Separately polynomial functions

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Abstract

It is known that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial in each variable, then f is a polynomial. We present generalizations of this fact, when $\mathbb{R}^2$ is replaced by $G \times H$, where G and H are topological Abelian groups. We show, e.g., that the conclusion holds (with generalized polynomials in place of polynomials) if G is a connected Baire space and H has a dense subgroup of finite rank or, for continuous functions, if G and H are connected Baire spaces. The condition of continuity can be omitted if G and H are locally compact or one of them is metrizable. We present several examples showing that the results are not far from being optimal.

Keywords Polynomials · Generalized polynomials · Functions on product spaces

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1 Introduction

It was proved by Carroll in [3] that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial in each variable, then f is a polynomial. Our aim is to find generalizations of this fact, when $\mathbb{R}^2$ is replaced by the product of two topological Abelian groups.

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On topological Abelian groups we have to distinguish between the class of polynomials and the wider class of generalized polynomials (see the next section for the definitions). The two classes coincide if the group contains a dense subgroup of finite rank. Now, the scalar product on the square of a Hilbert space is an example of a continuous function which is a polynomial in each variable, is a generalized polynomial on the product, but not a polynomial (see Example 1 below). Therefore, the appropriate problem is to find conditions on the groups G and H ensuring that whenever a function on $G \times H$ is a generalized polynomial in each variable, then it is a generalized polynomial.

This problem was considered already by Mazur and Orlicz in [10] in the case when G and H are topological vector spaces. They proved that if X, Y, Z are Banach spaces and the map $f \colon (X \times Y) \to Z$ is a generalized polynomial in each variable,[1] then f is a generalized polynomial [10, Satz IV]. They also considered the case when continuity is not assumed, and X, Y, Z are linear spaces without topology [10, Satz III] (see also [2, Lemma 1]). The topic has an extensive literature; see [14,16] and the references therein.

In this note we consider the analogous problem when G and H are topological Abelian groups. We show that if G is a connected Baire space, H has a dense subgroup of finite rank, and if a function $f \colon (G \times H) \to \mathbb{C}$ is a generalized polynomial in each variable, then f is a generalized polynomial on $G \times H$ (Theorem 1). The same conclusion holds if G and H are both connected Baire spaces, and one of them is metrizable or, if both are locally compact (Theorem 3).

If G and H are connected Baire spaces, $f \colon (G \times H) \to \mathbb{C}$ is a generalized polynomial in each variable, and if f has at least one point of joint continuity, then f is a generalized polynomial ((iii) of Theorem 2).

It is not clear if the extra condition of the existence of points of joint continuity can be omitted from this statement (Question 1). The problem is that a generalized polynomial must be continuous by definition, and a separately continuous function on the product of Baire spaces can be discontinuous everywhere, as it was shown recently in [11]. In our case, however, there are some extra conditions: the spaces are also connected, and the function in question is a generalized polynomial. It is conceivable that continuity follows under these conditions. As for the biadditive case, see [4].

There are several topological conditions implying that separately continuous functions on a product must have points of joint continuity. In fact, the topic has a vast literature starting with the paper [12]. See, e.g., the papers [5–7,13].

2 Preliminaries

Let G be a topological Abelian group. We denote the group operation by addition, and denote the unit by 0. The translation operator T_h and the difference operator Δ_h are

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[1] Actually Mazur and Orlicz only assume that X, Y, Z are F-spaces; that is, topological vector spaces whose topology is induced by a complete invariant metric.

defined by $T_h f(x) = f(x+h)$ and $\Delta_h f(x) = f(x+h) - f(x)$ for every $f : G \to \mathbb{C}$ and $h, x \in G$.

We say that a continuous function $f : G \to \mathbb{C}$ is a *generalized polynomial*, if there is an $n \geq 0$ such that $\Delta_{h_1} \ldots \Delta_{h_{n+1}} f = 0$ for every $h_1, \ldots, h_{n+1} \in G$. The smallest n with this property is the *degree of f*, denoted by $\deg f$. The degree of the identically zero function is -1. We denote by $\mathscr{GP} = \mathscr{GP}_G$ the set of generalized polynomials defined on G.

