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A note on the solution map for the periodic multi-dimensional Camassa–Holm-type system

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Abstract

Considered in this paper is the initial value problem of periodic multi-dimensional Camassa–Holm-type system. It is shown that the solution map of this problem is not uniformly continuous in Besov spaces $B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d) \times B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)$ with $d \in \mathbb{Z}^+$, $d \geq 1$. Based on the local well-posedness results, the method of approximate solutions is utilized.

Keywords Non-uniform dependence · Periodic multi-dimensional Camassa–Holm system · Besov spaces · Approximate solutions

Mathematics Subject Classification 35B30 · 35G25

1 Introduction

In this paper, we are concerned with the following initial value problem of the periodic multi-dimensional Camassa–Holm-type system with $\alpha = 1$:

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$$\begin{cases} m_t + u \cdot \nabla m + (\nabla u)^T \cdot m + m(\mathrm{div} u) + \rho \nabla \rho = 0, \quad t > 0, \quad x \in \mathbb{T}^d, \\ \rho_t + \mathrm{div}(\rho u) = 0, \quad t > 0, \quad x \in \mathbb{T}^d, \\ m = (1 - \alpha^2 \Delta) u, \quad t > 0, \quad x \in \mathbb{T}^d, \\ m(0, x) = m_0, \quad \rho(0, x) = \rho_0, \end{cases} \tag{1.1}$$

where $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$. The vector field $u = (u_1(t, x), u_2(t, x), \cdots, u_d(t, x))$ is the velocity of the fluid, m denotes the momentum of the fluid, and the function $\rho(t, x)$ represents the density or the total depth. The constant $\alpha > 0$ stands for the length scale and is called the dispersive parameter.

The system in (1.1) was proposed in [24,29]. Just as the authors in [13] stated that this system was presented as a framework for nonlinear shallow water waves, geophysical fluids and turbulence modeling, or recasting the geodesic flow on the diffeomorphism groups. The local well-posedness in Sobolev spaces $H^k(\mathbb{R}^d) \times H^{k-1}(\mathbb{R}^d)$ with $k > 3 + \frac{d}{2}$, $k, d \in \mathbb{Z}^+$, blow-up and global existence results with $d = 2, 3$ for the solutions of the initial value problem (1.1) were found in [13]. Recently, the local well-posedness in Besov spaces and blow-up phenomenon for the solutions of this problem with $d \geq 2$, $d \in \mathbb{Z}^+$ were investigated by Li and Yin [34].

In this paper, we consider the initial value problem (1.1) with $\alpha = 1$. Thanks to [34], we know that the solution map $z_0 \mapsto z(t)$ of this problem is continuous in Besov spaces $B^{1+\frac{d}{p}}_{p,1}(\mathbb{T}^d) \times B^{\frac{d}{p}}_{p,1}(\mathbb{T}^d)$ with $1 \leq p < 2d, d \geq 2, d \in \mathbb{Z}^+$. Owing to the local well-posedness results in [34], the non-uniform continuity of this solution map in Besov spaces $B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d) \times B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)$ is studied. Next, we establish that the solution map of this problem with $d = 1$ is not uniformly continuous in Besov spaces $B^{\frac{3}{2}}_{2,1}(\mathbb{T}) \times B^{\frac{1}{2}}_{2,1}(\mathbb{T})$, which is based on the local well-posedness results in [19].

When $\rho = 0$, the system in (1.1) reduces to the following classical mathematical model of the fully nonlinear shallow water wave system [25]

$$m_t + u \cdot \nabla m + (\nabla u)^T \cdot m + m(\mathrm{div} u) = 0, \; m = (1 - \alpha^2 \Delta) u. \tag{1.2}$$

For $d = 1, \alpha = 1$, Eq. (1.2) was regarded as the famous Camassa–Holm equation. It is called the Euler–Poincaré equations in the high dimensional case $d \geq 2$, $d \in \mathbb{Z}^+$.

The Camassa–Holm (CH) equation was firstly proposed in the context of hereditary symmetries studied by Fokas and Fuchssteiner in [15] and then was written explicitly as a shallow water wave equation by Camassa and Holm [5]. They also showed that the CH equation is completely integrable with a bi-Hamiltonian structure and infinitely many conservation laws in [5,6]. Moreover, they established that this equation admits peaked traveling waves which interact like solitons. In 2000, Constantin and Strauss claimed that these peakons are orbitally stable in [7]. The local well-posedness for the initial value problem associated with the CH equation was proved by many scholars in [10–12,31], who verified the fact that the solution map $u_0 \mapsto u(t)$ is continuous in Sobolev spaces $H^s(s > \frac{3}{2})$ and Besov spaces $B^s_{p,r}(s > \{\frac{3}{2}, 1 + \frac{1}{p}\}, 1 \leq p, r \leq 1)$. On the basis of these local well-posedness results, the non-uniform continuity of the solution map in corresponding energy spaces was investigated in [23,35] by the method of approximate solutions. Other results about wave breaking and persistence properties, we can refer to [2,3,22] and references therein.

For $d \geq 2,\ d \in \mathbb{Z}^+$, Chae and Liu established the local well-posedness in Hilbert spaces $m_0 \in H^{s+\frac{d}{2}} (s \geq 2)$, local existence of weak solutions in $W^{2,p}(\mathbb{R}^d)$, $p > d$ and a blow-up criterion of the Cauchy problem associated with Eq. (1.2) in [8]. Furthermore, Li et al. [30] revealed the fact that the solution of this problem containing non-zero dispersion with a large class of smooth initial data blows up in finite time and exists globally in time under some assumptions on initial data. For $\alpha = 1$, it is in [37] that Yan and Yin investigated the local well-posedness of the solutions in Besov spaces $B^s_{p,r}(\mathbb{R}^d)(s > \max\{1+\frac{d}{p}, \frac{3}{2}\}, 1 \leq p, r \leq \infty)$ and $B^{1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)(1 \leq p < 2d)$. A blow-up criterion in Besov spaces and analytic solutions were also given in this paper. In the light of the local well-posedness results in [37], the non-uniform continuity of the solution map of this problem in Besov spaces $B^s_{2,r}(\mathbb{T}^d)(s > 1 + \frac{d}{2}, 1 \leq r \leq \infty)$ was studied in [38].

When $\alpha = 1, d = 1$ and ρ is a non-constant function in the system of (1.1), it reduces to the celebrated two-component CH system [4]

$$\begin{cases} m_t + 2u_x m + u m_x + \rho\rho_x = 0, m = u - u_{xx}, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{1.3}$$

where u denotes the horizontal velocity of the fluid and ρ is related to the free surface elevation. The local well-posedness, wave breaking results, the global existence and analytic solutions for the initial value problem associated with System (1.3) have been extensively studied in the past decade by many scholars. For more details, we refer to [16–19] and references therein. According to the local well-posedness results in [17], Lv et al. [32] established the non-uniform continuity of the solution map of this problem in Sobolev spaces $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$.

After the phenomena of non-uniform continuity for some dispersive equations was studied by Kenig et al. [27], the issue of nonuniform dependence on the initial data has been the subject of many papers. At first, Koch and Tzvetkov [28] proved that the flow map of the Benjamin-Ono equation cannot be uniformly continuous on bounded sets of $H^s(\mathbb{R})$ for $s > 0$. Then Himonas and Misiołek [21] obtained the result on the non-uniform dependence for the CH equation in appropriate Sobolev spaces. For more results with respect to the non-uniform continuity of the solution map of the Cauchy problem associated with CH-type equations or systems such as the Degasperis–Procesi, Novikov, Hunter–Saxton and μ-b equations in energy spaces can be found in [14,20,26,32,33,36], etc.

In view of the properties of Besov spaces that $B^s_{2,2} \times B^{s-1}_{2,2} = H^s \times H^{s-1}$ and $B^s_{2,r} \times B^{s-1}_{2,r} \hookrightarrow B^{1+\frac{d}{2}}_{2,1} \times B^{\frac{d}{2}}_{2,1}(s > 1 + \frac{d}{2}, 1 \leq r \leq \infty)$, the non-uniform continuity of the solution map of the initial value problem associated with equations or systems in Besov spaces with critical index seems to be the better results. However, the non-uniform continuity of the solution map of the initial value problem associated with the high dimensional equations or systems in Besov spaces with critical index remains an open problem. Thus, we mainly consider this property in Besov spaces with critical index. Motivated by the method of approximate solutions in [23,35], our first goal is to prove that this solution map of the Cauchy problem (1.1) with $\alpha = 1, d \geq 2, d \in \mathbb{Z}^+$ is not uniformly continuous in Besov spaces $B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d) \times B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)$. In order to manage all terms especially those including ρ, we have to construct the appropriate approximate solutions which are different from other CH-type equations or systems in [35,36,38]. Next, the case $d = 1$ is taken into consideration in detail. The procedure of the proof is similar to the case $d \geq 2,\ d \in \mathbb{Z}^+$. However, we find that the results in this part cannot be obtained only by the properties of Besov spaces and transport equations. The Osgood Lemma is quite crucial in the process of estimations. Our main results are as follows:

Theorem 1.1 *If* $(u_0, \rho_0) = z_0 \in B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d) \times B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)$ *with* $d \geq 2, d \in \mathbb{Z}^+$, *then there exists a lower bound* T_0 *of the maximal existence time of the solutions such that the solution map* $z_0 \to z(t) = (u(t), \rho(t))$ *of the initial value problem* (1.1) *with* $\alpha = 1$ *is not uniformly continuous from any bounded subset of* $B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d) \times B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)$ *into* $C([0, T_0]; B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d) \times B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d))$. *More precisely, there exist two sequences of solutions* $(u^n(t), \rho_1^n(t))$ *and* $(v^n(t), \rho_2^n(t))$ *into* $C([0, T_0]; B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d) \times B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d))$ *such that*

$$\|u^n(t)\|_{B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d)} + \|v^n(t)\|_{B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d)} + \|\rho_1^n(t)\|_{B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)} + \|\rho_2^n(t)\|_{B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)} \lesssim 1,$$
$$\lim_{n\to\infty} \|u^n(0) - v^n(0)\|_{B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d)} = \lim_{n\to\infty} \|\rho_1^n(0) - \rho_2^n(0)\|_{B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)} = 0,$$

and

$$\liminf_{n\to\infty} \left(\|u^n(t) - v^n(t)\|_{B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d)} + \|\rho_1^n(t) - \rho_2^n(t)\|_{B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)}\right) \gtrsim \left|\sin t\right|,\ 0 \leq t \leq T_0.$$

Remark 1.1 If $\rho = 0$, the non-uniform continuous dependence on initial data for the periodic initial value problem associated with Eq. (1.2) with $\alpha = 1$ in Besov spaces $B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d)$ with $d \geq 2,\ d \in \mathbb{Z}^+$ holds.

