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On Landesman-Lazer conditions and the fundamental theorem of algebra

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Abstract
We give an elementary proof of a Landesman-Lazer type result for systems by means of a shooting argument and explore its connection with the fundamental theorem of algebra.

Keywords Landesman-Lazer conditions · Systems of ODEs · Periodic solutions · Resonant problems

Mathematics Subject Classification 34C25 · 34B15

1 Introduction and main results

In the well known paper Landesman and Lazer [2], gave a sufficient condition for the existence of solutions of a nonlinear scalar equation under resonance at a simple eigenvalue. Although the original result was devoted to a second order elliptic problem, an extremely simplified first order analogue is the periodic problem

$$u'(t) + g(u(t)) = p(t), \qquad u(t + T) = u(t)$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth bounded function with limits $g_\pm$ at $\pm\infty$ and $p \in C(\mathbb{R})$ is T-periodic. Here, the Landesman-Lazer condition reads

$$g_- < \overline{p} < g_+$$

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or

$$g_+ < \overline{p} < g_-$$

where $\overline{p}$ denotes the average of p, namely $\overline{p} := \frac{1}{T}\int_0^T p(t)\,dt$. Thus, the Landesman-Lazer conditions express in fact two different things, that can be summarized as follows:

1. $g_\pm \neq \overline{p}$.
2. The mapping $\Gamma : \{-1, 1\} \to \mathbb{R}$ given by $\Gamma(\pm 1) = g_\pm$ wraps around $\overline{p}$, in the sense that $(\Gamma(-1) - \overline{p})$ and $(\Gamma(1) - \overline{p})$ have different signs.

In order to extend this idea for a system of differential equations, assume now that $g : \mathbb{R}^2 \to \mathbb{R}^2$ is bounded and the radial limits

$$g_v := \lim_{r\to+\infty} g(rv),$$

exist and are uniform for $v \in \partial B$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we may define the curve $\Gamma(\theta) := g_{v(\theta)}$, with $v(\theta) = e^{i\theta}$ and $\theta \in [0, 2\pi]$. The continuity of Γ follows in a straightforward manner: for example, given $\varepsilon > 0$ we may fix a constant $r > 0$ such that $|g(re^{i\theta}) - \Gamma(\theta)| < \frac{\varepsilon}{3}$ for all θ. Then

$$|\Gamma(\theta) - \Gamma(\tilde{\theta})| \leq |\Gamma(\theta) - g(re^{i\theta})| + |g(re^{i\tilde{\theta}}) - \Gamma(\tilde{\theta})| + |g(re^{i\theta}) - g(re^{i\tilde{\theta}})|.$$

Using now the continuity of g, there exists $\delta > 0$ such that $|g(re^{i\theta}) - g(re^{i\tilde{\theta}})| < \frac{\varepsilon}{3}$, whence $|\Gamma(\theta) - \Gamma(\tilde{\theta})| < \varepsilon$. In this setting, the following result due to Nirenberg [4] may be considered as a natural extension of the Landesman-Lazer theorem for a system. The condition that Γ 'wraps around' $\overline{p}$ shall be obviously expressed in terms of the winding number $I(\Gamma, \overline{p})$:

Theorem 1 *In the previous situation, assume that*

1. $g_v \neq \overline{p}$ *for all* $v \in \partial B$.
2. $I(\Gamma, \overline{p}) \neq 0$.

Then the problem

$$u'(t) + g(u(t)) = p(t) \tag{1}$$

has at least one T-periodic solution.

In the interesting paper [5], Ortega and Sánchez observe that the Nirenberg condition does not hold for the so-called *vanishing nonlinearities*, that is, when $g(u) \to \overline{p}$ as $|u| \to \infty$ and propose to assume instead that $g(u) \neq \overline{p}$ for $|u| \gg 0$ and the limits

$$q_v := \lim_{r\to+\infty} \frac{g(rv) - \overline{p}}{|g(rv) - \overline{p}|}$$

exist and are uniform for $v \in \partial B$. In this case, Nirenberg's result is retrieved by defining now the (continuous) curve $\Gamma_q(\theta) := q_{v(\theta)}$.

Theorem 2 *In the previous context, assume that*

1. $q_v \neq \overline{p}$ *for all* $v \in \partial B$.
2. $I(\Gamma_q, 0) \neq 0$.

Then the system (1) has at least one T-periodic solution.