A function $f : G \to \mathbb{C}$ is said to be a *polynomial*, if there are continuous additive functions $a_1, \ldots, a_n : G \to \mathbb{C}$ and there is a $P \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f = P(a_1, \ldots, a_n)$. It is well-known that every polynomial is a generalized polynomial. It is also easy to see that the linear span of the translates of a polynomial is of finite dimension. More precisely, a function is a polynomial if and only if it is a generalized polynomial, and the linear span of its translates is of finite dimension (see [9, Proposition 5]). We denote by $\mathscr{P} = \mathscr{P}_G$ the set of polynomials defined on G.

Let f be a complex valued function defined on $X \times Y$. The *sections* $f_x : Y \to \mathbb{C}$ and $f^y : X \to \mathbb{C}$ of f are defined by $f_x(y) = f^y(x) = f(x, y)$ $(x \in X, y \in Y)$.

Let G, H be topological Abelian groups. A function $f : (G \times H) \to \mathbb{C}$ is a *separately polynomial function* if $f_x \in \mathscr{P}_H$ for every $x \in G$ and $f^y \in \mathscr{P}_G$ for every $y \in H$. Similarly, we say that $f : (G \times H) \to \mathbb{C}$ is a *separately generalized polynomial function* if $f_x \in \mathscr{GP}_H$ for every $x \in G$ and $f^y \in \mathscr{GP}_G$ for every $y \in H$.

In general we cannot expect that every separately polynomial function on $G \times H$ is a polynomial; not even if $G = H$ is a Hilbert space.

Example 1 Let G be the additive group of an infinite dimensional Hilbert space. Then the scalar product $f(x, y) = \langle x, y \rangle$ on G^2 is a separately polynomial function, since its sections are continuous additive functions. In fact, f^y is a linear functional and f_x is a conjugate linear functional for every $x, y \in G$. Thus the sections of f are polynomials.

Now, while the scalar product is a generalized polynomial (of degree 2) on G^2, it is not a polynomial on G^2, because the dimension of the linear span of its translates is infinite. Indeed, let $g(x) = \langle x, x \rangle = \|x\|^2$ for every $x \in G$. Then $\Delta_h g(x) = 2\langle h, x \rangle + \|h\|^2$ for every $h \in G$. It is easy to see that the functions $\langle h, x \rangle$ $(h \in G)$ generate a linear space of infinite dimension, and then the same is true for the translates of g and then for those of f as well.

Therefore, the best we can expect is that, under suitable conditions on G and H, every separately generalized polynomial function on $G \times H$ is a generalized polynomial.

We denote by $r_0(G)$ the torsion free rank of the group G; that is, the cardinality of a maximal independent system of elements of G of infinite order. Thus $r_0(G) = 0$ if and only if G is torsion. In the sequel by the rank of a group we shall mean the torsion free rank. It is known that if G has a dense subgroup of finite rank, then the classes of polynomials and of generalized polynomials on G coincide (see [9, Theorem 9]).

The set of roots of a function $f : G \to \mathbb{C}$ is denoted by Z_f. That is, $Z_f = \{x \in G : f(x) = 0\}$. We put

$$\mathscr{N}_P = \mathscr{N}_P(G) = \{A \subset G : \exists p \in \mathscr{P}_G, \ p \neq 0, \ A \subset Z_p\}$$

and

$$\mathcal{N}_{GP} = \mathcal{N}_{GP}(G) = \{A \subset G \colon \exists p \in \mathcal{GP}_G, \ p \neq 0, \ A \subset Z_p\}.$$

It is easy to see that $\mathcal{N}_P$ and $\mathcal{N}_{GP}$ are proper ideals of subsets of G. Let $\mathcal{N}_P^\sigma$ and $\mathcal{N}_{GP}^\sigma$ denote the σ-ideals generated by $\mathcal{N}_P$ and $\mathcal{N}_{GP}$, respectively. Note that $\mathcal{N}_P \subset \mathcal{N}_{GP}$ and $\mathcal{N}_P^\sigma \subset \mathcal{N}_{GP}^\sigma$.

