for each $i \in I$ that $\nu_i \circ \phi$ is the identity on $\mathbb{Q}_\chi \mathbf{e}_i$. This readily implies that ϕ is injective and that its image $R' := \phi(R)$ is pure in V. And since R is τ-homogeneous, so is R', as required.

(h) $\Rightarrow$ (c): Let J be a rank-1 pure subgroup of G. If J is contained in D, then J must be divisible, and hence a summand of G.

If J is not contained in D, then it follows that $J \cap D = 0$, so there is a decomposition $G = R \oplus D$, where R is reduced and $J \subseteq R$. If we can show that $J = \mathbb{Q}_\chi \mathbf{v}$ is a summand of R, then it automatically is also a summand of G.

So, we may assume $G = R$ is a pure τ-homogeneous subgroup of $V := \prod_{i\in I} \mathbb{Q}_\chi \mathbf{e}_i$; for each $i \in I$, we let $\nu_i : V \to \mathbb{Q}_\chi \mathbf{e}_i$ be the usual projection.

Choose some index $j \in I$ such that $\nu_j(J) \neq 0$. Since $J \cong \nu_j(J)$ and $\mathbb{Q}_\chi \mathbf{e}_i$ have the same type, for all but an infinite number of primes p_j we have

$$|\nu_i(\mathbf{v})|_{p_j} = \chi_j = |\mathbf{e}_i|_{p_j}$$

(where the heights are computed in either G or V). If $p_1, \ldots, p_k$ are the primes where this does not hold, then we can conclude that χ_m is finite for $m = 1, \ldots, k$. For each of this finite collection of primes, by the purity of R in V, we can find an index $j_m \in I$ such that

$$|\nu_{j_m}(\mathbf{v})|_{p_m} = \chi_m = |\mathbf{e}_{j_m}|_{p_m}.$$

Let $S = \{j, j_1, \ldots, j_m\} \subseteq I$ and $\nu_S : V \to \prod_{i\in S} \mathbb{Q}_\chi \mathbf{e_i} = \bigoplus_{i\in S} \mathbb{Q}_\chi \mathbf{e_i} := W$ be the usual projection. We have defined W so that $\nu_S(J)$ is pure in W. Since W is a finite rank completely decomposable homogeneous group, by Lemma 86.8 of [6], it follows that $\nu_S(J)$ will be a summand of W. Let $\pi : W \to \nu_S(J)$ be the corresponding projection.

Consider the composition

$$R \subseteq V \xrightarrow{\nu_S} W \xrightarrow{\pi} \nu_S(J) \xrightarrow{\nu_S^{-1}} J.$$

It is elementary to verify that this is the identity when restricted to J. So the group J must be a summand of the group R, and we thus have verified that point (c) holds, as claimed. □

Curiously, we also extract the following statement, which definitely can be viewed as a common expansion of [8, Lemmas 3.14,3.19; Theorem 3.20].

Corollary 2.2 *Universally fully transitive groups are always both fully transitive and transitive. The converse implication does not hold in general.*

Proof The property of being fully transitive follows directly from the Definition 1.3. Applying now Theorem 2.1 (g), the reduced universally fully transitive groups are just separable homogeneous, and the latter ones are known to be transitive (see, for example, [14, §40]).

About the failure of the reverse part, just take all algebraically compact torsion-free groups. In accordance with [7] they are simultaneously transitive and fully transitive, but definitely *not* separable. □

There is one situation where the requirement in Theorem 2.1(h) that R be τ-homogeneous can be dropped.

Corollary 2.3 *Suppose $\tau = \overline{\chi}$ has the property that for every prime p_j, χ_j is either 0 or ∞ (i.e., τ is "idempotent"). If $G \cong R \oplus D$, where D is divisible, R is reduced and G has a pure rank-1 subgroup of type τ, then G is universally fully transitive if, and only if, R is isomorphic to a pure subgroup of a vector group of the form $\prod_{i \in I} \mathbb{Q}_\chi \mathbf{e}_i$.*

Proof The condition on τ makes it easy to see that any pure subgroup of such a vector group will actually be τ-homogeneous. In fact, $\mathbb{Q}_\chi$ will be a ring and any pure subgroup of such a vector group will actually be a $\mathbb{Q}_\chi$-module whose rank-1 submodules are isomorphic to $\mathbb{Q}_\chi$. □

An immediate consequence of Corollary 2.3 is the following:

Corollary 2.4 *If G is a pure subgroup of a direct product of copies of $\mathbb{Z}$, then G is universally fully transitive. In particular, the Baer-Specker group is universally fully transitive.*

If τ is not idempotent, the last two results clearly fail. In fact, for a type $\tau = \overline{\chi}$, it is readily verified that $G = \prod_{i \in I} \mathbb{Q}_\chi \mathbf{e}_i$ is homogeneous if, and only if, τ is idempotent.

We next use some homological methods to show that the general situation of Theorem 2.1 (h) can, up to a categorical equivalence, be reduced to the more straightforward, but apparently more specialized situation of Corollary 2.3.

We fix some notation we will use for the remainder of the section. Suppose $\tau = \overline{\chi}$ and $\mathbb{Q}_\chi$ is the corresponding subgroup of $\mathbb{Q}$. Let ρ be the characteristic defined by

$$\rho_k = \begin{cases} \infty, & \text{when } \chi_k = \infty \\ 0, & \text{when } \chi_k \text{ is finite,} \end{cases}$$

and $\sigma = \overline{\rho}$ be the corresponding idempotent type. There is clearly a natural isomorphism $\mathbb{Q}_\rho \cong \mathrm{Hom}(\mathbb{Q}_\chi, \mathbb{Q}_\chi)$ where $1 \in \mathbb{Q}_\rho$ corresponds to the identity function in the later endomorphism ring. In fact, we will tend to identify these two rings.