Another result of non-uniform continuous dependence on the initial data with $d = 1$ reads:

Theorem 1.2 *If* $(u_0, \rho_0) = z_0 \in B^{\frac{3}{2}}_{2,1}(\mathbb{T}) \times B^{\frac{1}{2}}_{2,1}(\mathbb{T})$, *then there exists a lower bound* T_1 *of the maximal existence time of the solutions such that the solution map* $z_0 \to z(t) = (u(t), \rho(t))$ *of the initial value problem* (1.1) *with* $d = 1, \alpha = 1$ *is not uniformly continuous from any bounded subset of* $B^{\frac{3}{2}}_{2,1}(\mathbb{T}) \times B^{\frac{1}{2}}_{2,1}(\mathbb{T})$ *into* $C([0, T_2]; B^{\frac{3}{2}}_{2,1}(\mathbb{T}) \times B^{\frac{1}{2}}_{2,1}(\mathbb{T}))$ *with* $0 \leq T_2 < T_1$. *More precisely, there exist two sequences of solutions* $(u^n(t), \rho_1^n(t))$ *and* $(v^n(t), \rho_2^n(t))$ *into* $C([0, T_1]; B^{\frac{3}{2}}_{2,1}(\mathbb{T}) \times B^{\frac{1}{2}}_{2,1}(\mathbb{T}))$ *such that*

$$\|u^n(t)\|_{B^{\frac{3}{2}}_{2,1}(\mathbb{T})} + \|v^n(t)\|_{B^{\frac{3}{2}}_{2,1}(\mathbb{T})} + \|\rho_1^n(t)\|_{B^{\frac{1}{2}}_{2,1}(\mathbb{T})} + \|\rho_2^n(t)\|_{B^{\frac{1}{2}}_{2,1}(\mathbb{T})} \lesssim 1,$$

$$\lim_{n\to\infty}\|u^n(0)-v^n(0)\|_{B^{\frac{3}{2}}_{2,1}(\mathbb{T})}=\lim_{n\to\infty}\|\rho_1^n(0)-\rho_2^n(0)\|_{B^{\frac{1}{2}}_{2,1}(\mathbb{T})}=0,$$

and

$$\liminf_{n\to\infty}\left(\|u^n(t)-v^n(t)\|_{B^{\frac{3}{2}}_{2,1}(\mathbb{T})}+\|\rho_1^n(t)-\rho_2^n(t)\|_{B^{\frac{1}{2}}_{2,1}(\mathbb{T})}\right)\gtrsim\left|\sin t\right|,\ 0\le t\le T_2<T_1,$$

where T_2 *satisfies* $\exp(-ct)\geq 1-\delta\ \left(0<\delta<\frac{1}{3}, 0\le t\le T_2<T_1\right)$ *and* $c>0$ *is a constant.*

The paper is organized as follows. In Sect. 2, we present some facts about Littlewood–Paley decomposition, the definition and properties of Besov spaces, the transport equation theories and the Osgood Lemma. In Sect. 3, the non-uniform continuity of the solution map of the initial value problem (1.1) with $\alpha=1$ in Besov spaces $B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d)\times B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)$ is proved when $d\geq 2, d\in\mathbb{Z}^+$. At first, we give the local well-posedness results which is necessary for our proof. Next, we construct the appropriate approximate solutions and calculate the error. And then we solve the Cauchy problem for the periodic Camassa–Holm-type system with initial data given by the approximate solutions evaluated at $t=0$. In the following, we estimate the difference between actual and approximate solutions. Finally, we complete the proof of the non-uniform continuity of the solution map in the case $d\geq 2, d\in\mathbb{Z}^+$. In Sect. 4, we consider the non-uniform continuity of the solution map in the case $d=1$. The specific procedure is similar to that of Sect. 3.

Notation. In the following, for a given Banach space Z, we denote its norm by $\|\cdot\|_Z$. We denote $A\lesssim B$ if $A\le cB$ and $A\gtrsim B$ if $A\geq cB$, where c is a positive constant. Let $\|z(t)\|_{B^s_{p,r}\times B^{s-1}_{p,r}}=\|u(t)\|_{B^s_{p,r}}+\|\rho(t)\|_{B^{s-1}_{p,r}}=\sum_{i=1}^d\|u_i(t)\|_{B^s_{p,r}}+\|\rho(t)\|_{B^{s-1}_{p,r}}$, if $z=(u,\rho)=(u_1,u_2,\cdots,u_d,\rho)$. For convenience, let $u=(u_1,u_2,\ldots,u_d)$, $v=(v_1,v_2,\ldots,v_d)$ be vector fields, and $A=(a_{ij})_{d\times d}$, $B=(b_{ij})_{d\times d}$ be $d\times d$ matrices. Then

(1) $u\cdot\nabla v=u(\nabla v)^T=\sum_{j=1}^d u_j\partial_j v$, here $\nabla u=(\nabla u_1,\nabla u_2,\cdots,\nabla u_d)^T$ and $\cdot^T$ denotes the transpose of $\cdot$. Moreover, $\nabla v\cdot u=u\cdot\nabla v$.

(2)$\mathrm{div}u=\sum_{j=1}^d\partial_j u_j$, while $\mathrm{div}A=(\mathrm{div}A_1,\mathrm{div}A_2,\ldots,\mathrm{div}A_d)$ with $A=(A_1,A_2,\cdots,A_d)^T$ and each component $A_j=(a_{j1},a_{j2},a_{j3},\ldots,a_{jd})$.

(3)$A:B=\sum_{i,j=1}^d a_{ij}b_{ij}$ and $|A|=(A:A)^{1/2}$.

(4)$A=(A_1,A_2,\ldots,A_{d-1},A_d)=(A_j)_{1\le j\le d}$.

2 Preliminaries

In this section, some facts about Littlewood–Paley decomposition, the definition and properties of the nonhomogeneous Besov spaces will be recalled in the first place. For more details, the readers can refer to [1,10,12].

Proposition 2.1 [1,10,12] (Littlewood–Paley decomposition) *Let* $\mathcal{B}\overset{def}{=}\{\xi\in\mathbb{R}^d,\ |\xi|\le\frac{4}{3}\}$ *and* $\mathcal{C}\overset{def}{=}\{\xi\in\mathbb{R}^d,\ \frac{3}{4}\le|\xi|\le\frac{8}{3}\}$. *There exist two radial func-*

tions $\chi \in C_c^\infty(\mathcal{B})$ *and* $\varphi \in C_c^\infty(\mathcal{C})$ *such that*

$$\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1, \quad \text{for all} \;\; \xi \in \mathbb{R},$$
$$|q - q'| \geq 2 \Rightarrow \text{Supp}\, \varphi(2^{-q}\cdot) \cap \text{Supp}\, \varphi(2^{-q'}\cdot) = \varnothing,$$
$$q \geq 1 \Rightarrow \text{Supp}\, \chi(\cdot) \cap \text{Supp}\, \varphi(2^{-q}\cdot) = \varnothing,$$

and

$$\frac{1}{3} \leq \chi(\xi)^2 + \sum_{q\geq 0} \varphi((2^{-q}\xi))^2 \leq 1, \quad \text{for all} \;\; \xi \in \mathbb{R}^d.$$

In the periodic setting, we decompose the functions on the circle $\mathbb{T}^d$ in Fourier series:

$$u(x) = \sum_{\xi \in \mathbb{Z}^d} \hat{u}(\xi) e^{ix\cdot\xi} \quad \text{where} \quad \hat{u}(\xi) = \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} u(\xi) e^{-ix\cdot\xi} dx.$$

The periodic dyadic blocks can be defined as

$$\Delta_q u \stackrel{def}{=} 0 \quad \text{for} \quad q \leq -1, \quad \Delta_{-1} u \stackrel{def}{=} \sum_{\xi \in \mathbb{Z}^d} \chi(\xi) \hat{u}(\xi) e^{ix\cdot\xi},$$
$$\Delta_q u \stackrel{def}{=} \sum_{\xi \in \mathbb{Z}^d} \varphi(2^{-q}\xi) \hat{u}(\xi) e^{ix\cdot\xi} \quad \text{for} \quad q \geq 0.$$

We also use the notation $S_q u \stackrel{def}{=} \sum_{p\leq q-1} \Delta_p u$. The formal equality

$$u = \sum_{q\geq -1} \Delta_q u$$

holds in $\mathcal{S}'(\mathbb{T}^d)$ and is called the Littlewood–Paley decomposition.

Definition 2.1 [1,10,12] (Besov space) Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. The inhomogenous Besov space $B^s_{p,r}(\mathbb{T}^d)$ ($B^s_{p,r}$ for short) is defined by

$$B^s_{p,r} \stackrel{def}{=} \{f \in \mathcal{S}'(\mathbb{T}^d); \quad \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} \stackrel{def}{=} \begin{cases} \left(\sum_{q\in\mathbb{Z}} 2^{qsr} \|\Delta_q f\|^r_{L^p} \right)^{\frac{1}{r}}, & \text{for} \quad r < \infty, \\ \sup_{q\in\mathbb{Z}} 2^{qs} \|\Delta_q f\|_{L^p}, & \text{for} \quad r = \infty. \end{cases}$$

If $s = \infty$, $B^\infty_{p,r} := \bigcap_{s\in\mathbb{R}} B^s_{p,r}$.

Proposition 2.2 [1,10,12] *The following properties hold.*

(i) Density: if $p,\ r < \infty$, then $\mathcal{S}(\mathbb{T}^d)$ is dense in $B^s_{p,r}(\mathbb{T}^d)$, where $\mathcal{S}$ denotes the Schwartz space.

(ii) Besov embededings: if $p_1 \le p_2$ and $r_1 \le r_2$, then $B^s_{p_1,r_1} \hookrightarrow B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$. If $s_1 < s_2$, $1 \le p \le +\infty$ and $1 \le r_1,\ r_2 \le +\infty$, then the embedding $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ is locally compact.

(iii) Algebraic properties: for $s > 0$, $B^s_{p,r} \cap L^\infty$ is an algebra. Moreover, $B^s_{p,r}$ is an algebra $\Longleftrightarrow B^s_{p,r} \hookrightarrow L^\infty \Longleftrightarrow s > \frac{d}{p}$ or $s \ge \frac{d}{p}$ and $r = 1$.

(iv) Fatou property: if $(u^{(n)})_{n\in\mathbb{N}}$ is a bounded sequence of $B^s_{p,r}$ which tends to u in $\mathcal{S}'$, then $u \in B^s_{p,r}$ and

$$\|u\|_{B^s_{p,r}} \le \lim\inf_{n\to\infty} \|u^{(n)}\|_{B^s_{p,r}}.$$

(v) Complex interpolation: if $u \in B^s_{p,r} \cap B^{\tilde{s}}_{p,r}$ and $\theta \in [0,1]$, $1 \le p, r \le \infty$, then $u \in B^{\theta s+(1-\theta)\tilde{s}}_{p,r}$ and $\|u\|_{B^{\theta s+(1-\theta)\tilde{s}}_{p,r}} \le \|u\|^{\theta}_{B^s_{p,r}} \|u\|^{1-\theta}_{B^{\tilde{s}}_{p,r}}$.

(vi) Real interpolation: if $u \in B^s_{p,\infty} \cap B^{\tilde{s}}_{p,\infty}$ and $s < \tilde{s}$ then $u \in B^{\theta s+(1-\theta)\tilde{s}}_{p,1}$ for all $\theta \in (0,1)$ and there exists a universal constant C such that

$$\|u\|_{B^{\theta s+(1-\theta)\tilde{s}}_{p,1}} \le \frac{C}{\theta(1-\theta)(\tilde{s}-s)} \|u\|^{\theta}_{B^s_{p,\infty}} \|u\|^{1-\theta}_{B^{\tilde{s}}_{p,\infty}}.$$

(vii) Let $n \in \mathbb{R}$ and f be a S^n-multiplier (that is, $f : \mathbb{T}^d \to \mathbb{R}$ is smooth and satisfies that for all multi-index α, there exists a constant C_α such that for any $\xi \in \mathbb{S}^d$, $|\partial^\alpha f(\xi)| \le C_\alpha(1+|\xi|)^{n-|\alpha|}$. Then for all $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-n}_{p,r}$.

Lemma 2.1 [1] *Assume that $1 \le p,\ r \le +\infty$, the following estimates hold:*

(i) for $s > 0$, $\|fg\|_{B^s_{p,r}} \le C(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|g\|_{B^s_{p,r}}\|f\|_{L^\infty})$;

(ii) for $s_1 \le \frac{d}{p}$, $s_2 > \frac{d}{p}$ ($s_2 \ge \frac{d}{p}$ if $r = 1$) and $s_1 + s_2 > \max\{0, \frac{2d}{p} - d\}$,

$$\|fg\|_{B^{s_1}_{p,r}} \le C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}},$$

where the constant C is independent of f and g.

Lemma 2.2 [38] *Let $\sigma, \alpha \in \mathbb{R}$. If $n \in \mathbb{Z}^+$, $1 \le r \le \infty$ and $n \gg 1$, then for $i, j = 1, 2, 3, \cdots, d$ we have*

$$\|\sin(nx_i - \alpha)\|_{B^\sigma_{2,r}(\mathbb{T}^d)} = \|\cos(nx_i - \alpha)\|_{B^\sigma_{2,r}(\mathbb{T}^d)} \approx n^\sigma,$$
$$\|\sin(nx_i - \alpha)\cos(nx_j - \alpha)\|_{B^\sigma_{2,r}(\mathbb{T}^d)} \approx n^\sigma.$$

Lemma 2.3 [10] *For any $f \in B^{1/2}_{2,1}$, $g \in B^{-1/2}_{2,1}$, there holds the product estimate*

$$\|fg\|_{B^{-1/2}_{2,\infty}} \le \|f\|_{B^{1/2}_{2,1}}\|g\|_{B^{-1/2}_{2,1}}.$$

Lemma 2.4 [12] *There is a constant $C > 0$ such that for $s \in \mathbb{R}$, $\varepsilon > 0$ and $1 \le p \le \infty$,*

$$\|f\|_{B^s_{p,1}} \le C\frac{1+\varepsilon}{\varepsilon}\|f\|_{B^s_{p,\infty}} \ln\left(e + \frac{\|f\|_{B^{s+\varepsilon}_{p,\infty}}}{\|f\|_{B^s_{p,\infty}}}\right).$$

Next, we shall list the properties about the transport equation which play an important role in our work.

