Monatshefte für Mathematik (2021) 194:425–458
https://doi.org/10.1007/s00605-020-01476-7

Well-posedness and regularity for fractional damped wave equations

Yong Zhou[1,2] · **Jia Wei He**[3]

Received: 20 February 2020 / Accepted: 3 November 2020 / Published online: 20 November 2020

Abstract

In this paper, we study the well-posedness and regularity of mild solutions for a class of time fractional damped wave equations, which the fractional derivatives in time are taken in the sense of Caputo type. A concept of mild solutions is introduced to prove the existence for the linear problem, as well as the regularity of the solution. We also establish a well-posed result for nonlinear problem. By applying finite dimensional approximation method, a compact result of solution operators is presented, following this, an existence criterion shows that the Lipschitz condition or smoothness of nonlinear force functions in some literatures can be removed. As an application, we discuss a case of time fractional telegraph equations.

Keywords Damped wave equations · Caputo's fractional derivative · mild solutions · Well-posedness · Regularity

Mathematics Subject Classification 35L05 · 35R11 · 26A33

Communicated by Ansgar Jüngel.

✉ Yong Zhou
yzhou@xtu.edu.cn

Jia Wei He
jwhe@gxu.edu.cn

1 Faculty of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China

2 Faculty of Information Technology, Macau University of Science and Technology, Macau 999078, China

3 College of Mathematics and Information Science, Guangxi University, Nanning 530004, China

1 Introduction

Fractional differential equations have gained considerable importance due to their widespread applications in a variety of fields such as physics, chemistry, engineering, biology, geophysics and hydrology. In recent years, partial differential equations with fractional derivatives have been investigated extensively. For details and examples, we refer the reader to a series of papers [1,2,4–10,12,15–18,20–24,26–29,31,32] and the references cited therein. The main purpose of this paper is to investigate the initial/boundary value problems for time fractional damped wave equation

$$\partial_t^\beta u + \partial_t^\alpha u = \Delta u + f(u), \quad t > 0, \tag{1.1}$$

subject to Dirichlet's boundary condition

$$u(t, x) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

and initial value conditions

$$u(0, x) = \phi(x), \quad \partial_t u(0, x) = \psi(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with the sufficiently smooth boundary $\partial\Omega$, ∂_t^β, ∂_t^α are standard fractional derivatives in the sense of Caputo type of order $\beta \in (1, 2]$ and $\alpha \in (0, 1]$, respectively. f is an appropriate force function which will be special later. Taking the case of $\beta = 2$ and $\alpha = 1$ in (1.1), it becomes the standard damped wave equation, which is an important mathematical model in studying many physic problems. Readers can easily find a large number of related researches that are focused on the well-posedness of some linear or nonlinear Cauchy problems. In addition, various papers have considered to establish the asymptotic behavior and regularity estimates of the solutions, we refer to [3,11,14] and the references therein. Observe that, if $\beta = 2\alpha$ for $\alpha \in (1/2, 1]$ associated with (1.1), this equation contains a typical time fractional telegraph equation, which is derived from the law of the iterated Brownian motion and Brownian time for the telegraph process, see e.g. [23].

A strong motivation for investigating the Eq. (1.1) comes from physical phenomena. The time fractional diffusion equation $\partial_t^\beta u = \Delta u$ of order $\beta \in (0, 1)$ can be used to model anomalous diffusion phenomena, which is driven by fractional Brownian motion and it represents the subdiffusion behavior [32], while the time fractional wave equation of order $\beta \in (1, 2)$ will interpolate between the heat equation $(\beta = 1)$ and the wave equation $(\beta = 2)$ that govern intermediate processes between diffusion and wave propagation, and it further is interpreted as the superdiffusion behavior. Moreover, fractional wave equations also can model a cable made with special smart materials or a vibrating string in presence of a fractional friction with power-law memory kernel. From these physical points of view, some partial differential equations with fractional derivative will be better suitable to describe in practical problems. As for the current problem, in fact, without the term $\partial_t^\beta u$ associated with (1.1), there are more researches concerning with this fractional diffusion equation, the analysis of well-posedness, asymptotic analysis, decay estimates, blow-up solutions have been studied in [1,6,15,17]. Without the forcing term f and damped term $\partial_t^\alpha u$ associated with (1.1), the analysis theories of fractional wave equations have been studied by Luchko [20], Mainardi [21,22], Sakamoto and Yamamoto [26], Schneider and Wyss [27], etc. Recently, concerning with fractional wave equations, Kian and Yamamoto [16] have investigated the existence of weak solution and some Strichartz estimates under the case of semilinear force function on bounded domain. The well-posedness results associated with a Dirichlet space have been considered by Alvarez et al. [4]. In addition, Otarola and Salgado [24] have studied the regularity of weak solutions, and also discussed the spatial-time regularities of the solution for an extended problem. Djida et al. [8] have concerned with the well-posedness results on whole space $\mathbb{R}^N$ and they derived some $L^p - L^r$ estimates of solution. Associated with an extra damping term in fractional wave equation, that can describe the interaction between the vector electric field and the electric and magnetic properties of the material (see e.g. [13]), we observe that there are still few researches addressing the following wave equation with damping

$$\partial_t^\beta u + \partial_t^\alpha u - \Delta u = 0.$$

In 2005, Alaimia and Tatar [2], Tatar [29] have investigated the blow up for the wave equation with a fractional damping. In one dimensional unbounded domain, Stojanovic and Gorenflo [28] obtained an upper viscosity solution for the case $\beta \in (1, 2)$ and $\alpha \in (0, 1)$, while on a bounded domain, Lin and Nakamura [18] investigated the Carleman estimate that give the unique continuation property of solutions for an anomalous diffusion equation with multi-terms time fractional Caputo derivative, as well as the case for fractional diffusion equation [19]. Consequently, it is natural to discuss more general fractional wave equations with damping term.

Motivated by the above mentioned works, in this paper, we will focus on the well-posedness and regularity of linear fractional damped wave equations, one reason to consider these properties is that there are few papers to establish the qualitative theory of damped wave equations in the sense of fractional versions. Especially in nonlinear problem, there is an urgent need for existence results to extend some known conclusions. The second reason is that the Laplacian operator associated with Dirichlet's boundary condition on a bounded domain with the sufficiently smooth boundary on $L^2(\Omega)$ can be expressed as a spectrum problem, and this will lead to the relative solution operators are compact and are uniformly continuous on their domains, following these properties, we get a general existence result without the Lipschitz condition or the smoothness assumption on nonlinear function.

This paper is organized as follows. Section 2 recalls some concepts and known results which will be useful throughout this paper. In Sect. 3, we first introduce a suitable definition of mild solution for the linear problem, and then we obtain some existence and regularity of mild solutions. In Sect. 4, some exact upper bounds of several Mittag–Leffler functions are obtained. Under the local Lipschitz condition of nonlinear force function, a well-posed result of problem (1.1)–(1.3) is established. Next, we show the continuation and blow-up alternative of the solution. In addition, we also prove the compactness of the solution operator, which allows us to study the existence of mild solutions by removing the Lipschitz condition or higher regularity hypothesis of force function. Finally, an application is introduced to verify our main results.

2 Preliminary results

In this section, we will provide some notations and preliminary lemmas.

Let us first recall the Riemann–Liouville fractional integral of order $\beta \in \mathbb{R}_+$ with the lower limit zero for a function $v \in L^1(0, T; X)$ with X a Banach space defined by

$$J_t^\beta v(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds, \quad t \geq 0,$$

where $\Gamma(\beta)$ is the usual gamma function of order β.

Definition 2.1 Let $\alpha \in (0, 1)$, $\beta \in (1, 2)$ and $T > 0$. Consider a function $v \in L^1(0, T; X)$ such that $J_t^{1-\alpha} v \in W^{1,1}(0, T; X)$ or $J_t^{2-\beta} v \in W^{2,1}(0, T; X)$. The representations

$$\partial_t^\alpha v(t) = \partial_t \big(J_t^{1-\alpha}(v(t) - v(0))\big),$$

and

$$\partial_t^\beta v(t) = \partial_{tt}^2 \big(J_t^{2-\beta}(v(t) - v(0) - t\, \partial_t v(0))\big),$$

are called the Caputo fractional derivative of order α and β, respectively.

In particular, when $\rho = 0$, one finds that $J_t^\rho v(t) = v(t)$. Hence, if $\alpha = 1$ or $\beta = 2$, then the Caputo fractional derivatives commute with integer order derivatives, respectively.

2.1 Mittag–Leffler functions

In what follows, the Mittag–Leffler function $E_{\mu,\nu}(z)$ is defined by

$$E_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \nu)}, \quad \mu > 0, \nu \in \mathbb{R},\ z \in \mathbb{C}.$$

From the properties of power series, one can see that $E_{\mu,\nu}(z)$ is an entire function. Moreover, it is well known that $E_{\mu,1}(-t)$ is a positive and completely monotonic function for $\mu \in (0, 1)$, $t > 0$, that is, for all $t > 0$, $k \in \mathbb{N}_0$, we have $(-1)^k \big(\frac{d}{dt}\big)^k E_{\mu,1}(-t) \geq 0$. Additionally, one can find that $\omega(t) := E_{\mu,1}(\lambda t^\mu)$ is a solution of equation $\partial_t^\mu \omega(t) = \lambda \omega(t)$, $\lambda \in \mathbb{R}$, $\mu \in (0, 2)$. We use the notation $a \lesssim b$ that stands for $a \leq Cb$, with a positive constant C that does not depend on a, b. The following lemmas will be frequently used and can be found in [25].