Let $\mathcal{I}$ be the collection of primes p_j such that $\chi_j = \rho_j = \infty$, i.e., such that $\mathbb{Q}_\chi$ is p_j-divisible. So we can think of $\mathbb{Q}_\rho$ as the integers localized at $\mathcal{P} \setminus \mathcal{I}$.

If M is any group, then we can identify the divisible hull of M with $D := \mathbb{Q} \otimes M$, and our arguments will often take place in D. Our next observation is very well-known, but we include it here to remind the reader of a standard construction.

Lemma 2.5 *If M is a group and $x \in \mathbb{Q}_\chi \otimes M$, then $x = a \otimes m$ for some $a \in \mathbb{Q}_\chi$ and $m \in M$. If D is a divisible hull of M, then we can identify $\mathbb{Q}_\chi \otimes M$ with $\mathbb{Q}_\chi M \subseteq D$.*

Proof Let $x = \sum(a_i \otimes m_i)$. It follows that there is a single $a \in \mathbb{Q}_\chi$ and integers k_i such that $a_i = k_i a$ for each i. If $m = \sum k_i m_i$, then

$$x = \sum (a_i \otimes m_i) = \sum (k_i a \otimes m_i) = a \otimes \left(\sum k_i m_i\right) = a \otimes m,$$

as stated.

Now, mapping $a \otimes m \mapsto am \in D$ clearly gives the desired identification. □

If M is a group, let $\phi(M) = \mathbb{Q}_\chi \otimes M$ and $\mu(M) = \mathrm{Hom}(\mathbb{Q}_\chi, M)$. We have observed that we can identify $\phi(M)$ with $\mathbb{Q}_\chi M \subseteq D$. Similarly, a homomorphism $f \in \mu(M) = \mathrm{Hom}(\mathbb{Q}_\chi, M)$ will be completely determined by $x := f(1)$, and such an x leads to such an f precisely when $x \in p^\chi M$. In other words, $\mu(M)$ can be identified with $p^\chi M \subseteq M$.

We will say that M is $\mathcal{I}$-divisible if it is p-divisible for all $p \in \mathcal{I}$; so M is $\mathcal{I}$-divisible if, and only if, it is an $\mathbb{Q}_\rho$-module.

Lemma 2.6 *Suppose M is an $\mathcal{I}$-divisible group. If D is a divisible hull for M and $\mathbb{Q}_\chi \otimes M = \mathbb{Q}_\chi M \subseteq D$, then there is a short exact sequence*

$$0 \to M \to \mathbb{Q}_\chi M \to T \to 0,$$

where $T = \bigoplus_{p_k \notin \mathcal{I}} T_{p_k}$ is a torsion group with $p_k^{\chi_k} T_{p_k} = 0$.

Proof We have a short exact sequence

$$0 \to \mathbb{Q}_\rho \to \mathbb{Q}_\chi \to \oplus_{p_k \notin \mathcal{I}} (p_k^{-\chi_k} \mathbb{Q}_\rho / \mathbb{Q}_\rho) \to 0$$

Tensoring with the (flat $\mathbb{Q}_\rho$-module) M gives the result. □

The next two technical assertions are pivotal for our homological result stated as Theorem 2.9.

Lemma 2.7 *If M is a group, then*

$$\Gamma_M : M \to \mu(\phi(M)) = \mathrm{Hom}(\mathbb{Q}_\chi, \mathbb{Q}_\chi \otimes M).$$

given by $[\Gamma_M(m)](a) = a \otimes m$ for all $a \in \mathbb{Q}_\chi$ and $m \in M$ is an isomorphism if, and only if, M is $\mathcal{I}$-divisible.

Proof If $p \in \mathcal{I}$, then it is easy to see that $\mathrm{Hom}(\mathbb{Q}_\chi, \mathbb{Q}_\chi \otimes M)$ is also p-divisible; so if Γ_M is an isomorphism, then M is $\mathcal{I}$-divisible.

Conversely, assume that M is $\mathcal{I}$-divisible. Using the above interpretations of ϕ and μ, what we need to show is that in D we have $M = p^\chi(\mathbb{Q}_\chi M)$. The inclusion $\subseteq$ being reasonably clear, we consider the reverse. Note that in the sequence of Lemma 2.6 we have that $p^\chi(\mathbb{Q}_\chi M)$ maps to

$$\bigcap_{p_k \notin \mathcal{I}} p_k^{\chi_k} T = 0.$$

This, in turn, implies that $p^{\chi}(\mathbb{Q}_{\chi}M) \subseteq M$, and thus completes the proof. $\square$

Continuing with the above notation, recall that for the group N we have $N(\tau) = N$ if, and only if, $N/p^{\chi}N$ is torsion. For example, if M is any group, it is readily checked that $N := \phi(M) = \mathbb{Q}_{\chi} \otimes M$ satisfies this condition.

In the following proof we will use the observation that if B is any subgroup of $\mathbb{Q}$ of type $\geq \tau$, then $\mathbb{Q}_{\chi} \cap B$ also has type τ, so that there is an integer s such that $\mathbb{Q}_{\chi} \cap B = s\mathbb{Q}_{\chi}$.

