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Pseudocompact Δ-spaces are often scattered

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Abstract
Given a pseudocompact Δ-space X, we establish that countable subsets of X must be scattered. This implies that pseudocompact Δ-spaces of countable tightness are scattered. If a pseudocompact Δ-space has the Souslin property, then it is separable and has a dense set of isolated points. It is shown that adding a countable subspace to a pseudocompact Δ-space can destroy the Δ-property. However, if X is countably compact and $Y \subset X$ is a Δ-space for some $Y \subset X$ such that $|X \setminus Y| \leq \omega$, then X is a Δ-space. We also show that monotonically normal Δ-spaces must be hereditarily paracompact. Besides, if X is a subspace of an ordinal with its order topology, then X is hereditarily paracompact if and only if it has the Δ-property.

Keywords Eberlein compact space · Pseudocompact space · Δ-space · Monotonically normal space · GO space · Subspace of ordinals

Mathematics Subject Classification 54C35 · 54G12 · 54H05

1 Introduction

In 1975, Reed defined a set $D \subset \mathbb{R}$ to be a Δ-*set* if D is uncountable and, for any decreasing sequence $\{H_n : n \in \omega\}$ of subsets of D, if $\bigcap_{n\in\omega} H_n = \emptyset$, then there exists a sequence $\{V_n : n \in \omega\}$ of G_δ-subsets of D such that $H_n \subset V_n$ for each $n \in \omega$ and $\bigcap_{n\in\omega} V_n = \emptyset$. Przymusinski proved in [13] that existence of a Δ-set is equivalent

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to existence of a countably paracompact, separable non-normal Moore space. Reed mentions in his paper [14], that, in the definition of a Δ-set, the term "G_δ-sets" can be replaced with "open sets"; this was pointed out by van Douwen.

In [9], the essence of the definition of a Δ-set was the base for introducing general Δ-spaces by saying that X is a Δ-space if, for any decreasing sequence $\mathcal{S} = \{X_n : n \in \omega\}$ of subsets of X with empty intersection, there exists a sequence $\{U_n : n \in \omega\}$ of open subsets of X with empty intersection such that $X_n \subset U_n$ for each $n \in \omega$. According to this definition, a Δ-space need not be uncountable and a Δ-set is an uncountable Δ-subspace of $\mathbb{R}$.

The paper [9] features a systematic study of general Δ-spaces. One of its results states that X is a Δ-space if and only if it has the following *Δ-property*: for any disjoint family $\mathcal{A} = \{A_n : n \in \omega\}$ of subsets of X, there is a point-finite open expansion of $\mathcal{A}$, i.e., there exists a point-finite family $\{U_n : n \in \omega\}$ of open subsets of X such that $A_n \subset U_n$ for each $n \in \omega$. It was also proved in [9] that Čech-complete Δ-spaces must be scattered and every scattered Eberlein compact space must have the Δ-property. Besides, a Tychonoff space X has the Δ-property if and only if $C_p(X)$ is distinguished.

In [10] the authors proved that any σ-product of Eberlein compact spaces must be a Δ-space and established that each countably compact Δ-space is scattered. As a natural attempt to look for a generalization of this result, they asked whether every pseudocompact Δ-space is scattered; this question was an inspiration for the authors of this paper to study the Δ-property in compact-like spaces.

In this article, we show that, in a pseudocompact Δ-space, all countable subsets must be scattered. Therefore pseudocompact Δ-spaces of countable tightness are scattered. We also establish that adding a countable set to a pseudocompact Δ-space can destroy its Δ-property. On the other hand, a countably compact space X must be a Δ-space if it has a Δ-subspace Y such that $X \backslash Y$ is countable. It turns out that monotonically normal Δ-spaces are hereditarily paracompact while a subspace X of an ordinal has the Δ-property if and only if X is hereditarily paracompact. A by-product of this study is the fact that any pseudocompact crowded space of countable tightness must be ω-resolvable.