Lemma 2.5 [1,10] *Suppose that $(p, r) \in [1, +\infty]^2$ and $s > -\frac{d}{p}$. Let v be a vector field such that ∇v belongs to $L^1([0, T]; B^{s-1}_{p,r})$ if $s > 1 + \frac{d}{p}$ or to $L^1([0, T]; B^{\frac{d}{p}}_{p,r} \cap L^\infty)$ otherwise. Suppose also that $f_0 \in B^s_{p,r}$, $F \in L^1([0, T]; B^s_{p,r})$ and $f \in L^\infty([0, T]; B^s_{p,r}) \cap C([0, T]; \mathcal{S}')$ solves the d-dimensional linear transport equations*

$$(T) \qquad \begin{cases} \partial_t f + v\cdot\nabla f = F, \\ f|_{t=0} = f_0. \end{cases}$$

Then there exists a constant C depending only on s, p and d such that the following statements hold:

(1) If $r = 1$ or $s \ne 1 + \frac{d}{p}$, then

$$\|f\|_{B^s_{p,r}} \le \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}} d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{B^s_{p,r}} d\tau,$$

or

$$\|f\|_{B^s_{p,r}} \le e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}} d\tau\right) \tag{2.1}$$

hold, where $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{\frac{d}{p}}_{p,r}\cap L^\infty} d\tau$ if $s < 1 + \frac{d}{p}$ and $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{s-1}_{p,r}} d\tau$ else.

(2) If $f = v$, then for all $s > 0$, the estimate (2.1) holds with $V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$.

Lemma 2.6 [34] *Suppose that $(p, r) \in [1, +\infty]^2$ and $s > -\min d\{\frac{1}{p}, 1-\frac{1}{p}\}$. Assume $f_0 \in B^s_{p,r}(\mathbb{R}^d)$, $F \in L^1([0, T]; B^s_{p,r}(\mathbb{R}^d))$, and*

$$\begin{cases} \nabla v \in L^1([0, T]; B^{s-1}_{p,r}(\mathbb{R}^d)); \quad if\ s > 1 + \frac{d}{p} \quad \left(s = 1 + \frac{d}{p}, r = 1\right), \\ \nabla v \in L^1([0, T]; B^s_{p,r}(\mathbb{R}^d)); \quad if\ s = 1 + \frac{d}{p}, r > 1, \\ \nabla v \in L^1([0, T]; B^{\frac{d}{p}}_{p,r}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)), \quad if\ s < 1 + \frac{d}{p}. \end{cases}$$

If $f \in L^\infty([0,T]; B^s_{p,r}(\mathbb{R}^d)) \cap C([0,T];\mathcal{S}')$ *solves* (T)*, then there exists a constant* C*, such that the following statements hold:*

$$\|f\|_{B^s_{p,r}} \le \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}} d\tau \\ + C\int_0^t (\|f(\tau)\|_{B^s_{p,r}} \|\nabla v(\tau)\|_{L^\infty} + \|\nabla v(\tau)\|_{B^{s-1}_{p,r}} \|\nabla f(\tau)\|_{L^\infty}) d\tau.$$

Lemma 2.7 [34] *Suppose that* $\sigma = \frac{d}{p} - 1, r = 1, 1 \le p < 2d, d \ge 2$. *Assume that* $f_0 \in B^\sigma_{p,r}$, $F \in L^1([0,T]; B^\sigma_{p,r})$ *and* $v \in L^1([0,T]; B^{\sigma+2}_{p,r})$. *If* $f \in L^\infty([0,T]; B^\sigma_{p,r}) \cap C([0,T];\mathcal{S}')$ *solves* (T)*, then there exists a constant* C *such that*

$$\|f\|_{B^\sigma_{p,r}} \le Ce^{CV(t)} \left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B^\sigma_{p,r}} d\tau \right),$$

where $V(t) = \int_0^t \|v(\tau)\|_{B^{\sigma+2}_{p,r}} d\tau$.

For $d = 1$ in (T), we also have

Lemma 2.8 [9] *Suppose that* $v \in L^\rho([0,T], B^{-M}_{\infty,\infty})$ *for some* $\rho > 1, M > 0$ *and* $v_x \in L^1([0,T], B^{1/2}_{2,1})$. *Denote* $V(t) = \int_0^t \|v_x(\tau)\|_{B^{1/2}_{2,1}} d\tau$. *If* $f_0 \in B^{-1/2}_{2,\infty}$, $F \in L^1([0,T]; B^{-1/2}_{2,\infty})$*, then* (T) *has a unique solution* $f \in C([0,T]; B^{-1/2}_{2,\infty})$. *Moreover, we have for* $t \in [0,T]$,

$$\|f\|_{B^{-1/2}_{2,\infty}} \le e^{CV(t)} \left(\|f_0\|_{B^{-1/2}_{2,1}} + \int_0^t \|F(\tau)\|_{B^{-1/2}_{2,\infty}} d\tau \right).$$

Finally, the Osgood Lemma which is essential in the proof of Theorem 1.2 will be presented.

Lemma 2.9 [1](Osgood Lemma) *Let* $\rho \ge 0$ *be a measurable function,* $\gamma > 0$ *be a locally integrable function and* μ *be a continuous and increasing function. Assume that, for some nonnegative real number* c*, the function* ρ *satisfies*

$$\rho(t) \le c + \int_{t_0}^t \gamma(\tau)\mu(\rho(\tau))\, d\tau.$$

If $c > 0$*, then* $-\mathcal{M}(\rho(t)) + \mathcal{M}(A) \le \int_{t_0}^t \gamma(\tau)\, d\tau$ *with* $\mathcal{M} = \int_x^1 \frac{dr}{\mu(r)}$. *If* $c = 0$ *and* μ *stasfies* $\int_0^1 \frac{dr}{\mu(r)} = +\infty$*, then the function* $\rho = 0$.

Remark 2.1 Setting $\mu(r) = r(1 - lnr)$, we obtain $\mathcal{M}(x) = ln(1 - lnx)$ by the simple calculation. Moreover, for all $c > 0$ Lemma 2.9 shows us that

$$\rho(t) \lesssim c^{\exp(\int_{t_0}^t -\gamma(\tau)\, d\tau)}.$$

3 The proof of Theorem 1.1

In this section, we are going to discuss the non-uniform continuity of the solution map $z_0 \mapsto z(t)$ of the initial value problem (1.1) with $\alpha = 1$ in Besov spaces $B^{1+\frac{d}{2}}_{2,1}(\mathbb{T}^d) \times B^{\frac{d}{2}}_{2,1}(\mathbb{T}^d)$ when $d \geq 2,\ d \in \mathbb{Z}^+$. Since all spaces of functions are over $\mathbb{T}^d$ in this part, we drop $\mathbb{T}^d$ if there is no ambiguity. At first, the Cauchy problem (1.1) with $\alpha = 1$ can be rewritten as follows:

$$\begin{cases} u_t + u \cdot \nabla u + f(u, \rho) + g(u, \rho) = 0, \quad t > 0, \quad x \in \mathbb{T}^d, \\ \rho_t + u \cdot \nabla \rho + \rho \mathrm{div} u = 0, \quad t > 0 \quad x \in \mathbb{T}^d, \\ u(0, x) = u_0, \quad \rho(0, x) = \rho_0, \end{cases} \tag{3.1}$$

where

$$f(u, \rho) = (I - \Delta)^{-1} \mathrm{div} \left(\nabla u (\nabla u + (\nabla u)^T) - (\nabla u)^T \nabla u - \nabla u (\mathrm{div} u) + \frac{1}{2} I |\nabla u|^2) \right),$$

$$g(u, \rho) = (I - \Delta)^{-1} \left(u \mathrm{div} u + u \cdot (\nabla u)^T + \frac{1}{2} \nabla \rho^2 \right).$$

Next, we give the local well-posedness results for the Cauchy problem (3.1). For $T > 0$, $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we define

$$E^s_{p,r}(T) \overset{def}{=} \begin{cases} C([0, T]; B^s_{p,r}(\mathbb{T}^d)) \cap C^1([0, T]; B^{s-1}_{p,r}(\mathbb{T}^d)), & r < \infty, \\ L^\infty([0, T]; B^s_{p,\infty}(\mathbb{T}^d)) \cap Lip([0, T]; B^{s-1}_{p,\infty}(\mathbb{T}^d)), & r = \infty. \end{cases}$$

Lemma 3.1 *Suppose $z_0 = (u_0, \rho_0) \in B^{1+\frac{d}{p}}_{p,1} \times B^{\frac{d}{p}}_{p,1}$ with $1 \leq p < 2d, d \geq 2$ and $d \in \mathbb{Z}^+$, then there exists a time $T = T(z_0) > 0$ such that $(u(t), \rho(t)) = z(t) \in E^{1+\frac{d}{p}}_{p,1}(T) \times E^{\frac{d}{p}}_{p,1}(T)$ is the unique solution to the initial value problem (3.1), and the solution depends continuously on the initial data, that is, the solution map $z_0 \mapsto z(t)$ is continuous from $B^{1+\frac{d}{p}}_{p,1} \times B^{\frac{d}{p}}_{p,1}$ into $C([0, T]; B^{1+\frac{d}{p}}_{p,1} \times B^{\frac{d}{p}}_{p,1})$. Furthermore, the solution $z(t)$ satisfies the following estimate*

$$\|z(t)\|_{B^{1+\frac{d}{p}}_{p,1} \times B^{\frac{d}{p}}_{p,1}} \leq 2C \|z_0\|_{B^{1+\frac{d}{p}}_{p,1} \times B^{\frac{d}{p}}_{p,1}}, \quad 0 \leq t \leq T_0 := \frac{1}{4C^2 \|z_0\|_{B^{1+\frac{d}{p}}_{p,1} \times B^{\frac{d}{p}}_{p,1}}}, \tag{3.2}$$

where $C \geq 1$ is a constant independent of z_0.

Proof The proof of existence, uniqueness and continuity of the solution map can be found in [34]. Therefore, our main goal is to establish (3.2). From the proof of Theorem 3.2 in [34], we know that there exists a constant $C \geq 1$ and T satisfying

$2C^2T\|z_0\|_{B^{1+\frac{d}{p}}_{p,1}\times B^{\frac{d}{p}}_{p,1}} < 1$ such that for every $t\in[0,T]$, we have

$$\|z^n(t)\|_{B^{1+\frac{d}{p}}_{p,1}\times B^{\frac{d}{p}}_{p,1}} \leq \frac{C\|z_0\|_{B^{1+\frac{d}{p}}_{p,1}\times B^{\frac{d}{p}}_{p,1}}}{1-2C^2t\|z_0\|_{B^{1+\frac{d}{p}}_{p,1}\times B^{\frac{d}{p}}_{p,1}}}. \tag{3.3}$$

Putting $T_0 := \frac{1}{4C^2\|z_0\|_{B^{1+\frac{d}{p}}_{p,1}\times B^{\frac{d}{p}}_{p,1}}}$ into (3.3) yields

$$\|z^n(t)\|_{B^{1+\frac{d}{p}}_{p,1}\times B^{\frac{d}{p}}_{p,1}} \leq 2C\|z_0\|_{B^{1+\frac{d}{p}}_{p,1}\times B^{\frac{d}{p}}_{p,1}}, \quad 0\leq t\leq T_0 := \frac{1}{4C^2\|z_0\|_{B^{1+\frac{d}{p}}_{p,1}\times B^{\frac{d}{p}}_{p,1}}},$$

Since $z^n(t)$ converges to $z(t)$ and $z(t)$ is the solution to the initial value problem (3.1), (3.3) holds. This completes the proof of Lemma 3.1. □

In the following, we prove Theorem 1.1. Motivated by the approach in [23,35], we construct the approximate solutions in two cases:

(1) If $d\geq 2$ is even, we define the approximate solutions as:

$$u^{\omega,n} = (\omega n^{-1} + n^{-1-\frac{d}{2}}\cos\alpha_i)_{1\leq i\leq d}, \quad \rho^{\omega,n} = \omega n^{-1} + n^{-\frac{d}{2}}\sum_{i=1}^{d}\cos\alpha_i,$$

where $\alpha_i = nx_{d+1-i} - \omega t$, $d\geq 2$, $\omega=\pm 1$, $n, d\in\mathbb{Z}^+$, $n\gg 1$.