Lemma 2.1 *For $\lambda > 0$, $\mu > 0$, $\nu \in \mathbb{R}$ and any arbitrary positive number m, we have*

$$\left(\frac{d}{dt}\right)^m \left(t^{\nu-1}E_{\mu,\nu}(-\lambda t^\mu)\right) = t^{\nu-m-1}E_{\mu,\nu-m}(-\lambda t^\mu), \quad t > 0.$$

in particular,

$$\frac{d}{dt}E_{\mu,1}(-\lambda t^\mu) = -\lambda t^{\mu-1}E_{\mu,\mu}(-\lambda t^\mu), \quad t > 0.$$

Lemma 2.2 *If $0 < \mu < 2$, $\nu \in \mathbb{R}$, $\pi\mu/2 < \theta < \min(\pi, \pi\mu)$, then*

$$\left|E_{\mu,\nu}(z)\right| \lesssim \frac{1}{1+|z|}, \quad z \in \mathbb{C}, \quad \theta \le |\arg z| \le \pi.$$

Lemma 2.3 *If $0 < \mu < 2$, $\nu \in \mathbb{R}$, θ is such that $\pi\mu/2 < \theta < \min(\pi, \pi\mu)$, then*

$$|E_{\mu,\nu}(z)| \lesssim (1+|z|)^{(1-\nu)/\mu} \exp\left(Re(z^{1/\mu})\right) + \frac{1}{1+|z|}, \quad z \in \mathbb{C}, \quad |\arg z| \le \theta.$$

By the fractional order term-by-term integration of the series, there is a more general relationship obtained as follows

$$\frac{1}{\Gamma(\vartheta)}\int_0^t (t-s)^{\vartheta-1}s^{\nu-1}E_{\mu,\nu}(\lambda s^\nu)ds = t^{\nu+\vartheta-1}E_{\mu,\nu+\vartheta}(\lambda t^\nu), \ \vartheta > 0, \ \nu > 0, \ t > 0. \tag{2.1}$$

Lemma 2.4 [4] *Let $1 < \beta < 2$, $\beta' \in \mathbb{R}$ and $\lambda > 0$. Then the following estimates hold.*

(i) *Let $0 \le \mu \le 1$, $0 < \nu < \beta$. Then* $\left|\lambda^\mu t^\nu E_{\beta,\beta'}(-\lambda t^\beta)\right| \lesssim t^{\nu-\beta\mu}, \quad t > 0.$

(ii) *Let $0 \le \nu \le 1$. Then* $\left|\lambda^{1-\nu}t^{\beta-2}E_{\beta,\beta'}(-\lambda t^\beta)\right| \lesssim t^{\beta\nu-2}, \quad t > 0.$

2.2 Fractional power spaces

Let $L^2(\Omega)$ be the standard real Hilbert space with the norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$. $H^l(\Omega)$ and $H_0^m(\Omega)$ denote the usual Sobolev spaces for $l, m \ge 0$. Let X be a Banach space equipped with norm $\|\cdot\|_X$ and $\mathcal{B}(X)$ stands for the spaces of all bounded linear operators from X into itself. Let $C([0,T];X)$ be the Banach space of all continuous functions from $[0,T]$ into X equipped with the supremum norm $\|u\|_{\mathcal{C}} = \sup_{t\in[0,T]}\|u(t)\|_X$. The symbol $L^p(0,T;X)$ denotes the Banach space of all p-integrable measurable functions u such that:

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p dt\right)^p < \infty, & \text{if } 1 \le p < \infty, \\ \text{ess}\sup\limits_{0\le t\le T} \|u(t)\|_X < \infty, & \text{if } p = \infty. \end{cases}$$

The symbol $W^{k,p}(0,T;X)$ $(k \geq 1)$ stands for the Banach space of all k-times differentiable functions u such that:

$$\|u\|_{W^{k,p}(0,T;X)} = \sum_{n=0}^{k} \|\partial_t^n u\|_{L^p(0,T;X)} < \infty.$$

It is well-known that the Laplacian operator $A = -\Delta$ is nonegative and self-adjoint in Sobolev space $H_0^1(\Omega)$, and there exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions $\{e_n\}_{n=1}^{\infty} \subset H_0^1(\Omega)$, which are corresponding to the discrete positive eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ for every $n \in \mathbb{N}$, here $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$ satisfing

$$Ae_n = \lambda_n e_n, \quad \text{in } \Omega\,; \qquad e_n = 0, \quad \text{on } \partial\Omega.$$

For any $\gamma \geq 0$, let fractional power operator A^γ possess the following representation:

$$A^\gamma u = \sum_{n=1}^{\infty} \lambda_n^\gamma (u, e_n) e_n, \quad u \in D(A^\gamma),$$

where

$$D(A^\gamma) = \left\{ u \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(u, e_n)|^2 < \infty \right\},$$

as a Hilbert space of functions

$$u(t,x) := \sum_{n=1}^{\infty} u_n(t) e_n(x) = \sum_{n=1}^{\infty} (u, e_n) e_n(x) \in L^2(\Omega),$$

equipped with the norm

$$\|u\|_\gamma := \|u\|_{D(A^\gamma)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(u, e_n)|^2 \right)^{\frac{1}{2}}, \quad u \in D(A^\gamma).$$

By using the so called Gelfand triple, we denote the duality space of $D(A^\gamma)$ by $D(A^{-\gamma})$. It can be seen that $D(A^{-\gamma})$ is a Hilbert space endowed with the norm

$$\|u\|_{\gamma^*} := \|u\|_{D(A^{-\gamma})} = \left(\sum_{n=1}^{\infty} \lambda_n^{-2\gamma} |\langle u, e_n \rangle_{-\gamma,\gamma}|^2 \right)^{\frac{1}{2}}, \quad u \in D(A^{-\gamma}),$$

under the duality bracket $\langle \cdot, \cdot \rangle_{-\gamma,\gamma}$. Furthermore, we notice that

$$\langle u, v \rangle_{-\gamma,\gamma} = (u, v), \quad \text{for } u \in L^2(\Omega),\ v \in D(A^\gamma).$$

Specially, one has $D(A^\gamma) \subset H^{2\gamma}(\Omega)$ for $\gamma > 0$, $D(A^0) = L^2(\Omega)$, $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$, see e.g. [26].

3 Linear problems

In this sequel, consider the following linear problem

$$\partial_t^\beta u(t,x) + \partial_t^\alpha u(t,x) = \Delta u(t,x) + f(t,x), \quad x \in \Omega,\ t \in (0,T), \tag{3.1}$$
$$u(t,x) = 0, \quad x \in \partial\Omega,\ t \in (0,T), \tag{3.2}$$
$$u(0,x) = \phi(x),\ \partial_t u(0,x) = \psi(x), \quad x \in \Omega. \tag{3.3}$$

Next, a suitable definition of mild solutions will be introduced to study the above linear problem, furthermore, the existence and regularity of solutions are discussed.

3.1 Solution representation formula

Let $\phi \in D(A^\gamma)$, $\psi \in L^2(\Omega)$, with the aid of the spectrum property of operator A, observe that, the equation

$$\partial_t^\beta u(t,x) = -Au(t,x) + f(t,x), \quad t > 0,\ x \in \Omega, \tag{3.4}$$

associated with initial/boundary value conditions (3.2)–(3.3) can be converted into

$$\begin{cases} \partial_t^\beta u_n(t) = -\lambda_n u_n(t) + f_n(t), \\ u_n(0) = \phi_n,\ \partial_t u_n(0) = \psi_n, \end{cases}$$

where $\phi_n = (\phi, e_n)$, $\psi_n = (\psi, e_n)$, $f_n(t) = (f(t,\cdot), e_n)$ and the solutions $u_n(t)$ are explicitly expressed as follows, (see e.g. [4,16])

$$u_n(t) = E_{\beta,1}(-\lambda_n t^\beta)\phi_n + tE_{\beta,2}(-\lambda_n t^\beta)\psi_n + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-\lambda_n (t-s)^\beta) f_n(s)ds,$$

for all $t \geq 0$. With the help of the identities in Lemma 2.1, one can derive that the formulas of first order derivative with respect to t of $u_n(t)$ is equal to

$$-\lambda_n t^{\beta-1} E_{\beta,\beta}(-\lambda_n t^\beta)\phi_n + E_{\beta,1}(-\lambda_n t^\beta)\psi_n + \int_0^t (t-s)^{\beta-2} E_{\beta,\beta-1}(-\lambda_n (t-s)^\beta) f_n(s)ds.$$

It is notice that the definition of Caputo's fractional derivative and (2.1), by changing the order of integration, one has

$$\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-\lambda_n(t-s)^\beta)\partial_s^\alpha u_n(s)ds$$
$$= \int_0^t (t-s)^{\beta-\alpha}E_{\beta,\beta+1-\alpha}(-\lambda_n(t-s)^\beta)\partial_s u_n(s)ds.$$

Therefore, after integration by parts in s, it follows that

$$\int_0^t (t-s)^{\beta-\alpha}E_{\beta,\beta+1-\alpha}(-\lambda_n(t-s)^\beta)\partial_s u_n(s)ds$$
$$= \int_0^t (t-s)^{\beta+1-\alpha}E_{\beta,\beta-\alpha}(-\lambda_n(t-s)^\beta)u_n(s)ds$$
$$- t^{\beta-\alpha}E_{\beta,\beta+1-\alpha}(-\lambda_n t^\beta)u_n(0). \tag{3.5}$$

By using the eigenfunction expansions we set the operators

$$\mathcal{S}_\beta(t)v = \sum_{n=1}^{\infty} E_{\beta,1}(-\lambda_n t^\beta)(v, e_n)e_n, \quad \mathcal{P}_\beta(t)v = \sum_{n=1}^{\infty} tE_{\beta,2}(-\lambda_n t^\beta)(v, e_n)e_n,$$

and

$$\mathcal{T}_\beta(t)v = \sum_{n=1}^{\infty} t^{\beta-1}E_{\beta,\beta}(-\lambda_n t^\beta)(v, e_n)e_n,$$

for all $v \in L^2(\Omega)$ and $t \geq 0$. In order to simple the representation of solution, we just focus on the dependence of time variable t and sometimes omit the space variable x, writing $u(t) = u(t, \cdot)$, $f(t) = f(t, \cdot)$, and so on. Following above arguments, (3.4) associated with (3.2)–(3.3) have an equivalent integral form as follows

$$u(t) = \mathcal{S}_\beta(t)\phi + \mathcal{P}_\beta(t)\psi + \int_0^t \mathcal{T}_\beta(t-s)f(s)ds.$$

Hence, linear problem (3.1)–(3.3) has a representation of formal solution as follows

$$u(t) = \mathcal{S}_\beta(t)\phi + \mathcal{R}_\beta(t)\phi + \mathcal{P}_\beta(t)\psi - \int_0^t \mathcal{R}'_\beta(t-s)u(s)ds + \int_0^t \mathcal{T}_\beta(t-s)f(s)ds, \tag{3.6}$$

for $t \geq 0$, where

$$\mathcal{R}_\beta(t)\phi = \sum_{n=1}^{\infty} t^{\beta-\alpha} E_{\beta,\beta+1-\alpha}(-\lambda_n t^\beta)(\phi, e_n)e_n,$$
$$\mathcal{R}'_\beta(t)u = \sum_{n=1}^{\infty} t^{\beta-\alpha-1} E_{\beta,\beta-\alpha}(-\lambda_n t^\beta)(u, e_n)e_n.$$

As we saw, the damped term in linear problem can be regarded as a nonlinear term in nonlinear problem, that will avoid a lot of computations to check the properties of solution, for instance, when it is converted into a fundamental solution, see an application below, it is not easy to discuss the existence and regularities of solution, and especially for considering nonlinear problem (1.1)–(1.3). For this reason, it is worth to consider that such damped term converts into an integral representation at nonlinear term. Next, we shall introduce a suitable definition of mild solutions to problem (3.1)–(3.3) which involves the Mittag–Leffler functions from above arguments.