Lemma 2.8 *If N is a group, then*

$$\Lambda_N : \phi(\mu(N)) = \mathbb{Q}_{\chi} \otimes \mathrm{Hom}\left(\mathbb{Q}_{\chi}, N\right) \to N,$$

given by $\Lambda_N(a \otimes f) = f(a)$ *for all* $a \in \mathbb{Q}_{\chi}$ *and* $f \in \mathrm{Hom}(\mathbb{Q}_{\chi}, N)$*, is an isomorphism if, and only if,* $N(\tau) = N$.

Proof It follows immediately from the above discussion that if Λ_N is an isomorphism, then $N(\tau) = N$. So, assume that $N(\tau) = N$. It is easy to check that Λ_N must be injective; in fact, it corresponds to the inclusion $\mathbb{Q}_{\chi}(p^{\chi}N) \subseteq N$. Therefore, it is the other containment that concerns us.

So, suppose $x \in N$ is non-zero. Since $\mathbb{Q}_{\chi}x \subseteq \mathbb{Q}N$ has type τ and $N(\tau) = N$, it follows that $\mathbb{Q}_{\chi}x \cap N$ has the same type as $\mathbb{Q}_{\chi}$ and $\mathbb{Q}_{\chi}x$, namely τ. So for some integer s we have $s\mathbb{Q}_{\chi}x = \mathbb{Q}_{\chi}x \cap N$.

Since $x \in \mathbb{Q}_{\chi}x \cap N$, there is an $a \in \mathbb{Q}_{\chi}$ such that $sax = x$. Since $\mathbb{Q}_{\chi}(sx) = s\mathbb{Q}_{\chi}x \subseteq N$, we can conclude that $sx \in p^{\chi}N$. And since $a(sx) = sax = x$, we can conclude that $x \in \mathbb{Q}_{\chi}(p^{\chi}N)$, as required. $\square$

The last two lemmas can be combined into the following general result.

Theorem 2.9 *Let* $\mathcal{M}_{\rho}$ *be the class of all groups that are* $\mathcal{I}$*-divisible, i.e., the class of all* $\mathbb{Q}_{\rho}$*-modules, and* $\mathcal{N}_{\tau}$ *be the class of all groups N for which* $N = N(\tau)$*. Then* $\phi : \mathcal{M}_{\rho} \to \mathcal{N}_{\tau}$ *and* $\mu : \mathcal{N}_{\tau} \to \mathcal{M}_{\rho}$ *are inverse categorical equivalences.*

Again, working in D, what this result is saying is that if M is $\mathcal{I}$-divisible, then $p^{\chi}(\mathbb{Q}_{\chi}M) = M$; and if $N(\tau) = N$, then $\mathbb{Q}_{\chi}(p^{\chi}N) = N$. We now apply this categorical equivalence to the class of universally fully transitive groups.

Theorem 2.10 *The functors* $M \mapsto \mathbb{Q}_{\chi} \otimes M = \mathbb{Q}_{\chi}M$ *and* $N \mapsto \mathrm{Hom}(\mathbb{Q}_{\chi}, N) = p^{\chi}N$ *give categorical equivalences between the classes of universally fully transitive groups whose reduced parts are* $\sigma = \overline{\rho}$*-homogeneous (and separable) and those whose reduced parts are* $\tau = \overline{\chi}$*-homogeneous (and separable).*

Proof Clearly, if Q is divisible, then there are natural isomorphisms $\mathbb{Q}_{\chi}Q \cong Q \cong p^{\chi}Q$, so our functors preserve the divisible parts of our groups.

In addition, on rank-1 groups this categorical equivalence correspond to the natural isomorphisms $\mathbb{Q}_{\chi}(\mathbb{Q}_{\rho}\mathbf{e}) \cong \mathbb{Q}_{\chi}\mathbf{e}$ and $p^{\chi}(\mathbb{Q}_{\chi}\mathbf{e}) \cong \mathbb{Q}_{\rho}\mathbf{e}$. And since they also preserve direct sum decompositions, the result is clear. $\square$

Notice the fact that Theorem 2.10 states, up to a categorical equivalence, that the study of torsion-free universally transitive groups reduces to the case where such groups are separable and homogeneous of idempotent type σ, which we have seen in Corollary 2.3 is somewhat more straightforward.

Because of the importance of this special case, we discuss it in a bit more depth. For example, the following observation follows directly from Corollary 2.3 and parallels Corollary 4.1 from [4].

Corollary 2.11 *The class of σ-homogeneous universally fully transitive groups is the smallest class containing $\mathbb{Q}_\rho$ that is closed with respect to direct products and pure subgroups.*

If A is a $\mathbb{Q}_\rho$-module, then clearly $A^\bullet := \mathrm{Hom}(A, \mathbb{Q}_\rho)$ is also a $\mathbb{Q}_\rho$-module. Before proving our next characterizing result about torsion-free $\mathbb{Q}_\rho$-modules, we need the following observation.

Lemma 2.12 *If A is an $\mathbb{Q}_\rho$-module, then $A^\bullet$ will be a reduced σ-homogeneous universally fully transitive $\mathbb{Q}_\rho$-module.*

Proof Let

$$0 \to K \to F \to A \to 0$$

be a $\mathbb{Q}_\rho$-free resolution of A. Suppose $K = \bigoplus_{j\in J} \mathbb{Q}_\rho$ and $F = \bigoplus_{i\in I} \mathbb{Q}_\rho$. Since $\mathrm{Hom}(\mathbb{Q}_\rho, \mathbb{Q}_\rho) \cong \mathbb{Q}_\rho$, we can conclude that $K^\bullet \cong \prod_{j\in J} \mathbb{Q}_\rho$ and $F^\bullet \cong \prod_{i\in I} \mathbb{Q}_\rho$. Therefore, the left-exact sequence $0 \to A^\bullet \to F^\bullet \to K^\bullet$ is equivalent to

$$0 \to A^\bullet \to \prod_{i\in I} \mathbb{Q}_\rho \to \prod_{j\in J} \mathbb{Q}_\rho .$$