If G is discrete, then $\mathcal{N}_P^\sigma$ and $\mathcal{N}_{GP}^\sigma$ are not proper σ-ideals (except when G is torsion), according to the next observation.

Proposition 1 *Let G be a discrete Abelian group. If G is not torsion, then $G \in \mathcal{N}_P^\sigma$.*

Proof Let $a \in G$ be an element of infinite order. Then $\phi(na) = n$ $(n \in \mathbb{Z})$ defines a homomorphism from the subgroup generated by a into $\mathbb{Q}$, the additive group of the rationals. Since $\mathbb{Q}$ is divisible, ϕ can be extended to G as a homomorphism from G into $\mathbb{Q}$. Let ψ be such an extension.

Then $p_r = \psi + r$ is a nonzero polynomial on G for every $r \in \mathbb{Q}$. If $x \in G$, then x is the root of p_r, where $r = -\psi(x) \in \mathbb{Q}$. Therefore, $G = \bigcup_{r \in \mathbb{Q}} Z_{p_r} \in \mathcal{N}_P^\sigma$. □

A simple sufficient condition for $G \notin \mathcal{N}_{GP}^\sigma$ is given by the next result.

Lemma 1 *If G is a connected Baire space, then the σ-ideals $\mathcal{N}_P^\sigma$ and $\mathcal{N}_{GP}^\sigma$ are proper; that is, $G \notin \mathcal{N}_P^\sigma$ and $G \notin \mathcal{N}_{GP}^\sigma$.*

Proof It is enough to prove that every element of $\mathcal{N}_{GP}$ is nowhere dense. Suppose $A \in \mathcal{N}_{GP}$ is dense in a nonempty open set U. Let $p \in \mathcal{GP}_G$ be a nonzero generalized polynomial vanishing on A. Since $A \subset Z_p$ and Z_p is closed, we have $U \subset Z_p$. Since G is connected, every neighbourhood of the origin generates G. It is known that in such a group, if a generalized polynomial vanishes on a nonempty open set, then it vanishes everywhere (see [15, Theorem 3.2, p. 33]). This implies that p is identically zero, which is impossible. □

3 Main results

Theorem 1 *Let G, H be topological Abelian groups, and suppose that*

(i) $\mathcal{N}_{GP}^\sigma(G)$ *is a proper σ-ideal in G, and*
(ii) H *has a dense subgroup of finite rank.*

If $f \colon (G \times H) \to \mathbb{C}$ is a separately generalized polynomial function, then f is a generalized polynomial on $G \times H$.

Theorem 2 *Let G, H be topological Abelian groups, and suppose that $\mathcal{N}_{GP}^\sigma(G)$ is a proper σ-ideal in G, and $\mathcal{N}_{GP}^\sigma(H)$ is a proper σ-ideal in H. Then the following statements are true.*

(i) *If $f \colon (G \times H) \to \mathbb{C}$ is a separately generalized polynomial function, then f is a generalized polynomial on $G \times H$ with respect to the discrete topology.*

(ii) *Every joint continuous separately generalized polynomial function $f : (G \times H) \to$ $\mathbb{C}$ is a generalized polynomial on $G \times H$.*

(iii) *If G and H are connected and a separately generalized polynomial function $f : (G \times H) \to \mathbb{C}$ has at least one point of joint continuity, then f is a generalized polynomial on $G \times H$.*

By Lemma 1, (i) of Theorem 1 can be replaced by the condition that G is a connected Baire space. Similarly, the condition of Theorem 2 can be replaced by the condition that G and H are connected Baire spaces.

As for (iii) of Theorem 2 note the following facts.

- If X, Y are nonempty topological spaces, X is Baire, Y is first countable and $f : (X \times Y) \to \mathbb{C}$ is separately continuous, then f has at least one point of joint continuity. (See, e.g. [17, p. 441].)
- A topological group is first countable if and only if it is metrizable.
- If X, Y are nonempty locally compact and σ-compact topological spaces, $f : (X \times Y) \to \mathbb{C}$ is separately continuous, then f has at least one point of joint continuity. (See [12, Theorem 1.2]).
- Every connected and locally compact topological group is σ-compact.