(2) If $d\geq 3$ is odd, the approximate solutions are defined as:

$$\begin{aligned} u^{\omega,n} &= (\omega n^{-1} + n^{-1-\frac{d}{2}}\cos\beta_1, \omega n^{-1} + n^{-1-\frac{d}{2}}\cos\beta_2, \ldots, \omega n^{-1} \\ &\quad + n^{-1-\frac{d}{2}}\cos\beta_{d-1}, 0), \\ \rho^{\omega,n} &= \omega n^{-1} + n^{-\frac{d}{2}}\sum_{i=1}^{d-1}\cos\beta_i, \end{aligned}$$

where $\beta_i = nx_{d-i} - \omega t$, $1\leq i\leq d-1$, $d\geq 3$, $\omega=\pm 1$, $n, d\in\mathbb{Z}^+$, $n\gg 1$.

3.1 Estimating the error of approximate solutions

Lemma 3.2 *When $\omega=-1, 1$, $n\gg 1$, $d\geq 2$, $n, d\in\mathbb{Z}^+$, we have*

$$\|E(t)\|_{B^{\frac{d}{2}}_{2,1}}, \|F(t)\|_{B^{\frac{d}{2}-1}_{2,1}} \lesssim n^{-2}, \quad 0\leq t\leq T_0.$$

Proof We only give the proof of the case that $d \geq 2$ is even. For the case that $d \geq 3$ is odd, the proof is similar. Substituting $u^{\omega,n}$ and $\rho^{\omega,n}$ into the equations in (3.1), we obtain

$$\begin{aligned} E(t) &= \partial_t u^{\omega,n} + u^{\omega,n} \cdot \nabla u^{\omega,n} + f(u^{\omega,n}, \rho^{\omega,n}) + g(u^{\omega,n}, \rho^{\omega,n}) \\ &:= E_1(t) + (1-\Delta)^{-1} E_2(t) + (1-\Delta)^{-1} E_3(t), \\ F(t) &= \partial_t \rho^{\omega,n} + u^{\omega,n} \cdot \nabla \rho^{\omega,n} + \rho^{\omega,n} \mathrm{div} u^{\omega,n}, \end{aligned}$$

where

$$\begin{aligned} &E_1(t) = \partial_t u^{\omega,n} + u^{\omega,n} \cdot \nabla u^{\omega,n}, \\ &(1-\Delta)^{-1} E_2(t) = f(u^{\omega,n}, \rho^{\omega,n}), \quad (1-\Delta)^{-1} E_3(t) = g(u^{\omega,n}, \rho^{\omega,n}). \end{aligned}$$

Owing to the definition of $u^{\omega,n}$, we obtain that

$$\begin{aligned} \partial_t u^{\omega,n} &= (\omega n^{-1-\frac{d}{2}} \sin \alpha_i)_{1 \leq i \leq d}, \\ \nabla u^{\omega,n} &= -n^{-\frac{d}{2}} \begin{bmatrix} 0 & 0 & \dots & 0 & \sin\alpha_1 \\ 0 & 0 & \dots & \sin\alpha_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \sin\alpha_{d-1} & \dots & \vdots & \vdots \\ \sin\alpha_d & 0 & \dots & 0 & 0 \end{bmatrix}_{d\times d}, \end{aligned}$$

In conclusion,

$$E_1(t) = \partial_t u^{\omega,n} + u^{\omega,n} (\nabla u^{\omega,n})^T = (-n^{-1-d} \sin\alpha_i \cos\alpha_{d+1-i})_{1\leq i \leq d}.$$

A few simple calculations yield

$$\begin{aligned} &\nabla u^{\omega,n} (\nabla u^{\omega,n} + (\nabla u^{\omega,n})^T) - (\nabla u^{\omega,n})^T \nabla u^{\omega,n} \\ &= n^{-d} \begin{pmatrix} \begin{bmatrix} \sin\alpha_1(\sin\alpha_1 + \sin\alpha_d) \\ - \sin^2\alpha_d \end{bmatrix} & 0 & \dots & 0 \\ 0 & \begin{bmatrix} \sin\alpha_2(\sin\alpha_2 + \sin\alpha_{d-1}) \\ - \sin^2\alpha_{d-1} \end{bmatrix} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \begin{bmatrix} \sin\alpha_d(\sin\alpha_d + \sin\alpha_1) \\ - \sin^2\alpha_1 \end{bmatrix} \end{pmatrix}, \\ &\frac{1}{2} I |\nabla u^{\omega,n}|^2 = \left(\frac{1}{2} n^{-d} \sum_{i=1}^{d} \sin^2 \alpha_i \right) I = \frac{1}{2} n^{-d} h I, \end{aligned}$$

where I is a d-order unit matrix and $h = \sum_{i=1}^d \sin^2 \alpha_i$. Consequently, combining above equalities and the fact that $\mathrm{div} u^{\omega,n} = 0$ yields

$$
\begin{aligned}
E_2(t) &= \operatorname{div}\left(\nabla u^{\omega,n}(\nabla u^{\omega,n}+(\nabla u^{\omega,n})^T)-(\nabla u^{\omega,n})^T\nabla u^{\omega,n}+\frac{1}{2}I|\nabla u^{\omega,n}|^2)\right)\\
&= n^{-d}\operatorname{div}\\
&\begin{pmatrix}
\begin{bmatrix}\sin\alpha_1(\sin\alpha_1+\sin\alpha_d)\\ -\sin^2\alpha_d+\frac{h}{2}\end{bmatrix} & 0 & \cdots & 0\\
0 & \begin{bmatrix}\sin\alpha_2(\sin\alpha_2+\sin\alpha_{d-1})\\ -\sin^2\alpha_{d-1}+\frac{h}{2}\end{bmatrix} & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & \begin{bmatrix}\sin\alpha_d(\sin\alpha_d+\sin\alpha_1)\\ -\sin^2\alpha_1+\frac{h}{2}\end{bmatrix}
\end{pmatrix}\\
&= n^{-d+1}\left(\sin\alpha_i\cos\alpha_{d+1-i}-\frac{1}{2}\sin 2\alpha_{d+1-i}\right)_{1\le i\le d}.
\end{aligned}
$$

By the definition of $\rho^{\omega,n}$, we derive that

$$
\begin{aligned}
\nabla\rho^{\omega,n} &= (-n^{-\frac{d}{2}+1}\sin\alpha_{d+1-i})_{1\le i\le d},\\
\nabla(\rho^{\omega,n})^2 &= \left(-2\omega n^{-\frac{d}{2}}\sin\alpha_{d+1-i}-2n^{-d+1}\sin\alpha_{d+1-i}\sum_{i=1}^{d}\cos\alpha_i\right)_{1\le i\le d},\\
\partial_t\rho^{\omega,n} &= \omega n^{-\frac{d}{2}}\sum_{i=1}^{d}\sin\alpha_i.
\end{aligned}
$$

As a result, we deduce that

$$
\begin{aligned}
E_3(t) &= u^{\omega,n}\nabla u^{\omega,n}+\frac{1}{2}\nabla(\rho^{\omega,n})^2\\
&= \left(-\omega\left(n^{-\frac{d}{2}-1}+n^{-\frac{d}{2}}\right)\sin\alpha_{d+1-i}\right)_{1\le i\le d}\\
&\quad+\left(-\frac{1}{2}n^{-1-d}\sin 2\alpha_{d+1-i}-n^{-d+1}\sin\alpha_{d+1-i}\sum_{i=1}^{d}\cos\alpha_i\right)_{1\le i\le d},\\
F(t) &= \partial_t\rho^{\omega,n}+u^{\omega,n}(\nabla\rho^{\omega,n})^T = -n^{-d}\sum_{i=1}^{d}\cos\alpha_i\sin\alpha_{d+1-i}.
\end{aligned}
$$

Estimating the $\|E_1(t)\|_{B^{\frac{d}{2}}_{2,1}}$, $\|f(u^{\omega,n},\rho^{\omega,n})\|_{B^{\frac{d}{2}}_{2,1}}$, $\|g(u^{\omega,n},\rho^{\omega,n})\|_{B^{\frac{d}{2}}_{2,1}}$, $\|F(t)\|_{B^{\frac{d}{2}-1}_{2,1}}$ by using Lemma 2.2 yields

$$
\begin{aligned}
\|E_1(t)\|_{B^{\frac{d}{2}}_{2,1}} &= \left\|(-n^{-1-d}\sin\alpha_i\cos\alpha_{d+1-i})_{1\le i\le d}\right\|_{B^{\frac{d}{2}}_{2,1}}\\
&\lesssim n^{-1-d}\sum_{i=1}^{d}\|\sin\alpha_i\cos\alpha_{d+1-i}\|_{B^{\frac{d}{2}}_{2,1}}\lesssim n^{-1-\frac{d}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\|f(u^{\omega,n},\rho^{\omega,n})\|_{B^{\frac{d}{2}}_{2,1}} &= \Bigg\|(1-\Delta)^{-1} \\
&\qquad \left(n^{-d+1}\left(\sin\alpha_i\cos\alpha_{d+1-i}-\frac{1}{2}\sin 2\alpha_{d+1-i}\right)_{1\le i\le d}\right)\Bigg\|_{B^{\frac{d}{2}}_{2,1}} \\
&\lesssim n^{-d+1}\left(\sum_{i=1}^{d}\|\sin\alpha_i\cos\alpha_{d+1-i}\|_{B^{\frac{d}{2}-2}_{2,1}}+\sum_{i=1}^{d}\|\sin 2\alpha_{d+1-i}\|_{B^{\frac{d}{2}-2}_{2,1}}\right) \\
&\lesssim n^{-1-\frac{d}{2}}, \\
\|g(u^{\omega,n},\rho^{\omega,n})\|_{B^{\frac{d}{2}}_{2,1}} &\lesssim \left\|(1-\Delta)^{-1}\left(n^{-\frac{d}{2}-1}+n^{-\frac{d}{2}}\right)(\sin\alpha_{d+1-i})_{1\le i\le d}\right\|_{B^{\frac{d}{2}}_{2,1}} \\
&\quad+\left\|(1-\Delta)^{-1}\left(n^{-1-d}(\sin 2\alpha_{d+1-i})_{1\le i\le d}\right)\right\|_{B^{\frac{d}{2}}_{2,1}} \\
&\quad+\left\|(1-\Delta)^{-1}\left(n^{-d+1}\sin\alpha_{d+1-i}\sum_{i=1}^{d}\cos\alpha_i\right)_{1\le i\le d}\right\|_{B^{\frac{d}{2}}_{2,1}} \\
&\lesssim \left(n^{-\frac{d}{2}-1}+n^{-\frac{d}{2}}\right)\sum_{i=1}^{d}\|\sin\alpha_{d+1-i}\|_{B^{\frac{d}{2}-2}_{2,1}} \\
&\quad+n^{-1-d}\sum_{i=1}^{d}\|\sin 2\alpha_{d+1-i}\|_{B^{\frac{d}{2}-2}_{2,1}} \\
&\quad+n^{-d+1}\sum_{j=1}^{d}\sum_{i=1}^{d}\|\sin\alpha_{d+1-j}\cos\alpha_i\|_{B^{\frac{d}{2}-2}_{2,1}} \\
&\lesssim n^{-3}+n^{-2}+n^{-3-\frac{d}{2}}+n^{-1-\frac{d}{2}}\lesssim n^{-2}, \\
\|F(t)\|_{B^{\frac{d}{2}-1}_{2,1}} &\lesssim n^{-d}\sum_{i=1}^{d}\|\cos\alpha_i\sin\alpha_{d+1-i}\|_{B^{\frac{d}{2}-1}_{2,1}}\lesssim n^{-1-\frac{d}{2}}.
\end{aligned}
$$

Putting the above estimates together yields

$$
\begin{aligned}
&\|E(t)\|_{B^{\frac{d}{2}}_{2,1}} \lesssim \|E_1(t)\|_{B^{\frac{d}{2}}_{2,1}}+\|f(u^{\omega,n},\rho^{\omega,n})\|_{B^{\frac{d}{2}}_{2,1}}+\|g(u^{\omega,n},\rho^{\omega,n})\|_{B^{\frac{d}{2}}_{2,1}}\lesssim n^{-2}, \\
&\|F(t)\|_{B^{\frac{d}{2}-1}_{2,1}} \lesssim n^{-1-\frac{d}{2}}\lesssim n^{-2}.
\end{aligned}
$$