Since $\prod_{j\in J} \mathbb{Q}_\rho$ is torsion-free, $A^\bullet$ is pure in $\prod_{i\in I} \mathbb{Q}_\rho$. So, by Corollary 2.11, this means that $A^\bullet$ is a reduced σ-homogeneous universally fully transitive $\mathbb{Q}_\rho$-module, as required. □

If A is a $\mathbb{Q}_\rho$-module, then in the usual way, the map $\phi_A : A \to A^{\bullet\bullet}$ given by $\phi_A(x) = f(x)$, for each $f \in A^\bullet$, defines a natural homomorphism from A to its second dual. This brings us to the following characterization:

Theorem 2.13 *Suppose $\sigma = \overline{\rho}$ is an idempotent type. If G is a $\mathbb{Q}_\rho$-module and $G = R \oplus D$, where D is divisible and R is reduced, then the following are equivalent:*

(a) G is universally fully transitive and if $R \neq 0$, then it has a pure submodule isomorphic to $\mathbb{Q}_\rho$;
(b) G is universally Krylov transitive and if $R \neq 0$, then it has a pure submodule isomorphic to $\mathbb{Q}_\rho$;
(c) R is isomorphic to a pure submodule of some direct product $\prod_{i\in I} \mathbb{Q}_\rho$;
(d) Under the natural map $\phi_R : R \to R^{\bullet\bullet}$, R maps to a pure submodule of $R^{\bullet\bullet}$.

Proof Notice that, by Theorem 2.1(g), points (a) and (b) are equivalent to requiring that R be a σ-homogeneous separable $\mathbb{Q}_\rho$-module. And by using Corollary 2.11, these are also equivalent to condition (c). So, we need to verify that conditions (a), (b), (c) are equivalent to (d).

To that goal, note first that, if R satisfies (d), then by Lemma 2.12, $R^{\bullet\bullet} = (R^\bullet)^\bullet$ is a reduced σ-homogeneous separable $\mathbb{Q}_\rho$-module, so that with Corollary 2.11 at hand so is R, itself.

Conversely, suppose R is a reduced σ-homogeneous separable $\mathbb{Q}_\rho$-module. For each $x \in R$ there is a cyclic summand $K \cong \mathbb{Q}_\rho$ of R with $x \in K$; suppose $R = K \oplus H$. It follows that there are isomorphisms $R^\bullet \cong K^\bullet \oplus H^\bullet$ and

$$K \oplus H \cong R \to R^{\bullet\bullet} \cong K^{\bullet\bullet} \oplus H^{\bullet\bullet}.$$

Clearly, $K^{\bullet\bullet} \cong \mathbb{Q}_\rho{}^{\bullet\bullet} \cong \mathbb{Q}_\rho \cong K$, which shows that $\phi_R(x) \in R^{\bullet\bullet}$ has the same height sequence as $x \in R$. Therefore, $\phi_R(R)$ will be pure in $R^{\bullet\bullet}$, as required. □

The following example illustrates that the purity hypotheses in Theorem 2.13(c) cannot be ignored at all.

Example 2.14 Given such a ring $\mathbb{Q}_\rho$, let $p \in \mathbb{Q}_\rho$ be a prime. Let $P = \prod_{i\in\mathbb{N}} \mathbb{Q}_\rho \mathbf{e}_i$ and $S = \oplus_{i\in\mathbb{N}}\mathbb{Q}_\rho\mathbf{e}_i \subseteq P$. Consider $G := pP + S$. It follows from the theory of slender groups (see, e.g., [7]) that $\phi_P : P \to P^{\bullet\bullet}$ is an isomorphism. If we compose $\phi_G : G \to G^{\bullet\bullet}$ with the natural homomorphism $G^{\bullet\bullet} \to P^{\bullet\bullet} \cong P$, the result is equivalent to the embedding $G \subseteq P$. It follows now that ϕ_G embeds G as a subgroup of $G^{\bullet\bullet}$. On the other hand, G is clearly not separable (indeed, it can be shown that the vector $x = (p, p, p, \dots) \in G$ does not embed in a direct summand of G, whose proof we leave to the interested reader). Comparing with Theorem 2.13, this is equivalent to the observation that the subgroup $G \cong \phi_G(G)$ is not pure in $G \cong G^{\bullet\bullet}$.

Classifying the modules in the classes described in Theorem 2.13 is clearly impossible. Even in the case of where $\mathbb{Q}_\rho = \mathbb{Z}$, we are highly unlikely to come even remotely close to classifying all pure subgroups of a direct product of copies of the integers, such as the Baer-Specker group.

Remark 2.15 In [3] were considered those groups whose pure endomorphic images are always direct summands (see [12] as well). Comparing them with the defined above universally fully transitive groups we found that there is some similarity in the two classes, though they are distinct, however. In fact, if A and B are reduced torsion-free rank-1 groups of incomparable types, then the direct sum $A \oplus B$ will have the property that every pure image is a direct summand, but it will definitely *not* be universally fully transitive.

In the other direction, if G is a universally fully transitive torsion-free group whose reduced part has finite rank, then since it must be homogeneous and completely decomposable, it certainly will have the property that every pure endomorphic image is a direct summand. This does not hold in general, however. For example, suppose A is a direct sum of a countable number of copies of the integers, and B be the corresponding direct product (i.e., the Baer-Specker group). In accordance with Theorem 2.1 the direct sum $G = A \oplus B$ will be universally fully transitive, but it will *not* have the property that every pure image is a direct summand – indeed, just map A isomorphically onto a pure subgroup of B and, moreover, map B to 0. Since B does not have a free summand of infinite rank, this image will not be a summand.