Definition 3.1 Let $T > 0$. If a function $u \in C([0, T]; L^2(\Omega))$ satisfies (3.6), then we say, u is a mild solution of problem (3.1)–(3.3).

3.2 Existence and regularity

In this subsection, we will prove the existence and regularity of linear problem (3.1)–(3.3). The first result is concerned with the existence of the mild solution, the regularities of the solution are given in the rest of results.

Theorem 3.1 *Let $(\phi, \psi) \in D(A^\gamma) \times L^2(\Omega)$ for $\gamma \in (0, 1)$ and let $f \in L^1(0, T; L^2(\Omega))$. Then there exists a unique mild solution u to problem (3.1)–(3.3). Moreover,*

$$\|u(t)\| \lesssim \|\phi\|_\gamma + \|\psi\| + \|f\|_{L^1(0,T;L^2(\Omega))}. \tag{3.7}$$

The hidden constant, in above inequality, is independent of t, γ but may be dependent on T.

Proof Let us first denote an operator $\mathcal{Q}$ on $C([0, T]; L^2(\Omega))$ as follows

$$(\mathcal{Q}u)(t) = \mathcal{S}_\beta(t)\phi + \mathcal{R}_\beta(t)\phi + \mathcal{P}_\beta(t)\psi \\ - \int_0^t \mathcal{R}'_\beta(t-s)u(s)ds + \int_0^t \mathcal{T}_\beta(t-s)f(s)ds.$$

Clearly, there exists a mild solution of problem (3.1)–(3.3) if and only if operator $\mathcal{Q}$ has a fixed point on $C([0, T]; L^2(\Omega))$. In what follows, we shall show that operator $\mathcal{Q}$ is well-defined in $C([0, T]; L^2(\Omega))$. Firstly, for any $\epsilon > 0$, let $0 \leq t < t + \epsilon \leq T$, we have

$$
\begin{aligned}
(\mathcal{Q}u)(t+\epsilon)-(\mathcal{Q}u)(t) =&\, \mathcal{S}_\beta(t+\epsilon)\phi-\mathcal{S}_\beta(t)\phi \\
&+\mathcal{R}_\beta(t+\epsilon)\phi-\mathcal{R}_\beta(t)\phi+\mathcal{P}_\beta(t+\epsilon)\psi-\mathcal{P}_\beta(t)\psi \\
&-\int_0^{t+\epsilon}\mathcal{R}'_\beta(t+\epsilon-s)u(s)ds+\int_0^t\mathcal{R}'_\beta(t-s)u(s)ds \\
&+\int_0^{t+\epsilon}\mathcal{T}_\beta(t+\epsilon-s)f(s)ds-\int_0^t\mathcal{T}_\beta(t-s)f(s)ds.
\end{aligned} \tag{3.8}
$$

In view of Lemma 2.1 and (i) in Lemma 2.4, we know that

$$
\begin{aligned}
\|\mathcal{S}_\beta(t+\epsilon)\phi-\mathcal{S}_\beta(t)\phi\| &= \left(\sum_{n=1}^\infty\left|\int_t^{t+\epsilon}-\lambda_n s^{\beta-1}E_{\beta,\beta}(-\lambda_n s^\beta)ds\right|^2|\phi_n|^2\right)^{1/2} \\
&=\left(\sum_{n=1}^\infty\left(\int_t^{t+\epsilon}\lambda_n^{1-\gamma}s^{\beta-1}E_{\beta,\beta}(-\lambda_n s^\beta)ds\right)^2\lambda_n^{2\gamma}|\phi_n|^2\right)^{1/2} \\
&\lesssim\left((t+\epsilon)^{\beta\gamma}-t^{\beta\gamma}\right)\|\phi\|_\gamma.
\end{aligned}
$$

From the definition of the fractional power space $D(A^\gamma)$ for $\gamma>0$, in view of the Sobolev embedding $D(A^\gamma)\subset L^2(\Omega)$, it can be checked inequality $\|\phi\|\lesssim\|\phi\|_\gamma$. Moreover,

$$
\begin{aligned}
\|\mathcal{R}_\beta(t+\epsilon)\phi-\mathcal{R}_\beta(t)\phi\| &= \left(\sum_{n=1}^\infty\left|\int_t^{t+\epsilon}s^{\beta-\alpha-1}E_{\beta,\beta-\alpha}(-\lambda_n s^\beta)ds\right|^2|\phi_n|^2\right)^{1/2} \\
&\lesssim\left((t+\epsilon)^{\beta-\alpha}-t^{\beta-\alpha}\right)\|\phi\|_\gamma.
\end{aligned}
$$

Lemmas 2.1–2.2 imply

$$
\|\mathcal{P}_\beta(t+\epsilon)\psi-\mathcal{P}_\beta(t)\psi\| = \left(\sum_{n=1}^\infty\left|\int_t^{t+\epsilon}E_{\beta,1}(-\lambda_n s^\beta)ds\right|^2|\psi_n|^2\right)^{1/2}\lesssim\epsilon\,\|\psi\|.
$$

By virtue of Lemma 2.2 again, one obtains

$$
\|\mathcal{R}_\beta(t)v\|\lesssim t^{\beta-\alpha}\|v\|,\quad \|\mathcal{R}'_\beta(t)v\|\lesssim t^{\beta-\alpha-1}\|v\|,\quad v\in L^2(\Omega). \tag{3.9}
$$

Therefore, it yields

$$
\int_t^{t+\epsilon}\|\mathcal{R}'_\beta(t+\epsilon-s)u(s)\|ds\lesssim\epsilon^{\beta-\alpha}\,\|u\|_{\mathcal{C}}.
$$

Moreover, by (i) in Lemma 2.4 with respect to $\mu=1-\frac{1}{\beta}$, we have

$$
\int_0^t\|(\mathcal{R}'_\beta(t+\epsilon-s)-\mathcal{R}'_\beta(t-s))u(s)\|ds
$$

$$= \int_0^t \left(\sum_{n=1}^{\infty} \left| \int_{t-s}^{t+\epsilon-s} \tau^{\beta-\alpha-2} E_{\beta,\beta-\alpha-1}(-\lambda_n \tau^{\beta}) d\tau \right|^2 (u(s), e_n)^2 \right)^{1/2} ds$$
$$\lesssim \int_0^t \left| \int_{t-s}^{t+\epsilon-s} \tau^{-\alpha-1} d\tau \right| ds \, \|u\|_{\mathcal{C}}$$
$$\lesssim \left(\epsilon^{1-\alpha} + t^{1-\alpha} - (t+\epsilon)^{1-\alpha} \right) \|u\|_{\mathcal{C}}.$$

Noting that

$$\|\mathcal{S}_\beta(t)v\| \lesssim \|v\|, \quad \|\mathcal{P}_\beta(t)v\| \lesssim t\,\|v\|, \quad \|\mathcal{T}_\beta(t)v\| \lesssim t^{\beta-1}\|v\|, \tag{3.10}$$

for all $v \in L^2(\Omega)$, and then

$$\int_t^{t+\epsilon} \|\mathcal{T}_\beta(t+\epsilon-s)f(s)\| ds \lesssim \int_t^{t+\epsilon} (t+\epsilon-s)^{\beta-1} \|f(s)\| ds$$
$$\lesssim \epsilon^{\beta-1} \, \|f\|_{L^1(0,T;L^2(\Omega))}.$$

With the aid of Lemma 2.1, we deduce that

$$\int_0^t \|(\mathcal{T}_\beta(t+\epsilon-s) - \mathcal{T}_\beta(t-s))f(s)\| ds$$
$$\lesssim \int_0^t \left((t+\epsilon-s)^{\beta-1} - (t-s)^{\beta-1} \right) \|f(s)\| ds$$
$$\lesssim \epsilon^{\beta-1} \, \|f\|_{L^1(0,T;L^2(\Omega))},$$

where we use the following inequality

$$\xi_1^{\mu} - \xi_2^{\mu} \leq (\xi_1 - \xi_2)^{\mu}, \quad \mu \in (0,1], \ \xi_1, \xi_2 \in \mathbb{R}, \ \text{and } 0 \leq \xi_2 \leq \xi_1, \tag{3.11}$$

Therefore, together the triangle inequality and the above estimates, we conclude that $\|(\mathcal{Q}u)(t+\epsilon) - (\mathcal{Q}u)(t)\| \to 0$ as ϵ tends to zero. An analogous argument can show that $\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t-\epsilon)\| \to 0$ as ϵ tends to zero for $0 \leq t-\epsilon < t \leq T$. Consequently, we obtain that $\mathcal{Q}u \in C([0,T]; L^2(\Omega))$ for any $u \in C([0,T]; L^2(\Omega))$.