This completes the proof of Lemma 3.2. □

3.2 Difference between approximate and actual solutions

In this section, we also only consider the case that $d \ge 2$ is even. The proof of the case that $d \ge 3$ is odd is similar. Let $z_{\omega,n}(t,x) = (u_{\omega,n}(t,x), \rho_{\omega,n}(t,x))$ be the solution of the system (3.1) with initial data given by the approximate solution $z^{\omega,n}(t,x) = (u^{\omega,n}(t,x), \rho^{\omega,n}(t,x))$ evaluated at the initial time. That means $z_{\omega,n}(t,x)$ solves the following Cauchy problem

$$
\begin{cases}
\partial_t u_{\omega,n} + u_{\omega,n} \cdot \nabla u_{\omega,n} + f(u_{\omega,n}, \rho_{\omega,n}) + g(u_{\omega,n}, \rho_{\omega,n}) = 0, \quad t > 0, \quad x \in \mathbb{T}^d, \\
\partial_t \rho_{\omega,n} + u_{\omega,n} \cdot \nabla \rho_{\omega,n} + \rho_{\omega,n} \mathrm{div} u_{\omega,n} = 0, \quad t > 0, \quad x \in \mathbb{T}^d, \\
u_{\omega,n}(0,x) = u^{\omega,n}(0,x) = (\omega n^{-1} + n^{-1-\frac{d}{2}} \cos n x_{d+1-i})_{1 \le i \le d}, \\
\rho_{\omega,n}(0,x) = \rho^{\omega,n}(0,x) = \omega n^{-1} + n^{-\frac{d}{2}} \sum_{i=1}^d \cos n x_{d+1-i},
\end{cases}
\tag{3.4}
$$

where

$$
\begin{aligned}
f(u_{\omega,n}, \rho_{\omega,n}) &= (I-\Delta)^{-1} \mathrm{div} \left(\nabla u_{\omega,n} (\nabla u_{\omega,n} + (\nabla u_{\omega,n})^T) - (\nabla u_{\omega,n})^T \nabla u_{\omega,n} \right) \\
&\quad + (I-\Delta)^{-1} \mathrm{div} \left(-\nabla u_{\omega,n} (\mathrm{div} u_{\omega,n}) + \frac{1}{2} I |\nabla u_{\omega,n}|^2) \right), \\
g(u_{\omega,n}, \rho_{\omega,n}) &= (I-\Delta)^{-1} \left(u_{\omega,n} \mathrm{div} u_{\omega,n} + u_{\omega,n} \cdot (\nabla u_{\omega,n})^T + \frac{1}{2} \nabla (\rho_{\omega,n})^2 \right).
\end{aligned}
$$

Due to Lemma 2.2, we deduce that

$$
\begin{aligned}
\|u_{\omega,n}(0)\|_{B^{1+\frac{d}{2}}_{2,1}} &\lesssim \sum_{i=1}^d \|\omega n^{-1} + n^{-1-\frac{d}{2}} \cos n x_{d+1-i}\|_{B^{1+\frac{d}{2}}_{2,1}} \lesssim 1, \\
\|\rho_{\omega,n}(0)\|_{B^{\frac{d}{2}}_{2,1}} &\lesssim n^{-1} + \sum_{i=1}^d \|n^{-\frac{d}{2}} \cos n x_{d+1-i}\|_{B^{\frac{d}{2}}_{2,1}} \lesssim 1.
\end{aligned}
$$

According to Lemma 3.1, we derive that $(u_{\omega,n}, \rho_{\omega,n})$ is the unique solution of the initial value problem (3.4) with the maximal existence time

$$
T > T_0 := \frac{1}{4C^2 \|z_{\omega,n}(0)\|_{B^{1+\frac{d}{2}}_{2,1} \times B^{\frac{d}{2}}_{2,1}}} = \frac{1}{4C^2 \left(\|u_{\omega,n}(0)\|_{B^{1+\frac{d}{2}}_{2,1}} + \|\rho_{\omega,n}(0)\|_{B^{\frac{d}{2}}_{2,1}} \right)}.
$$

To estimate the difference between approximate and actual solutions, letting $\sigma = u^{\omega,n} - u_{\omega,n}$, $\tau = \rho^{\omega,n} - \rho_{\omega,n}$ yields

$$
\begin{cases}
\partial_t \sigma + u^{\omega,n} \cdot \nabla \sigma + \sigma \cdot \nabla u_{\omega,n} - E(t) + f(u^{\omega,n}, \rho^{\omega,n}) \\
\qquad - f(u_{\omega,n}, \rho_{\omega,n}) + g(u^{\omega,n}, \rho^{\omega,n}) - g(u_{\omega,n}, \rho_{\omega,n}) = 0, \\
\partial_t \tau + u_{\omega,n} \cdot \nabla \tau + \sigma \cdot \nabla \rho^{\omega,n} + \tau \mathrm{div} u^{\omega,n} + \rho_{\omega,n} \mathrm{div} \sigma - F(t) = 0, \\
\sigma(0,x) = \sigma_0 = 0, \quad \tau(0,x) = \tau_0 = 0,
\end{cases}
\tag{3.5}
$$

where

$$
\begin{aligned}
& f(u^{\omega,n}, \rho^{\omega,n}) - f(u_{\omega,n}, \rho_{\omega,n}) \\
&\quad = (I-\Delta)^{-1} \mathrm{div} \left(\nabla (u^{\omega,n} + u_{\omega,n}) \nabla \sigma + \nabla \sigma (\nabla u^{\omega,n})^T + \nabla u_{\omega,n} (\nabla \sigma)^T \right) \\
&\qquad + (I-\Delta)^{-1} \mathrm{div} \left(-(\nabla \sigma)^T \nabla u^{\omega,n} - (\nabla u_{\omega,n})^T \nabla \sigma - \nabla \sigma \mathrm{div} u^{\omega,n} - \nabla u_{\omega,n} \mathrm{div} \sigma \right)
\end{aligned}
$$

$$+(I-\Delta)^{-1}\mathrm{div}\left(\frac{1}{2}\nabla(u^{\omega,n}+u_{\omega,n}):\nabla\sigma\right),$$

$$g(u^{\omega,n},\rho^{\omega,n})-g(u_{\omega,n},\rho_{\omega,n})$$

$$=(I-\Delta)^{-1}\left(\sigma\mathrm{div}u^{\omega,n}+u_{\omega,n}\mathrm{div}\sigma+\sigma\nabla u^{\omega,n}+u_{\omega,n}\nabla\sigma+\frac{1}{2}\nabla((\rho^{\omega,n}+\rho_{\omega,n})\tau)\right).$$

Lemma 3.3 *When $\omega=-1,1$, $n\gg 1$, $d\geq 2$, $n,d\in\mathbb{Z}^+$, we have*

$$\|\sigma(t)\|_{B^{\frac{d}{2}}_{2,1}},\ \|\tau(t)\|_{B^{\frac{d}{2}-1}_{2,1}}\lesssim n^{-2},\quad 0\leq t\leq T_0. \tag{3.6}$$

Proof Dealing the first and second equation in (3.5) with Lemmas 2.5 and 2.7 yields

$$\begin{aligned}\|\sigma(t)\|_{B^{\frac{d}{2}}_{2,1}}\lesssim{}&\exp\left(\int_0^t\|\nabla u^{\omega,n}(\eta)\|_{B^{\frac{d}{2}}_{2,1}\cap L^\infty}d\eta\right)\\&\times\left(\|\sigma_0\|_{B^{\frac{d}{2}}_{2,1}}+\int_0^t\left(\|\sigma\cdot\nabla u_{\omega,n}\|_{B^{\frac{d}{2}}_{2,1}}+\|E(\eta)\|_{B^{\frac{d}{2}}_{2,1}}\right)d\eta\right.\\&+\int_0^t\left(\|f(u^{\omega,n},\rho^{\omega,n})-f(u_{\omega,n},\rho_{\omega,n})\|_{B^{\frac{d}{2}}_{2,1}}\right.\\&\left.\left.+\|g(u^{\omega,n},\rho^{\omega,n})-g(u_{\omega,n},\rho_{\omega,n})\|_{B^{\frac{d}{2}}_{2,1}}\right)d\eta\right),\end{aligned}\tag{3.7}$$

$$\begin{aligned}\|\tau(t)\|_{B^{\frac{d}{2}-1}_{2,1}}\lesssim{}&\exp\left(\int_0^t\|u_{\omega,n}(\eta)\|_{B^{\frac{d}{2}+1}_{2,1}}d\eta\right)\\&\times\left(\|\tau_0\|_{B^{\frac{d}{2}-1}_{2,1}}+\int_0^t\|F(\eta)\|_{B^{\frac{d}{2}-1}_{2,1}}d\eta\right.\\&\left.+\int_0^t\|\sigma\cdot\nabla\rho^{\omega,n}+\tau\mathrm{div}u^{\omega,n}+\rho_{\omega,n}\mathrm{div}\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}d\eta\right).\end{aligned}\tag{3.8}$$

From Lemmas 2.2 and 3.1, we deduce that

$$\|u^{\omega,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}\lesssim\sum_{i=1}^d\|\omega n^{-1}+n^{-1-\frac{d}{2}}\cos\alpha_i\|_{B^{1+\frac{d}{2}}_{2,1}}\lesssim 1,$$

$$\|\rho^{\omega,n}(t)\|_{B^{\frac{d}{2}}_{2,1}}\lesssim n^{-1}+\sum_{i=1}^d\|n^{-\frac{d}{2}}\cos\alpha_i\|_{B^{\frac{d}{2}}_{2,1}}\lesssim 1,$$

$$\|\rho_{\omega,n}(t)\|_{B^{\frac{d}{2}}_{2,1}},\ \|u_{\omega,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}\lesssim\|u^{\omega,n}(0)\|_{B^{1+\frac{d}{2}}_{2,1}}+\|\rho^{\omega,n}(0)\|_{B^{\frac{d}{2}}_{2,1}}\lesssim 1.$$

Using the properties (ii), (iii) in Proposition 2.2 and Lemma 2.1 yields

$$\|\nabla u^{\omega,n}\|_{B^{\frac{d}{2}}_{2,1}\cap L^\infty}\lesssim\|u^{\omega,n}\|_{B^{1+\frac{d}{2}}_{2,1}}\lesssim 1,\quad\|u_{\omega,n}\|_{B^{\frac{d}{2}+1}_{2,1}}\lesssim 1,$$