Finally, all homogeneous transitive groups are clearly fully transitive. As to the converse, in [1, Corollary 7.20] it was noted that for finite rank homogeneous groups the concepts of being transitive and fully transitive do coincide, but there are homogeneous fully transitive groups of infinite rank which are surely *not* transitive (cf. [1, §4]).

3 H-fully and H-Krylov transitive groups

We begin with the following observation.

Proposition 3.1 *Suppose H is a subgroup of G.*

(a) If the reduced part of H is separable and homogeneous, then G is H-fully transitive.
(b) If G is completely decomposable and homogeneous of idempotent type, then G is H-fully transitive.

Proof Clearly (a) follows immediately from Theorem 2.1. Regarding (b), suppose G is $\tau = \overline{\chi}$-homogeneous, so that it is a free $\mathbb{Q}_\chi$-module. Suppose $x \in H$ and $y \in G$ with $|x|_H \leq |y|_G$. Clearly $F := \mathbb{Q}_\chi H \subseteq G$ will also be a free $\mathbb{Q}_\chi$-module, and hence universally fully transitive. Since we clearly have $|x|_F \leq |y|_G$, it follows that there is a homomorphism $\phi : F \to G$ with $\phi(x) = y$. Restricting ϕ to H shows that G is also H-fully transitive.

In particular, if G has rank-1, then any subgroup H of G satisfies Proposition 3.1(a), so that G is H-fully transitive.

In the introduction it was observed that if G is fully or Krylov transitive, then it is, respectively, H-fully or H-Krylov transitive for any pure subgroup H. The next example shows that the hypothesis of purity is necessary.

Example 3.2 There if a fully transitive group G with a (non-pure) subgroup H such that G is *not* H-fully transitive.

Proof Define $\mathbb{Q}_{2^{-\infty}}, \mathbb{Q}_{2^{-\infty},3^{-\infty}} \subseteq \mathbb{Q}$ in the usual way and let $\hat{\mathbb{Q}} = \langle p^{-\infty} : 5 \leq p \in \mathcal{P}\rangle \subseteq \mathbb{Q}$. We let

$$G = \left(\mathbb{Q}_{2^{-\infty},3^{-\infty}}\, \mathbf{a}\right) \oplus \left(\hat{\mathbb{Q}}\, \mathbf{b}\right) := A \oplus B \subseteq \mathbb{Q}\,\mathbf{a} \oplus \mathbb{Q}\,\mathbf{b}.$$

We leave it to the reader to verify that G is fully transitive. Let

$$H = (\mathbb{Q}_{2^{-\infty}}\, \mathbf{a}) \oplus B := A' \oplus B \subseteq G.$$

If $x = \mathbf{a} + 3\mathbf{b} \in H$ and $y = \mathbf{b} \in G$, then for every prime p one checks that $|x|_{p,H} = 0 \leq |y|_p$. On the other hand, for any homomorphism $\gamma : H \to G$ we must have $\gamma(A') \subseteq A$ and $\gamma(B) \subseteq B$. This implies that $\gamma : B \to B$ satisfies $\gamma(3\mathbf{b}) = \mathbf{b}$, which cannot be true. □

By Proposition 3.1(b), if D is divisible, then D is H-fully transitive for any subgroup $H \subseteq D$. On the other hand, we have the following elementary observation, whose proof we lead to the reader.

Proposition 3.3 *Suppose G is a group that is contained in the divisible group D. Then G is universally fully transitive if, and only if, every group of the form $M \oplus D$ is G-fully transitive.*

As noticed above, there are G-Krylov transitive groups that are not G-fully transitive. For the existence in the p-primary case of a Krylov transitive group which is neither fully transitive nor transitive (in this situation p necessarily equals 2), we refer the interested reader to [2] for a more detailed information.

However, in our torsion-free case, we need a construction for a proper subgroup. Specifically, the following is true:

Example 3.4 There exists a group G with a subgroup H such that G is H-Krylov transitive, but *not* H-fully transitive.

Proof Let $G = A \oplus B$ be a reduced group, where A, B are subgroups of G with $r(A) = r(B) = 1$, $\tau(A) > \tau(B) > \tau(\mathbb{Z})$ and $\mathbb{Z}$ is the infinite cyclic group. Assume that $\pi : G \to B$ is the usual projection. Also, let $H = A \oplus C$, where $0 \neq C \leq B$ and $\tau(C) < \tau(B)$. Then $\chi_G(g) = \chi_H(y)$ for some $0 \neq g \in G$ and, moreover, $y \in H$ if and only if $g, y \in A$. Therefore, there exists $f \in \mathrm{Hom}\,(H, G)$ with $f(y) = g$, i.e., G is H-Krylov transitive, as asserted.

Suppose now $pA \neq A$ for a prime p and $0 \neq a \in A \setminus pA$, $0 \neq c \in C \setminus pC$. Thus $\tau_H(a + pc) < \tau(B)$. So, there will exist $0 \neq b \in B \setminus pB$ with $\chi_B(b) > \chi_H(a + pc)$. Assume without loss of generality that $\varphi \in \mathrm{Hom}\,(H, G)$ is chosen such that $\varphi(a + pc) = b$. Consequently, $\pi\varphi(a) + p\pi\varphi(c) = b$. But $\tau(\pi\varphi(a)) > \tau(b)$, whence $\tau(\pi\varphi(a)) = 0$ and $p\pi\varphi(c) = b$ which contradicts $b \notin pB$. This shows that G is not H-fully transitive, as claimed. □

Nevertheless, an easier way to construct a proper subgroup with the asked property in the non-reduced case is as follows: Letting H be transitive and not fully transitive, we could consider the group $G = H \oplus D$, where $D \neq 0$ is a divisible subgroup. The above example treats the reduced case, however.