We claim that $\mathcal{Q}$ has a unique fixed point. Indeed, for any $u_1, u_2 \in C([0,T]; L^2(\Omega))$, by (3.9), we have

$$\|(\mathcal{Q}u_1)(t) - (\mathcal{Q}u_2)(t)\| \lesssim \int_0^t \|\mathcal{R}'_\beta(t-s)(u_1(s) - u_2(s))\| ds$$
$$\lesssim \int_0^t (t-s)^{\beta-\alpha-1} \|u_1(s) - u_2(s)\| ds$$
$$\lesssim \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} \|u_1 - u_2\|_{\mathcal{C}},$$

By mathematical induction, it follows that

$$\|(\mathcal{Q}^j u_1)(t) - (\mathcal{Q}^j u_2)(t)\| \lesssim \frac{(\Gamma(\beta-\alpha))^j}{\Gamma(j(\beta-\alpha)+1)} t^{j(\beta-\alpha)} \|u_1 - u_2\|_{\mathcal{C}}. \tag{3.12}$$

If $j = 1$, it have been proved. Assume that (3.12) holds for any $j > 1$. We will show that (3.12) also holds for $j + 1$. For this purpose, by using (3.9) again, one has

$$\begin{aligned}
\|(\mathcal{Q}^{j+1} u_1)(t) - (\mathcal{Q}^{j+1} u_2)(t)\| &\lesssim \int_0^t \|\mathcal{R}'_\beta(t-\tau)((\mathcal{Q}^j u_1)(\tau) - (\mathcal{Q}^j u_2)(\tau))\| d\tau \\
&\lesssim \int_0^t (t-\tau)^{\beta-\alpha-1} \|(\mathcal{Q}^j u_1)(\tau) - (\mathcal{Q}^j u_2)(\tau)\| d\tau \\
&\lesssim \frac{(\Gamma(\beta-\alpha))^j}{\Gamma(j(\beta-\alpha)+1)} \int_0^t (t-\tau)^{\beta-\alpha-1} \tau^{j(\beta-\alpha)} d\tau \|u_1 - u_2\|_{\mathcal{C}} \\
&= \frac{(\Gamma(\beta-\alpha))^{j+1}}{\Gamma((j+1)(\beta-\alpha)+1)} t^{(j+1)(\beta-\alpha)} \|u_1 - u_2\|_{\mathcal{C}}.
\end{aligned}$$

Thus, the inequality (3.12) follows for any $j + 1$ and there exists a constant $C > 0$ such that

$$\|(\mathcal{Q}^{j+1} u_1)(t) - (\mathcal{Q}^{j+1} u_2)(t)\| \leq \frac{C(\Gamma(\beta-\alpha))^{j+1}}{\Gamma((j+1)(\beta-\alpha)+1)} t^{(j+1)(\beta-\alpha)} \|u_1 - u_2\|_{\mathcal{C}}.$$

Let us choose $j = \hat{j}$ large enough so that

$$\varsigma := \frac{C(\Gamma(\beta-\alpha))^{\hat{j}}}{\Gamma(\hat{j}(\beta-\alpha)+1)} T^{\hat{j}(\beta-\alpha)} < 1.$$

Therefore, one has

$$\|\mathcal{Q}^{\hat{j}} u_1 - \mathcal{Q}^{\hat{j}} u_2\|_{\mathcal{C}} \leq \varsigma \|u_1 - u_2\|_{\mathcal{C}}.$$

The contractility of $\mathcal{Q}^{\hat{j}}$ follows, and then $\mathcal{Q}^{\hat{j}}$ has a unique fixed point u^* on $C([0,T]; L^2(\Omega))$. Since $\mathcal{Q}\mathcal{Q}^{\hat{j}} = \mathcal{Q}^{\hat{j}+1} = \mathcal{Q}^{\hat{j}}\mathcal{Q}$, one can see that $\mathcal{Q}^{\hat{j}}(\mathcal{Q}u^*) = \mathcal{Q}(\mathcal{Q}^{\hat{j}} u^*) = \mathcal{Q}u^*$ which deduce that $\mathcal{Q}u^*$ is the fixed point of $\mathcal{Q}^{\hat{j}}$. By virtue of the uniqueness, we conclude that $\mathcal{Q}u^* = u^*$. Consequently, there exists a unique mild solution.

Let us check (3.7). It follows from (3.9) and (3.10) that $\|\mathcal{S}_\beta(t)\phi\| \lesssim \|\phi\|_\gamma$, $\|\mathcal{P}_\beta(t)\psi\| \lesssim t\,\|\psi\|$ and $\|\mathcal{R}_\beta(t)\phi\| \lesssim t^{\beta-\alpha}\,\|\phi\|_\gamma$. Hence, one obtains

$$\begin{aligned}
\|u(t)\| \lesssim \|\phi\|_\gamma + t^{\beta-\alpha}\,\|\phi\|_\gamma + t\|\psi\| + \int_0^t (t-\tau)^{\beta-\alpha-1} \|u(\tau)\| d\tau \\
+ \int_0^t (t-s)^{\beta-1} \|f(s)\| ds.
\end{aligned}$$

From the generalized Gronwall's inequality (see e.g. [31, Corollary 2]) and Lemma 2.3, there exists a positive constant C such that

$$\|u(t)\| \le \varrho(t) \exp\left(\left(C\Gamma(\beta-\alpha)\right)^{\frac{1}{\beta-\alpha}} t \right),$$

where

$$\varrho(t) \lesssim \|\phi\|_\gamma + t^{\beta-\alpha}\|\phi\|_\gamma + t\|\psi\| + t^{\beta-1}\|f\|_{L^1(0,T;L^2(\Omega))}.$$

Thus, the main conclusion is obtained. We have completed this proof. □

In what follows, we are in position to show the regularity of solution.

Theorem 3.2 *Let $(\phi, \psi) \in D(A^\gamma) \times L^2(\Omega)$ for $\gamma \in (0,1)$ and let $\beta - 1 > \alpha$, $f \in L^p(0,T;L^2(\Omega))$ for $p > \frac{1}{\beta-1}$. Then, the solution u of problem (3.1)–(3.3) satisfies*

$$\|\partial_t u(t)\| \lesssim \begin{cases} \|\phi\|_\gamma + \|\psi\| + \|f\|_{L^p(0,T;L^2(\Omega))}, & \beta\gamma \ge 1, \\ t^{\beta\gamma-1}\left(\|\phi\|_\gamma + \|\psi\| + \|f\|_{L^p(0,T;L^2(\Omega))}\right), & \beta\gamma < 1. \end{cases} \tag{3.13}$$

Proof Theorem 3.1 ensures a mild solution of problem (3.1)–(3.3). Hence, it remains to check (3.13). For any $v \in L^2(\Omega)$, let

$$\mathcal{S}'_\beta(t)v = \sum_{n=1}^{\infty} -\lambda_n t^{\beta-1} E_{\beta,\beta}(-\lambda_n t^\beta)(v, e_n)e_n,$$

$$\mathcal{T}'_\beta(t)v = \sum_{n=1}^{\infty} t^{\beta-2} E_{\beta,\beta-1}(-\lambda_n t^\beta)(v, e_n)e_n,$$

and

$$\mathcal{R}''_\beta(t)v = \sum_{n=1}^{\infty} t^{\beta-\alpha-2} E_{\beta,\beta-\alpha-1}(-\lambda_n t^\beta)(v, e_n)e_n.$$

It is not difficult to check that $\|\mathcal{S}'_\beta(t)\phi\| \lesssim t^{\beta\gamma-1}\|\phi\|_\gamma$ and $\|\mathcal{T}'_\beta(t)f(\cdot)\| \lesssim t^{\beta-2}\|f(\cdot)\|$, respectively. It follows from (3.10) that $\|\mathcal{S}_\beta(t)\psi\| \lesssim \|\psi\|$. As the same argument, Lemma 2.2 shows that $\|\mathcal{R}''_\beta(t)u(\cdot)\| \lesssim t^{\beta-\alpha-2}\|u(\cdot)\|$. In view of Lemma 2.1, we have

$$\partial_t u(t) = \mathcal{S}'_\beta(t)\phi + \mathcal{R}'_\beta(t)\phi + \mathcal{S}_\beta(t)\psi - \int_0^t \mathcal{R}''_\beta(t-s)u(s)ds + \int_0^t \mathcal{T}'_\beta(t-s)f(s)ds, \tag{3.14}$$

Therefore, it follows that

$$\|\partial_t u(t)\| \lesssim t^{\beta\gamma-1}\|\phi\|_\gamma + t^{\beta-\alpha-1}\|\phi\|_\gamma + \|\psi\| + \int_0^t (t-s)^{\beta-\alpha-2}\|u(s)\|ds + t^{\beta-1-\frac{1}{p}}\|f\|_{L^p(0,T;L^2(\Omega))}.$$

Substituting (3.7) into the above inequality, we thus obtain the desired result. □

Theorem 3.3 *Let $(\phi,\psi) \in D(A^\gamma) \times L^2(\Omega)$ for $\gamma \in (0,1)$ satisfying $\frac{1}{\beta} \le \gamma < \frac{\beta-\alpha}{\beta}$ and let $f \in L^p(0,T;L^2(\Omega))$ for $p > \frac{1}{\beta-\beta\gamma}$. Then the solution u belongs to $C((0,T];H^{2\gamma}(\Omega))$ and satisfies*

$$\|u(t)\|_{H^{2\gamma}(\Omega)} \lesssim t^{-(\beta\gamma-1)}\left(\|\phi\|_\gamma + \|\psi\| + \|f\|_{L^p(0,T;L^2(\Omega))}\right). \tag{3.15}$$

Proof To begin with this theorem, from the assumption of $1 \le \beta\gamma \le \beta-\alpha$, one can see that there exists a mild solution u such that $u(t)$, $\partial_t u(t) \in L^2(\Omega)$ for $t \in [0,T]$ by Theorems 3.1–3.2. Consequently, by virtue of (3.5) and transposition of term we have

$$\int_0^t \mathcal{R}'_\beta(t-s)u(s)ds = \mathcal{R}_\beta(t)\phi + \int_0^t \mathcal{R}_\beta(t-s)\partial_s u(s)ds. \tag{3.16}$$