Comparing these with (iii) of Theorem 2 we obtain the following.

Theorem 3 *Suppose that the topological Abelian groups G, H are connected and Baire, and either*

(i) *at least one of G and H is metrizable, or*
(ii) *G and H are locally compact.*

If $f : (G \times H) \to \mathbb{C}$ is a separately generalized polynomial function, then f is a generalized polynomial on $G \times H$.

Question 1 Are the conditions (i) and (ii) necessary in the statement of Theorem 3? (See the introduction.)

We prove Theorems 1 and 2 in the next section. In Sect. 5 we present examples showing that some of the conditions appearing in Theorems 1 and 2 cannot be omitted.

4 Proof of Theorems 1 and 2

Lemma 2 *Let H be a topological Abelian group, and suppose that H has a dense subgroup of finite rank. Then, for every positive integer d, there are finitely many points $x_1, \ldots, x_s \in H$ and there are generalized polynomials $q_1, \ldots, q_s \in \mathscr{GP}_H$ of degree $< d$ such that $p = \sum_{i=1}^{s} p(x_i) \cdot q_i$ for every $p \in \mathscr{GP}_H$ with $\deg p < d$.*

Proof Let $\mathscr{GP}^{<d}$ denote the set of generalized polynomials $f \in \mathscr{GP}_H$ of degree $< d$. Clearly, $\mathscr{GP}^{<d}$ is a linear space over $\mathbb{C}$.

Let K be a dense subgroup of H with $r_0(K) = N < \infty$. Let $\{h_1, \ldots, h_N\}$ be a maximal set of independent elements of K of infinite order, and let L denote the

subgroup of K generated by the elements $h_1, \ldots, h_N$. If $k = (k_1, \ldots, k_N) \in \mathbb{Z}^N$, then we put $\|k\| = \max_{1 \le i \le N} |k_i|$. We abbreviate the sum $\sum_{i=1}^{N} k_i \cdot h_i$ by $\langle k, h \rangle$. Then we have $L = \{\langle k, h \rangle \colon k \in \mathbb{Z}^N\}$. We put

$$A = \{\langle k, h \rangle \colon k \in \mathbb{Z}^N, \ \|k\| \le [d/2]\}.$$

First we prove that if $p \in \mathscr{GP}^{<d}$ vanishes on A, then $p = 0$.

Suppose $p \ne 0$. Since p is continuous and K is dense in H, there is an $x_0 \in K$ such that $p(x_0) \ne 0$. The maximality of the system $\{h_1, \ldots, h_N\}$ implies that $n x_0 \in L$ with a suitable nonzero integer n. It is easy to see that there is a polynomial $P \in \mathbb{C}[x]$ such that $p(m x_0) = P(m)$ for every integer m. Since $P(1) = p(x_0) \ne 0$, it follows that $P \ne 0$, hence P only has a finite number of roots. Thus $p(mn x_0) = P(mn) \ne 0$ for all but a finite number of integers m. Fix such an m. Then $mn x_0 \in L$, and thus $mn x_0 = \langle k, h \rangle$ with a suitable $k \in \mathbb{Z}^N$. We find that $p(\langle k, h \rangle) \ne 0$ for some $k \in \mathbb{Z}^N$.

Let $k = (k_1, \ldots, k_N) \in \mathbb{Z}^N$ be such that $p(\langle k, h \rangle) \ne 0$ and $\|k\|$ is minimal. If $\|k\| \le [d/2]$, then $\langle k, h \rangle \in A$, and we have $p(\langle k, h \rangle) = 0$ by assumption. Thus we have $\|k\| > [d/2]$. Put $\ell = (\ell_1, \ldots, \ell_N)$, where

$$\ell_i = \begin{cases} 1 & \text{if } k_i > [d/2], \\ 0 & \text{if } |k_i| \le [d/2], \\ -1 & \text{if } k_i < -[d/2] \end{cases} \qquad (i = 1, \ldots, N).$$

Then we have $\|k - j\ell\| < \|k\|$ for every $j = 1, \ldots, d$. By the minimality of $\|k\|$ we have $p(\langle k - j\ell, h \rangle) = 0$ for every $j = 1, \ldots, d$.