For the remainder of this section, by analogy with the approach demonstrated in [4], we shall consider a more global version of Definition 1.4 by *not* considering H to be necessarily a subgroup of the group G; in fact, H will allowed to be an arbitrary fixed group. With this in mind, we can offer the following construction which is closely related to the last example. Indeed, we will construct the wanted group as a subspace of the vector space V over the field of all rational numbers $\mathbb{Q}$. To that goal, let $H = \langle p_1^{-\infty}a, p_2^{-\infty}b, p_3^{-\infty}(a + b)\rangle$ and $G = \langle p_3^{-\infty}q^{-\infty}c\rangle$, where p_1, p_2, p_3, q are different prime numbers and a, b, c are independent elements from V. Then one verifies that $\tau(H) \cap \tau(G) = \varnothing$. So, G is H-Krylov transitive, but definitely it is *not* H-fully transitive, as asked for. In fact, $\chi_H(a + b) < \chi_G(c)$ and there will not exist $f \in \mathrm{Hom}\,(H, G)$ with $f(a + b) = c$, because $f(a) = f(b) = 0$ for any $f \in \mathrm{Hom}\,(H, G)$. This completes the desired example.

We shall say now that the torsion-free groups G_1 and G_2 (the equality $G_1 = G_2$ is also possible) form *a completely transitive pair* of the corresponding subgroups

$H_1 \leq G_1$ and $H_2 \leq G_2$ provided, for each $x \in H_i$,$y \in G_j$ with $i, j \in \{1, 2\}$ satisfying the condition $\chi_{H_i}(x) \leq \chi_{G_j}(y)$, there exists $\alpha \in \mathrm{Hom}(H_i, G_j)$ such that $\alpha(x) = y$, i.e., G_1 and G_2 are H_1-fully transitive and H_2-fully transitive.

Moreover, in conjunction with [10], we shall say that the system $\{G_i\}_{i\in I}$ of torsion-free groups satisfies *the monotonic condition with respect to the characteristics* of the subgroups $\{H_i \leq G_i\}_{i\in I}$ if, for every $0 \neq y \in G_i$, the condition $\chi_{H_{i_1}}(a_1) \wedge \cdots \wedge \chi_{H_{i_m}}(a_m) \leq \chi_{G_i}(y)$, where $a_j \in H_{i_j}$; $i_j \neq i_s$ if $j \neq s$; $j, s = 1, \ldots, m$ implies the existence of the elements $b_1, \ldots, b_r \in G_i$ equipped with the properties $b_1 + \cdots + b_r = y$ and, for any b_l with $(l = 1, \ldots, r)$ among the elements $a_1, \ldots, a_m$, there exists at least one element a_s such that $\chi_{G_i}(b_l) \geq \chi_{H_{i_s}}(a_s)$.

A non-trivial concrete example of the last concept is the following one: If all torsion-free groups G_i are homogeneous of the same type, then the system $\{G_i\}_{i\in I}$ satisfies the required property of monotonic condition with respect to the characteristics of the subgroups $\{\pi_i(H)\}_{i\in I}$ for any $H \leq G$, where $\pi_i : \bigoplus_{i\in I} G_i \to G_i$ are the corresponding projections for all indices i from the index set I.

The next statement somewhat refines [10, Lemma 2.19] like this:

Proposition 3.5 *Let $G = \bigoplus_{i\in I} G_i$ be a torsion-free group, where $\pi_i : G \to G_i$ are the corresponding projections, and let $H \leq G$ be an invariant subgroup, comparatively on the system $\{\pi_i\}_{i\in I}$, i.e., $H = \bigoplus_{i\in I} H_i$, where $H_i = \pi_i(H)$. Then G is a H-fully transitive if, and only if, the system $\{G_i\}_{i\in I}$ satisfies the monotonic condition with respect to the characteristics of the subgroups $\{H_i\}_{i\in I}$ and, for all indices $i, j \in I$, the groups G_i, G_j form a completely transitive pair of the corresponding subgroups H_i, H_j.*

Proof To prove necessity, assume that $\chi_{G_i}(y) \geq \chi_H(x)$ for some $y \in G_i$, $x \in H$. Assume also as in the given above notion that

$$\chi_{G_i}(y) \geq \chi_H(a_1 + \cdots + a_m) = \chi_{H_{i_1}}(a_1) \wedge \cdots \wedge \chi_{H_{i_m}}(a_m).$$

Therefore, if $\varphi(a_1 + \cdots + a_m) = y$ for $\varphi \in \mathrm{Hom}(H, G)$, then one sees that $\pi_i\varphi(a_j) \in G_i$ and $\chi_{G_i}(\pi_i\varphi(a_j)) \geq \chi_{H_j}(a_j)$, $j = 1, \ldots, m$, as required. Furthermore, the fact that the groups G_i and G_j form a completely transitive pair of the subgroups H_i and H_j is pretty evident, so we omit its verification.