Hence, we only need to consider $u \in C((0,T];H^{2\gamma}(\Omega))$ and further it satisfies (3.15). Initially, the Sobolev embedding $D(A^\gamma) \subset H^{2\gamma}(\Omega)$ for $\gamma > 0$ implies that if u belongs to $D(A^\gamma)$ then one has u belonging to $H^{2\gamma}(\Omega)$. Repeating the existence proof process of Theorem 3.1, we can verify $u \in C((0,T];D(A^\gamma))$. It means $u \in C((0,T];H^{2\gamma}(\Omega))$. Thus, it is sufficient to check the estimate (3.15). Now, from the definition of fractional power operators, it follows that

$$A^\gamma \mathcal{T}_\beta(t)v = t^{\beta-1}\sum_{n=1}^\infty \lambda_n^\gamma E_{\beta,\beta}(-\lambda_n t^\beta)(v,e_n)e_n, \quad t \ge 0,$$

for any $v \in L^2(\Omega)$ and $\gamma \in (0,1)$. By Lemma 2.4, we have

$$\|A^\gamma \mathcal{T}_\beta(t)v\| \lesssim t^{\beta-\beta\gamma-1}\|v\|, \quad t > 0. \tag{3.17}$$

For $t \in (0,T]$, we set

$$\chi(t) = \int_0^t A^\gamma \mathcal{R}_\beta(t-s)\partial_s u(s)ds, \ \varphi(t) = \int_0^t A^\gamma \mathcal{T}_\beta(t-s)f(s)ds.$$

Applying (3.6) and (3.16), we conclude that

$$A^\gamma u(t) = A^\gamma \mathcal{S}_\beta(t)\phi + A^\gamma \mathcal{P}_\beta(t)\psi - \chi(t) + \varphi(t), \quad t \in (0,T].$$

On the other hand, Sobolev embedding theorem shows that $\|u(t)\|_{H^{2\gamma}(\Omega)} \lesssim \|u(t)\|_\gamma$. Thus, it is sufficient to estimate these terms $\|\mathcal{S}_\beta(t)\phi\|_\gamma$, $\|\mathcal{P}_\beta(t)\psi\|_\gamma$, $\|\chi(t)\|$ and $\|\varphi(t)\|$.

Lemma 2.2 implies

$$\|\mathcal{S}_\beta(t)\phi\|_\gamma = \left(\sum_{n=1}^{\infty} |E_{\beta,1}(-\lambda_n t^\beta)|^2 \lambda_n^{2\gamma} |\phi_n|^2\right)^{1/2} \lesssim \|\phi\|_\gamma. \tag{3.18}$$

With the help of (i) in Lemma 2.4, we get

$$\|\mathcal{P}_\beta(t)\psi\|_\gamma \leq \left(\sum_{n=1}^{\infty} \left|\lambda_n^\gamma t E_{\beta,2}(-\lambda_n t^\beta)\right|^2 |\psi_n|^2\right)^{1/2} \lesssim t^{-(\beta\gamma-1)}\|\psi\|. \tag{3.19}$$

For the fourth term containing φ, by applying (3.17), we have the estimate

$$\|\varphi(t)\| \lesssim \int_0^t (t-s)^{\beta-\beta\gamma-1}\|f(s)\|ds \lesssim t^{\beta-\beta\gamma-\frac{1}{p}}\|f\|_{L^p(0,T;L^2(\Omega))}. \tag{3.20}$$

Therefore, it remains to verify the third term containing χ. Obviously, we get the inequality

$$\|A^\gamma \mathcal{R}'_\beta(t)v\| \lesssim t^{\beta-\beta\gamma-\alpha-1}\|v\|, \quad v \in L^2(\Omega).$$

Therefore, one can see

$$\|\chi(t)\| \lesssim \int_0^t (t-s)^{\beta-\beta\gamma-\alpha-1}\|\partial_s u(s)\|ds.$$

By virtue of (3.13) and $\beta\gamma \geq 1$, the following estimate is established

$$\|\chi(t)\| \lesssim t^{\beta-\beta\gamma-\alpha}\left(\|\phi\|_\gamma + \|\psi\| + \|f\|_{L^p(0,T;L^2(\Omega))}\right). \tag{3.21}$$

Together (3.18) to (3.21), the proof is completed. □

Using a similar argument as in Theorems 3.1–3.2, we can deduce the following conclusion.

Theorem 3.4 *Let $(\phi, \psi) \in D(A^\gamma) \times L^2(\Omega)$ for $\gamma \in (0, 1)$ satisfying $\gamma \leq \frac{\beta-1}{2\beta}$ and let $f \in L^\infty(0, T; D(A^{-\gamma}))$. Then the solution u of problem (3.1)–(3.3) belongs to $L^\infty(0, T; H^{2\gamma}(\Omega))$. Moreover*

$$\|u\|_{L^\infty(0,T;H^{2\gamma}(\Omega))} \lesssim \|\phi\|_\gamma + \|\psi\| + \|f\|_{L^\infty(0,T;D(A^{-\gamma}))}. \tag{3.22}$$

Proof Indeed noting that, in view of the assumption $2\beta\gamma \leq \beta - 1$, we get $0 < \beta\gamma < 1/2$, $\beta - \beta\gamma - 1 > 0$ and

$$\|\mathcal{T}_\beta(t)v\| \leq \left(\sum_{n=1}^{\infty} \left|\lambda_n^\gamma t^{\beta-1} E_{\beta,\beta}(-\lambda_n t^\beta)\right|^2 \lambda_n^{-2\gamma} |(v, e_n)|^2\right)^{1/2}$$

$$\lesssim t^{\beta-\beta\gamma-1}\|v\|_{\gamma*}, \quad v \in D(A^{-\gamma}).$$

From the assumption of f, by applying an analogous method of existence proof in Theorem 3.1, one can easily check that there exists a unique mild solution u, which satisfies

$$\|u(t)\| \lesssim \|\phi\|_\gamma + \|\psi\| + \|f\|_{L^\infty(0,T;D(A^{-\gamma}))}. \tag{3.23}$$

On the other hand, by virtue of Lemma 2.2, we see that

$$\|A^\gamma \mathcal{T}_\beta(t) f(\cdot)\| \lesssim t^{\beta-2\beta\gamma-1}\|f(\cdot)\|_{\gamma*}.$$

Therefore, one has

$$\begin{aligned}\left\|A^\gamma \mathcal{S}_\beta(t)\phi + A^\gamma \mathcal{P}_\beta(t)\psi + \varphi(t)\right\| &\lesssim \|\phi\|_\gamma + t^{1-\beta\gamma}\|\psi\| \\ &\quad + t^{\beta-2\beta\gamma}\|f\|_{L^\infty(0,T;D(A^{-\gamma}))},\end{aligned}$$

where φ is defined in Theorem 3.3. Hence, it remains to estimate

$$A^\gamma \mathcal{R}_\beta(t)\phi - \int_0^t A^\gamma \mathcal{R}'_\beta(t-s)u(s)ds. \tag{3.24}$$

It is easy to estimate $\|A^\gamma \mathcal{R}_\beta(t)\phi\| \lesssim t^{\beta-\alpha}\|\phi\|_\gamma$. Now, we estimate another term of (3.24). Indeed, by (ii) in Lemma 2.4, we have

$$\begin{aligned}\int_0^t \|A^\gamma \mathcal{R}'_\beta(t-s)u(s)\|ds &= \int_0^t \left(\sum_{n=1}^\infty |\lambda_n^\gamma (t-s)^{\beta-\alpha-1} E_{\beta,\beta-\alpha}(-\lambda_n(t-s)^\beta)|^2 |u_n(s)|^2\right)^{\frac{1}{2}} ds \\ &\lesssim \int_0^t (t-s)^{\beta-\beta\gamma-\alpha-1}\|u(s)\|ds.\end{aligned}$$

Noting that the assumption $2\beta\gamma \le \beta - 1$ and $\alpha \in (0, 1]$, by substituting (3.23) to the above inequality, we thus immediately conclude that the assertion of (3.22) is satisfied. This completes the proof. □

In [26], the authors considered a fractional diffusion-wave problem, and further they obtained that the regularity property in time is of infinity order which means that $u \in C^\infty$ for $t > 0$. In [24], the authors derived some time regularity estimates for a weak solution of fractional wave equations, they also corrected some papers including numerical technique, which ignores the situation that the solution will blow up at point $t = 0$ for the time regularity $u \in C^3$. Inspired by these works, we establish the following regularity results for time fractional damped wave equations.