2 Notation and terminology

All spaces are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X) \backslash \{\emptyset\}$; let $\tau(x, X) = \{U \in \tau(X) : x \in U\}$ for any point $x \in X$. The set $\mathbb{R}$ is the real line with its usual topology while $\mathbb{I} = [0, 1] \subset \mathbb{R}$ and $\mathbb{D}$ is the two-point set $\{0, 1\} \subset \mathbb{R}$. If κ is a cardinal, then $[X]^{\leq\kappa} = \{A \subset X : |A| \leq \kappa\}$. Given a family $\mathcal{F}$ of subsets of a space X, an *open expansion of* $\mathcal{F}$ is a family $\{U_F : F \in \mathcal{F}\} \subset \tau(X)$ such that $F \subset U_F$ for each $F \in \mathcal{F}$. If $A \subset X$, then a family $\mathcal{U} \subset \tau(X)$ is an open expansion of the set A if it is an open expansion of the family $\{\{x\} : x \in A\}$. Recall that $S \subset X$ is *a free ω_1-sequence in a space* X if $S = \{x_\alpha : \alpha < \omega_1\}$ and $\overline{\{x_\gamma : \gamma < \alpha\}} \cap \overline{\{x_\gamma : \gamma \geq \alpha\}} = \emptyset$ for any $\alpha < \omega_1$.

The space X is *Fréchet–Urysohn* provided that, for any $A \subset X$, if $x \in \overline{A}$, there is a sequence $\{a_n : n \in \omega\} \subset A$ that converges to x. If, for any non-closed set $A \subset X$, there exists a sequence $\{a_n : n \in \omega\} \subset A$ such that $a_n \to x \notin A$, then the space X is called *sequential*.

The cardinal $t(X) = \min\{\kappa : \overline{A} = \bigcup\{\overline{B} : B \in [A]^{\leq\kappa}\}$ for every $A \subset X\}$ is the *tightness* of the space X. A family $\mathcal{U}$ of subsets of X is *point-finite* if every $x \in X$ belongs only to finitely many elements of $\mathcal{U}$. If $x \in X$, then the cardinal $\psi(x, X) = min\{|\mathcal{U}| : \mathcal{U} \subset \tau(X)$ and $\bigcap \mathcal{U} = \{x\}\}$ is called the *pseudocharacter of* x *in* X and $\psi(X) = sup\{\psi(x, X) : x \in X\}$ is the *pseudocharacter of* X. The minimal cardinality of a local base at a point $x \in X$ is called the *character of* X *at* x; it is denoted by $\chi(x, X)$ and $\chi(X) = \sup\{\chi(x, X) : x \in X\}$.

A space X is called *monotonically normal* if it admits an operator O (called the *monotone normality operator*) that assigns to any point $x \in X$ and any $U \in \tau(x, X)$ a set $O(x, U) \in \tau(x, X)$ such that $O(x, U) \subset U$ and for any points $x, y \in X$ and sets $U, V \in \tau(X)$ such that $x \in U$ and $y \in V$, it follows from $O(x, U) \cap O(y, V) \neq \emptyset$ that $x \in V$ or $y \in U$. Recall that X is a *GO space* if it embeds into a linearly ordered space. Given a cardinal κ, a space X is said to be κ*-resolvable* if it is possible to find κ-many disjoint dense subspaces in X. Furthermore, X is a P*-space* if every G_δ-subset of X is open.

A space X is *crowded* if X has no isolated points; if every non-empty subspace of a space X has an isolated point, the space X is called *scattered*.

We recall some basic facts about Eberlein compact spaces. A space K is an Eberlein compact space if K is homeomorphic to a weakly compact subset of a Banach space. Equivalently, a compact space K is Eberlein if and only if K can be homeomorphically embedded into

$$c_0(T) = \left\{x \in \prod_{t\in T} I_t : \text{ the set } \{t : |x(t)| > \epsilon\} \text{ is finite for every } \epsilon > 0\right\},$$

where I_t denotes a copy of the closed united interval $[0, 1]$, for each $t \in T$.

A compact space K is a scattered Eberlein compact space if and only if K can be homeomorphically embedded into

$$\sigma(T) = \left\{x \in \prod_{t\in T} D_t : \text{ the set } \{t : x(t) \neq 0\} \text{ is finite}\right\},$$

where D_t denotes a copy of the two-points set $\{0, 1\}$, for each $t \in T$.