To prove sufficiency, assume $\chi_G(y) \geq \chi_H(x)$ for some $0 \neq y \in G$, $0 \neq x \in H$. We have $x = x_1 + \cdots + x_m$, $y = y_1 + \cdots + y_n$, where $x_j \in H_{i_j}$, $y_q \in G_{i_q}$ with $j = 1, \ldots, m; q = 1, \ldots, n$. We will show that, for each y_q, there will exist $\varphi_q \in \mathrm{Hom}(H, G)$ such that $\varphi_q(x) = y_q$. If so, it will follow that $(\varphi_1 + \cdots + \varphi_n)(x) = y$, as required.

In fact, to establish that, if for y_q there exists such a x_j having the property $\chi_G(y_q) \geq \chi_H(x_j)$, then $\varphi_j(x_j) = y_q$ for some $\varphi_j \in \mathrm{Hom}(H_{i_j}, G_{i_q})$. Setting $\varphi_q \mid H_j = \varphi_j$ and $\varphi_q(H_{i_s}) = 0$ for $i_s \neq i_j$, we obtain that $\varphi_q(x) = y_q$, as asked for.

So, assume now that $\chi_G(y_q) \ngeq \chi_H(x_j)$ for all $j = 1, \ldots, m$. Since $\chi_G(y_q) \geq \chi_G(y) \geq \chi_H(x) = \chi_H(x_1) \wedge \cdots \wedge \chi_H(x_m)$, by what we stated above there exist such elements $b_1, \ldots, b_r \in G_{i_q}$ that $b_1 + \cdots + b_r = y_q$, where, for every b_l with $l = 1, \ldots, r$, there is such a x_s with $s \in \{1, \ldots, m\}$ that $\chi_G(b_l) \geq \chi_H(x_s)$. Hence $\psi_s(x_s) = b_l$ for some $\psi_s \in \mathrm{Hom}(H_{i_s}, G_{i_l})$. By putting $\overline{\psi}_s(H_{i_j}) = 0$ when $i_j \neq i_s$, we deduce that $\overline{\psi}_s(x) = b_l$, where $\overline{\psi}_s \in \mathrm{Hom}(H, G)$. As this holds for any element b_l in the decomposition $y_q = b_1 + \cdots + b_r$, we therefore elementarily derive that y_q is a homomorphic image of x, as desired. □

As consequences to the last statement, we obtain the following ones:

Corollary 3.6 *Let $G = \bigoplus_{i\in I} G_i$ be a torsion-free group and $H = \bigoplus_{i\in I} H_i$, where $H_i \leq G_i$. If G is a H-fully transitive group, then, for every $\varnothing \neq J \subseteq I$, the group $\bigoplus_{j\in J} G_j$ is $\bigoplus_{j\in J} H_i$-fully transitive.*

Corollary 3.7 *Suppose $G = D(G) \oplus A$, where $D(G)$ is the divisible part of G and suppose $H = F \oplus R$ is a subgroup of G, where $F \leq D(G), R \leq A$. If G is H-fully transitive, then A is both R-fully transitive and F-fully transitive. The converse implication fails in general.*

Proof The first part-half follows directly from the previous proposition. As for the failure of the reverse part, suppose F and R are reduced rank-1 groups of different types. It follows by application of Theorem 2.1 that $H = F \oplus R$ is not universally fully transitive. Therefore, there is a group C and elements $x \in H$, $y \in C$ such that $|x|_H \leq |y|_C$, but no homomorphism $H \to C$ will exist taking x to y. Moreover, letting D be the divisible hull of F (and so $F \cong \mathbb{Q}$) and $A = R \oplus C$, one obtains that $G = D \oplus A$. Since both F and R are trivially universally fully transitive, one has that A is both F-fully transitive and R-fully transitive. However, the same homomorphism that shows C is not H-fully transitive also shows that G is not H-fully transitive, as wanted.

Corollary 3.8 *Let $G = \bigoplus_{i\in I} G_i$ be a torsion-free group and $H = \bigoplus_{i\in I} H_i$, where $H_i \leq G_i$. If G is a H-fully transitive group and* $\mathrm{Hom}(H_j, G_i) = 0$, *then, for every prime number p, if $pH_j \neq H_j$ then every element from H_i has infinite p-height in G_i for all $i \in I$.*

Proof Assume that $|x|_{H_i,p} = 0$ and that $|y|_{H_j,p} = 0$ for some $x \in H_i$ and $y \in H_j$. Hence $\chi_{G_i}(x) \geq \chi_H(p^n x) \wedge \chi_H(y)$ for every $n \geq 1$. By assumption, $x = b_1 + \cdots + b_r$ and, for each $b_l \in G_i$ with $(l = 1, \ldots, r)$, it follows that either $\chi_{G_i}(b_l) \geq \chi_{H_i}(p^n x)$ or $\chi_{G_i}(b_l) \geq \chi_{H_j}(y)$. But $\chi_{G_i}(b_l) \not\geq \chi_{H_j}(y)$ since $\mathrm{Hom}(H_j, G_i) = 0$, so that $\chi_{G_i}(b_l) \geq \chi_{H_i}(p^n x)$ for every $l = 1, \ldots, r$. Consequently, $|x|_{G,p}$ is infinite, as claimed. □

Corollary 3.9 *Let $G = \bigoplus_{i\in I} G_i$ be a torsion-free group and $H = \bigoplus_{i\in I} H_i$, where $H_i \leq G_i$ and G_i are homogeneous of the same type. Then G is H-fully transitive if, and only if, the groups G_i and G_j form a completely transitive pair of the corresponding subgroups H_i and H_j for all $i, j \in I$.*

Corollary 3.10 *Let $G = \bigoplus_{i\in I} G_i$ be a torsion-free homogeneous group and $H = \bigoplus_{i\in I} H_i$, where $H_i \leq G_i$ and $G_i \cong A$, $H_i \cong B$ for some groups A, B. Then G is H-fully transitive if, and only if, A is B-fully transitive.*

We shall deal now with vector groups. To that goal, let $G = \prod_{i\in I} G_i$ be a vector group, where $r(G_i) = 1$ for all indices i. From the already proved assertions above, it follows that $\tau(G_i) = \tau(G_j)$ for all indexes $i, j \in I$. If I is infinite and $\tau(G_i)$ is a non idempotent, then there exists such an element $x \in G$ that $\tau(x) < \tau(G_i)$. But if now $r(H) = 1$ and $\tau(H) = \tau(x)$, then Hom $(G_i, H) = \{0\}$. So, appealing to [6, Lemma 96.1], we deduce that Hom $(G, H) = \{0\}$, i.e., G is non universally fully transitive.

