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Generalized harmonic functions and Schwarz lemma for biharmonic mappings

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Abstract

In this paper, we establish some Schwarz type lemmas for mappings Φ satisfying the inhomogeneous biharmonic Dirichlet problem $\Delta(\Delta(\Phi)) = g$ in $\mathbb{D}$, $\Phi = f$ on $\mathbb{T}$ and $\partial_n \Phi = h$ on $\mathbb{T}$, where g is a continuous function on $\overline{\mathbb{D}}$, f, h are continuous functions on $\mathbb{T}$, where $\mathbb{D}$ is the unit disc of the complex plane $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$ is the unit circle. To reach our aim, we start by investigating some properties of generalized harmonic functions called T_α-harmonic functions. Finally, we prove a Landau-type theorem for this class of functions, when $\alpha > 0$.

Keywords Schwarz's lemma · Boundary Schwarz's lemma · Landau theorem · Biharmonic equations · T_α-harmonic mappings

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1 Preliminaries and main results

Let $\mathbb{C}$ denote the complex plane and $\mathbb{D}$ the open unit disk in $\mathbb{C}$. Let $\mathbb{T} = \partial\mathbb{D}$ be the boundary of $\mathbb{D}$, and $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$, the closure of $\mathbb{D}$. Furthermore, we denote by $\mathcal{C}^m(\Omega)$ the set of all complex-valued $m-$times continuously differentiable functions from Ω into $\mathbb{C}$, where Ω stands for a domain of $\mathbb{C}$ and $m \in \mathbb{N}$. In particular, $\mathcal{C}(\Omega) := \mathcal{C}^0(\Omega)$ denotes the set of all continuous functions in Ω.

For a real 2×2 matrix A, we use the matrix norm

$$\|A\| = \sup\{|Az| : |z| = 1\},$$

and the matrix function

$$\lambda(A) = \inf\{|Az| : |z| = 1\}.$$

For $z = x + iy \in \mathbb{C}$, the formal derivative of a complex-valued function $\Phi = u + iv$ is given by

$$D_\Phi = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

so that

$$\|D_\Phi\| = |\Phi_z| + |\Phi_{\overline{z}}| \quad \text{and} \quad \lambda(D_\Phi) = \big||\Phi_z| - |\Phi_{\overline{z}}|\big|,$$

where

$$\Phi_z = \frac{1}{2}(\Phi_x - i\Phi_y) \quad \text{and} \quad \Phi_{\overline{z}} = \frac{1}{2}(\Phi_x + i\Phi_y).$$

We use

$$J_\Phi := \det D_\Phi = |\Phi_z|^2 - |\Phi_{\overline{z}}|^2.$$

The main objective of this paper is to establish a Schwarz-type lemma for the solutions to the following inhomogeneous biharmonic Dirichlet problem (briefly, IBDP):

$$\begin{cases} \Delta^2\Phi = g & \text{in } \mathbb{D}, \\ \Phi = f & \text{on } \mathbb{T}, \\ \partial_n\Phi = h & \text{on } \mathbb{T}. \end{cases} \tag{1.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

denotes the standard Laplacian and ∂_n denotes the differentiation in the inward normal direction, $g \in \mathcal{C}(\overline{\mathbb{D}})$ and the boundary data f and $h \in \mathcal{C}(\mathbb{T})$.

We would like to mention that in [13,14] the authors have considered similar inhomogeneous biharmonic equations but with different boundaries conditions.

In order to state our main results, we introduce some necessary terminologies. For $z, w \in \mathbb{D}$, let

$$G(z, w) = |z - w|^2 \log \left| \frac{1 - z\overline{w}}{z - w} \right|^2 - (1 - |z|^2)(1 - |w|^2),$$

and

$$P(z) = \frac{1 - |z|^2}{|1 - z|^2},$$

denote the biharmonic Green function and the harmonic Poisson kernel, respectively. For $\varphi \in L^1(\mathbb{T})$, we denote by $P[\varphi]$ the Poisson extension of φ, defined on $\mathbb{D}$ by

$$P[\varphi](z) = \frac{1}{2\pi} \int_0^{2\pi} P(ze^{-i\theta})\varphi(e^{i\theta})d\theta.$$

Riesz representation of (super-)biharmonic functions started with Abkar and Hedenmalm [2]. By [26, Theorem 1.1], we see that all solutions of IBDP (1.1) are given by

$$\Phi(z) = F_0[f](z) + H_0[h](z) - G[g](z),$$

where

$$F_0[f](z) = \frac{1}{2\pi} \int_0^{2\pi} F_0(ze^{-i\theta}) f(e^{i\theta})d\theta, \quad H_0[h](z) = \frac{1}{2\pi} \int_0^{2\pi} H_0(ze^{-i\theta})h(e^{i\theta})d\theta,$$
$$\text{and} \quad G[g](z) = \frac{1}{16} \int_{\mathbb{D}} G(z, \omega)g(\omega)dA(\omega),$$

where $dA(\omega)$ denotes the Lebesgue area measure in $\mathbb{D}$. Here the kernels H_0 and F_0 are given by

$$F_0(z) = H_0(z) + K_2(z),$$
$$H_0(z) = \frac{1}{2}(1 - |z|^2)P(z),$$
$$K_2(z) = \frac{1}{2} \frac{(1 - |z|^2)^3}{|1 - z|^4}.$$

Thus, the solutions of the equation (1.1) are given by

$$\Phi(z) = \frac{1}{2}(1 - |z|^2)P[f + h](z) + K_2[f](z) - G[g](z).$$

Obviously $P[f+h]$ is a bounded harmonic function, and Heinz [19] proved the Schwarz lemma for planar harmonic functions: if Φ is a harmonic mapping from $\mathbb{D}$ into itself with $\Phi(0)=0$, then for $z\in\mathbb{D}$,

$$|\Phi(z)| \le \frac{4}{\pi}\arctan|z|.$$

Hethcote [20] and Pavlović [34, Theorem 3.6.1] improved Heinz's result, by removing the assumption $\Phi(0)=0$, and proved the following.

Theorem A *Let $\Phi:\mathbb{D}\to\mathbb{D}$ be a harmonic function from the unit disc to itself, then*

$$\left|\Phi(z) - \frac{1-|z|^2}{1+|z|^2}\Phi(0)\right| \le \frac{4}{\pi}\arctan|z|, \qquad z\in\mathbb{D}. \tag{1.2}$$

A higher dimensional version for harmonic functions is proved in [21].

We remark that $K_2[f]$ is a bounded T_2-harmonic which is a special type of biharmonic functions. So naturally our first aim is to study the class of T_α-harmonic functions [31]. These functions can be seen as generalized harmonic functions as T_0-harmonic functions coincide with classical harmonic functions. Other variants of generalized (or weighted) harmonic functions and their properties can be found in [32,33].

First, let us recall the definition of T_α-harmonic functions.

Definition 1 [31] Let $\alpha\in\mathbb{R}$, and let $f\in\mathcal{C}^2(\mathbb{D})$. We say that f is T_α-harmonic if f satisfies

$$T_\alpha(f)=0 \quad \text{in } \mathbb{D},$$

where the T_α-Laplacian operator is defined by

$$T_\alpha = -\frac{\alpha^2}{4}(1-|z|^2)^{-(\alpha+1)} + \frac{1}{2}L_\alpha + \frac{1}{2}\overline{L_\alpha},$$

with the weighted Laplacian operator L_α is defined by

$$L_\alpha = \frac{\partial}{\partial\overline{z}}(1-|z|^2)^{-\alpha}\frac{\partial}{\partial z}.$$

Remark 1.1 Let f be a T_α-harmonic function.

(1) If $\alpha=0$, then f is harmonic.
(2) If $\alpha=2n$, then f is $(n+1)$-harmonic, where $n\in\mathbb{N}$, see [1,5,31,32].

The homogeneous expansion of T_α-harmonic functions is giving by

Theorem B [31] *Let $\alpha \in \mathbb{R}$ and $f \in \mathcal{C}^2(\mathbb{D})$. Then f is T_α-harmonic if and only if it has a series expansion of the form*

$$f(z)=\sum_{k=0}^{\infty} c_k F(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; |z|^2) z^k+\sum_{k=1}^{\infty} c_{-k} F(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; |z|^2) \bar{z}^k, \tag{1.3}$$

for some sequence $\{c_k\}$ of complex numbers satisfying $\limsup_{|k|\to\infty} |c_k|^{\frac{1}{|k|}} \le 1$, *where F is the Gauss hypergeometric function.*

For $\alpha > -1$, a Poisson type integral representation for T_α-harmonic mappings is provided by the following theorem.