Put $v = \langle \ell, h \rangle$. Since $\deg p < d$, it follows that $\Delta_{-v}^d p(x) = 0$ for every $x \in H$. Now we have

$$0 = \Delta_{-v}^d p(\langle k, h \rangle) = \sum_{j=0}^{d} (-1)^{d-j} \binom{d}{j} p(\langle k, h \rangle - jv) =$$

$$= (-1)^d p(\langle k, h \rangle) + \sum_{j=1}^{d} (-1)^{d-j} \binom{d}{j} p(\langle k - j\ell, h \rangle) =$$

$$= (-1)^d p(\langle k, h \rangle),$$

which is impossible. This proves $p = 0$.

The set of functions $V = \{p|_A \colon p \in \mathscr{GP}^{<d}\}$ is a finite dimensional linear space over $\mathbb{C}$. The map $p \mapsto p|_A$ is linear from $\mathscr{GP}^{<d}$ onto V and, as we proved above, it is injective. Therefore $\mathscr{GP}^{<d}$ is of finite dimension.

Let $b_1, \ldots, b_s$ be a basis of $\mathscr{GP}^{<d}$. Since the functions $b_1, \ldots, b_s$ are linearly independent, there are elements $x_1, \ldots, x_s$ such that the determinant $\det |b_i(x_j)|$ is nonzero (see [1, Lemma 1, p. 229]). Put $X = \{x_1, \ldots, x_s\}$. Then $b_1|_X, \ldots, b_s|_X$ are linearly independent, and thus the map $f \mapsto f|_X$ is bijective and linear from $\mathscr{GP}^{<d}$ onto the set of functions $f \colon X \to \mathbb{C}$.

Then there are functions $q_1, \ldots, q_s \in \mathscr{GP}^{<d}$ such that $q_i(x_i) = 1$ and $q_i(x_j) = 0$ for every $i, j = 1, \ldots, s, \ i \ne j$.

Let $p \in \mathscr{GP}^{<d}$ be given. Then $p - \sum_{i=1}^{s} p(x_j)q_j$ is a generalized polynomial of degree $< d$ vanishing on X, hence on H. That is, we have $p = \sum_{i=1}^{s} p(x_j)q_j$. □

Proof of Theorem 1 Let $f: (G \times H) \to \mathbb{C}$ be a separately generalized polynomial function. Put $G_n = \{x \in G: \deg f_x < n\}$ $(n = 1, 2, \ldots)$. Since $\mathscr{N}_{GP}^{\sigma}(G)$ is a proper σ-ideal in G, there is an n such that $G_n \notin \mathscr{N}_{GP}(G)$. Fix such an n.

By Lemma 2, there are points $y_1, \ldots, y_s \in H$ and generalized polynomials $q_1, \ldots, q_s \in \mathscr{GP}_H$ such that $p = \sum_{i=1}^{s} p(y_i) \cdot q_i$ for every $p \in \mathscr{GP}_H$ with $\deg p < n$. Therefore, we have

$$f(x, y) = \sum_{i=1}^{s} f(x, y_i)q_i(y)$$

for every $x \in G_n$ and $y \in H$. If $y \in H$ is fixed, then $f(x, y) - \sum_{i=1}^{s} f(x, y_i)q_i(y)$ is a generalized polynomial on G vanishing on G_n. Since $G_n \notin \mathscr{N}_{GP}(G)$, it follows that $f(x, y) = \sum_{i=1}^{s} f(x, y_i)q_i(y)$ for every $(x, y) \in G \times H$. By $f^{y_i} \in \mathscr{GP}_G$ and $q_i \in \mathscr{GP}_H$, we obtain $f \in \mathscr{GP}_{G \times H}$. □

Lemma 3 *Let G, H be discrete Abelian groups. A function $f: (G \times H) \to \mathbb{C}$ is a generalized polynomial if and only if the sections f_x $(x \in G)$ and f^y $(y \in H)$ are generalized polynomials of bounded degree.*