We, thereby, arrive at the following assertion:

Proposition 3.11 *A vector group is universally fully transitive if, and only if, all of the rank 1 groups forming the direct product are of the same idempotent type.*

Let us notice also that this can be directly deduced from Corollary 2.3 and the ensuing statement: *For a type $\tau = \overline{\chi}$, it is readily verified that $G = \prod_{i \in I} \mathbb{Q}_\chi e_i$ is homogeneous if, and only if, τ is an idempotent.*

Proposition 3.12 *Let G be a reduced group and H a fully transitive separable pure subgroup of G. Then G is H-fully transitive if, and only if, for every $\tau \in \tau(G)$ with $\tau \geq \tau_1 \wedge \cdots \wedge \tau_n$, where $\tau_1, \ldots, \tau_n \in \Omega(H)$, the inclusion $G(\tau) \subseteq G(\tau_1) + \cdots + G(\tau_n)$ holds.*

Proof To prove necessity, let $g \in G$, $0 \neq x_1, \ldots, x_n \in H$ and $x_i \in X_i$, where X_i are rank 1 direct summands of H, $\tau(x_i) = \tau_i$ for $i = 1, \ldots, n$ and . It is possible to consider the case when $\tau_i \neq \tau_j$ with $i \neq j$. Since H is fully transitive, by what we have already commented above, one deduces that $\pi(X_i) \cap \pi(X_j) = \varnothing$ when $i \neq j$. So, $X_1 \oplus \cdots \oplus X_n$ is a direct summand of G. Let $\tau(x) \leq \tau = \tau(g)$, where $x = x_1 + \cdots + x_n$. It is also possible to choose x_i such that $\chi(x) \leq \chi(g)$. Then, one follows that $f(x_1 + \cdots + x_n) = g$ for some $f \in$ Hom (H, G). Thus, $g = f(x_1) + \cdots + f(x_n) \in G(\tau_1) + \cdots + G(\tau_n)$, and hence $G(\tau) \subseteq G(\tau_1) + \cdots + G(\tau_n)$, as asserted.

To prove sufficiency, let $\chi(x) \leq \chi(g)$ for some $0 \neq x \in H$, $g \in G$. Since H is separable, one has that $x = x_1 + \cdots + x_n$, where $x_i \in X_i$ and $X_1 \oplus \cdots \oplus X_n$ is a direct summand of G. If $\tau(x_i) = \tau_i$, then $\tau = \tau(g) \geq \tau(x) = \tau_1 \wedge \cdots \wedge \tau_n$. Indeed, for example, if $\tau(x_1) = \tau(x_2)$, then the sum $x_1 + x_2$ can be embedded in a rank 1 direct summand of H. Furthermore, it is possible to get that $\tau_i \neq \tau_j$ when $i \neq j$. Under the stated condition $g = y_1 + \cdots + y_n$, where $y_i \in G(\tau_i)$ for $i = 1, \ldots, n$, some of the elements y_i could be zero. However, note that if $y_i \neq 0$, then $\tau(y_i) \in \{\tau_1, \ldots, \tau_n\}$. In fact, if $\tau(y_i) \geq \tau'$ for some $\tau' \in \Omega(H) \setminus \{\tau_1, \ldots, \tau_n\}$ and $p \in \pi(\tau(y_i))$, then $p \notin \pi(\tau_i)$, but this disagrees the condition $\tau(y_i) \geq \tau_i$. So, the conditions $\tau(y_i) \geq \tau_i = \tau(x_i)$ and $\pi(\tau_i) \cap \pi(\tau_j) = \varnothing$ for $i \neq j$ with $i, j = 1, \ldots, n$ imply that $\chi(y_i) \geq \chi(x_i)$, whence there exists $f \in$ Hom (H, G) such that $f(x) = g$, as required. □

4 Concluding discussion and open problems

It is obvious that there are too many possible interesting questions in the subject, so that we conclude our discussion with some selection of them.

The first query is devoted to the p-mixed case, that is, the only torsion is p-torsion for some prime p. If, in addition, the group is q-divisible for any prime $q \neq p$, we have in mind p-locality.

Problem 1 Examine for any prime p the p-local universally fully transitive groups by determining their structure.

When every torsion-free element has at most a finite number of gaps in its height the problem can be settled without a big difficulty by following the methods developed by us in [4] and in the present work. However, it is not so clear what to do in the case where there are an infinite number of gaps in the height.

The next question is related to Corollary 2.2.

Problem 2 Does it follow, in general, that universally fully transitive groups (possibly p-primary or mixed) are always transitive?

Problem 3 Determine those subgroups $H \leq G$ for which the properties of being H-Krylov transitive and H-fully transitive coincide.

Problem 4 If G is a reduced group with the property that any two elements of G can be embedded in a countable (homogeneous completely decomposable) direct summand of G, is then G transitive?

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