Theorem C ([31] Theorem 3.3) *Let $\alpha > -1$ and u be a T_α-harmonic in $\mathbb{D}$. Assume that $\lim_{r\to 1} u_r = u^*$ in $\mathcal{D}'(\mathbb{T})$. Then u has a form of a Poisson type integral*

$$u(z) = K_\alpha[u^*](z) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(z, e^{i\theta}) u^*(e^{i\theta})\, d\theta.$$

The integral is understood in the sense of distribution theory and

$$K_\alpha(z, e^{i\theta}) = c_\alpha \frac{(1-|z|^2)^{\alpha+1}}{|z-e^{i\theta}|^{\alpha+2}}, \quad c_\alpha = \frac{\Gamma(\alpha/2+1)^2}{\Gamma(\alpha+1)}.$$

The factor of normalization c_α is chosen in order to ensure that the integral means

$$M_\alpha(r) = \frac{1}{2\pi} \int_{\mathbb{T}} K_\alpha(r, e^{i\theta}) d\theta, \quad r \in [0, 1)$$

satisfies

$$\lim_{r\to 1} M_\alpha(r) = 1.$$

Moreover, the function M_α is increasing on $[0, 1)$, see [31, Theorem 3.1].

It is well known that the Schwarz lemma is one of the most influential results in many branches of mathematical research for more than a hundred years. We refer the reader to [6,13,22,29,30] for generalizations and applications of this lemma.

Define

$$U_\alpha(z) = K_\alpha[\chi_{\mathbb{T}^r} - \chi_{\mathbb{T}^l}](z), \tag{1.4}$$

where

$$\mathbb{T}^r = \{z \in \mathbb{T} : \operatorname{Re} z > 0\}, \text{ and } \mathbb{T}^l = \{z \in \mathbb{T} : \operatorname{Re} z < 0\}.$$

U_α is a T_α-harmonic function on $\mathbb{D}$ with values in $(-1, 1)$ such that $U_\alpha(0) = 0$.
First, we establish a Heniz-Hethcote theorem for T_α-harmonic functions.

Theorem 1 *Let $\alpha > -1$ and $u : \mathbb{D} \longrightarrow \mathbb{D}$ be a T_α-harmonic function, then*

$$\left| u(z) - \frac{(1-|z|^2)^{\alpha+1}}{(1+|z|^2)^{\frac{\alpha}{2}+1}} u(0) \right| \leq U_\alpha(|z|),$$

for all $z \in \mathbb{D}$, where U_α is the function defined in (1.4).

In particular, for T_2-harmonic functions, we obtain

Corollary 1.1 *Let $u : \mathbb{D} \longrightarrow \mathbb{D}$ be a T_2-harmonic function, then*

$$\left| u(z) - \frac{(1-|z|^2)^3}{(1+|z|^2)^2} u(0) \right| \leq \frac{2}{\pi} \left[\frac{|z|(1-|z|^2)}{1+|z|^2} + (1+|z|^2) \arctan |z| \right].$$

Next, we prove a sharp estimate of $D_u(0)$, where u is a T_α-harmonic function.

Theorem 2 *Let $\alpha > -1$ and $u : \mathbb{D} \longrightarrow \mathbb{D}$ be a T_α-harmonic function, then*

$$\| D_u(0) \| \leq \frac{2c_\alpha}{\pi} (\alpha + 2). \tag{1.5}$$

The inequality (1.5) is sharp and U_α is an extremal function, see (1.4).

Let $\mathcal{A}(\mathbb{D})$ the set of all holomorphic functions Φ in $\mathbb{D}$ satisfying the standard normalization: $\Phi(0) = \Phi'(0) - 1 = 0$. Landau [23] showed that there is a constant $r > 0$, independent of elements in $\mathcal{A}(\mathbb{D})$, such that $\Phi(\mathbb{D})$ contains a disk of radius r. Later, Landau's theorem has become an important tool in geometric function theory. Indeed, many authors considered Landau type theorems for harmonic functions i.e., $\alpha = 0$ (cf. [7–10,12,28]), for biharmonic functions, $\alpha = 2$ (cf. [1,27]) and for polyharmonic functions $\alpha = 2(n-1)$ (see [4,11]), and in [12], the authors considered the case $\alpha \in (-1, 0)$.
Naturally, our next aim is to establish a Landau type theorem for T_α-harmonic functions, for $\alpha > 0$.

Theorem 3 *Let $\alpha > 0$, and $u \in \mathcal{C}^2(\mathbb{D})$ be a T_α-harmonic function satisfying $u(0) = J_u(0) - 1 = 0$ and $\sup_{z\in\mathbb{D}} |u(z)| \leq M$, where $M > 0$ and J_u is the Jacobian of u. Let $n \geq 1$ be an integer such that $n - 1 < \frac{\alpha}{2} \leq n$. Then u is univalent on D_{r_α}, where r_α satisfies the following equation*

$$\frac{2c_\alpha(\alpha+2)}{\pi} M \sigma_\alpha(r_\alpha) = 1. \tag{1.6}$$

Moreover, $u(\mathbb{D}_{r_\alpha})$ contains an univalent disk D_{R_α} with

$$R_\alpha \geq \frac{\sigma_\alpha(r_\alpha) r_\alpha}{2},$$

where

$$\sigma_\alpha(r) := \frac{4Mr}{\pi}\left[\frac{6a_\alpha}{(1-r)^3} + \frac{r}{(1-r)^2} + (\frac{2-\alpha}{4})r\right], \ \ \textit{if}\ n = 1.$$

$$\sigma_\alpha(r) := \frac{4Mr}{\pi}\left[\frac{24a_\alpha}{(1-r)^4} + 3\alpha a_\alpha\left(1 + \frac{4r}{3(1-r)^3}\right)r + 3a_\alpha(\alpha-2)r\right], \ \textit{if}\ n = 2.$$

$$\sigma_\alpha(r) := \frac{4Mr}{\pi}\left[\frac{(n+1)(n-2)}{2}(1+r) + \frac{a_\alpha(2n)!r^{n-2}}{(1-r)^{2n}} + \frac{\alpha a_\alpha(2n-1)!}{n!}\left(1 + \frac{2nr}{(n+1)(1-r)^{n+1}}\right)r^{n-1} + (\frac{\alpha-2}{4})r\right], \ \textit{if}\ n \geq 3.$$

with $a_\alpha = \dfrac{\Gamma(\frac{\alpha}{2}+1)}{\Gamma(\alpha+1)}$.

Remark 1.2 In particular, for $\alpha = 2$, we obtain

$$\sigma_2(r) = \frac{4Mr}{\pi(1-r)^2}\left[\frac{3}{1-r} + r\right]. \tag{1.7}$$

Now we are in the position to prove some results related to the Dirichlet problem (1.1).

Theorem 4 *Let $g \in \mathcal{C}(\overline{\mathbb{D}})$, $f, h \in \mathcal{C}(\mathbb{T})$ and suppose that $\Phi \in \mathcal{C}^4(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ satisfies (1.1). Then for $z \in \mathbb{D}$,*

$$\left|\Phi(z) - \frac{1}{2}\frac{(1-|z|^2)^3}{(1+|z|^2)^2}P[f](0) - \frac{1}{2}\frac{(1-|z|^2)^2}{1+|z|^2}P[f+h](0)\right| \leq \left[\frac{2}{\pi}(1-|z|^2)\arctan|z|\right]\|f+h\|_\infty + \frac{2}{\pi}\left[(1+|z|^2)\arctan|z| + |z|\frac{1-|z|^2}{1+|z|^2}\right]\|f\|_\infty + \frac{(1-|z|^2)^2}{64}\|g\|_\infty, \tag{1.8}$$

where $\|f\|_\infty = \sup_{\zeta\in\mathbb{T}}|f(\zeta)|$, $\|f+h\|_\infty = \sup_{\zeta\in\mathbb{T}}|f(\zeta)+h(\zeta)|$ *and* $\|g\|_\infty = \sup_{\zeta\in\mathbb{D}}|g(\zeta)|$.