Theorem 3.5 *Let $(\phi, \psi) \in D(A^\gamma) \times L^2(\Omega)$ for $\gamma \in (0, 1)$ satisfying $\frac{1}{\beta} \le \gamma$ and let $\beta - 1 > \alpha$. Assume that $f(0) \in L^2(\Omega)$ is finite and $f \in W^{1,p}(0, T; L^2(\Omega))$ for $p > \frac{1}{\beta-1}$. Then the mild solution u of problem (3.1)–(3.3) satisfies*

$$\left\|\partial_t^2 u(t)\right\| \lesssim t^{-1}\left(\|\phi\|_\gamma + \|\psi\| + \|f\|_{W^{1,p}(0,T;L^2(\Omega))} + \|f(0)\|\right).$$

Proof By Theorems 3.1–3.2, the mild solution belongs to $C^1([0,T];L^2(\Omega))$. Hence, we can find an u satisfying (3.14). Invoking the initial value conditions $u(0)=\phi$ and $\partial_t u(0)=\psi$, by changing of variable and taking the derivative with respect to t in (3.14), we conclude that for $t>0$

$$\partial_t^2 u(t) = \mathcal{S}_\beta''(t)\phi + \mathcal{S}_\beta'(t)\psi - \int_0^t \mathcal{R}_\beta''(s)\partial_t u(t-s)ds + \mathcal{T}_\beta'(t)f(0) + \int_0^t \mathcal{T}_\beta'(s)\partial_t f(t-s)ds.$$

On the other hand, Lemma 2.2 shows that

$$\|\mathcal{S}_\beta''(t)\phi\| \le t^{\beta-2}\left(\sum_{n=1}^{\infty}\left(\lambda_n^{1-\gamma}E_{\beta,\beta-1}(-\lambda_n t^\beta)\right)^2 \lambda_n^{2\gamma}|\phi_n|^2\right)^{1/2} \lesssim t^{\beta-2}\|\phi\|_\gamma.$$

For $t>0$, we have the following inequalities

$$\|\mathcal{S}_\beta''(t)\psi\| \lesssim t^{-1}\|\psi\|,\quad \|\mathcal{R}_\beta''(t)v\| \lesssim t^{\beta-\alpha-2}\|v\|,\quad \|\mathcal{T}_\beta'(t)v\| \lesssim t^{\beta-2}\|v\|,$$

for $v\in L^2(\Omega)$. Hence, we obtain the estimate

$$\left\|\partial_t^2 u(t)\right\| \lesssim t^{\beta-2}\|\phi\|_\gamma + t^{-1}\|\psi\| + \int_0^t s^{\beta-\alpha-2}\|\partial_t u(t-s)\|ds + t^{\beta-2}\|f(0)\| + \int_0^t s^{\beta-2}\|\partial_t f(t-s)\|ds,$$

which implies from (3.13) that the desired result. The proof is completed. □

Remark 3.1 Let us mention that the time regularity of mild solutions in the present problem just achieve the second time derivative under the assumptions of Theorem 3.5. This is difference between the previous papers [24,26] where they could establish more higher time regularity of solutions. Nevertheless, if we alter the initial value of ψ belonging to $D(A^\gamma)$, $f\in W^{2,p}(0,T;L^2(\Omega))$ such that $\partial_t f(0)\in L^2(\Omega)$, by the Sobolev embedding relationship $D(A^\gamma)\subset L^2(\Omega)$ for $\gamma\in(0,1)$, based on the existing assumptions in Theorem 3.5, we also establish a unique mild solution on $C^1([0,T];L^2(\Omega))$, the solution will possess the third time derivative

$$\left\|\partial_t^3 u(t)\right\| \lesssim t^{\beta-\alpha-3}\left(\|\phi\|_\gamma + \|\psi\|_\gamma + \|f\|_{W^{2,p}(0,T;L^2(\Omega))} + \|f(0)\| + \|\partial_t f(0)\|\right).$$

4 Nonlinear problems

In this section, we will take account of the nonlinear problem for fractional wave equation with damping. Initially, as before, we introduce a suitable definition of mild solutions to the nonlinear problem.

Definition 4.1 Let $T > 0$. A function $u \in C([0,T]; L^2(\Omega))$ is said to be a mild solution of problem (1.1)–(1.3), if u satisfies the following equation

$$u(t) = \mathcal{S}_\beta(t)\phi + \mathcal{R}_\beta(t)\phi + \mathcal{P}_\beta(t)\psi - \int_0^t \mathcal{R}'_\beta(t-s)u(s)ds \\ + \int_0^t \mathcal{T}_\beta(t-s)f(u(s))ds.$$

4.1 Well-posedness of problem

In this subsection, we shall infer that the present problem is well-posed. In general, the constant C in Lemma 2.2 is not easy to check, in order to overcome this difficulty, we will show some more exact estimates of Mittag–Leffler functions. In what follows, let us first state some properties of a function $M_\upsilon(\cdot)$ which is also called Mainardi's Wright-type function. This function is a special case of the Wright function that plays an important role in different areas of fractional calculus and it is introduced by Mainardi to characterize the solution of initial value problem for fractional diffusion-wave equations. More precisely, the function $M_\upsilon(\cdot) : \mathbb{C} \to \mathbb{C}$ is defined by

$$M_\upsilon(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-\upsilon(n+1))}, \quad \upsilon \in (0,1),\ z \in \mathbb{C}.$$

Clearly, it is an entire function. For $\theta > 0$, Mainardi's Wright-type function has the properties

$$M_\upsilon(\theta) \geq 0, \quad \int_0^\infty \theta^\delta M_\upsilon(\theta)d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\upsilon\delta)}, \quad for\ -1 < \delta < \infty. \tag{4.1}$$

Lemma 4.1 *Let $\beta \in (1,2)$. Then for $z \in \mathbb{C}$, there are important formulas between Mittag–Leffler functions, Mainardi's Wright-type functions and sine/cosine functions given by*

$$E_{\beta,1}(-z^2) = \int_0^\infty M_{\beta/2}(\theta)\cos(z\theta)d\theta, \quad E_{\beta,\beta}(-z^2) = \frac{\beta}{2z}\int_0^\infty \theta M_{\beta/2}(\theta)\sin(z\theta)d\theta.$$

Proof The first identity was proved in [22, p. 252]. Hence, it is sufficient to verify the second identity. Indeed, by developing the sine function in series, we get

$$\frac{1}{z}\int_0^\infty \theta M_{\beta/2}(\theta)\sin(z\theta)d\theta = \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k+1)!}\int_0^\infty \theta^{2k+2}M_{\beta/2}(\theta)d\theta, \quad \text{for } z \in \mathbb{C}.$$

Applying the formula in (4.1), it is easily seen that

$$\frac{1}{z}\int_0^\infty \theta M_{\beta/2}(\theta)\sin(z\theta)d\theta = \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}(2k+2)}{\Gamma(1+(n+1)\beta)} = \frac{2}{\beta}E_{\beta,\beta}(-z^2).$$

Consequently, we get the desired formulas. □

It is interesting to notice that the Mittag–Leffer function has a strong connection with the sine/cosine functions as well as the exponential function $\exp(z)$ (see e.g. [17,33]) such as

$$E_{\alpha,1}(z) = \int_0^\infty M_\alpha(\theta)\exp(z\theta)d\theta,$$
$$E_{\alpha,\alpha}(z) = \alpha\int_0^\infty \theta M_\alpha(\theta)\exp(z\theta)d\theta, \quad z\in\mathbb{C},\ \alpha\in(0,1).$$

This means that Mainardi' Wright type function acts as a bridge between the classical and fractional differential equations.

Lemma 4.2 *Let $\beta\in(1,2]$, $\alpha\in(0,1]$ and $\lambda>0$. Then the following estimates hold for $t\ge 0$:*

$$|E_{\beta,\beta'}(-\lambda t^\beta)| \le \frac{1}{\Gamma(\beta')},\ \textit{for}\ \beta'=1,2,\beta;\ |E_{\beta,\beta-\alpha}(-\lambda t^\beta)| \le \frac{1}{(\beta-1)\Gamma(\beta-\alpha)}.$$

Proof By properties of Mittag–Leffler function in series, the case of $t=0$ is obvious. Hence, for any $t>0$, $z\in\mathbb{R}^+$, from the fact $E_{2,1}(-z^2)=\cos(z)$ and $zE_{2,2}(-z^2)=\sin(z)$, by using the inequalities $|\cos(z)|\le 1$ and $\sin(z)\le z$, it is easy to check the first inequality for $\beta=2$, $\beta'=1,2$. By Lemma 4.1, it yields from (4.1) and $|\cos(z)|\le 1$ that $|E_{\beta,1}(-\lambda t^\beta)|\le 1$. Lemma 2.1 checks $\frac{d}{dt}(tE_{\beta,2}(-\lambda t^\beta)) = E_{\beta,1}(-\lambda t^\beta)$, hence $|E_{\beta,2}(-\lambda t^\beta)|\le 1$ follows. Lemma 4.1 implies

$$t^{\beta-1}E_{\beta,\beta}(-\lambda t^\beta) = \frac{1}{\sqrt{\lambda}}\frac{\beta}{2}t^{\frac{\beta}{2}-1}\int_0^\infty \theta M_{\beta/2}(\theta)\sin\left(\sqrt{\lambda}t^{\frac{\beta}{2}}\theta\right)d\theta. \tag{4.2}$$

We notice that the left side of above equation tends to zero when $t\to 0$ as well as the right-hand side of (4.2), because of the fact $\lim_{z\to 0}\frac{\sin(z)}{z}=1$. Consequently, by using $\sin(z)\le z$ for $z\in\mathbb{R}^+$ and (4.1), we get

$$|E_{\beta,\beta}(-\lambda t^\beta)| \le \frac{\beta}{2}\int_0^\infty \theta^2 M_{\beta/2}(\theta)d\theta = \frac{1}{\Gamma(\beta)}.$$

Taking the derivative with respect to t in (4.2), noting that

$$\begin{aligned}\frac{d}{dt}\left(t^{\beta-1}E_{\beta,\beta}(-\lambda t^\beta)\right) &= t^{\beta-2}\frac{\beta^2}{4}\int_0^\infty \theta^2 M_{\beta/2}(\theta)\cos\left(\sqrt{\lambda}t^{\frac{\beta}{2}}\theta\right)d\theta \\ &\quad - t^{\frac{\beta}{2}-2}\frac{1}{\sqrt{\lambda}}\frac{\beta}{2}\left(1-\frac{\beta}{2}\right)\int_0^\infty \theta M_{\beta/2}(\theta)\sin\left(\sqrt{\lambda}t^{\frac{\beta}{2}}\theta\right)d\theta,\end{aligned} \tag{4.3}$$

in view of (2.1) and Lemma 2.1, we find

$$t^{\beta-\alpha-1}E_{\beta,\beta-\alpha}(-\lambda t^{\beta}) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{d}{ds}\left(s^{\beta-1}E_{\beta,\beta}(-\lambda s^{\beta})\right)ds.$$

Substituting (4.3) into (4.2) implies

$$|t^{\beta-\alpha-1}E_{\beta,\beta-\alpha}(-\lambda t^{\beta})| \leq \frac{1}{(\beta-1)\Gamma(\beta-\alpha)}t^{\beta-\alpha-1}, \quad t>0.$$

Hence, it remains to check the case of $\beta = 2$, $\alpha \in (0, 1)$. Indeed, noting that

$$t^{1-\alpha}E_{2,2-\alpha}(-\lambda t^2) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}E_{2,1}(-\lambda s^2)ds,$$

which is easy to deduce that $|E_{2,2-\alpha}(-\lambda t^2)| \leq 1/\Gamma(2-\alpha)$. Thus, we obtain the desired results. □

On the basis of above arguments, we now show that the problem (1.1)–(1.3) is well-posed.