Proof Suppose $f: (G \times H) \to \mathbb{C}$ is a generalized polynomial of degree $< d$. Then $\Delta_{(x_1,0)} \ldots \Delta_{(x_d,0)} f = 0$ for every $x_1, \ldots, x_d \in G$. Then, for every $y \in H$, we have $\Delta_{x_1} \ldots \Delta_{x_d} f^y = 0$ for every $x_1, \ldots, x_d \in G$, and thus f^y is a generalized polynomial of degree $< d$ for every $y \in H$. A similar argument shows that f_x is a generalized polynomial of degree $< d$ for every $x \in G$, proving the "only if" statement.

Now suppose that $f: (G \times H) \to \mathbb{C}$ is such that f_x $(x \in G)$ and f^y $(y \in H)$ are generalized polynomials of degree $< d$. Then we have

$$\Delta_{(h_1,0)} \ldots \Delta_{(h_d,0)} f = 0 \tag{1}$$

for every $h_1, \ldots, h_d \in G$, and

$$\Delta_{(0,k_1)} \ldots \Delta_{(0,k_d)} f = 0 \tag{2}$$

for every $k_1, \ldots, k_d \in H$. In order to prove that f is a generalized polynomial of degree $< 2d$, it is enough to show that

$$\Delta_{(a_1,b_1)} \ldots \Delta_{(a_{2d},b_{2d})} f = 0 \tag{3}$$

for every $(a_i, b_i) \in G \times H$ $(i = 1, \ldots, 2d)$. The identity $\Delta_{u+v} = T_u \Delta_v + \Delta_u$ gives

$$\Delta_{(a_i,b_i)} = T_{(a_i,0)} \Delta_{(0,b_i)} + \Delta_{(a_i,0)}$$

for every i. Therefore, the left hand side of (3) is the sum of terms of the form $T_c \Delta_{c_1} \ldots \Delta_{c_{2d}} f$, where $c \in G \times \{0\}$, and $c_i \in (G \times \{0\}) \cup (\{0\} \times H)$ for every i. If

there are at least d indices i with $c_i \in (G \times \{0\})$, then (1) gives $\Delta_{c_1} \dots \Delta_{c_{2d}} f = 0$. Otherwise there are at least d indices i with $c_i \in (\{0\} \times H)$, and then (2) gives $\Delta_{c_1} \dots \Delta_{c_{2d}} f = 0$. This proves (3). $\square$

Proof of Theorem 2 (i) Suppose f satisfies the conditions. By Lemma 3, it is enough to show that the degrees $\deg f_x$ and f^y are bounded.

Put $A_n = \{x \in G \colon \deg f_x < n\}$. Then $G = \bigcup_{n=1}^{\infty} A_n$. Since $\mathcal{N}_{GP}^{\sigma}(G)$ is a proper σ-ideal, there is an n such that $A_n \notin \mathcal{N}_{GP}(G)$. We fix such an n, and prove that

$$\Delta_{(0,h_1)} \dots \Delta_{(0,h_n)} f = 0 \tag{4}$$

for every $h_1, \dots, h_n \in H$. Let g denote the left hand side of (4). Then $g(x, y) = \sum_{i=1}^{s} a_i f(x, y + b_i)$, where $s = 2^n$, $a_i = \pm 1$ and $b_i \in H$ for every i. Let $y \in H$ be fixed. Then $g^y = \sum_{i=1}^{s} a_i f^{y+b_i}$, and thus g^y is a generalized polynomial on G.

If $x \in A_n$, then $\deg f_x < n$, and thus $g_x = 0$. Therefore $g^y(x) = 0$ for every $x \in A_n$. Since g^y is a generalized polynomial and $A_n \notin \mathcal{N}_{GP}(G)$, it follows that $g^y = 0$. Since y was arbitrary, this proves (4). Thus $\deg f_x < n$ for every $x \in G$.

A similar argument shows that, for a suitable m, $\deg f^y < m$ for every $y \in H$. Statement (ii) of the theorem is clear from (i).