Theorem 5 *Let $g \in \mathcal{C}(\overline{\mathbb{D}})$, f and $h \in \mathcal{C}(\mathbb{T})$. Suppose that $\Phi \in \mathcal{C}^4(\mathbb{D})$ is satisfying (1.1). Then for all $z \in \mathbb{D}$,*

$$\|D_\Phi(z)\| \leq \frac{2+5|z|}{1-|z|^2}(1+|z|^2)\|f\|_\infty + \left(\frac{2}{\pi} + |z|\right)\|f+h\|_\infty + \frac{23}{48}\|g\|_\infty. \tag{1.9}$$

Moreover at $z = 0$, we have

$$\|D_\Phi(0)\| \leq \frac{4}{\pi}\|f\|_\infty + \frac{2}{\pi}\|f+h\|_\infty + \frac{23}{48}\|g\|_\infty. \tag{1.10}$$

The classical Schwarz lemma at the boundary is as follows.

Theorem D *Suppose $f : \mathbb{D} \longrightarrow \mathbb{D}$ is a holomorphic function with $f(0) = 0$, and further, f is analytic at $z = 1$ with $f(1) = 1$. Then, the following two conditions hold:*

(a) $f'(1) \geq 1$;

(b) $f'(1) = 1$ *if and only if* $f(z) = z$.

The previous theorem is known as the Schwarz lemma on the boundary, and its generalizations have important applications in geometric theory of functions (see, [18,24,35]). Among the recent papers devoted to this subject, for example, Burns and Krantz [6], Krantz [22], Liu and Tang [29] explored many versions of the Schwarz lemma at the boundary point of holomorphic functions, Dubinin also applied this latter for algebraic polynomials and rational functions (see [16,17]). In the present paper, we refine the Schwarz type lemma at the boundary for Φ satisfies (1.1) as an application of Theorem 4.

Theorem 6 *Suppose that $\Phi \in \mathcal{C}^4(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ satisfies (1.1), where $g \in \mathcal{C}(\overline{\mathbb{D}})$ and f, $h \in \mathcal{C}(\mathbb{T})$ such that $\|f\|_\infty \leq 1$, and $\|f + h\|_\infty \leq 1$. If $\lim_{r\to 1} |\Phi(r\eta)| = 1$ for $\eta \in \mathbb{T}$, then*

$$\liminf_{r\to 1} \frac{|\Phi(\eta) - \Phi(r\eta)|}{1 - r} \geq 1 - \|f + h\|_\infty.$$

In particular if $\|f + h\|_\infty = 0$, then $\liminf_{r\to 1} \frac{|\Phi(\eta)-\Phi(r\eta)|}{1-r} \geq 1$, and this estimate is sharp.

For $g \in \mathcal{C}(\overline{\mathbb{D}})$ and $h \in \mathcal{C}(\mathbb{T})$, let $\mathcal{BF}_{g,h}(\overline{\mathbb{D}})$ denote the class of all complex-valued functions $\Phi \in \mathcal{C}^4(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ satisfying (1.1) with the normalization $\Phi(0) = J_\Phi(0) - 1 = 0$.

We establish the following Landau-type theorem for $\Phi \in \mathcal{BF}_{g,h}(\overline{\mathbb{D}})$. In particular, if $g \equiv 0$, then $\Phi \in \mathcal{BF}_{g,h}(\overline{\mathbb{D}})$ is biharmonic. In this sense, the following result is a generalization of [1, Theorem 1 and 2].

Theorem 7 *Suppose that $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$ are constants, and suppose that $\Phi \in \mathcal{BF}_{g,h}(\overline{\mathbb{D}})$ satisfies the following conditions:*

$$\sup_{z\in\mathbb{T}} |f(z)| \leq M_1, \quad \sup_{z\in\mathbb{T}} |f(z) + h(z)| \leq M_2, \quad \textit{and} \ \sup_{z\in\mathbb{D}} |g(z)| \leq M_3.$$

Then Φ is univalent in $\mathbb{D}_{r_0}$ and $\Phi(\mathbb{D}_{r_0})$ contains a univalent disk $\mathbb{D}_{R_0}$, where r_0 satisfies the following equation:

$$\left(\frac{4}{\pi}M_1 + \frac{2}{\pi}M_2 + \frac{23}{48}M_3\right)\mu(r_0) = 1,$$

with

$$\mu(|z|) := (M_1 + M_2 + \frac{101}{120}M_3)|z| + \frac{2M_2|z|}{\pi}\left[\frac{(2 - |z|)(1 + |z|^2)}{(1 - |z|)^2} + |z|\right] + \frac{4M_1|z|}{\pi(1 - |z|)^3}(|z|^2(1 - |z|) + 3),$$

and

$$R_0 \geq \frac{r_0}{\frac{8}{\pi}M_1 + \frac{4}{\pi}M_2 + \frac{23}{24}M_3}.$$

2 Preliminaries

Here we collect some preliminary facts used in the sequel. The Gauss hypergeometric function is defined by the power series

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad |x| < 1,$$

for $a, b, c \in \mathbb{R}$, with $c \neq 0, -1, -2, \ldots,$ where $(a)_0 = 1$ and $(a)_n = a(a+1)\ldots(a+n-1)$ for $n = 1, 2, \ldots$ are the *Pochhammer* symbols.

We list few properties, see for instance [3, Chapter 2]

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \text{ if } c-a-b > 0. \tag{2.1}$$

$$F(a+1, b+1; c+1; 1) = \frac{c}{c-a-b-1} F(a, b; c; 1), \text{ if } c-a-b > 1. \tag{2.2}$$

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x). \tag{2.3}$$

$$\frac{d}{dx} F(a, b; c; x) = \frac{ab}{c} F(a+1, b+1; c+1, x). \tag{2.4}$$

The following lemma about the monotonicity of hypergeometric functions follows immediately from the properties (2.3) and (2.4).

Lemma 1 [31] *Let* $c > 0$, $a \leq c$, $b \leq c$ *and* $ab \leq 0$ *(*$ab \geq 0$*). Then the function* $F(a, b; c; \bullet)$ *is decreasing (increasing) on* $(0, 1)$.

The following results are useful to establish a Landau theorem for T_α-harmonic functions, when $\alpha > 0$.

Lemma 2 [25, Formula 5.2.2 (9) p. 697] *for* $n \geq 1$ *and* $|x| < 1$

$$\gamma_n(x) := \sum_{k=0}^{\infty} (k+1)(k+2)\ldots(k+n)x^k = \frac{n!}{(1-x)^{n+1}}. \tag{2.5}$$

As a direct application of Lemma 2, it yields

Lemma 3 *For* $r \in (0, 1)$*, and* $n \geq 1$*, define the sequence*

$$S_n(r) = \sum_{k \geq 1} (k+1)(k+2)\ldots(k+n)r^k.$$

Then,

$$S_n(r) = \frac{n!r}{(1-r)^{n+1}} \sum_{k=0}^{n} (1-r)^k.$$

In particular,

$$S_n(r) \leq \frac{n!r}{(1-r)^{n+1}} \Big[(n+1) - nr\Big] \leq \frac{(n+1)!r}{(1-r)^{n+1}}. \tag{2.6}$$

Proposition 2.1 *Let $n \geq 1$ and $n-1 < \frac{\alpha}{2} \leq n$. Then, we have the following two estimates*

$$(a) \sum_{k=n}^{\infty} \frac{\Gamma(k+\frac{\alpha}{2}+1)}{(k-1)!} r^{k-1} \leq r^{n-2} S_{2n-1}(r) \leq \frac{(2n)!r^{n-1}}{(1-r)^{2n}}, \tag{2.7}$$

$$(b) \sum_{k=n}^{\infty} \frac{\Gamma(k+\frac{\alpha}{2}+1)(k-\frac{\alpha}{2})}{(k+1)!} r^{k+1} \leq \frac{(2n)!}{(n+1)!} \frac{r^{n+1}}{(1-r)^{n+1}}, \tag{2.8}$$

for $r \in [0, 1)$.