Theorem 4.1 *Let $\gamma \in (0, 1)$. Assume that there exist two positive constants a, b such that the nonlinear function $f \in L^1(\mathbb{R}; L^2(\Omega))$ satisfies the following conditions*

$$\|f(u) - f(v)\| \leq a\left(\|u\|^{\vartheta-1} + \|v\|^{\vartheta-1}\right)\|u-v\|,$$
$$\|f(u)\| \leq b\left(1 + \|u\|^{\vartheta}\right),$$

for each $u, v \in L^2(\Omega)$, where $\vartheta \geq 1$ is a constant. Then for $\phi \in D(A^{\gamma})$, $\psi \in L^2(\Omega)$, problem (1.1)–(1.3) possesses a unique mild solution on $C([0, T_0], L^2(\Omega))$ for some $T_0 \in (0, T]$. Moreover, the solutions $u, \widetilde{u}$ depend continuously on the initial functions $\widetilde{\phi}$ and $\widetilde{\psi}$ which correspond to the mild solution $\widetilde{u}$ in the sense that

$$\|u(t) - \widetilde{u}(t)\| \lesssim \|\phi - \widetilde{\phi}\|_{\gamma} + \|\psi - \widetilde{\psi}\|. \tag{4.4}$$

Proof For fixed $r > 0$, let us introduce a metric space

$$B_r(\phi,\psi) = \left\{u \in C([0,T]; L^2(\Omega)) :\ \rho_T\left(u, \mathcal{S}_{\beta}(t)\phi + \mathcal{R}_{\beta}(t)\phi + \mathcal{P}_{\beta}(t)\psi\right) \leq r\right\},$$

where

$$\rho_T(u_1,u_2) = \sup_{t\in[0,T]}\|u_1(t) - u_2(t)\|.$$

It is not difficult to check that $B_r(\phi, \psi)$ is a complete metric space with the above metric.

Let us consider an operator $\mathcal{Q}$ given by

$$(\mathcal{Q}u)(t) = \mathcal{S}_\beta(t)\phi + \mathcal{P}_\beta(t)\psi + \mathcal{R}_\beta(t)\phi - \int_0^t \mathcal{R}'_\beta(t-s)u(s)ds \\ + \int_0^t \mathcal{T}_\beta(t-s)f(u(s))ds, \tag{4.5}$$

for any $u \in B_r(\phi, \psi)$. Clearly, $\mathcal{Q}$ is well-defined in $C([0, T]; L^2(\Omega))$, as it follows from the assumptions of f. Next, we are planing to show the existence and uniqueness. It is sufficient to verify that $\mathcal{Q}$ has a unique fixed point in $B_r(\phi, \psi)$.

In view of Lemma 4.2, for $t \in [0, T]$, we get some exact upper bounds

$$\|\mathcal{S}_\beta(t)v\| \le \|v\|, \ \|\mathcal{P}_\beta(t)v\| \le t\,\|v\|, \ \|\mathcal{T}_\beta(t)v\| \le \frac{t^{\beta-1}}{\Gamma(\beta)}\|v\|, \tag{4.6}$$

and $\|\mathcal{R}'_\beta(t)v\| \le \sigma\, t^{\beta-\alpha-1}\|v\|$, $\|\mathcal{R}_\beta(t)v\| \le \varrho\, t^{\beta-\alpha}\|v\|$ for any $v \in L^2(\Omega)$, where $\sigma := 1/((\beta-1)\Gamma(\beta-\alpha))$, $\varrho = \sigma/(\beta-\alpha)$. In view of the Sobolev embedding $D(A^\gamma) \subset L^2(\Omega)$, $\gamma \in (0, 1)$, one find that $\|\phi\| \le \lambda_1^{-\gamma}\|\phi\|_\gamma$, where λ_1 is the first eigenvalue of operator A. Hence, taking

$$L_r := r + (1 + \varrho T^{\beta-\alpha})\lambda_1^{-\gamma}\|\phi\|_\gamma + T\|\psi\|, \tag{4.7}$$

the following estimate is established

$$\|u(t)\| \le \left\|u(t) - \mathcal{S}_\beta(t)\phi - \mathcal{R}_\beta(t)\phi - \mathcal{P}_\beta(t)\psi\right\| + \|\mathcal{S}_\beta(t)\phi\| \\ + \|\mathcal{R}_\beta(t)\phi\| + \|\mathcal{P}_\beta(t)\psi\| \le L_r.$$

Choose $T_0 \in (0, T]$ such that

$$\varrho\, L_r T_0^{\beta-\alpha} + \frac{b}{\Gamma(\beta+1)}\left(1 + L_r^{\vartheta-1}\right) T_0^\beta \le r, \tag{4.8}$$

and

$$\varrho T_0^{\beta-\alpha} + \frac{2a}{\Gamma(\beta+1)} L_r^{\vartheta-1} T_0^\beta \le \frac{1}{2}. \tag{4.9}$$

Therefore, from (4.8) we get

$$\begin{aligned} &\|(\mathcal{Q}u)(t) - \mathcal{S}_\beta(t)\phi - \mathcal{R}_\beta(t)\phi - \mathcal{P}_\beta(t)\psi\| \\ &\le \int_0^t \|\mathcal{R}'_\beta(t-s)u(s)\|ds + \int_0^t \|\mathcal{T}_\beta(t-s)f(u(s))\|ds \\ &\le \sigma\int_0^t (t-s)^{\beta-\alpha-1}\|u(s)\|ds + \frac{b}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(1 + \|u(s)\|^\rho)ds \\ &\le \varrho L_r T_0^{\beta-\alpha} + \frac{b}{\Gamma(\beta+1)}\left(1 + L_r^{\vartheta-1}\right) T_0^\beta \le r. \end{aligned}$$

This implies that $\mathcal{Q}$ maps $B_r(\phi, \psi)$ into itself. In addition, for any $u, v \in B_r(\phi, \psi)$, by the assumption of f, we have

$$
\begin{aligned}
\|(\mathcal{Q}u)(t) - (\mathcal{Q}v)(t)\| \leq & \left\| \int_0^t \mathcal{R}'_\beta(t-s)\big(u(s)-v(s)\big)ds \right\| \\
& + \left\| \int_0^t \mathcal{T}_\beta(t-s)(f(u(s)) - f(v(s)))ds \right\| \\
\leq & \, \sigma \int_0^t (t-s)^{\beta-\alpha-1} \|u(s)-v(s)\| \, ds \\
& + \frac{a}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(\|u(s)\|^{\vartheta-1} + \|v(s)\|^{\vartheta-1} \right) \|u(s)-v(s)\| ds \\
\leq & \, \varrho t^{\beta-\alpha} \rho_t(u,v) + \frac{2a}{\Gamma(\beta+1)} L_r^{\vartheta-1} t^\beta \rho_t(u,v) \\
\leq & \left(\varrho T_0^{\beta-\alpha} + \frac{2a}{\Gamma(\beta+1)} L_r^{\vartheta-1} T_0^\beta \right) \rho_{T_0}(u,v).
\end{aligned}
$$

Hence, in view of (4.9), we conclude that $\mathcal{Q}$ is a contraction on $B_r(\phi, \psi)$. Thus, according to Banach's fixed point theorem, the operator $\mathcal{Q}$ has a unique fixed point that is the mild solution of problem (1.1)–(1.3) on $[0, T_0]$.

We next show the continuous dependence of the mild solution on the initial data. Let $(u, \widetilde{u})$ be two mild solutions of problem (1.1)–(1.3) associated with the initial conditions $(\phi, \widetilde{\phi})$ and $(\psi, \widetilde{\psi})$. Then, one obtains

$$
\begin{aligned}
\|u(t) - \widetilde{u}(t)\| \leq & \|\mathcal{S}_\beta(t)\phi - \mathcal{S}_\beta(t)\widetilde{\phi}\| + \|\mathcal{R}_\beta(t)\phi - \mathcal{R}_\beta(t)\widetilde{\phi}\| + \|\mathcal{P}_\beta(t)\psi - \mathcal{P}_\beta(t)\widetilde{\psi}\| \\
& + \int_0^t \left\| \mathcal{R}'_\beta(t-s)\big(u(s) - \widetilde{u}(s)\big) \right\| ds \\
& + \int_0^t \left\| \mathcal{T}_\beta(t-s)(f(u(s)) - f(\widetilde{u}(s))) \right\| ds \\
\leq & \, \zeta(t) \|\phi - \widetilde{\phi}\|_\gamma + t \|\psi - \widetilde{\psi}\| + \sigma \int_0^t (t-s)^{\beta-\alpha-1} \|u(s) - \widetilde{u}(s)\| \, ds \\
& + \frac{a}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(\|u(s)\|^{\vartheta-1} + \|\widetilde{u}(s)\|^{\vartheta-1} \right) \|u(s) - \widetilde{u}(s)\| ds \\
\leq & \, \Phi(t) + \Phi_r(t) \int_0^t \|u(s) - \widetilde{u}(s)\| ds,
\end{aligned}
$$

where $\zeta(t) = \lambda_1^{-\gamma} \left(1 + \varrho t^{\beta-\alpha}\right)$ and

$$
\Phi(t) = \zeta(t) \|\phi - \widetilde{\phi}\|_\gamma + t \|\psi - \widetilde{\psi}\|, \quad \Phi_r(t) = \left(\sigma + \frac{2a}{\Gamma(\beta)} L_r^{\vartheta-1} t^\alpha \right).
$$

Thus, the generalized Gronwall's inequality imply that

$$
\|u(t) - \widetilde{u}(t)\| \lesssim \Phi(t) \exp\left(\big(\Phi_r(t)\Gamma(\beta-\alpha)\big)^{\frac{1}{\beta-\alpha}} t \right),
$$

which means that the solution is continuous dependence on the initial conditions for any positive real number r. We thus have proved this theorem. □

Remark 4.1 Noting that, if the assumptions of nonlinear function f is replaced by another local Lipshitz condition: there exists a nondecreasing function $L_f(\cdot) \in L^\infty(\mathbb{R}_+)$ such that the nonlinear mapping f is continuous with respect to t and satisfies the condition

$$\|f(u) - f(v)\| \leq L_f(r)\|u - v\|, \quad r > 0,$$

for each $u, v \in L^2(\Omega)$ satisfying $\|u\|, \|v\| \leq r$. Then, for some $T_0 \in (0, T)$, we get an analogous result of Theorem 4.1 on the following Banach space:

$$B_r(T_0, \phi) = \left\{ u \in C([0, T_0]; L^2(\Omega)) : \sup_{t\in[0,T_0]} \|u\| \leq r \right\},$$

is that, for $\phi \in D(A^\gamma)$, $\psi \in L^2(\Omega)$, $\gamma \in (0, 1)$, problem (1.1)–(1.3) possesses a unique mild solution on $C([0, T_0], L^2(\Omega))$. Moreover, solutions $u, \widetilde{u}$ depend continuously on the initial conditions $\widetilde{\phi}$ and $\widetilde{\psi}$ which correspond to the mild solution $\widetilde{u}$ in the sense of (4.4).