More generally, let $\{M_t : t \in T\}$ be any family of topological spaces, and $M = \prod_{t\in T} M_t$ be a topological product. For any fixed point $a \in M$ we denote the σ-product

$$\sigma(M, a) = \{x \in M : \text{ the set } \{t \in T : x(t) \neq a(t)\} \text{ is finite}\}.$$

Denote also $\sigma_n(M, a) = \{x \in M : |\{t \in T : x(t) \neq a(t)\}| \leq n\}$, for every $n \in \omega$. Evidently, $\sigma(M, a) = \bigcup\{\sigma_n(M, a) : n \in \omega\}$.

The rest of our notation is standard and follows the book [5]. All relevant information on cardinal invariants can be found in the paper of Hodel [8].

3 Compactness-like properties in Δ-spaces

We will show that pseudocompact Δ-spaces of countable tightness are scattered and monotonically normal Δ-spaces must be hereditarily paracompact.

Theorem 3.1 *If X is a pseudocompact Δ-space, then every countable subspace of X is scattered.*

Proof If some countable subset of X is not scattered, then there is a countable crowded set $A \subset X$; let $\{a_n : n \in \omega\}$ be a faithful enumeration of A. By the Δ-property of X, there exists a point-finite open expansion $\mathcal{U} = \{U_n : n \in \omega\}$ of the set A. Let $k_0 = 0$ and pick a set $V_0 \in \tau(a_{k_0}, X)$ such that $\overline{V}_0 \subset U_{k_0}$.

Proceeding by induction, assume that, for some $n \in \omega$, we have open sets $V_0, \ldots, V_n$ in the space X together with $k_0, \ldots, k_n \in \omega$ with the following properties:

(1) $\overline{V}_{i+1} \subset V_i$ and $k_i < k_{i+1}$ whenever $0 \leq i < n$;

(2) $a_{k_i} \in V_i \subset \overline{V}_i \subset U_{k_0} \cap \ldots \cap U_{k_i}$ for every $i \leq n$.

The set $V_n \cap A$ being infinite, we can find a number $k_{n+1} \in \omega$ such that $k_{n+1} > k_n$ and $a_{k_{n+1}} \in V_n$. There exists a set $V_{n+1} \in \tau(a_{k_{n+1}}, X)$ such that $\overline{V}_{n+1} \subset V_n \cap U_{k_{n+1}}$. It is straightforward that (1) and (2) now hold if replace n with $n + 1$ so our inductive procedure can be continued to construct a family $\{V_n : n \in \omega\}$ and a sequence $\{k_n : n \in \omega\}$ such that the conditions (1) and (2) are satisfied for each $n \in \omega$.

The property (1), together with pseudocompactness of the space X implies that $F = \bigcap_{n\in\omega} V_n \neq \emptyset$; take a point $x \in F$. The property (2) shows that $x \in U_{k_n}$ for every $n \in \omega$ and hence the family $\mathcal{U}$ is not point-finite. This contradiction proves that every countable subset of X is scattered. □

It is conjectured in [10, Problem 3.21] that every pseudocompact Δ-space X is scattered. We will show that this is true if X has countable tightness.

Corollary 3.2 *Any pseudocompact Δ-space of countable tightness must be scattered.*

Proof Let X be a pseudocompact Δ-space with $t(X) \leq \omega$. If $A \subset X$ and $x \in \overline{A}$, then take a countable $B \subset A$ such that $x \in \overline{B}$. The set B is scattered by Theorem 3.1 so there is a discrete set $D \subset B$ such that $B \subset \overline{D}$ and, in particular, $x \in \overline{D}$. This shows that

(3) if $A \subset X$ and $x \in \overline{A}$, then there exists a countable discrete set $D \subset A$ such that $x \in \overline{D}$.

If X is not scattered, then fix a crowded subset $Y \subset X$. It is standard that there exists a countably infinite discrete subset $D_0 \subset Y$. Proceeding by induction, assume that $n \in \omega$ and we have disjoint countable discrete subsets $D_0, \ldots, D_n$ in the space Y such that

(4) $D_0 \cup \ldots \cup D_i \subset \overline{D}_{i+1}$ for any $i < n$.