Suppose that G and H are connected. Now we use the fact that if f is a discrete generalized polynomial on an Abelian group which is generated by every neighbourhood of the origin, and if f has a point of continuity, then f is continuous everywhere. (See [15, Theorem 3.6]) or, for topological vector spaces, [2, Theorem 1].) In our case the group $G \times H$ is connected, so the condition is satisfied, and we conclude that f is continuous everywhere on $G \times H$. Thus (iii) follows from (ii). $\square$

5 Examples

In Theorem 1 none of the conditions on G and H can be omitted. First we show that without condition (i) the conclusion of Theorem 1 may fail. We shall need the easy direction of Lemma 3.

Example 2 Let G, H be discrete Abelian groups. We show that if none of G and H is torsion, then there is a separately polynomial function $f \colon (G \times H) \to \mathbb{C}$ such that f is not a generalized polynomial on $G \times H$.

By Proposition 1, $\mathcal{N}_P^{\sigma}(G)$ is not a proper σ-ideal; that is, $G = \bigcup_{n=1}^{\infty} A_n$, where $A_n \neq \emptyset$ and $A_n \in \mathcal{N}_P(G)$ for every n. Let $p_n \in \mathscr{P}_G$ be such that $p_n \neq 0$ and $A_n \subset Z_{p_n}$. Then p_n is not constant; that is, $\deg p_n \geq 1$.

Let $P_n = p_1 \cdots p_n$; then $P_n(x) = 0$ for every $x \in \bigcup_{i=1}^{n} A_i$, and we have $0 < \deg P_1 < \deg P_2 < \dots$. (Here we use the fact that $\deg pq = \deg p + \deg q$ for every $p, q \in \mathscr{GP}_G$, $p, q \neq 0$.) Note that for every $x \in G$ we have $P_n(x) = 0$ for all but a finite number of indices n.

Similarly, we find polynomials $Q_n \in \mathscr{P}_H$ such that $0 < \deg Q_1 < \deg Q_2 < \dots$, and for every $y \in H$ we have $Q_n(y) = 0$ for all but a finite number of indices n.

We put $f(x, y) = \sum_{n=1}^{\infty} P_n(x) Q_n(y)$ for every $x \in G$ and $y \in H$. If $y \in H$ is fixed, then the sum defining f is finite, and thus $f^y \in \mathscr{P}_G$. Similarly, we have $f_x \in \mathscr{P}_H$ for every $x \in G$.

The degrees $\deg f^y$ $(y \in H)$ are not bounded. Indeed, for every N, there is an $y \in H$ such that $Q_N(y) \neq 0$. Then $f^y = \sum_{n=1}^{M} Q_n(y) \cdot P_n$ with an $M \geq N$, where the coefficients $Q_n(y)$ are nonzero if $n \leq N$. Therefore, $\deg f^y \geq \deg P_N \geq N$, proving that the set $\{\deg f^y : y \in H\}$ is not bounded. By Lemma 3, it follows that f is a not a generalized polynomial.

By the example above, if G and H are discrete Abelian groups of positive and finite rank, then the conclusion of Theorem 1 fails. That is, $G \notin \mathscr{N}_{GP}^{\sigma}(G)$ cannot be omitted from the conditions of Theorem 1.

Next we show that the condition on H cannot be omitted either.

Example 3 Let H be a discrete Abelian group of infinite rank. We show that if G is a topological Abelian group such that $\mathscr{P}_G$ contains nonconstant polynomials, then there is a continuous separately polynomial function f on $G \times H$ such that f is not a generalized polynomial.

Let h_α $(\alpha < \kappa)$ be a maximal set of independent elements of H of infinite order, where $\kappa \geq \omega$. Let K denote the subgroup of H generated by the elements h_α $(\alpha < \kappa)$. Every element of K is of the form $\sum_{\alpha < \kappa} k_\alpha h_\alpha$, where $k_\alpha \in \mathbb{Z}$ for every α, and all but a finite number of the coefficients k_α equal zero.