Proof The inequality (2.7) follows immediately from (2.6).
Now we prove the inequality (2.8). By assumption, we have

$$\begin{aligned}\sum_{k=n}^{\infty} \frac{\Gamma(k+\frac{\alpha}{2}+1)(k-\frac{\alpha}{2})}{(k+1)!} r^{k+1} &\leq \sum_{k=n}^{\infty} \frac{\Gamma(k+n+1)(k-n+1)}{(k+1)!} r^{k+1} \\ &= r^{n+1} \sum_{k=0}^{\infty} (k+1).(k+n+2)(k+n+3)\ldots(k+2n) r^k.\end{aligned}$$

Clearly, for $k \geq 0$ and all $2 \leq j \leq n$, we have

$$k+j+n \leq \frac{j+n}{j}(k+j).$$

Thus

$$\sum_{k=0}^{\infty} (k+1).(k+n+2)(k+n+3)\ldots(k+2n) r^k \leq \frac{\gamma_n(r)}{n!} \frac{(2n)!}{(n+1)!}.$$

Therefore, by Lemma 2, it yields

$$\sum_{k=n}^{\infty} \frac{\Gamma(k+\frac{\alpha}{2}+1)(k-\frac{\alpha}{2})}{(k+1)!} r^{k+1} \leq \frac{(2n)!}{(n+1)!} \frac{r^{n+1}}{(1-r)^{n+1}}.$$

□

3 Schwarz and Landau type lemmas for T_α-harmonic functions

3.1 Schwarz type lemma for T_α-harmonic functions

The main purpose of this section is to prove a Schwarz type lemma for T_α-harmonic functions.

Proof of Theorem 1 Let $0 \le r = |z| < 1$. As u is a T_α-harmonic function, then

$$u(z) = K_\alpha[u^*](z) = \frac{1}{2\pi}\int_0^{2\pi} c_\alpha \frac{(1-r^2)^{\alpha+1}}{|1-ze^{-i\theta}|^{\alpha+2}} u^*(e^{i\theta})d\theta,$$

where $u^* \in L^\infty(\mathbb{T})$. Thus

$$\begin{aligned}\left|u(z) - \frac{(1-r^2)^{\alpha+1}}{(1+r^2)^{\frac{\alpha}{2}+1}}u(0)\right| &\le \frac{c_\alpha}{2\pi}\int_{\mathbb{T}}\left|\frac{(1-r^2)^{\alpha+1}}{(1+r^2-2r\cos\theta)^{\frac{\alpha}{2}+1}} - \frac{(1-r^2)^{\alpha+1}}{(1+r^2)^{\frac{\alpha}{2}+1}}\right|d\theta \\ &= \frac{c_\alpha}{2\pi}\left[\int_{-\pi/2}^{\pi/2}\frac{(1-r^2)^{\alpha+1}}{(1+r^2-2r\cos\theta)^{\frac{\alpha}{2}+1}} - \frac{(1-r^2)^{\alpha+1}}{(1+r^2)^{\frac{\alpha}{2}+1}}\,d\theta\right. \\ &\quad\left. - \int_{\pi/2}^{3\pi/2}\frac{(1-r^2)^{\alpha+1}}{(1+r^2-2r\cos\theta)^{\frac{\alpha}{2}+1}} - \frac{(1-r^2)^{\alpha+1}}{(1+r^2)\frac{\alpha}{2}+1}\,d\theta\right] \\ &= K_\alpha[\chi_{\mathbb{T}^r} - \chi_{\mathbb{T}^l}](|z|).\end{aligned}$$

□

To compute U_2, we need to evaluate the following integral.

$$J(\theta) := \int_0^\theta \frac{(1-r^2)^3}{(1+r^2-2r\cos\varphi)^2}d\varphi.$$

Easy but tedious computations show that

Lemma 4 *For $0 \le \theta < \pi$, and $r \in [0,1)$, we have*

$$J(\theta) := \int_0^\theta \frac{(1-r^2)^3}{(1+r^2-2r\cos\varphi)^2}d\varphi = \frac{2r(1-r^2)\sin\theta}{1+r^2-2r\cos\theta} + 2(1+r^2)\arctan\left(\frac{(1+r)\tan\theta/2}{1-r}\right),$$

and $J(\pi) = \lim_{\theta\to\pi} J(\theta) = \pi(1+r^2)$.

Proof of Corollary 1.1 By Lemma 4, and using the fact that $\arctan(\frac{1+r}{1-r}) - \frac{\pi}{4} = \arctan r$, we have

$$\begin{aligned} U_2(r) &= \frac{1}{2\pi}\Big[2J(\pi/2) - J(\pi)\Big] \\ &= \frac{1}{2\pi}\Big[4\frac{r(1-r^2)}{1+r^2} + 4(1+r^2)\arctan(\frac{1+r}{1-r}) - \pi(1+r^2)\Big] \\ &= \frac{2}{\pi}\Big[\frac{r(1-r^2)}{1+r^2} + (1+r^2)\arctan r\Big]. \end{aligned}$$

□

Proof of Theorem 2 Near 0, we have

$$\frac{(1-r^2)^{\alpha+1}}{(1+r^2-2r\cos\theta)^{\frac{\alpha}{2}+1}} = 1 + (\alpha+2)\cos\theta r + O(r^2),$$

$$U_\alpha(r) = \frac{2c_\alpha}{\pi}(\alpha+2)r + O(r^2) \text{ and } \frac{(1-r^2)^3}{(1+r^2)^2} = 1 + O(r^2).$$

Hence from Theorem 1 and (3.1), we get

$$|u(z) - u(0)| \leq \frac{2c_\alpha}{\pi}(\alpha+2)|z| + O(|z|^2). \tag{3.1}$$

Thus

$$\|D_u(0)\| \leq \frac{2c_\alpha}{\pi}(\alpha+2).$$

To show that the last estimate is sharp. Let us consider the T_α-harmonic mapping defined by

$$U_\alpha(z) = K_\alpha[\chi_{\mathbb{T}^r} - \chi_{\mathbb{T}^l}](z).$$

By [31, Theorem 1.1], we have

$$\frac{\partial}{\partial z}K_\alpha(ze^{-it}) = c_\alpha\frac{(1-|z|^2)^\alpha}{|1-ze^{-it}|^{2+\alpha}}\left(-\frac{\alpha}{2}\overline{z}e^{it} + \frac{2+\alpha}{2}\frac{1-\overline{ze^{-it}}}{1-ze^{-it}}\right)e^{-it}.$$

Hence

$$\frac{\partial}{\partial z}K_\alpha(ze^{-it})_{|z=0} = \frac{c_\alpha}{2}(2+\alpha)e^{-it}.$$

As

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta}(\chi_{\mathbb{T}^r}-\chi_{\mathbb{T}^l})(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-i\theta}d\theta - \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} e^{-i\theta}d\theta = \frac{2}{\pi},$$

we conclude that

$$|\nabla U_\alpha(0)| = 2|\frac{\partial U_\alpha}{\partial z}(0)| = \frac{2c_\alpha}{\pi}(\alpha+2).$$

□

3.2 Proof of Theorem 3

First, we need the following theorem which provides some estimates on the coefficients of T_α-harmonic mappings.