The set $D' = D_0 \cup \ldots \cup D_n$ is nowhere dense in Y so $Y \setminus D'$ is dense in Y. Let $\{O_x : x \in D_n\}$ be a disjoint open expansion of D_n in the space Y. The property (3) shows that there exists a countable discrete set $E_x \subset O_x \setminus D'$ such that $x \in \overline{E}_x$ for every $x \in D_n$. The set $D_{n+1} = \bigcup\{E_x : x \in D_n\}$ is discrete, disjoint from D' and

$D_n \subset \overline{D}_{n+1}$. This shows that our inductive procedure can be continued to construct a disjoint family $\{D_n : n \in \omega\}$ of countably infinite discrete subsets of Y such that the condition (4) is satisfied for all $n \in \omega$. The same condition (4) easily implies that $D = \bigcup\{D_n : n \in \omega\}$ is a countable crowded subspace of X; this contradiction with Theorem 3.1 proves that X is scattered. □

Proposition 2.3 of [9] states that the Δ-property is preserved if we add a finite set to a Δ-space. However, the same conclusion cannot be made if we add a countable set to a space because Example 59 of the paper [6] shows that Michael line is not a Δ-space and hence the Δ-property can be destroyed by adding a countable set to a discrete space. Our next example shows that even the Δ-property of a pseudocompact space can be lost if we add a countable set to the space.

Example 3.3 Recall that M is a Mrowka space if it is pseudocompact, the set I of isolated points of M is countable and dense in M and, additionally, $D = M \setminus I$ is closed, discrete and uncountable. Let M be a Mrowka space for which there is a continuous onto map $f : M \to \mathbb{I}$ (see [16, Fact 2 of S.154]). If $K = \beta M$, then there exists a countable crowded set $A \subset K \setminus M$ and hence $X = M \cup A$ is not a Δ-space. Since M is a Δ-space by Corollary 3.9 of [9], it is possible to destroy the Δ-property of a pseudocompact space by adding a countable set.

Proof Take a continuous map $g : K \to \mathbb{I}$ such that $g|M = f$; it is clear that $g(K) = \mathbb{I}$ and hence K is not scattered by Problem 129 of [17]. Take a closed crowded subspace $Z \subset K$. Then $Z \subset K \setminus I$ because every point of I is isolated in K. The subspace $D \cap Z$ is discrete and hence nowhere dense in Z so we can find a crowded compact set $Z_0 \subset Z \setminus D \subset K \setminus M$. It is standard that every compact crowded space contains a countable crowded space so pick a countable set $A \subset Z_0$ which is dense in itself. We already saw that M is a Δ-space so all is left is to note that the space $X = M \cup A$ does not have the Δ-property by Theorem 3.1. □

Our next step is to show that there are many situations where adding a countable set to a Δ-space preserves the Δ-property.

Proposition 3.4 *Given a space X, assume that Y is a Δ-subspace of X, the set $A = X \setminus Y$ is countable and has a point-finite open expansion in X. Then X is a Δ-space.*

Proof If the set A is finite, then X is a Δ-space by Proposition 2.3 of [9] so we can assume that A is infinite; let $\{a_n : n \in \omega\}$ be a faithful enumeration of the set A. By our hypothesis, there exists a point-finite expansion $\{O_n : n \in \omega\}$ of the set A.

Take any disjoint collection $\mathcal{H} = \{X_n : n \in \omega\}$ of subsets of X and apply the Δ-property of Y to find a point-finite open expansion $\{U'_n : n \in \omega\}$ of the family $\mathcal{G} = \{X_n \cap Y : n \in \omega\}$ in the space Y. Pick a set $U_n \in \tau(X)$ such that $U_n \cap Y = U'_n$ and consider the set $V_n = U_n \setminus \{a_0, \ldots, a_n\}$ for every $n \in \omega$. It is immediate that the family $\{V_n : n \in \omega\}$ is a point-finite open expansion of $\mathcal{G}$ in the space X.