$$
\begin{aligned}
&\|\sigma\cdot\nabla u_{\omega,n}\|_{B^{\frac{d}{2}}_{2,1}}\lesssim\|\sigma\|_{B^{\frac{d}{2}}_{2,1}}\|(\nabla u_{\omega,n})^T\|_{B^{\frac{d}{2}}_{2,1}}\lesssim\|\sigma\|_{B^{\frac{d}{2}}_{2,1}}\|u_{\omega,n}\|_{B^{1+\frac{d}{2}}_{2,1}}\lesssim\|\sigma\|_{B^{\frac{d}{2}}_{2,1}},\\
&\|f(u^{\omega,n},\rho^{\omega,n})-f(u_{\omega,n},\rho_{\omega,n})\|_{B^{\frac{d}{2}}_{2,1}}\\
&\quad\lesssim\|\nabla(u^{\omega,n}+u_{\omega,n})\nabla\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}+\|\nabla\sigma(\nabla u^{\omega,n})^T\|_{B^{\frac{d}{2}-1}_{2,1}}+\|\nabla u_{\omega,n}(\nabla\sigma)^T\|_{B^{\frac{d}{2}-1}_{2,1}}\\
&\qquad+\|(\nabla\sigma)^T\nabla u^{\omega,n}\|_{B^{\frac{d}{2}-1}_{2,1}}+\|(\nabla u_{\omega,n})^T\nabla\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}+\|\nabla\sigma\,\mathrm{div}u^{\omega,n}\|_{B^{\frac{d}{2}-1}_{2,1}}\\
&\qquad+\|\nabla u_{\omega,n}\mathrm{div}\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}+\|\nabla(u^{\omega,n}+u_{\omega,n}):\nabla\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}\\
&\quad\lesssim\|\nabla(u^{\omega,n}+u_{\omega,n})\|_{B^{\frac{d}{2}}_{2,1}}\|\nabla\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}+\|\nabla\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}\|(\nabla u^{\omega,n})^T\|_{B^{\frac{d}{2}}_{2,1}}\\
&\qquad+\|\nabla u_{\omega,n}\|_{B^{\frac{d}{2}}_{2,1}}\|(\nabla\sigma)^T\|_{B^{\frac{d}{2}-1}_{2,1}}+\|(\nabla\sigma)^T\|_{B^{\frac{d}{2}-1}_{2,1}}\|\nabla u^{\omega,n}\|_{B^{\frac{d}{2}}_{2,1}}\\
&\qquad+\|(\nabla u_{\omega,n})^T\|_{B^{\frac{d}{2}}_{2,1}}\|\nabla\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}+\|\nabla\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}\|\mathrm{div}u^{\omega,n}\|_{B^{\frac{d}{2}}_{2,1}}\\
&\qquad+\|\nabla u_{\omega,n}\|_{B^{\frac{d}{2}}_{2,1}}\|\mathrm{div}\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}\\
&\quad\lesssim\left(\|u^{\omega,n}\|_{B^{1+\frac{d}{2}}_{2,1}}+\|u_{\omega,n}\|_{B^{1+\frac{d}{2}}_{2,1}}\right)\|\sigma\|_{B^{\frac{d}{2}}_{2,1}}\lesssim\|\sigma\|_{B^{\frac{d}{2}}_{2,1}},\\
&\|g(u^{\omega,n},\rho^{\omega,n})-g(u_{\omega,n},\rho_{\omega,n})\|_{B^{\frac{d}{2}}_{2,1}}\\
&\quad\lesssim\|\sigma\,\mathrm{div}u^{\omega,n}\|_{B^{\frac{d}{2}-2}_{2,1}}+\|u_{\omega,n}\mathrm{div}\sigma\|_{B^{\frac{d}{2}-2}_{2,1}}+\|\sigma\nabla u^{\omega,n}\|_{B^{\frac{d}{2}-2}_{2,1}}+\|u_{\omega,n}\nabla\sigma\|_{B^{\frac{d}{2}-2}_{2,1}}\\
&\qquad+\|\nabla((\rho^{\omega,n}+\rho_{\omega,n})\tau)\|_{B^{\frac{d}{2}-2}_{2,1}}\\
&\quad\lesssim\|\sigma\,\mathrm{div}u^{\omega,n}\|_{B^{\frac{d}{2}-1}_{2,1}}+\|u_{\omega,n}\mathrm{div}\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}+\|\sigma\nabla u^{\omega,n}\|_{B^{\frac{d}{2}-1}_{2,1}}+\|u_{\omega,n}\nabla\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}\\
&\qquad+\|(\rho^{\omega,n}+\rho_{\omega,n})\tau\|_{B^{\frac{d}{2}-1}_{2,1}}\\
&\quad\lesssim\left(\|u^{\omega,n}\|_{B^{1+\frac{d}{2}}_{2,1}}+\|u_{\omega,n}\|_{B^{1+\frac{d}{2}}_{2,1}}+\|\rho^{\omega,n}\|_{B^{\frac{d}{2}}_{2,1}}+\|\rho_{\omega,n}\|_{B^{\frac{d}{2}}_{2,1}}\right)\left(\|\sigma\|_{B^{\frac{d}{2}}_{2,1}}+\|\tau\|_{B^{\frac{d}{2}-1}_{2,1}}\right)\\
&\quad\lesssim\|\sigma\|_{B^{\frac{d}{2}}_{2,1}}+\|\tau\|_{B^{\frac{d}{2}-1}_{2,1}}.
\end{aligned}
$$

In the same manner, we obtain

$$
\begin{aligned}
&\|\sigma\cdot\nabla\rho^{\omega,n}+\tau\,\mathrm{div}u^{\omega,n}+\rho_{\omega,n}\mathrm{div}\sigma\|_{B^{\frac{d}{2}-1}_{2,1}}\\
&\quad\lesssim\left(\|u^{\omega,n}\|_{B^{1+\frac{d}{2}}_{2,1}}+\|\rho^{\omega,n}\|_{B^{\frac{d}{2}}_{2,1}}+\|\rho_{\omega,n}\|_{B^{\frac{d}{2}}_{2,1}}\right)\left(\|\sigma\|_{B^{\frac{d}{2}}_{2,1}}+\|\tau\|_{B^{\frac{d}{2}-1}_{2,1}}\right)\\
&\quad\lesssim\|\sigma\|_{B^{\frac{d}{2}}_{2,1}}+\|\tau\|_{B^{\frac{d}{2}-1}_{2,1}}.
\end{aligned}
$$

As a consequence of $\|\sigma_0\|_{B^{\frac{d}{2}-1}_{2,1}} = \|\tau_0\|_{B^{\frac{d}{2}}_{2,1}} = 0$, (3.7) and (3.8) reduce to

$$\|\sigma(t)\|_{B^{\frac{d}{2}}_{2,1}} \lesssim \|E(t)\|_{B^{\frac{d}{2}}_{2,1}} + \int_0^t \left(\|\sigma(\eta)\|_{B^{\frac{d}{2}}_{2,1}} + \|\tau(\eta)\|_{B^{\frac{d}{2}-1}_{2,1}}\right) d\eta, \tag{3.9}$$

$$\|\tau(t)\|_{B^{\frac{d}{2}-1}_{2,1}} \lesssim \|F(t)\|_{B^{\frac{d}{2}-1}_{2,1}} + \int_0^t \left(\|\sigma(\eta)\|_{B^{\frac{d}{2}}_{2,1}} + \|\tau(\eta)\|_{B^{\frac{d}{2}-1}_{2,1}}\right) d\eta. \tag{3.10}$$

Combining (3.9). (3.10) and Lemma 3.2 yields

$$\|\hat{z}(t)\|_{B^{\frac{d}{2}}_{2,1}\times B^{\frac{d}{2}-1}_{2,1}} = \|\sigma(t)\|_{B^{\frac{d}{2}}_{2,1}} + \|\tau(t)\|_{B^{\frac{d}{2}-1}_{2,1}} \lesssim n^{-2} + \int_0^t \|\hat{z}(\eta)\|_{B^{\frac{d}{2}}_{2,1}\times B^{\frac{d}{2}-1}_{2,1}} d\eta.$$

Using the Gronwall's inequality, we obtain (3.6). This completes the proof of Lemma 3.3. □

3.3 Non-uniform dependence

In this section, we only discuss the case that $d \geq 2$ is even in detail. The proof of the case that $d \geq 3$ is odd is similar, so we omit it here. Let $z_{1,n}(t,x)$ and $z_{-1,n}(t,x)$ are the solutions to the system (3.1) with the initial data $z^{1,n}(0,x)$ and $z^{-1,n}(0,x)$, respectively. Using Lemma 2.2, we obtain

$$\|z_{1,n}(0)\|_{B^{1+\frac{d}{2}}_{2,1}\times B^{\frac{d}{2}}_{2,1}} \lesssim \|z^{1,n}(0)\|_{B^{1+\frac{d}{2}}_{2,1}\times B^{\frac{d}{2}}_{2,1}} \lesssim 1,$$

and

$$\|z_{-1,n}(0)\|_{B^{1+\frac{d}{2}}_{2,1}\times B^{\frac{d}{2}}_{2,1}} \lesssim \|z^{-1,n}(0)\|_{B^{1+\frac{d}{2}}_{2,1}\times B^{\frac{d}{2}}_{2,1}} \lesssim 1.$$

According to Lemma 3.1, we know that $z_{1,n}(t,x)$ and $z_{-1,n}(t,x)$ are the unique solutions to the initial value problem (3.1) with the initial value $z^{1,n}(0,x)$ and $z_{-1,n}(0,x)$, respectively. And the maximal existence time T is independent of n. In order to achieve our goals, we have to prove the following lemma.

Lemma 3.4 *When $n \gg 1, d \geq 2$, $n, d \in \mathbb{Z}^+$, we have*

$$\|u^{\pm1,n}(t) - u_{\pm1,n}(t)\|_{B^{2+\frac{d}{2}}_{2,1}}, \|\rho^{\pm1,n}(t) - \rho_{\pm1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}} \lesssim n, \quad 0 \leq t \leq T_0.$$

Proof For the equations in (3.4), using Lemma 2.6 yields

$$\begin{aligned}\|u_{\omega,n}(t)\|_{B^{2+\frac{d}{2}}_{2,1}} &\lesssim \|u_{\omega,n}(0)\|_{B^{2+\frac{d}{2}}_{2,1}} \\ &\quad + \int_0^t \left(\|f(u_{\omega,n},\rho_{\omega,n})\|_{B^{2+\frac{d}{2}}_{2,1}} + \|g(u_{\omega,n},\rho_{\omega,n})\|_{B^{2+\frac{d}{2}}_{2,1}}\right) d\eta\end{aligned}$$

$$+\int_0^t \|u_{\omega,n}\|_{B_{2,1}^{2+\frac{d}{2}}} \|u_{\omega,n}\|_{B_{2,1}^{1+\frac{d}{2}}} d\eta,$$

$$\begin{aligned}\|\rho_{\omega,n}(t)\|_{B_{2,1}^{1+\frac{d}{2}}} \lesssim &\|\rho_{\omega,n}(0)\|_{B_{2,1}^{1+\frac{d}{2}}}\\ &+\int_0^t \left(\|\rho_{\omega,n}\|_{B_{2,1}^{1+\frac{d}{2}}} \|\nabla u_{\omega,n}\|_{L^\infty} + \|\nabla u_{\omega,n}\|_{B_{2,1}^{\frac{d}{2}}} \|\nabla \rho_{\omega,n}\|_{L^\infty}\right) d\eta\\ &+\int_0^t \|\rho_{\omega,n}\mathrm{div} u_{\omega,n}\|_{B_{2,1}^{1+\frac{d}{2}}} d\eta.\end{aligned}$$

From Lemma 2.1, we deduce that

$$\begin{aligned}\|u_{\omega,n}(t)\|_{B_{2,1}^{2+\frac{d}{2}}} \lesssim &\|u_{\omega,n}(0)\|_{B_{2,1}^{2+\frac{d}{2}}}\\ &+\int_0^t \left(\|u_{\omega,n}(\eta)\|_{B_{2,1}^{1+\frac{d}{2}}} + \|\rho_{\omega,n}(\eta)\|_{B_{2,1}^{\frac{d}{2}}}\right)\\ &\left(\|u_{\omega,n}(\eta)\|_{B_{2,1}^{2+\frac{d}{2}}} + \|\rho_{\omega,n}(\eta)\|_{B_{2,1}^{1+\frac{d}{2}}}\right) d\eta, \end{aligned} \tag{3.11}$$

$$\begin{aligned}\|\rho_{\omega,n}(t)\|_{B_{2,1}^{1+\frac{d}{2}}} \lesssim &\|\rho_{\omega,n}(0)\|_{B_{2,1}^{1+\frac{d}{2}}}\\ &+\int_0^t \left(\|u_{\omega,n}(\eta)\|_{B_{2,1}^{1+\frac{d}{2}}} + \|\rho_{\omega,n}(\eta)\|_{B_{2,1}^{\frac{d}{2}}}\right)\\ &\left(\|u_{\omega,n}(\eta)\|_{B_{2,1}^{2+\frac{d}{2}}} + \|\rho_{\omega,n}(\eta)\|_{B_{2,1}^{1+\frac{d}{2}}}\right) d\eta. \end{aligned} \tag{3.12}$$

Combining $\|u_{\omega,n}\|_{B_{2,1}^{1+\frac{d}{2}}}, \|\rho_{\omega,n}\|_{B_{2,1}^{\frac{d}{2}}} \lesssim 1$, (3.11) and (3.12) yields

$$\begin{aligned}&\|u_{\omega,n}(t)\|_{B_{2,1}^{2+\frac{d}{2}}} + \|\rho_{\omega,n}(t)\|_{B_{2,1}^{1+\frac{d}{2}}}\\ &= \|z_{\omega,n}(t)\|_{B_{2,1}^{2+\frac{d}{2}} \times B_{2,1}^{1+\frac{d}{2}}}\\ &\lesssim \|z_{\omega,n}(0)\|_{B_{2,1}^{2+\frac{d}{2}} \times B_{2,1}^{1+\frac{d}{2}}} + \int_0^t \|z_{\omega,n}(\eta)\|_{B_{2,1}^{2+\frac{d}{2}} \times B_{2,1}^{1+\frac{d}{2}}} d\eta.\end{aligned}$$