Theorem E [12] *For $\alpha > -1$, let $u \in \mathcal{C}^2(\mathbb{D})$ be a T_α-harmonic function with the series expansion of the form (1.3) and $\sup_{z\in\mathbb{D}}|u(z)| \le M$, where $M > 0$. Then, for $k \in \{1, 2, \ldots\}$,*

$$(|c_k| + |c_{-k}|)\, F(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1) \le \frac{4M}{\pi}, \tag{3.2}$$

and

$$|c_0| F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1) \le M. \tag{3.3}$$

Therefore for $k \ge 1$ and $\alpha > -1$, using (2.1), we have

$$F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1\big) = \frac{k!\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}+1)\Gamma(k+\frac{\alpha}{2}+1)}.$$

Thus if u is T_α-harmonic such that $|u(z)| \le M$, then by (3.2) it yields

$$|c_k| + |c_{-k}| \le \frac{4Ma_\alpha}{\pi}\,\frac{\Gamma(k+\frac{\alpha}{2}+1)}{k!} \quad \text{for } k \ge 1. \tag{3.4}$$

Proof of Theorem 3 Let us compute u_z and $u_{\overline{z}}$, for u is a T_α-harmonic with $\alpha > 0$ and $u(0) = c_0 = 0$. The power series expansion is provided by

$$u(z) = \sum_{k=1}^{\infty} c_k F(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; |z|^2)z^k + \sum_{k=1}^{\infty} c_{-k} F(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; |z|^2)\overline{z}^k,$$

and the series converges for $\mathcal{C}^\infty$-topology. Hence

$$\begin{aligned} u_z(z) &= \sum_{k=2}^{\infty} k c_k F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; \omega\big) z^{k-1} + \sum_{k=1}^{\infty} c_k \frac{d}{d\omega} F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; \omega\big)\overline{z} z^k \\ &\quad + \sum_{k=1}^{\infty} c_{-k} \frac{d}{d\omega} F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; \omega\big)\overline{z}^{k+1} + c_1 F\big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; \omega\big), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} u_{\overline{z}}(z) &= \sum_{k=1}^{\infty} c_k \frac{d}{d\omega} F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; \omega\big) z^{k+1} + \sum_{k=2}^{\infty} c_{-k} k F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; \omega\big)\overline{z}^{k-1} \\ &\quad + \sum_{k=1}^{\infty} c_{-k} \frac{d}{d\omega} F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; \omega\big) z\overline{z}^{k} + c_{-1} F\big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; \omega\big), \end{aligned} \tag{3.6}$$

where $\omega = |z|^2$.

We have $u_z(0) = c_1 F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 0\big) = c_1$, and similarly $u_{\overline{z}}(0) = c_{-1}$. Thus combining (3.5) and (3.6), we obtain

$$\begin{aligned} |u_z(z) - u_z(0)| + |u_{\overline{z}}(z) - u_{\overline{z}}(0)| &\le \sum_{k=2}^{\infty} k(|c_k| + |c_{-k}|) F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; \omega\big)|z|^{k-1} \\ &\quad + 2\sum_{k=1}^{\infty} (|c_k| + |c_{-k}|) \Big|\frac{d}{d\omega} F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; \omega\big)\Big| |z|^{k+1} \\ &\quad + (|c_1| + |c_{-1}|)\Big| F\big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; \omega\big) - 1\Big|. \end{aligned}$$

By (2.4) and (2.3), we see that

$$\Big|\frac{d}{d\omega} F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; \omega\big)\Big| = \frac{\alpha\big|\frac{\alpha}{2} - k\big|}{2(k+1)} F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; \omega\big),$$

as the mapping $F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; \bullet\big)$ is positive. We denote

$$E_\alpha(r) := \sum_{k=2}^{\infty} k(|c_k| + |c_{-k}|) F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; r^2\big) r^{k-1}, \tag{3.7}$$

$$F_\alpha(r) := \sum_{k=1}^{\infty} (|c_k| + |c_{-k}|) \frac{\alpha\big|\frac{\alpha}{2} - k\big|}{(k+1)} F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; r^2\big) r^{k+1}, \tag{3.8}$$

$$G_\alpha(r) := (|c_1| + |c_{-1}|)\Big| F\big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; r^2\big) - 1\Big|. \tag{3.9}$$

In the sequel, we will estimate each of these expressions. □

Estimate of $E_\alpha(r)$

Lemma 5 *Let $n \in \mathbb{N}$, $n \geq 1$ and $\frac{\alpha}{2} \in (n-1, n]$.*
If $n = 1$, then

$$E_\alpha(r) \leq \frac{4Ma_\alpha}{\pi} S_2(r) \leq \frac{24Ma_\alpha r}{\pi(1-r)^3}. \tag{3.10}$$

If $n \geq 2$, then

$$E_\alpha(r) \leq \frac{4M}{\pi}\left[\frac{(n+1)(n-2)}{2} r + \frac{a_\alpha (2n)! r^{n-1}}{(1-r)^{2n}}\right]. \tag{3.11}$$

Proof A straightforward application of Lemma 1 implies that the monotonicity properties of $F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; \bullet\big)$ depends on $\alpha\big(\dfrac{\alpha}{2}-k\big)$. Therefore the function $F(-\dfrac{\alpha}{2}, k-\dfrac{\alpha}{2}; k+1; \bullet)$ is decreasing on $[0, 1)$ when $\alpha \in (0, 4]$ and $k \geq 2$. Thus for $\omega \in [0, 1)$,

$$F\Big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; \omega\Big) \leq F\Big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 0\Big) = 1.$$

First, we estimate $E_\alpha(r)$ for $\alpha \in (0, 4]$, then we will consider the case $\alpha > 4$.

Case 1. $0 < \alpha \leq 4$

The decreasing property of $F(-\dfrac{\alpha}{2}, k-\dfrac{\alpha}{2}; k+1; \bullet)$ and (3.4) imply that

$$\begin{aligned}
E_\alpha(r) &= \sum_{k=2}^{\infty} k(|c_k| + |c_{-k}|) F(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; r^2) r^{k-1} \\
&\leq \sum_{k=2}^{\infty} k(|c_k| + |c_{-k}|) r^{k-1} \\
&\leq \frac{4Ma_\alpha}{\pi} \sum_{k=2}^{\infty} \frac{\Gamma(k+\frac{\alpha}{2}+1)}{(k-1)!} r^{k-1}.
\end{aligned}$$

Subcase 1. $0 < \alpha \leq 2$

Remark that $\Gamma(k+\frac{\alpha}{2}+2) \leq \Gamma(k+3)$. Thus

$$\sum_{k=2}^{\infty} \frac{\Gamma(k+\frac{\alpha}{2}+1)}{(k-1)!} r^{k-1} = \sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{\alpha}{2}+2)}{k!} r^k \leq \sum_{k=1}^{\infty} \frac{\Gamma(k+3)}{k!} r^k = S_2(r).$$

Hence

$$E_\alpha(r) \leq \frac{4Ma_\alpha}{\pi} S_2(r) \leq \frac{24Mra_\alpha}{\pi(1-r)^3}.$$

Subcase 2. $2 < \alpha \leq 4$

By Proposition 2.1, for $n = 2$, it yields $\sum_{k=2}^{\infty} \frac{\Gamma(k + \frac{\alpha}{2} + 1)}{(k-1)!} r^k \leq S_3(r)$. Therefore,

$$E_\alpha(r) \leq \frac{4Ma_\alpha}{\pi} S_3(r) \leq \frac{96Ma_\alpha r}{(1-r)^4}.$$

Case 2. $2(n-1) < \alpha \leq 2n, n \geq 3$, that is, $n - 1 = \lceil \frac{\alpha}{2} \rceil$.

According to the discussion at the beginning of the proof, we see that the function $F\big(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; \bullet\big)$ is increasing for $2 \leq k \leq n-1$, and, decreasing, for $k \geq n$ on $[0, 1)$.

Consequently, we split E_α in two sums according to the monotonicity of $F\big(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; \bullet\big)$. On one hand, we have

$$\begin{aligned} \sum_{k=2}^{n-1} k(|c_k| + |c_{-k}|) F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; r^2) r^{k-1} &\leq \frac{4M}{\pi} \sum_{k=2}^{n-1} k r^{k-1} \\ &\leq \frac{2Mr}{\pi} (n+1)(n-2). \end{aligned}$$

On the other hand, using the estimate of the coefficients (3.4), we have

$$\begin{aligned} \sum_{k=n}^{\infty} k(|c_k| + |c_{-k}|) F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; r^2) r^{k-1} &\leq \sum_{k=n}^{\infty} k(|c_k| + |c_{-k}|) r^{k-1} \\ &\leq \frac{4Ma_\alpha}{\pi} \sum_{k=n}^{\infty} \frac{\Gamma(k + \frac{\alpha}{2} + 1)}{(k-1)!} r^{k-1}. \end{aligned}$$

Finally, by Proposition 2.1 (a), we conclude

$$E_\alpha(r) \leq \frac{4M}{\pi} \Big[\frac{(n+1)(n-2)}{2} r + a_\alpha \frac{(2n)! r^{n-1}}{(1-r)^{2n}} \Big].$$

We remark that this formula is still valid for $n = 2$. □

Estimate of $F_\alpha(r)$

Lemma 6 *Let* $n \in \mathbb{N}$, $n \geq 1$ *and* $\frac{\alpha}{2} \in (n-1, n]$.
If $n = 1$, *then*

$$F_\alpha(r) \leq \frac{4Mr^2}{\pi(1-r)} \Big(\frac{1}{1-r} - \frac{\alpha}{2} \Big) \leq \frac{4Mr^2}{\pi(1-r)^2}. \tag{3.12}$$

If $n \geq 2$*, then*

$$F_\alpha(r) \leq \frac{2M}{\pi}(n-2)(n+1)r^2 + \frac{4M\alpha a_\alpha (2n-1)!}{\pi n!}\left(1 + \frac{2nr}{(n+1)(1-r)^{n+1}}\right)r^n. \tag{3.13}$$

Proof We start by investigating the monotonicity of $F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; \bullet\big)$. By Lemma 1, we infer that its monotonicity depends on

$$\big(-\frac{\alpha}{2}+1\big)(k-\frac{\alpha}{2}+1), \qquad k \geq 1, \quad \alpha > 0.$$

Therefore the function $F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; \bullet\big)$ is increasing for $0 < \alpha \leq 2$, and decreasing for $2 \leq \alpha < 4$ on $[0, 1)$.