Given any $n \in \omega$, let $W_n = \bigcup\{O_i : a_i \in X_n\}$; we omit a straightforward proof of the fact that $\{W_n : n \in \omega\}$ is a point-finite open expansion of the family $\{X_n \cap A : n \in \omega\}$. As a consequence, $\{V_n \cup W_n : n \in \omega\}$ is a point-finite open expansion of $\mathcal{H}$ so X is a Δ-space. □

Corollary 3.5 *Suppose that Y is a Δ-subspace of a space X such that $X\setminus Y$ is countable and scattered. Then X is a Δ-space.*

Proof By [19, Theorem 3.1], the set $X\setminus Y$ has a point-finite open expansion in X; Proposition 3.4 does the rest. □

Proposition 3.6 *If X is a countably compact Δ-space and $f : X \to Y$ is a continuous onto map of X onto a sequential space Y, then Y is a Δ-space.*

Proof It is well known that any continuous map of a countably compact space onto a sequential space is closed so f is closed and hence Y is a Δ-space by Theorem 2.1 of [10]. □

Corollary 3.7 *If X is a countably compact Δ-space and $f : X \to M$ is a continuous onto map of X onto a second countable space M, then M is countable.*

Proof Just note that M is a second countable compact space which has the Δ-property by Proposition 3.6. Therefore M is countable by Proposition 3.5 of [9]. □

There is an important case when a countable complement of a Δ-subspace is scattered automatically.

Theorem 3.8 *If X is a countably compact space and Y is a Δ-subspace of X such that $A = X\setminus Y$ is countable, then X is a Δ-space.*

Proof Assume that the space X is not scattered. Then there is a continuous onto map $f : X \to \mathbb{I}$ (see Problem 133 of [17]). Since $f(A) \subset \mathbb{I}$ is countable, we can find an uncountable compact set $K \subset \mathbb{I}\setminus f(A)$. Then $L = f^{-1}(K)$ is a closed subset of X contained in Y so L is a countably compact Δ-space that maps continuously onto K which is impossible by Corollary 3.7; this contradiction shows that X must be scattered and hence so is A. Finally, apply Corollary 3.5 to conclude that X is a Δ-space. □

It was proved in [11] that there exists a compact space X that fails to be a Δ-space, but there is a discrete uncountable subset $Y \subset X$ such that $A = X\setminus Y$ is a scattered Eberlein compact. So, in the above Corollary 3.5 one cannot assume that $A = X\setminus Y$ is a compact Δ-space.

Proposition 2.12 of the paper [10] states that the Δ-property is preserved by inverse images of continuous maps with finite fibers. It turns out that if the respective map is open, then the Δ-property is preserved in both directions.

Proposition 3.9 *Given spaces X and Y, let $f : X \to Y$ be a continuous open onto map with finite fibers. Then X is a Δ-space if and only if so is Y.*

Proof If Y is a Δ-space, then so is X by Proposition 2.12 of [10]; here we don't even need the map f to be open.

Now, if X is a Δ-space, then take any disjoint family $\mathcal{F} = \{P_n : n \in \omega\}$ of subsets of Y. Then $\{f^{-1}(P_n) : n \in \omega\}$ is a disjoint family of subsets of X so it has a point-finite open expansion $\{U_n : n \in \omega\}$. Then $\mathcal{G} = \{f(U_n) : n \in \omega\}$ is an open expansion of the family $\mathcal{F}$ and it is standard to deduce that $\mathcal{G}$ is point-finite from the fact that $f^{-1}(y)$ is finite for every $y \in Y$. □

It was proved in [10, Proposition 2.10] that the unions of σ-locally finite families of closed Δ-subspaces have the Δ-property. The result that follows shows when the Δ-property is preserved by the unions of families of open sets.

Corollary 3.10 *If a space X has a point-finite open cover $\mathcal{U}$ such that every $U \in \mathcal{U}$ is a Δ-space, then X is a Δ-space.*

Proof Let $Z = \bigoplus\{U : U \in \mathcal{U}\}$; if $x \in U \in \mathcal{U}$, then, letting $\varphi(x) = x \in X$, we obtain a continuous open surjective map $\varphi : Z \to X$. The family $\mathcal{U}$ being point-finite, every fiber of the map φ is finite. It is an easy exercise that direct sums preserve the Δ-property so Z is a Δ-space and hence so is the space X by Proposition 3.9. □

Theorem 3.11 *Assume that M_t is a space of countable pseudocharacter for every $t \in T$ and $a \in M = \prod_{t\in T} M_t$. Then every Lindelöf scattered subspace X of the σ-product $\sigma(M, a)$ is the union of countably many scattered Eberlein compact subspaces and, in particular, X has the Δ-property.*

Proof Let $p_t : M \to M_t$ be the projection map for every $t \in T$ and denote by X_ω the set X with the topology generated by all G_δ-subsets of the space X. Then X_ω is a Lindelöf P-space (see Problem 128 of [17]) and X is a continuous image of X_ω. Therefore $p_t(X)$ is also a continuous image of X_ω. The Lindelöf P-property of X_ω together with countable pseudocharacter of $p_t(X)$ imply that $p_t(X)$ is countable and hence we can assume, without loss of generality, that M_t is countable for each $t \in T$.