Using the Gronwall's inequality and Lemma 2.2 yields

$$\begin{aligned}&\|u_{\omega,n}(t)\|_{B_{2,1}^{2+\frac{d}{2}}}, \|\rho_{\omega,n}(t)\|_{B_{2,1}^{1+\frac{d}{2}}}\\ &\lesssim \|z_{\omega,n}(0)\|_{B_{2,1}^{2+\frac{d}{2}} \times B_{2,1}^{1+\frac{d}{2}}} \lesssim \|z^{\omega,n}(0)\|_{B_{2,1}^{2+\frac{d}{2}} \times B_{2,1}^{1+\frac{d}{2}}} \lesssim n, \quad 0 \le t \le T_0.\end{aligned}$$

Furthermore,

$$\|u^{\pm 1,n}(t)-u_{\pm 1,n}(t)\|_{B^{2+\frac{d}{2}}_{2,1}} \lesssim \|u^{\pm 1,n}(t)\|_{B^{2+\frac{d}{2}}_{2,1}} + \|u_{\pm 1,n}(t)\|_{B^{2+\frac{d}{2}}_{2,1}} \lesssim n,$$
$$\|\rho^{\pm 1,n}(t)-\rho_{\pm 1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}} \lesssim \|\rho^{\pm 1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}} + \|\rho_{\pm 1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}} \lesssim n, \quad 0 \le t \le T_0.$$

This completes the proof of Lemma 3.4. □

By using Lemmas 3.3, 3.4 and the property $v)$ in Proposition 2.2, we have

$$\begin{aligned}\|u^{\pm 1,n}(t)-u_{\pm 1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}} &\lesssim \|u^{\pm,n}(t)-u_{\pm,n}(t)\|^{\frac{1}{2}}_{B^{2+\frac{d}{2}}_{2,1}} \|u^{\pm,n}(t)-u_{\pm,n}(t)\|^{\frac{1}{2}}_{B^{\frac{d}{2}}_{2,1}}\\ &\lesssim n^{-\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}\|\rho^{\pm 1,n}(t)-\rho_{\pm 1,n}(t)\|_{B^{\frac{d}{2}}_{2,1}} &\lesssim \|\rho^{\pm,n}(t)-\rho_{\pm,n}(t)\|^{\frac{1}{2}}_{B^{1+\frac{d}{2}}_{2,1}} \|u^{\pm,n}(t)-u_{\pm,n}(t)\|^{\frac{1}{2}}_{B^{\frac{d}{2}-1}_{2,1}}\\ &\lesssim n^{-\frac{1}{2}}.\end{aligned}$$

Using Lemma 2.2, we obtain

$$\|u_{-1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}} + \|u_{1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}} + \|\rho_{-1,n}(t)\|_{B^{\frac{d}{2}}_{2,1}} + \|\rho_{1,n}(t)\|_{B^{\frac{d}{2}}_{2,1}} \lesssim 1,$$
$$\|u_{1,n}(0)-u_{-1,n}(0)\|_{B^{1+\frac{d}{2}}_{2,1}},\ \|\rho_{1,n}(0)-\rho_{-1,n}(0)\|_{B^{\frac{d}{2}}_{2,1}} \lesssim n^{-1}.$$

Hence we have

$$\lim_{n\to\infty} \|u_{1,n}(0)-u_{-1,n}(0)\|_{B^{1+\frac{d}{2}}_{2,1}} = \lim_{n\to\infty} \|\rho_{1,n}(0)-\rho_{-1,n}(0)\|_{B^{\frac{d}{2}}_{2,1}} = 0.$$

Note that

$$\begin{aligned}&\lim_{n\to\infty}\left(\|u_{1,n}(t)-u_{-1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}} + \|\rho_{1,n}(t)-\rho_{-1,n}(t)\|_{B^{\frac{d}{2}}_{2,1}}\right)\\ &\gtrsim \lim_{n\to\infty} \|u_{1,n}(t)-u_{-1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}.\end{aligned}$$

From the formula

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2},$$

it follows that

$$\|u^{1,n}(t)-u^{-1,n}(t)\|_{B^{\frac{d}{2}+1}_{2,1}} \gtrsim n^{-1-\frac{d}{2}}\|\sin nx_{d+1-i}\|_{B^{1+\frac{d}{2}}_{2,1}}|\sin t| - n^{-1}.$$

Moreover, we have

$$\liminf_{n\to\infty}\|u^{1,n}(t)-u^{-1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}} \gtrsim \Big|\sin t\Big|.$$

Altogether, for all $0 \le t \le T_0$ we have

$$\begin{aligned}
&\|u_{1,n}(t)-u_{-1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}\\
&\quad= \|u^{1,n}(t)-u^{-1,n}(t)+u_{1,n}(t)-u^{1,n}(t)-u_{-1,n}(t)+u^{-1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}\\
&\quad\gtrsim \|u^{1,n}(t)-u^{-1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}\\
&\qquad -\left(\|u_{1,n}(t)-u^{1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}+\|u_{-1,n}(t)-u^{-1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}\right)\\
&\quad\gtrsim \|u^{1,n}(t)-u^{-1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}} - n^{-\frac{1}{2}}.
\end{aligned}$$

Consequently, we obtain

$$\liminf_{n\to\infty}\|u_{1,n}(t)-u_{-1,n}(t)\|_{B^{1+\frac{d}{2}}_{2,1}} \gtrsim |\sin t|.$$

This completes the proof of Theorem 1.1.

4 The proof of Theorem 1.2

In this section, we will pay attention to the initial value problem (3.1) with $d=1$. The non-uniform continuity of the solution map $z_0 \to z(t)$ in Besov spaces $B^{\frac{3}{2}}_{2,1}(\mathbb{T}) \times B^{\frac{1}{2}}_{2,1}(\mathbb{T})$ would be considered in detail. Since all spaces of functions are over $\mathbb{T}$, we drop $\mathbb{T}$ if there is no ambiguity. The problem can be rewritten as follows:

$$\begin{cases} u_t + uu_x + \partial_x(1-\partial_x^2)^{-1}\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right)=0, & t>0, \quad x\in\mathbb{T},\\ \rho_t + u\rho_x + \rho u_x = 0, \quad t>0 \quad x\in\mathbb{T}, & \\ u(0,x)=u_0, \quad \rho(0,x)=\rho_0. & \end{cases} \tag{4.1}$$

The local well-posedness results for initial value problem (4.1) is stated as follows:

Lemma 4.1 *Let $z_0 = (u_0, \rho_0) \in B^{\frac{3}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1}$, then there exists a time $T = T(z_0) > 0$ such that $z(t) \in E^{\frac{3}{2}}_{2,1}(T) \times E^{\frac{1}{2}}_{2,1}(T)$ is the unique solution to the initial value problem (4.1), and the solution depends continuously on the initial data, that is, the solution map $z_0 \mapsto z(t)$ is continuous from $B^{\frac{3}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1}$ into $C([0, T]; B^{\frac{3}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1})$. Furthermore, the solution $z(t)$ satisfies the following estimate*

$$\|z(t)\|_{B^{\frac{3}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1}} \leq 2C\|z_0\|_{B^{\frac{3}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1}}, \quad 0 \leq t \leq T_1 := \frac{1}{4C^2\|z_0\|_{B^{\frac{3}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1}}}, \tag{4.2}$$

where $C \geq 1$ is a constant independent of z_0 and $(u(t), \rho(t)) = z(t)$.

Proof The proof of existence, uniqueness and continuity of the solution map can be found in [19]. Therefore, our main goal is to establish (4.2). From the proof of Theorem 3.3 in [19], we know that there exist a constant $C \geq 1$ and a time T satisfying $2C^2T\|z_0\|_{B^{\frac{3}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1}} < 1$ such that for every $t \in [0, T]$, we have

$$\|z^n(t)\|_{B^{\frac{3}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1}} \leq \frac{C\|z_0\|_{B^{\frac{3}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1}}}{1 - 2C^2t\|z_0\|_{B^{\frac{3}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1}}}.$$

The rest proof is similar to Lemma 3.1. This completes the proof of Lemma 4.1. □

Next, Theorem 1.2 will be proved by a similar way. The approximate solutions can be given as follows:

$$u^{\omega,n} = \omega n^{-1} + n^{-\frac{3}{2}} \cos(nx - \omega t), \;\; \rho^{\omega,n} = \omega n^{-1} + n^{-\frac{1}{2}} \cos(nx - \omega t),$$

where $\omega = \pm 1$, $n \in \mathbb{Z}^+, n \gg 1$. Substituting $u^{\omega,n}$ and $\rho^{\omega,n}$ into the equations in (3.1), we obtain

$$G(t) = \partial_t u^{\omega,n} + u^{\omega,n}\partial_x u^{\omega,n} + \partial_x(1 - \partial_x^2)^{-1}\left((u^{\omega,n})^2 + \frac{1}{2}(\partial_x u^{\omega,n})^2 + \frac{1}{2}(\rho^{\omega,n})^2\right),$$
$$H(t) = \partial_t \rho^{\omega,n} + u^{\omega,n}\partial_x \rho^{\omega,n} + \rho^{\omega,n}\partial_x u^{\omega,n}.$$

A few simple calculations and estimations yield

$$\begin{aligned}\|G(t)\|_{B^{\frac{1}{2}}_{2,\infty}} &\lesssim \|n^{-2}\sin(2nx - 2\omega t)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|n^{-\frac{3}{2}}\sin(nx - \omega t)\|_{B^{-\frac{3}{2}}_{2,\infty}} \\ &\quad + \|n^{-2}\sin(nx - \omega t)\|_{B^{-\frac{3}{2}}_{2,\infty}} + \|n^{-\frac{1}{2}}\sin(2nx - 2\omega t)\|_{B^{-\frac{3}{2}}_{2,\infty}} \\ &\lesssim n^{-\frac{3}{2}} + n^{-3} + n^{-\frac{7}{2}} + n^{-2} \lesssim n^{-\frac{3}{2}},\end{aligned}$$

$$\|H(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \lesssim \|n^{-1}\sin(2nx-2\omega t)\|_{B^{-\frac{1}{2}}_{2,\infty}} + \|n^{-\frac{3}{2}}\sin(nx-\omega t)\|_{B^{-\frac{1}{2}}_{2,\infty}}$$
$$\lesssim n^{-\frac{3}{2}} + n^{-2} \lesssim n^{-\frac{3}{2}}.$$

Consequently, we have the following lemma.