Remark 1 Observe that the second condition is analogous to the second condition in Theorem 1 due to the obvious fact that $I(\Gamma, \overline{p}) = I(\Gamma - \overline{p}, 0)$.

As a corollary, it follows that if (1) is a *gradient system*, that is

$$u'(t) = \nabla G(u(t)) + p(t),$$

then the condition that $g = \nabla G$ is bounded can be dropped. The reason of this, as we shall see, is the fact that if u is a T-periodic solution then multiplying the system by $u'(t)$ it is obtained, upon integration:

$$\int_0^T |u'(t)|^2\, dt = \int_0^T (G \circ u)'(t)\, dt + \int_0^T \langle p(t), u'(t)\rangle\, dt$$

whence

$$\|u'\|_{L^2} \leq \|p\|_{L^2}.$$

Corollary 1 *Assume that* $g = \nabla G$ *and that conditions 1. and 2. of Theorem 2 are satisfied. Then the system (1) has at least one T-periodic solution.*

A particular instance of Corollary 1 is the complex equation

$$z'(t) = f(\overline{z}(t)) + p(t), \tag{2}$$

where f is a polynomial. Indeed, in this case the Ortega-Sánchez condition follows trivially since

$$\lim_{r\to+\infty} \frac{f(re^{-i\theta}) - \overline{p}}{|f(re^{-i\theta}) - \overline{p}|} = \frac{a_n}{|a_n|} e^{-in\theta},$$

uniformly on θ, where a_n is the leading coefficient of f. This implies that Γ_q performs n clockwise turns around the origin and the conditions 1. and 2. in Theorem 2 are fulfilled.

The fact that (2) is a gradient system follows from the Cauchy-Riemann conditions: if $f = a + ib$ and $F = A + iB$ is a (complex) primitive of f, then

$$[A(\overline{z})]_x = A_x(\overline{z}) = a(\overline{z}), \qquad [A(\overline{z})]_y = -A_y(\overline{z}) = b(\overline{z}).$$

Alternatively, we may multiply the equation by $\overline{z}'(t)$ to obtain

$$|z'(t)|^2 = z'(t)\overline{z}'(t) = G(\overline{z}(t))' + p(t)\overline{z}'(t);$$

thus, if z is a T-periodic solution it follows, as before, that

$$\|z'\|_{L^2} \leq \|p\|_{L^2}. \tag{3}$$

This explains why a general version of the preceding result is interpreted by Mawhin in [3] as an extension of the Fundamental Theorem of Algebra: indeed, taking $p = 0$, the inequality (3) implies that the periodic solutions are constants and, consequently, roots of f.

To conclude this introduction, let us mention that most of the literature concerning Landesman-Lazer theorem and its extensions involves second order problems, which sometimes go beyond the context of the standard semilinear problems. For instance, using the saddle point theorem and other classical results of the calculus of variations, Landesman-Lazer type condition were obtained for a p-Laplacian Neumann problem in [1], and (p, q)-Laplacian Neumann problems in [6], among other works. On the other hand, Nirenberg's original result was formulated for a more general abstract problem, from which the version presented in Theorem 1 follows easily. It is not difficult to adapt the ideas in the present paper to the second order case, although the first order system is simpler and already captures the geometrical meaning of the conditions described above.

2 Proofs and discussion

In order to give elementary proofs of the preceding results, it proves convenient to recall a useful property of the winding number, which follows straightforwardly from the homotopy invariance: if $F : \overline{B_r(0)} \to \mathbb{R}^2$ is continuous and $I(\gamma, 0) \neq 0$, where $\gamma(\theta) := F(re^{i\theta})$, then F vanishes in $B_r(0)$.

Proof of Theorem 1 Without loss of generality, we may assume that $\overline{p} = 0$. Let $u(t)$ be a solution of (1) with initial value $u(0) = u_0$, then

$$|u(t) - u_0| = \left| \int_0^t (g(u(s)) + p(s))\, ds \right| \leq M := T(\|g\|_\infty + \|p\|_\infty).$$