As we have noted in Sect. 2, $\sigma(M, a) = \bigcup\{\sigma_n(M, a) : n \in \omega\}$. Hence, it suffices to show that every $X \cap \sigma_n(M, a)$ is the union of countably many Eberlein compact subspaces. Observe that if every space M_t is a finite space with the same size $m \in \omega$, then every $\sigma_n(M, a)$ is a scattered Eberlein compact space (see, for example, [2]). Now choose an enumeration $\{q_n^t : n \in \omega\}$ of the set M_t such that $a(t) = q_0^t$ and let $M_t^n = \{q_i^t : i \le n\}$ for every $t \in T$. If $Q_n = \prod_{t\in T} M_t^n$, then $a \in Q_n$ for each $n \in \omega$ and $\sigma(M, a) = \bigcup\{\sigma(Q_k, a) : k \in \omega\}$.

All finite sets M_t^n have the same size, this fact implies that every $\sigma(Q_k, a) = \bigcup\{\sigma_n(Q_k, a) : n \in \omega\}$ is represenatble as the union of countably many Eberlein compact subspaces.

Write $\sigma(Q_n, a) = \bigcup\{K_m : m \in \omega\}$, where each K_m is a scattered Eberlein compact space. Theorem 3.7 of [18] implies that $X \cap K_m$ is σ-compact for every $m \in \omega$ and therefore $X_n = X \cap \sigma(Q_n, a)$ is σ-compact as well for every $n \in \omega$. Thus, $X = \bigcup_{n\in\omega} X_n$ is the union of countably many scattered Eberlein compact subspaces. Finally, recalling that every scattered Eberlein compact space has the Δ-property by Theorem 49 of [6] and Theorem 2.1 of [9], we conclude that X is a Δ-space by Proposition 2.2 of [10]. □

Corollary 3.12 *Suppose that X is a scattered compact space. If X embeds in a σ-product of spaces of countable pseudocharacter, then X is a Δ-space.*

Corollary 3.13 *Suppose that X is a scattered Lindelöf space. If X embeds in a σ-product of real lines, then X is a Δ-space.*

It was proved in [9] that the space $\omega_1 + 1$ does not have the Δ-property. This result was strengthened in [11] where it was established that a set $X \subset \omega_1$ is a Δ-space if and only if it is not stationary. Our next group of results describe the behavior of the Δ-property in a more general context.

Proposition 3.14 *If $S \subset \kappa$ is a stationary subset of an uncountable regular cardinal κ, then S is not a Δ-space.*

Proof It follows from Theorem 9 of [15] that there exists a disjoint partition $\mathcal{F} = \{S_n : n \in \omega\}$ of the set S such that every S_n is also stationary. If $\{U_n : n \in \omega\}$ is an open expansion of $\mathcal{F}$ in the space κ, then $|\kappa \backslash U_n| < \kappa$ for every $n \in \omega$ so there is $\alpha < \kappa$ such that $[\alpha, \kappa) \subset \bigcap_{n\in\omega} U_n$ and therefore $S \cap \bigcap_{n\in\omega} U_n \neq \emptyset$ which in turn implies that the partition $\mathcal{F}$ has no point-finite open expansion in the space S. □

Corollary 3.15 *If X is a monotonically normal Δ-space, then X must be hereditarily paracompact.*

Proof Assume that a subspace $Y \subset X$ is not paracompact. Since Y is also monotonically normal, we can apply Theorem 4.5 of [3] to see that there is a closed subset F in the space Y which is homeomorphic to a stationary set in an uncountable regular cardinal. Then F is a Δ-space which is a contradiction with Proposition 3.14. □

Corollary 3.16 *If X is a pseudocompact monotonically normal Δ-space, then X is a compact Fréchet–Urysohn space and $\overline{A}$ is countable whenever A is a countable subset of X.*

Proof The space X must be compact because it is paracompact according to Corollary 3.15. If $A \subset X$ is countable, then $\overline{A}$ is hereditarily Lindelöf being a separable monotonically normal space (see Theorem A of [7]). This, together with compactness of $\overline{A}$, shows that $\chi(\overline{A}) \leq \omega$. Recalling that $\overline{A}$ is also a Δ-space, we conclude that $\overline{A}$ is countable (see Proposition 3.5 of [9]).