Lemma 4.2 *When* $\omega = -1, 1$*,* $n \gg 1$*, we have*

$$\|G(t)\|_{B^{\frac{1}{2}}_{2,\infty}}, \|H(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \lesssim n^{-\frac{3}{2}}, \quad 0 \le t \le T_1.$$

In the following, the difference between approximate solution and actual solution will be taken into consideration as in Sect. 3.2. The Osgood Lemma is necessary during the process of estimations. Let $z_{\omega,n} = (u_{\omega,n}, \rho_{\omega,n})$ solve the following Cauchy problem:

$$\begin{cases} \partial_t u_{\omega,n} + u_{\omega,n}\partial_x u_{\omega,n} \\ \quad +\partial_x(1-\partial_x^2)^{-1}\left((u_{\omega,n})^2 + \frac{1}{2}(\partial_x u_{\omega,n})^2 + \frac{1}{2}(\rho_{\omega,n})^2\right) = 0, \quad t>0, \quad x\in\mathbb{T}, \\ \partial_t\rho_{\omega,n} + u_{\omega,n}\partial_x\rho_{\omega,n} + \rho_{\omega,n}\partial_x u_{\omega,n} = 0, \quad t>0, \quad x\in\mathbb{T}, \\ u_{\omega,n}(0,x) = u^{\omega,n}(0,x) = \omega n^{-1} + n^{-\frac{3}{2}}\cos nx, \\ \rho_{\omega,n}(0,x) = \rho^{\omega,n}(0,x) = \omega n^{-1} + n^{-\frac{1}{2}}\cos nx. \end{cases} \tag{4.3}$$

Since $\|u_{\omega,n}(0)\|_{B^{\frac{3}{2}}_{2,1}}, \|\rho_{\omega,n}(0)\|_{B^{\frac{1}{2}}_{2,1}} \lesssim 1$. From Lemma 4.1, we know the existence, uniqueness of $z_{\omega,n}$ and the lifespan satisfies

$$T > T_1 := \frac{1}{4C^2\|z_{\omega,n}(0)\|_{B^{\frac{3}{2}}_{2,1}\times B^{\frac{1}{2}}_{2,1}}}.$$

Similar to Theorem 1.1, letting $\sigma = u^{\omega,n} - u_{\omega,n}$, $\tau = \rho^{\omega,n} - \rho_{\omega,n}$ yields

$$\begin{cases} \partial_t\sigma + u^{\omega,n}\partial_x\sigma + \sigma\partial_x u_{\omega,n} + \partial_x(1-\partial_x^2)^{-1}\Bigg((u^{\omega,n}+u_{\omega,n})\sigma \\ \quad +\frac{1}{2}\partial_x(u^{\omega,n}+u_{\omega,n})\partial_x\sigma + \frac{1}{2}(\rho^{\omega,n}+\rho_{\omega,n})\tau\Bigg) - G(t) = 0, \quad t>0, \quad x\in\mathbb{T}, \\ \partial_t\tau + u_{\omega,n}\partial_x\tau + \sigma\partial_x\rho^{\omega,n} + \tau\partial_x u^{\omega,n} + \rho_{\omega,n}\partial_x\sigma - H(t) = 0, \quad t>0 \quad x\in\mathbb{T}, \\ \sigma(0,x) = \sigma_0 = \tau_0 = \tau(0,x) = 0. \end{cases} \tag{4.4}$$

Lemma 4.3 *When* $\omega = -1, 1$*,* $n \gg 1$*, we have*

$$\|\sigma(t)\|_{B^{\frac{1}{2}}_{2,\infty}}, \|\tau(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \lesssim n^{-\frac{3}{2}\exp(-ct)}, \tag{4.5}$$

$$\|\sigma(t)\|_{B^{\frac{5}{2}}_{2,1}}, \|\tau(t)\|_{B^{\frac{3}{2}}_{2,1}} \lesssim n, \quad 0 \le t \le T_1. \tag{4.6}$$

Proof Dealing the first and second equation in (4.4) with Lemmas 2.5 and 2.8 yields

$$
\begin{aligned}
\|\sigma(t)\|_{B^{\frac{1}{2}}_{2,\infty}} \lesssim& \exp\left(\int_0^t \|\partial_x u^{\omega,n}(\eta)\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\eta\right)\\
&\times\left(\|\sigma_0\|_{B^{\frac{1}{2}}_{2,\infty}} + \int_0^t \left(\|\sigma\partial_x u_{\omega,n}\|_{B^{\frac{1}{2}}_{2,\infty}} + \|G(\eta)\|_{B^{\frac{1}{2}}_{2,\infty}}\right) d\eta\right.\\
&+\int_0^t \left\|\partial_x(1-\partial_x^2)^{-1}\left((u^{\omega,n}+u_{\omega,n})\sigma + \frac{1}{2}\partial_x(u^{\omega,n}+u_{\omega,n})\partial_x\sigma\right.\right.\\
&\quad\left.\left.\left.+\frac{1}{2}(\rho^{\omega,n}+\rho_{\omega,n})\tau\right)\right\|_{B^{\frac{1}{2}}_{2,\infty}} d\eta\right),
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
\|\tau(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \lesssim& \exp\left(\int_0^t \|\partial_x u_{\omega,n}(\eta)\|_{B^{\frac{1}{2}}_{2,1}} d\eta\right)\\
&\times\left(\|\tau_0\|_{B^{-\frac{1}{2}}_{2,1}} + \int_0^t \|H(\eta)\|_{B^{-\frac{1}{2}}_{2,\infty}} d\eta + \int_0^t \|\sigma\partial_x\rho^{\omega,n} + \tau\partial_x u^{\omega,n} + \rho^{\omega,n}\partial_x\sigma\|_{B^{-\frac{1}{2}}_{2,\infty}} d\eta\right).
\end{aligned}
\tag{4.8}
$$

From Lemmas 2.2 and 4.1, we deduce that

$$
\|u_{\omega,n}(t)\|_{B^{\frac{3}{2}}_{2,1}}, \|\rho_{\omega,n}(t)\|_{B^{\frac{1}{2}}_{2,1}} \lesssim \|z_{\omega,n}(t)\|_{B^{\frac{3}{2}}_{2,1}\times B^{\frac{1}{2}}_{2,1}} \lesssim \|z^{\omega,n}(0)\|_{B^{\frac{3}{2}}_{2,1}\times B^{\frac{1}{2}}_{2,1}} \lesssim 1.
$$

Using Lemmas 2.1, 2.3 and 4.1 yields

$$
\begin{aligned}
&\|\partial_x u^{\omega,n}\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} \lesssim \|u^{\omega,n}\|_{B^{\frac{3}{2}}_{2,1}} \lesssim 1,\ \|\partial_x u_{\omega,n}\|_{B^{\frac{1}{2}}_{2,1}} \lesssim \|u_{\omega,n}\|_{B^{\frac{3}{2}}_{2,1}} \lesssim 1,\\
&\|\sigma\partial_x u_{\omega,n}\|_{B^{\frac{1}{2}}_{2,\infty}} \lesssim \|\sigma\partial_x u_{\omega,n}\|_{B^{\frac{1}{2}}_{2,1}} \lesssim \|\sigma\|_{B^{\frac{1}{2}}_{2,1}}\|\partial_x u_{\omega,n}\|_{B^{\frac{1}{2}}_{2,1}} \lesssim \|\sigma\|_{B^{\frac{1}{2}}_{2,1}},\\
&\left\|\partial_x(1-\partial_x^2)^{-1}\left((u^{\omega,n}+u_{\omega,n})\sigma + \frac{1}{2}\partial_x(u^{\omega,n}+u_{\omega,n})\partial_x\sigma + \frac{1}{2}(\rho^{\omega,n}+\rho_{\omega,n})\tau\right)\right\|_{B^{\frac{1}{2}}_{2,\infty}}\\
&\quad\lesssim \|(u^{\omega,n}+u_{\omega,n})\sigma\|_{B^{-\frac{1}{2}}_{2,\infty}} + \|\partial_x(u^{\omega,n}+u_{\omega,n})\partial_x\sigma\|_{B^{-\frac{1}{2}}_{2,\infty}} + \|(\rho^{\omega,n}+\rho_{\omega,n})\tau\|_{B^{-\frac{1}{2}}_{2,\infty}}\\
&\quad\lesssim \|\sigma\|_{B^{\frac{1}{2}}_{2,\infty}} + \|\tau\|_{B^{-\frac{1}{2}}_{2,\infty}},\\
&\|\sigma\partial_x\rho^{\omega,n} + \tau\partial_x u^{\omega,n} + \rho_{\omega,n}\partial_x\sigma\|_{B^{-\frac{1}{2}}_{2,\infty}} \lesssim \|\sigma\|_{B^{\frac{1}{2}}_{2,\infty}} + \|\tau\|_{B^{-\frac{1}{2}}_{2,\infty}}.
\end{aligned}
$$

Owing to Lemma 4.2 and $\|\sigma_0\|_{B^{\frac{1}{2}}_{2,\infty}} = \|\tau_0\|_{B^{-\frac{1}{2}}_{2,1}} = 0$, (4.7) and (4.8) reduce to

$$
\|\sigma(t)\|_{B^{\frac{1}{2}}_{2,\infty}} \lesssim n^{-\frac{3}{2}} + \int_0^t \left(\|\sigma(\eta)\|_{B^{\frac{1}{2}}_{2,1}} + \|\tau(\eta)\|_{B^{-\frac{1}{2}}_{2,1}}\right) d\eta,
$$

$$\|\tau(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \lesssim n^{-\frac{3}{2}} + \int_0^t \left(\|\sigma(\eta)\|_{B^{\frac{1}{2}}_{2,1}} + \|\tau(\eta)\|_{B^{-\frac{1}{2}}_{2,1}} \right) d\eta.$$

Let $\|\tilde{z}(t)\|_{B^{\frac{3}{2}}_{2,\infty}\times B^{\frac{1}{2}}_{2,\infty}} = \|\sigma\|_{B^{\frac{3}{2}}_{2,\infty}} + \|\tau\|_{B^{\frac{1}{2}}_{2,\infty}} \leq M$, and then using Lemma 2.4 yields

$$\|\sigma(t)\|_{B^{\frac{1}{2}}_{2,\infty}}, \|\tau(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \lesssim n^{-\frac{3}{2}} + \int_0^t \left(\|\sigma(\eta)\|_{B^{\frac{1}{2}}_{2,\infty}} \ln\left(e + \frac{M}{\|\sigma(\eta)\|_{B^{\frac{1}{2}}_{2,\infty}}}\right) \right.$$
$$\left. + \|\tau(\eta)\|_{B^{-\frac{1}{2}}_{2,\infty}} \ln\left(e + \frac{M}{\|\tau(\eta)\|_{B^{-\frac{1}{2}}_{2,\infty}}}\right) \right) d\eta.$$

Due to the fact that $x\ln(e + \frac{M}{x})$ is nondecreasing when $x > 0$ and $\ln(x + \frac{M}{x}) \leq (1 - ln\frac{x}{M})\ln(e+1)$ if $x \in (0, M]$, we have

$$\frac{\|\tilde{z}(t)\|_{B^{\frac{1}{2}}_{2,\infty}\times B^{-\frac{1}{2}}_{2,\infty}}}{M} \lesssim \frac{n^{-\frac{3}{2}}}{M} + \int_0^t \frac{\|\tilde{z}(t)\|_{B^{\frac{1}{2}}_{2,\infty}\times B^{-\frac{1}{2}}_{2,\infty}}}{M} \left(1 - \ln\left(\frac{\|\tilde{z}(t)\|_{B^{\frac{1}{2}}_{2,\infty}\times B^{-\frac{1}{2}}_{2,\infty}}}{M}\right)\right) d\eta.$$

Using Remark 2.1, (4.5) holds. The proof of (4.6) is similar to Lemma 3.4. This completes the proof of Lemma 4.3. □

Finally, we shall deal with the problem as in Sect. 3.3. However, the estimations of $\|\sigma(t)\|_{B^{\frac{3}{2}}_{2,1}}, \|\tau(t)\|_{B^{\frac{1}{2}}_{2,1}}$ are different. The property $vi)$ in Proposition 2.2 instead of $v)$ should be used in this part. With Lemma 4.3 in hand, the specific process is as follows:

$$\|\sigma(t)\|_{B^{\frac{3}{2}}_{2,1}} \lesssim \|\sigma(t)\|^{\frac{1}{2}}_{B^{\frac{1}{2}}_{2,\infty}} \|\sigma(t)\|^{\frac{1}{2}}_{B^{\frac{5}{2}}_{2,\infty}} \lesssim \|\sigma(t)\|^{\frac{1}{2}}_{B^{\frac{1}{2}}_{2,\infty}} \|\sigma(t)\|^{\frac{1}{2}}_{B^{\frac{5}{2}}_{2,1}} \lesssim n^{\frac{1}{2} - \frac{3\exp(-ct)}{4}},$$
$$\|\tau(t)\|_{B^{\frac{1}{2}}_{2,1}} \lesssim \|\tau(t)\|^{\frac{1}{2}}_{B^{-\frac{1}{2}}_{2,\infty}} \|\tau(t)\|^{\frac{1}{2}}_{B^{\frac{3}{2}}_{2,\infty}} \lesssim \|\tau(t)\|^{\frac{1}{2}}_{B^{-\frac{1}{2}}_{2,\infty}} \|\tau(t)\|^{\frac{1}{2}}_{B^{\frac{3}{2}}_{2,1}} \lesssim n^{\frac{1}{2} - \frac{3\exp(-ct)}{4}}.$$

Choosing $0 < T_2 < T_1$ and $0 \leq t \leq T_2$ such that $\exp(-ct) \geq 1 - \delta$ $(0 < \delta < \frac{1}{3})$, we obtain

$$\|\sigma(t)\|_{B^{\frac{3}{2}}_{2,1}}, \|\tau(t)\|_{B^{\frac{1}{2}}_{2,1}} \lesssim n^{\frac{3\delta-1}{4}}, \ 0 \leq t \leq T_2,$$

where $3\delta - 1 < 0$. The rest of proof is similar to Theorem 1.1. This completes the proof of Theorem 1.2.

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