Case 1. $0 < \alpha \leq 2$.

As the function $F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; \bullet\big)$ is increasing, we have

$$F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; \omega\big) \leq F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; 1\big)$$

According to (2.2), we obtain

$$F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1\big) = \frac{\alpha}{k+1}F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; 1\big) \tag{3.14}$$

Finally, using (3.2), (3.8) and (3.14), we have

$$\begin{aligned} F_\alpha(r) &\leq \sum_{k=1}^{\infty}(|c_k|+|c_{-k}|)\frac{\alpha\big|\frac{\alpha}{2}-k\big|}{(k+1)}F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; 1\big)r^{k+1}, \\ &= \sum_{k=1}^{\infty}(k-\frac{\alpha}{2})(|c_k|+|c_{-k}|)F\big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1\big)r^{k+1} \\ &\leq \frac{4M}{\pi}\sum_{k=1}^{\infty}(k-\frac{\alpha}{2})r^{k+1} = \frac{4Mr^2}{\pi(1-r)}\Big(\frac{1}{1-r}-\frac{\alpha}{2}\Big). \end{aligned}$$

Case 2. $2 < \alpha \leq 4$.

As the function $F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; \bullet\big)$ is decreasing on $[0, 1)$, and using (3.4), it follows

$$\begin{aligned} F_\alpha(r) &\leq \sum_{k=1}^{\infty}(|c_k|+|c_{-k}|)\frac{\alpha\big|k-\frac{\alpha}{2}\big|}{(k+1)}r^{k+1} \\ &= (|c_1|+|c_{-1}|)\frac{\alpha(\alpha-2)r^2}{4} + \sum_{k=2}^{\infty}(|c_k|+|c_{-k}|)\frac{\alpha\big(k-\frac{\alpha}{2}\big)}{(k+1)}r^{k+1} \\ &\leq \frac{12M\alpha a_\alpha r^2}{\pi} + \frac{4M\alpha a_\alpha}{\pi}\sum_{k=2}^{\infty}\frac{\Gamma(\frac{\alpha}{2}+k+1)(k-\frac{\alpha}{2})}{(k+1)!}r^{k+1}. \end{aligned}$$

Using Proposition 2.1 (b) for $n = 2$, we deduce that $F_\alpha(r) \le \dfrac{12M\alpha a_\alpha r^2}{\pi}\Big(1 + \dfrac{4r}{3(1-r)^3}\Big)$.

Case 3. $n-1 < \frac{\alpha}{2} \le n, n \ge 3$

By Lemma 1, we deduce that the function $F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; \bullet\big)$ is increasing for $1 \le k \le n-2$ and decreasing for $k \ge n-1$. We split the summation in F_α in two sums according to the monotonicity of $F\big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; \bullet\big)$.

Let us start the first sum. Using (3.14), we get

$$\begin{aligned}
\Sigma_1 &:= \sum_{k=1}^{n-2}(|c_k|+|c_{-k}|)\frac{\alpha\big|k-\frac{\alpha}{2}\big|}{(k+1)}F\Big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; r^2\Big)r^{k+1} \\
&\le \sum_{k=1}^{n-2}(|c_k|+|c_{-k}|)(\frac{\alpha}{2}-k)F\Big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1\Big)r^{k+1} \\
&\le \frac{4M}{\pi}\sum_{k=1}^{n-2}(\frac{\alpha}{2}-k)r^{k+1} \\
&\le \frac{4Mr^2}{\pi}\sum_{k=1}^{n-2}(n-k) = \frac{2Mr^2}{\pi}(n-2)(n+1).
\end{aligned}$$

For the second sum, using Proposition 2.1 (b), we have

$$\begin{aligned}
\Sigma_2 &:= \sum_{k=n-1}^{\infty}(|c_k|+|c_{-k}|)\frac{\alpha\big|k-\frac{\alpha}{2}\big|}{(k+1)}F\Big(-\frac{\alpha}{2}+1, k-\frac{\alpha}{2}+1; k+2; r^2\Big)r^{k+1} \\
&\le \sum_{k=n-1}^{\infty}(|c_k|+|c_{-k}|)\frac{\alpha\big|k-\frac{\alpha}{2}\big|}{(k+1)}r^{k+1} \\
&\le \frac{4M\alpha a_\alpha}{\pi}\sum_{k=n-1}^{\infty}|k-\frac{\alpha}{2}|\frac{\Gamma(k+\frac{\alpha}{2}+1)}{(k+1)!}r^{k+1} \\
&\le \frac{4M\alpha a_\alpha}{\pi}\left(\frac{(2n-1)!}{n!}r^n + \sum_{k=n}^{\infty}(k-\frac{\alpha}{2})\frac{\Gamma(k+\frac{\alpha}{2}+1)}{(k+1)!}r^{k+1}\right) \\
&\le \frac{4M\alpha a_\alpha}{\pi}\left(\frac{(2n-1)!}{n!}r^n + \frac{(2n)!}{(n+1)!}\frac{r^{n+1}}{(1-r)^{n+1}}\right).
\end{aligned}$$

Finally

$$F_\alpha(r) \le \frac{2M}{\pi}(n-2)(n+1)r^2 + \frac{4M\alpha a_\alpha(2n-1)!}{\pi n!}\left(1 + \frac{2nr}{(n+1)(1-r)^{n+1}}\right)r^n.$$

We remark that his inequality remains valid for $n = 2$. □

Estimate of $G_\alpha(r)$

Lemma 7 *Let $n \in \mathbb{N}$, $n \geq 1$ and $\frac{\alpha}{2} \in (n-1, n]$.*
If $n = 1$ or $n \geq 3$, then

$$G_\alpha(r) \leq \frac{2Mr^2|1-\frac{\alpha}{2}|}{\pi}. \tag{3.15}$$

If $n = 2$, then

$$G_\alpha(r) \leq \frac{24M}{\pi} a_\alpha(\frac{\alpha}{2}-1)r^2. \tag{3.16}$$

Proof By the mean value theorem, there exists $c \in (0, r^2)$ such that

$$\begin{aligned} G_\alpha(r) &= r^2(|c_1|+|c_{-1}|)\left|\frac{d}{d\omega}F\big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; c\big)\right| \\ &= r^2(|c_1|+|c_{-1}|)\frac{\alpha|\frac{\alpha}{2}-1|}{4}F\big(1-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; 3; c\big). \end{aligned}$$

Lemma 1 shows that the function $F\big(1-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; 3; \bullet\big)$ is increasing for $0 < \alpha \leq 2$ or $\alpha > 4$, and decreasing for $2 < \alpha \leq 4$.