This implies that the Poincaré map $u_0 \mapsto P(u_0) := u(T)$ is well defined, continuous and

$$P(u_0) - u_0 = \int_0^T g(u(t))\, dt.$$

Writing $u_0 = re^{i\theta}$ with $r > 0$, it follows that $u(t) = r[e^{i\theta} + a(t)]$ with $|a(t)| \leq \frac{M}{r}$ and hence

$$u(t) = r(t)e^{i\theta(t)}$$

where

$$|r(t) - r| \leq M, \qquad |\theta(t) - \theta| \leq \frac{M}{r}.$$

Thus, given $\varepsilon > 0$, for sufficiently large r we obtain

$$|g(u(t)) - \Gamma(\theta)| \leq |g(u(t)) - \Gamma(\theta(t))| + |\Gamma(\theta(t)) - \Gamma(\theta)| < \varepsilon$$

and hence

$$|P(u_0) - u_0 - T\Gamma(\theta)| \leq \int_0^T |g(u(t)) - \Gamma(\theta)|\, dt < T\varepsilon.$$

Choosing $\varepsilon < |\Gamma(\theta)|$ for all θ and setting $\gamma(\theta) := P(re^{i\theta}) - re^{i\theta}$, it follows that

$$|\gamma(\theta) - \Gamma(\theta)| < |\Gamma(\theta)|$$

for $r \gg 0$ which, in turn, implies that

$$I(\gamma, 0) = I(\Gamma, 0) \neq 0.$$

This proves the existence of u_0 such that $P(u_0) = u_0$, and the corresponding $u(t)$ is a T-periodic solution of the problem. □

The proof of Theorem 2 is essentially the same as the preceding one: assuming w.l.o.g. that $\overline{p} = 0$, for $r \gg 0$ it is seen that

$$I(\gamma_q, 0) = I(\Gamma_q, 0),$$

where $\gamma_q(\theta) := \frac{P(re^{i\theta}) - re^{i\theta}}{|g(re^{i\theta})|}$, and the result follows.

Proof of Corollary 1 We may assume again that $\overline{p} = 0$. With the aim of keeping the exposition at a very elementary level, let us assume for simplicity that ∇G is controlled by G, in the sense that

$$|\nabla G(u)| \leq \xi(G(u)) \tag{4}$$

for some continuous mapping $\xi : \mathbb{R} \to (0, +\infty)$. For example, this holds when G is a polynomial, or if $G(u) = r(|u|)$ with $r \nearrow +\infty$. In this case, we may replace G by a mapping $\hat{G}(u) := \varphi(G(u))$ with $\varphi : \mathbb{R} \to \mathbb{R}$ a smooth increasing function such that

$$\varphi'(s) = \begin{cases} 1 & \text{if } |s| \le R \\ \frac{1}{\xi(s)} & \text{if } |s| \ge 2R \end{cases}$$

for some R to be specified. Observe that $\nabla\hat{G}(u) = \varphi'(G(u))\nabla G(u)$ is bounded and $\frac{\nabla\hat{G}(u)}{|\nabla\hat{G}(u)|} = \frac{\nabla G(u)}{|\nabla G(u)|}$, so by Theorem 2 the problem $u'(t) = \nabla\hat{G}(u(t)) + p(t)$ has a T-periodic solution u. We claim that if R is large enough, then $\|u\|_\infty \le R$ and, consequently, u is a solution of the original problem. Indeed, as in the introduction it is verified that

$$\|u'\|_{L^2} \le \|p\|_{L^2}$$

and hence

$$|u(t) - u(0)| = \left| \int_0^t u'(s)\, ds \right| \le T^{1/2}\|p\|_{L^2} := M.$$

As before, fix $\varepsilon < |\Gamma_q(\theta)|$ for all θ and r_0 such that if $r \ge r_0$ then

$$\left| \frac{\nabla G(r(e^{i\theta} + a))}{|\nabla G(re^{i\theta})|} - \Gamma_q(\theta) \right| < \varepsilon$$

for $|a| \le \frac{M}{r}$. Because $\int_0^T \nabla\hat{G}(u(t))\, dt = 0$, if $u_0 = re^{i\theta}$ with $r \ge r_0$ then we deduce that

$$\Gamma_q(\theta) \int_0^T \varphi'(G(u(t))\, dt = \int_0^T \varphi'(G(u(t)) \left[\Gamma_q(\theta) - \frac{\nabla G(u(t))}{|\nabla G(re^{i\theta})|} \right] dt.$$

Thus

$$|\Gamma_q(\theta)| \int_0^T \varphi'(G(u(t))\, dt < \varepsilon \int_0^T \varphi'(G(u(t))\, dt,$$

a contradiction. Notice that r_0 depends only on G and M; thus, it suffices to take $R = r_0 + M$. □