As X is a compact Δ-space, it has countable tightness by a result obtained in [10]. If $A \subset X$ and $x \in \overline{A}$, then there is a countable set $B \subset A$ such that $x \in \overline{B}$. But $\chi(\overline{B}) \leq \omega$ so there exists a sequence $S = \{a_n : n \in \omega\} \subset B$ that converges to x. Therefore S witnesses the Fréchet-Urysohn property of X. □

It is worth noting that the converse of Corollary 3.15 is not true even for linearly ordered spaces: the real line $\mathbb{R}$ is a counterexample. We will show that, if a topology of a space X is generated by a well-order, then the Δ-property of X is equivalent to it hereditary paracompactness.

Theorem 3.17 *If X is a subspace of an ordinal with its order topology, then X has the Δ-property if and only if it is hereditarily paracompact.*

Proof Since necessity is an immediate consequence of Corollary 3.15, assume that the space X is hereditarily paracompact. Our proof will be by induction on the order type $ot(X)$ of the ordered set X. Observe that any countable space has the Δ-property so there is nothing to prove if $ot(X) < \omega_1$. Now, assume that $ot(X) = \beta$ and any subspace Y of an ordinal has the Δ-property whenever $ot(Y) < \beta$. Given any $x \in X$ let $L_x = \{y \in X : y \leq x\}$. It is easy to see that L_x is an open neighborhood of the point x and if $Y = \{x \in X : ot(L_x) < ot(X)\}$, then $X \backslash Y$ contains at most one point.

Thus, $\mathcal{L} = \{L_x : x \in Y\}$ is a cover of Y by open subsets with the Δ-property. By paracompactness of Y, there exists a locally finite open refinement $\mathcal{U}$ of the cover $\mathcal{L}$. Since every element of $\mathcal{U}$ has the Δ-property, Corollary 3.10 is applicable to conclude that Y is a Δ-space. Finally, observe that $|X\backslash Y| \leq 1$ so X is a Δ-space by Proposition 2.3 of [9]. □

Corollary 3.18 *If X is a pseudocompact GO space with the Δ-property, then X is countable.*

Proof Observe that every GO space is monotonically normal so X is compact and $t(X) \leq \omega$ by Corollary 3.16. Since tightness and character coincide in GO spaces (see Theorem 1.3.1 of [1]), we conclude that $\chi(X) \leq \omega$ and hence X is countable by Proposition 3.5 of [9]. □

Any compact Δ-space must have countable tightness [10]. It is still an open question whether compactness can be replaced with pseudocompactness in this result. However, the situation is quite different if we consider σ-compact Δ-spaces.

Example 3.19 For any uncountable cardinal κ, there exists a σ-compact Δ-space X such that $t(X) = \kappa$.

Proof Let $S = \{x \in \mathbb{D}^\kappa : |x^{-1}(1)| < \omega\}$ be the σ-product in $\mathbb{D}^\kappa$. It is a consequence of Corollary 2.5 of [10] that S is a σ-compact Δ-space. Let $u(\alpha) = 1$ for all $\alpha < \kappa$. Then $X = S \cup \{u\}$ is a σ-compact Δ-space by Proposition 2.3 of [9]. It is easy to see that the set S and the point u witness that $t(X) = \kappa$. □

The following fact was established in [11].