Case 1. $0 < \alpha \leq 2$ or $\alpha > 4$

As the function $F\big(1-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; 3; \bullet\big)$ is increasing on $[0, 1)$, and using (3.14), we get

$$F\big(1-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; 3; c\big) \leq F\big(1-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; 3; 1\big) = \frac{2}{\alpha}F\big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; 1\big).$$

By (3.2), we have

$$\begin{aligned} G_\alpha(r) &= r^2(|c_1|+|c_{-1}|)\frac{\alpha|\frac{2-\alpha}{2}|}{4}F\big(1-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; 3; c\big) \\ &\leq r^2(|c_1|+|c_{-1}|)\frac{\alpha|1-\frac{\alpha}{2}|}{4}\frac{2}{\alpha}F\big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; 1\big) \\ &= \frac{r^2|1-\frac{\alpha}{2}|}{2}(|c_1|+|c_{-1}|)F\big(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; 1\big) \\ &\leq \frac{2Mr^2|1-\frac{\alpha}{2}|}{\pi}. \end{aligned}$$

Case 2. $2 < \alpha \leq 4$

As the function $F\left(1-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; 3; \cdot\right)$ is decreasing on [0, 1) and using (3.4), we get

$$\begin{aligned} G_\alpha(r) &\leq r^2(|c_1|+|c_{-1}|)\frac{\alpha(\frac{\alpha}{2}-1)}{4} \\ &\leq r^2(|c_1|+|c_{-1}|)(\frac{\alpha}{2}-1) \\ &\leq \frac{24Ma_\alpha}{\pi}r^2(\frac{\alpha}{2}-1). \end{aligned}$$

□

Finally, combining (3.10–3.16), we conclude that

$$\|D_u(z)-D_u(0)\| \leq E_\alpha(|z|)+F_\alpha(|z|)+G_\alpha(|z|) \leq \sigma_\alpha(|z|), \tag{3.17}$$

where σ_α is defined in Theorem 3.

It is clear that σ_α is strictly increasing on [0, 1) for all $\alpha > 0$. Applying Theorem 2 (1.5), we get

$$1 = J_u(0) = \|D_u(0)\|\lambda_u(0) \leq \frac{2c_\alpha(\alpha+2)M}{\pi}\lambda_u(0).$$

Therefore,

$$\lambda_u(0) \geq \frac{\pi}{2Mc_\alpha(\alpha+2)}. \tag{3.18}$$

We will prove that u is univalent in $\mathbb{D}_{r_\alpha}$, where r_α satisfies the following equation:

$$\frac{2c_\alpha(\alpha+2)}{\pi}M\sigma(r_\alpha) = 1.$$

Indeed, let $z_1, z_2 \in \mathbb{D}_{r_\alpha}$ such that $z_1 \neq z_2$ and $[z_1, z_2]$ denote the line segment from z_1 to z_2, by using (3.17) and (3.18), we get

$$\begin{aligned} |u(z_1)-u(z_2)| &= \left|\int_{[z_1,z_2]} u_z(z)\,dz + u_{\overline{z}}(z)\,d\overline{z}\right| \\ &\geq \left|\int_{[z_1,z_2]} u_z(0)\,dz + u_{\overline{z}}(0)\,d\overline{z}\right| \\ &\quad - \left|\int_{[z_1,z_2]} (u_z(z)-u_z(0))\,dz + (u_{\overline{z}}(z)-u_{\overline{z}}(0))\,d\overline{z}\right| \\ &> |z_2-z_1|\left\{\frac{\pi}{2Mc_\alpha(\alpha+2)} - \sigma(r_\alpha)\right\} \\ &= 0. \end{aligned}$$

Thus $u(z_1) \neq u(z_2)$. The univalence of u follows from the arbitrariness of z_1 and z_2. This implies that u is univalent in $\mathbb{D}_{r_\alpha}$. As the mapping $\frac{\sigma_\alpha(|z|)}{|z|}$ is increasing, we deduce

$$\int_{[0,\xi]} \sigma_\alpha(|z|)|dz| \leq \frac{\sigma_\alpha(r_\alpha)r_\alpha}{2}.$$

For any $\xi \in \partial\mathbb{D}_{r_\alpha}$, we have

$$\begin{aligned}
|u(\xi)| &= \left| \int_{[0,\xi]} u_z(z)dz + u_{\overline{z}}(z)d\overline{z} \right| \\
&\geq \left| \int_{[0,\xi]} u_z(0)dz + u_{\overline{z}}(0)d\overline{z} \right| \\
&- \left| \int_{[0,\xi]} (u_z(z) - u_z(0))dz + (u_{\overline{z}}(z) - u_{\overline{z}}(0))d\overline{z} \right| \\
&\geq \lambda(D_u(0))r_\alpha - \int_{[0,\xi]} \sigma_\alpha(|z|)|dz| \\
&\geq \sigma_\alpha(r_\alpha)r_\alpha - \frac{\sigma_\alpha(r_\alpha)r_\alpha}{2} = \frac{\sigma_\alpha(r_\alpha)r_\alpha}{2}.
\end{aligned}$$

Hence $u(\mathbb{D}_{r_\alpha})$ contains a univalent disk $\mathbb{D}_{R_\alpha}$ with $R_\alpha \geq \frac{\sigma_\alpha(r_\alpha)r_\alpha}{2}$.

4 Schwarz-type lemmas for solutions to inhomogeneous biharmonic equations

Proof of Theorem 4 The solution of (1.1) can be written in the following form

$$\Phi(z) = \frac{1}{2}(1 - |z|^2)P[f + h](z) + K_2[f](z) - G[g](z).$$

As $z \longmapsto K_2[f](z)$ is T_2-harmonic function, then by Theorem 1, we have

$$\left| K_2[f](z) - \frac{(1 - |z|^2)^3}{(1 + |z|^2)^2} K_2[f](0) \right| \leq \frac{2}{\pi} \left[(1 + |z|^2) \arctan |z| + \frac{|z|(1 - |z|^2)}{1 + |z|^2} \right] \|f\|_\infty. \tag{4.1}$$

Using the estimate (1.2) for the harmonic mapping $P[f + h]$, we get

$$\left| P[f + h](z) - \frac{1 - |z|^2}{1 + |z|^2} P[f + h](0) \right| \leq \frac{4}{\pi} \arctan |z| \, \|f + h\|_\infty. \tag{4.2}$$

In addition, using [13, inequality 2.3], we obtain

$$|G[g](z)| \leq \frac{(1 - |z|^2)^2}{64} \|g\|_\infty. \tag{4.3}$$

Finally as $K_2[f](0) = \dfrac{1}{2}P[f](0)$, then the inequality (1.8) follows directly from (4.1–4.3). □

Proof of Theorem 5 The solution of (1.1) can be written in the following form

$$\Phi(z) = \frac{1}{2}(1-|z|^2)P[f+h](z) + K_2[f](z) - G[g](z).$$

Therefore,

$$\|D_\Phi(z)\| \leq \frac{1}{2}(1-|z|^2)\|D_{P[f+h]}(z)\| + |z||P[f+h](z)| + \|D_{K_2[f]}(z)\| + \|D_{G[g]}(z)\|.$$

By Colonna [15], we have

$$\|D_{P[f+h]}(z)\| \leq \frac{4}{\pi}\frac{1}{1-|z|^2}\|f+h\|_\infty. \tag{4.4}$$

It follows from [26, Lemma 2.5], that

$$\|D_{G[g]}(z)\| \leq \frac{23}{48}\|g\|_\infty, \tag{4.5}$$

since

$$\int_{\mathbb{D}} |G_z(z,\omega)g(\omega)|dA(\omega) \leq \frac{23}{6}\|g\|_\infty \text{ and } \int_{\mathbb{D}} |G_{\overline{z}}(z,\omega)g(\omega)|dA(\omega) \leq \frac{23}{6}\|g\|_\infty.$$

In addition by [12, Theorem 1], we have

$$\|D_{K_2[f]}(z)\| \leq \frac{(2+5|z|)(1+|z|^2)}{1-|z|^2}, \text{ for all } z \in \mathbb{D}. \tag{4.6}$$

Therefore, combining (4.4–4.6), we obtain

$$\begin{aligned}
\|D_\Phi(z)\| &\leq \frac{1}{2}(1-|z|^2)\|D_{P[f+h]}(z)\| + |z||P[f+h](z)| + \|D_{K_2[f]}(z)\| + \|D_{G[g]}(z)\| \\
&\leq \frac{2}{\pi}\|P[f+h]\|_\infty + |z|\|f+h\|_\infty + \frac{(2+5|z|)(1+|z|^2)}{1-|z|^2}\|f\|_\infty + \frac{23}{48}\|g\|_\infty \\
&\leq (\frac{2}{\pi} + |z|)\|f+h\|_\infty + \frac{(2+5|z|)(1+|z|^2)}{1-|z|^2}\|f\|_\infty + \frac{23}{48}\|g\|_\infty.
\end{aligned}$$