Let $p \in \mathscr{P}_G$ be a nonconstant polynomial. We define $f(x, y) = \sum_{i=1}^{\infty} k_i \cdot p^i(x)$ for every $x \in G$ and $y \in K$, $y = \sum_{\alpha < \kappa} k_\alpha h_\alpha$. (Note that the sum only contains a finite number of nonzero terms for every x and y.) In this way we defined f on $G \times K$ such that f_x is additive on K for every $x \in G$.

If $y \in H$, then there is a nonzero integer n such that $ny \in K$. Then we define $f(x, y) = \frac{1}{n} \cdot f(x, ny)$ for every $x \in G$. It is easy to see that $f(x, y)$ is well-defined on $G \times H$, and f_x is additive on H for every $x \in G$. Therefore, f_x is a polynomial on G for every $x \in G$.

If $y \in H$ and $ny \in K$ for a nonzero integer n, then f^y is of the form $\frac{1}{n} \cdot \sum_{i=1}^{N} k_i \cdot p^i$, and thus $f^y \in \mathscr{P}_G$. Since f^y is continuous for every $y \in H$ and H is discrete, it follows that f is continuous on $G \times H$.

Still, f is not a generalized polynomial on $G \times H$, as the set of degrees $\deg f^y$ $(y \in H)$ is not bounded: if $y = h_i$, then $f^y = p^i$, and $\deg p^i = i \cdot \deg p \geq i$ for every $(i = 1, 2, \ldots)$.

In the example above we may choose G in such a way that $G \notin \mathscr{N}_{GP}^{\sigma}(G)$ holds. (Take, e.g., $G = \mathbb{R}$.) In our next example this condition holds for both G and H.

Example 4 Let E be a Banach space of infinite dimension, and let G be the additive group of E equipped with the weak topology τ of E. It is well-known that every ball in E is nowhere dense w.r.t. τ, and thus G is of first category in itself.

Still, we show that $G \notin \mathscr{N}_{GP}^{\sigma}(G)$. Indeed, the original norm topology of E is stronger than τ, and makes E a connected Baire space. If a function is continuous w.r.t. τ, then it is also continuous w.r.t. the norm topology. Therefore, every polynomial $p \in \mathscr{P}(G)$ is also a polynomial on E, and thus $\mathscr{N}_P(G) \subset \mathscr{N}_P(E)$ and $\mathscr{N}_P^{\sigma}(G) \subset$

$\mathscr{N}_P^\sigma(E)$. Since $\mathscr{N}_P^\sigma(E)$ is proper by Lemma 1, it follows that $\mathscr{N}_P^\sigma(G)$ is proper. The same is true for $\mathscr{N}_{GP}^\sigma(G)$.

Now let H be an infinite dimensional Hilbert space, and let G be the additive group of H equipped with the weak topology of H. Let f be the scalar product on H^2. Since the linear functionals and conjugate linear functionals are continuous w.r.t. the weak topology, it follows that f is a separately polynomial function on G^2 (see Example 1).

However, f is not a generalized polynomial on G^2, since f is not continuous. In order to prove this, it is enough to show that $f(x, x) = \|x\|^2$ is not continuous on H w.r.t. the weak topology. Suppose it is. Then there is a neighbourhood U of 0 such that $\|x\| < 1$ for every $x \in U$. By the definition of the weak topology, there are linear functionals $L_1, \ldots, L_n$ and there is a $\delta > 0$ such that whenever $|L_i(x)| < \delta$ $(i = 1, \ldots, n)$, then $\|x\| < 1$.

Since H is of infinite dimension, there is an $x \neq 0$ such that $L_i(x) = 0$ for every $i = 1, \ldots, n$. (Otherwise every linear functional would be a linear combination of $L_1, \ldots, L_n$, and then $H = H^*$ would be finite dimensional.) Then $\lambda x \in U$ for every $\lambda \in \mathbb{C}$ and $\|\lambda x\| < 1$ for every $\lambda \in \mathbb{C}$, which is impossible.

The example above shows that in (ii) of Theorem 2 the condition of joint continuity cannot be omitted. Note also that the group G defined in Example 4 is a topological vector space, hence connected. This shows that in (iii) of Theorem 2 the condition of the existence of points of joint continuity cannot be omitted either.

Declarations

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