Proposition 3.20 *Let X be a crowded Baire space. If X is ω-resolvable, then it does not have the Δ-property.*

Corollary 3.21 *Let X be a pseudocompact Δ-space. If $c(X) = \omega$, then X is separable and has a dense set of isolated points.*

Proof Let D be the set of isolated points of X; if D is not dense in X, then take a non-empty open set $U \subset X\backslash D$. The space X is $\mathfrak{c}$-resolvable and hence ω-resolvable—this was proved in [12]. As an immediate consequence, the set U is ω-resolvable as well. Since U is a crowded Baire space, it does not have the Δ-property by Proposition 3.20. However, the Δ-property is hereditary so we have obtained a contradiction. Therefore D is dense in X and it follows from the Souslin property of X that D is countable so the space X is separable. □

Our last result was originally obtained as an auxiliary step in the proof that any pseudocompact Δ-space of countable tightness is scattered. After we discovered a different proof for a stronger fact, the theorem below is not needed to study the Δ-property but we decided to keep it here anyway because it seems to be interesting in itself.

Theorem 3.22 *Any pseudocompact crowded space of countable tightness must be ω-resolvable.*

Proof Let X be a pseudocompact crowded space with $t(X) \leq \omega$. It is standard to see that X has no countable open sets and hence
(5) if $U \in \tau^*(X)$ and $A \subset U$ is countable, then $U \setminus A$ is dense in U.

Given any $U \in \tau^*(X)$ take a countably infinite set $D_0 \subset U$. Using the property (5) and countable tightness of X, it is easy to construct by induction a sequence $\{D_n : n \in \omega\}$ of disjoint countable subsets of U such that
(6) $D_0 \cup \ldots \cup D_n \subset \overline{D}_{n+1}$ for any $n \in \omega$.

Let $D = \bigcup_{n \in \omega} D_n$ and choose a disjoint family $\{A_n : n \in \omega\}$ of infinite subsets of ω; it is an easy consequence of (6) that $E_n = \bigcup_{i \in A_n} D_n$ is dense in D for any $n \in \omega$. Therefore
(7) every non-empty open subset of X contains an ω-resolvable subspace.

Take a maximal disjoint subfamily $\mathcal{F}$ of the family of all ω-resolvable subsets of X. It is an immediate consequence of (7) that $Y = \bigcup \mathcal{F}$ is dense in X. It is standard that Y is ω-resolvable and hence X is ω-resolvable as well. □

4 Open questions

The Δ-property is a classical notion that has important applications both in descriptive set theory and functional analysis. The study of Δ-property in general topological spaces is comparatively recent so there are still numerous interesting open questions about its behavior as can be seen for the list below.

Question 4.1 *Suppose that X is a pseudocompact Δ-space. Is it true that $t(X) \leq \omega$?*

Dow and Vaughan showed in [4] that for every ordinal $\gamma < \mathfrak{t}^+$ (where $\mathfrak{t}$ is the *tower number*), there is a maximal almost disjoint family $\mathcal{A}$ on ω such that the Stone-Čech remainder of the corresponding Mrowka Ψ-space $X = \omega \cup \mathcal{A}$ is homeomorphic to $\gamma + 1$ with the order topology. Let p be the last point in $\gamma + 1$, where $\gamma = \omega_1$. We do not know if the space $X \cup \{p\}$, with the topology inherited from βX, has uncountable tightness at the point p. If this is true, then $X \cup \{p\}$ would be a counterexample to Question 4.1.

Question 4.2 *Is it true in ZFC that every countably compact Δ-space has countable tightness?*

Question 4.3 *Suppose that X is a compact Δ-space. Is it true in ZFC that X is sequential?*

Question 4.4 *Suppose that X is a separable pseudocompact Δ-space. Must X be scattered?*

Question 4.5 *Assume that X is a Lindelöf scattered subspace of an Eberlein compact space. Must X be a Δ-space?*

Question 4.6 *Is it true that every compact Δ-space is the countable union of Eberlein compact spaces?*

Question 4.7 *Is it true in ZFC that every countably compact Δ-space must be compact?*

Question 4.8 *Suppose that X is a countably compact Δ-space. Is it true that $X \times X$ is countably compact?*

Question 4.9 *Suppose that X is a countably compact Δ-space. Is it true that every continuous image of X is a Δ-space?*

Question 4.10 *Suppose that X is a space and there is a family $\{U_n : n \in \omega\}$ of open Δ-subspaces of X such that $X = \bigcup_{n\in\omega} U_n$. Is it true that X must be a Δ-space?*

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