□

Proof of Theorem 6 Suppose that $|z| = r$, it follows from Theorem 4 that

$$|\Phi(\eta) - \Phi(r\eta)| \geq 1 - \frac{1}{2}(1-r^2)\|f+h\|_\infty - \frac{2}{\pi}\left[(r^2+1)\arctan r + \frac{r(1-r^2)}{1+r^2}\right]$$
$$- \frac{\|g\|_\infty(1-r^2)^2}{64} - \frac{1}{2}\frac{(1-r^2)^3}{(1+r^2)^2}|P[f](0)| - \frac{1}{2}\frac{(1-r^2)^2}{1+r^2}|P[f+h](0)|.$$

Divide by $1-r$ and used the Hospital rule, we obtain

$$\liminf_{r\to 1}\frac{|\Phi(\eta)-\Phi(r\eta)|}{1-r} \geq \lim_{r\to 1}\frac{1-\frac{2}{\pi}(r^2+1)\arctan r}{1-r} - \lim_{r\to 1}\frac{2}{\pi}\frac{r(1-r^2)}{(1-r)(1+r^2)}$$
$$- \frac{1}{2}\lim_{r\to 1}(1+r)\|f+h\|_\infty$$
$$= \varphi'(1) - \frac{2}{\pi} - \|f+h\|_\infty,$$

where $\varphi(r) = \frac{2}{\pi}(r^2+1)\arctan r$. Hence $\liminf_{r\longrightarrow 1}\frac{|\Phi(\eta)-\Phi(r\eta)|}{1-r} \geq 1 - \|f+h\|_\infty$. □

5 A Landau-type theorem for solutions to inhomogeneous biharmonic equations

First, let us recall the following result.

Theorem F ([11], Lemma 1) *Suppose f is a harmonic mapping of $\mathbb{D}$ into $\mathbb{C}$ such that $|f(z)| \leq M$ for all $z \in \mathbb{D}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$.*
Then $|a_0| \leq M$ and for all $n \geq 1$, $|a_n| + |b_n| \leq \frac{4M}{\pi}$.

Proof of Theorem 7 The solution of (1.1) can be written in the following form

$$\Phi(z) = H_0[f+h](z) + K_2[f](z) - G[g](z),$$

where

$$H_0[f+h](z) = \frac{1}{2}(1-|z|^2)P[f+h](z). \tag{5.1}$$

Since $P[f+h]$ is harmonic in $\mathbb{D}$, we have $P[f+h](z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$. As $|P[f+h](z)| \leq M_2$ for all $z \in \mathbb{D}$, by Theorem F, we have

$$|a_n| + |b_n| \leq \frac{4M_2}{\pi} \quad \text{for } n \geq 1. \tag{5.2}$$

Using the chain rule and by (5.1) and (5.2), we have

$$\begin{aligned}
&\|D_{H_0[f+h]}(z) - D_{H_0[f+h]}(0)\| \\
&\quad \leq |z||P[f+h](z)| + \frac{1}{2}\|D_{P[f+h]}(z) - D_{P[f+h]}(0)\| + \frac{1}{2}|z|^2\|D_{P[f+h]}(z)\| \\
&\quad \leq M_2|z| + \frac{1}{2}\sum_{n\geq 2} n(|a_n| + |b_n|)|z|^{n-1} + \frac{1}{2}|z|^2\sum_{n\geq 1} n(|a_n| + |b_n|)|z|^{n-1} \\
&\quad \leq M_2|z| + \frac{1}{2}(1+|z|^2)\sum_{n\geq 2} n(|a_n| + |b_n|)|z|^{n-1} + \frac{1}{2}|z|^2(|a_1| + |b_1|) \\
&\quad \leq M_2|z| + \frac{2M_2|z|}{\pi}\left[\frac{(2-|z|)(1+|z|^2)}{(1-|z|)^2} + |z|\right].
\end{aligned} \tag{5.3}$$

Since K_2 is T_2-harmonic, then

$$K_2(z) = \sum_{k=0}^{\infty} c_k F(-1, k-1; k+1; |z|^2) z^k + \sum_{k=1}^{\infty} c_{-k} F(-1, k-1; k+1; |z|^2)\overline{z}^k.$$

Let us denote

$$K_2^0[f](z) := K_2(z) - c_0 F(-1, -1; 1; |z|^2) = K_2[f](z) - c_0(1+|z|^2).$$

Hence

$$K_2[f] = K_2^0[f](z) + c_0(1+|z|^2),$$

and

$$\|D_{K_2[f]}(z) - D_{K_2[f]}(0)\| \leq \|D_{K_2^0[f]}(z) - D_{K_2^0[f]}(0)\| + 2|c_0||z|.$$

By (3.3), we have $2|c_0| \leq M_1$. On the other hand, as $K_2^0(f)$ is a T_2-harmonic function with $K_2^0(0) = 0$, it yields

$$\|D_{K_2^0[f]}(z) - D_{K_2^0[f]}(0)\| \leq \sigma_2(r),$$

where σ_2 is defined by $\sigma_2(r) = \frac{4M_1 r}{\pi(1-r)^3}(r^2(1-r)+3)$, see Remark 1.2. Thus

$$\|D_{K_2[f]}(z) - D_{K_2[f]}(0)\| \leq \frac{4M_1|z|}{\pi(1-|z|)^3}\left(|z|^2(1-|z|)+3\right) + M_1|z|. \tag{5.4}$$

Let

$$\psi_1(z) = \left|\frac{1}{16\pi}\int_{\mathbb{D}} g(\omega)(G_z(z,\omega) - G_z(0,\omega))dA(\omega)\right|,$$

and

$$\psi_2(z) = \left| \frac{1}{16\pi} \int_{\mathbb{D}} g(\omega)(G_{\overline{z}}(z,\omega) - G_{\overline{z}}(0,\omega)) dA(\omega) \right|.$$

Then by [13, Inequality (3.6)], we have

$$\max(\psi_1(z), \psi_2(z)) \leq \left(\frac{1-|z|^2}{16} + \frac{43}{120} \right) \|g\|_\infty |z|. \tag{5.5}$$

Now, it follows from (5.3)–(5.5) that

$$\begin{aligned} \|D_\Phi(z) - D_\Phi(0)\| &\leq M_2|z| + \frac{2M_2|z|}{\pi}\left[\frac{(2-|z|)(1+|z|^2)}{(1-|z|)^2} + |z|\right] \\ &\quad + \sigma_2(|z|) + M_1|z| + \psi_1(z) + \psi_2(z) \\ &\leq \mu(|z|), \end{aligned}$$

where

$$\begin{aligned} \mu(|z|) &= (M_1 + M_2 + \frac{101}{120}M_3)|z| + \frac{2M_2|z|}{\pi}\left[\frac{(2-|z|)(1+|z|^2)}{(1-|z|)^2} + |z|\right] \\ &\quad + \frac{4M_1|z|}{\pi(1-|z|)^3}(|z|^2(1-|z|)+3). \end{aligned}$$

Remark that not only $\mu(|z|)$ is increasing but also $\dfrac{\mu(|z|)}{|z|}$ is increasing with respect to $|z|$ in $[0, 1)$. By Theorem 5, we obtain that

$$1 = J_\Phi(0) = \|D_\Phi(0)\|\lambda(D_\Phi(0)) \leq \lambda(D_\Phi(0))\left(\frac{4}{\pi}M_1 + \frac{2}{\pi}M_2 + \frac{23}{48}M_3\right)$$

yields $\lambda(D_\Phi(0)) \geq \frac{1}{\frac{4}{\pi}M_1 + \frac{2}{\pi}M_2 + \frac{23}{48}M_3}$. As in Theorem 3, we prove that Φ is univalent in $\mathbb{D}_{r_0}$, where r_0 satisfies $(\frac{4}{\pi}M_1 + \frac{2}{\pi}M_2 + \frac{23}{48}M_3)\mu(r_0) = 1$, and $\Phi(\mathbb{D}_{r_0})$ contains an univalent disk $\mathbb{D}_{R_0}$ with the radius R_0 satisfying $R_0 \geq \frac{r_0}{\frac{8}{\pi}M_1 + \frac{4}{\pi}M_2 + \frac{23}{24}M_3}$. □

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