Further comments

It is easy to see that Theorem 1 still holds when p is a bounded function depending also on u; however, one needs to guarantee that, for r large, the curve $\gamma(\theta) := P(re^{i\theta}) - re^{i\theta}$ wraps around $\frac{1}{T}\int_0^T p(t, u(t))\, dt$, which varies also with θ. This is achieved if for example we assume

$$\limsup_{|u|\to\infty} \frac{|p(t,u)|}{|g(u)|} = 0 \tag{5}$$

uniformly on t. It is readily verified that the same condition suffices also in the situations of Theorem 2 and Corollary 1, assuming now that the limits

$$q_v^0 := \lim_{r\to+\infty} \frac{g(rv)}{|g(rv)|} \tag{6}$$

exist uniformly for $v \in \partial B$ and replacing the curve Γ_q by

$$\Gamma_q^0(\theta) := q_{v(\theta)}^0$$

The results may be extended also for delay systems like

$$u'(t) = g(u(t)) + p(t, u(t), u(t-\tau)) \tag{7}$$

where $\tau > 0$ and p is bounded, continuous and T-periodic in the first coordinate:

Theorem 3 *Assume that g is bounded or $g = \nabla G$ such that the radial limits (6) exist uniformly on $v \in \partial B$. Further, assume that p is bounded with*

$$\limsup_{|u|\to\infty} \frac{|p(t,u,u)|}{|g(u)|} = 0 \tag{8}$$

uniformly on t. If $I(\Gamma_q^0, 0) \neq 0$, then problem (7) has at least one T-periodic solution.

It should be noticed that, in this case, the problem cannot be reduced to find a fixed point in a finite dimensional space and less elementary tools are required. However, the proof is still easy in the context of the Leray-Schauder degree, which yields the following continuation theorem:

Theorem 4 *Assume that*

1. *There exists $R > 0$ such that any T-periodic solution of the problem*

$$u'(t) = \lambda[g(u(t)) + p(t, u(t), u(t-\tau))]$$

   with $\lambda \in (0, 1]$ satisfies $\|u\|_\infty < R$.
2. *$I(\Gamma_R, 0) \neq 0$, where*

$$\Gamma_R(\theta) := g(re^{i\theta}) + \int_0^T p(t, re^{i\theta}, re^{i\theta})\, dt.$$

Then problem (7) has at least one T-periodic solution u with $\|u\|_\infty < R$.

Indeed, when g is bounded or $g = \nabla G$, it follows as before that $\|u'\|_{L^2}$ is bounded by a constant depending only on $\|p\|_\infty$. This, in turn, implies that $|u(t) - u(0)|$ is also bounded and the conditions of Theorem 4 are obtained under the assumptions of Theorem 2. It is clear that condition (4) is not necessary at all.

Analogous results may be obtained for larger systems: let $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $p : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be continuous, bounded and T-periodic in its first coordinate. Assume that g is bounded or $g = \nabla G$ and that the radial limits (6) exist uniformly for $v \in S^{n-1} \subset \mathbb{R}^n$. Furthermore, assume that (8) holds. Then the problem has at least one T-periodic solution, provided that the degree of the mapping $\Gamma_q^0 : S^{n-1} \to S^{n-1}$ given by $\Gamma_q^0(v) := q_v^0$ is different from 0. It is readily seen that the latter condition is equivalent to

$$\deg(g, B_R(0), 0) \neq 0 \text{ when } R \text{ is sufficiently large,} \tag{9}$$

where deg stands for the Brouwer degree. A more delicate argument given in [7] shows that, if (9) is fulfilled, then the existence of the limits (6) is not necessary when $g = \nabla G$ is coercive, that is $|\nabla G(u)| \to +\infty$ as $|u| \to +\infty$.

As a final remark, let us try to understand why the result does not hold for the equation

$$z'(t) = f(\overline{z}(t)) + p(t)$$

when f is an arbitrary entire function. Because g is analytic, the bounds for z' are obtained exactly as before; however, if f is not a polynomial then the curves $f(Re^{i\theta})$ with $R \gg 0$ are very badly behaved. This is clearly related to the fact that f has an essential singularity at ∞; for example, when $f(z) = e^z$ the problem has no solutions for $p = 0$ and

$$f(Re^{i\theta}) = e^{Re^{-i\theta}} = e^{R\cos(\theta)}e^{-iR\sin(\theta)},$$

which has zero winding number although for $R \gg 0$ it passes many times back and forth around the origin.

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Compliance with ethical standards

Conflict of interest The author declares that there is no conflict of interest.

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