4.2 Continuation and blow-up alternative

Given a mild solution $u \in C([0, T_0]; L^2(\Omega))$ of problem (1.1)–(1.3), we say that $\bar{u} : [0, T_0] \to L^2(\Omega)$ is a continuation of $\bar{u}$ in $[0, T_1]$ with $T_1 > T_0$ if $\bar{u}$ is a mild solution, and $u(t) = \bar{u}(t)$ whenever $t \in [0, T_0]$.

Theorem 4.2 *Let the assumptions of Theorem 4.1 hold and u be a mild solution of problem (1.1)–(1.3) on* $[0, T_0]$. *Then u can be uniquely continued up a time* T_1.

Proof Fix $R > 0$. Take $T_1 > T_0$ such that for $t \in [T_0, T_1]$, we denote a metric space

$$\mathcal{B}_R = \left\{ v \in C([0, T_1]; L^2(\Omega)) : \ \rho_{T_1}(v, u(T_0)) \leq R, \ \text{and } v(t) = u(t), \ t \in [0, T_0] \right\},$$

equipped with the metric

$$\rho_{T_1}(v, u) = \sup_{t\in[0,T_1]} \|v(t) - u(t)\|.$$

It is not difficult to check that $\mathcal{B}_R$ is a complete metric space. Let us define $\mathcal{G} : \mathcal{B}_R \to \mathcal{B}_R$ by

$$(\mathcal{G}v)(t) = (\mathcal{G}_1 v)(t) + (\mathcal{G}_2 v)(t),$$

where

$$(\mathcal{G}_1 v)(t) = \mathcal{S}_\beta(t)\phi + \mathcal{R}_\beta(t)\phi + \mathcal{P}_\beta(t)\psi,$$

$$(\mathcal{G}_2 v)(t) = -\int_0^t \mathcal{R}'_\beta(t-s)v(s)ds + \int_0^t \mathcal{T}_\beta(t-s)f(v(s))ds.$$

If $v \in \mathcal{B}_r$, it is clear to obtain that $\mathcal{G}v(t) = u(t)$ for any $t \in [0, T_0]$. Let $t \in [T_0, T_1]$, for any $v \in \mathcal{B}_r$, by some simple computations, we get

$$\begin{aligned}\|(\mathcal{G}v)(t) - u(T_0)\| \le \|(\mathcal{G}_1 v)(t) - (\mathcal{G}_1 v)(T_0)\| \\ + \int_0^{T_0} \|(\mathcal{R}'_\beta(t-s) - \mathcal{R}'_\beta(T_0-s))u(s)\|ds \\ + \int_0^{T_0} \|(\mathcal{T}_\beta(t-s) - \mathcal{T}_\beta(T_0-s))f(u(s))\|ds \\ + \int_{T_0}^t \|\mathcal{R}'_\beta(t-s)v(s)\|ds + \int_{T_0}^t \|\mathcal{T}_\beta(t-s)f(v(s))\|ds.\end{aligned}$$

Since the mappings $t \mapsto \mathcal{S}_\beta(t)\phi$, $t \mapsto \mathcal{R}_\beta(t)\phi$ and $t \mapsto \mathcal{P}_\beta(t)\psi$ belong to $C([0, T]; L^2(\Omega))$ for every $t \in [0, T]$ with $T > T_1$, it means that we can pick $T_a \in [T_0, T)$ such that for $t \in [T_0, T_a]$

$$\|(\mathcal{G}_1 v)(t) - (\mathcal{G}_1 v)(T_0)\| \le \frac{R}{3}.$$

Processing as the proof of Theorem 3.1, one see that for $t \in [T_0, T)$, Lemma 2.2 and by (i) in Lemmas 2.4 with respect to $\mu = 1 - \frac{1}{\beta}$, we have

$$\begin{aligned}&\int_0^{T_0} \|(\mathcal{R}'_\beta(t-s) - \mathcal{R}'_\beta(T_0-s))u(s)\|ds + \int_0^{T_0} \|(\mathcal{T}_\beta(t-s) - \mathcal{T}_\beta(T_0-s))f(u(s))\|ds \\ &\lesssim \int_0^{T_0} \left| \int_{T_0-s}^{t-s} \tau^{-\alpha-1}d\tau \right| \|u(s)\|ds + \int_0^{T_0} \left| \int_{T_0-s}^{t-s} \tau^{\beta-2}d\tau \right| \|f(u(s))\|ds \\ &\lesssim r\left(T_0^{1-\alpha} + (t-T_0)^{1-\alpha} - t^{1-\alpha}\right) + (t-T_0)^\beta L_r^\vartheta \\ &\to 0, \quad \text{as } t \to T_0,\end{aligned}$$

where r is picked as in Theorem 4.1 and L_r is defined in (4.7). Therefore, we can choose $T_b \in [T_0, T)$ such that for $t \in [T_0, T_b]$

$$\begin{aligned}&\int_0^{T_0} \|(\mathcal{R}'_\beta(t-s) - \mathcal{R}'_\beta(T_0-s))u(s)\|ds \\ &\quad + \int_0^{T_0} \|(\mathcal{T}_\beta(t-s) - \mathcal{T}_\beta(T_0-s))f(u(s))\|ds \le \frac{R}{3}.\end{aligned}$$

On the other hand, it is easy to see that

$$\int_{T_0}^t \|\mathcal{R}'_\beta(t-s)v(s)\|ds + \int_{T_0}^t \|\mathcal{T}_\beta(t-s)f(v(s))\|ds$$

$$\leq \sigma \int_{T_0}^{t} (t-s)^{\beta-\alpha-1} \|v(s)\| ds + \frac{1}{\Gamma(\beta)} \int_{T_0}^{t} (t-s)^{\beta-1} \|f(v(s))\| ds$$
$$\leq \varrho (t-T_0)^{\beta-\alpha} (R + L_r) + \frac{b}{\Gamma(\beta)} (t-T_0)^{\beta} (1 + L_r^{\vartheta}).$$

where we use the fact $\|u(T_0)\| \leq L_r$. With the same argument, one can choose $T_c \in [T_0, T)$ such that for $t \in [T_0, T_c]$

$$\int_{T_0}^{t} \|\mathcal{R}'_{\beta}(t-s)v(s)\| ds + \int_{T_0}^{t} \|\mathcal{T}_{\beta}(t-s) f(v(s))\| ds \leq \frac{R}{3}.$$

Consequently, let $T_1 := \min\{T_a, T_b, T_c\}$ and then

$$\|(\mathcal{G}v)(t) - u(T_0)\| \leq R.$$

We thus prove that $\mathcal{G}$ maps $\mathcal{B}_R$ into itself. Now, for any $v, w \in \mathcal{B}_R$, one has

$$\begin{aligned} \|(\mathcal{G}v)(t) - (\mathcal{G}w)(t)\| &= \|(\mathcal{G}_2 v)(t) - (\mathcal{G}_2 w)(t)\| \\ &= \int_0^t \|\mathcal{R}'_{\beta}(t-s)(v(s) - w(s))\| ds \\ &\quad + \int_0^t \|\mathcal{T}_{\beta}(t-s)(f(v(s)) - f(w(s))\| ds. \end{aligned}$$

By the uniqueness, clearly for $t \in [0, T_0]$, $\mathcal{G}$ is contractive on $\mathcal{B}_R$. Let $t \in [T_0, T_1]$, from the assumption of f, we have

$$\begin{aligned} \|(\mathcal{G}v)(t) - (\mathcal{G}w)(t)\| \leq & \sigma \int_{T_0}^{t} (t-s)^{\beta-\alpha-1} \|v(s) - w(s)\| ds \\ & + \frac{a}{\Gamma(\beta)} \int_{T_0}^{t} (t-s)^{\beta-1} (\|v(s)\|^{\vartheta} + \|w(s)\|^{\vartheta}) \|v(s) - w(s)\| ds \\ \leq & \varrho (t-T_0)^{\beta-\alpha} \rho_{T_1}(v, w) + \frac{2a}{\Gamma(\beta)} (t-T_0)^{\beta} (R + L_r)^{\vartheta} \rho_{T_1}(v, w). \end{aligned}$$

Therefore, choosing T_1 such that

$$\varrho (t-T_0)^{\beta-\alpha} + \frac{2a}{\Gamma(\beta)} (t-T_0)^{\beta} (R + L_r)^{\vartheta} < 1,$$

we thus conclude that $\mathcal{G}$ is a contraction map on $\mathcal{B}_R$. This implies that $\mathcal{G}$ has a unique fixed point v on $\mathcal{B}_R$. We have finished this proof. □

Theorem 4.3 *Let the assumptions of Theorem 4.1 hold and $u \in C([0, T_{max}); L^2(\Omega))$ be a mild solution of problem (1.1)–(1.3) with the existence of maximal time T_{max}. Then $T_{max} = +\infty$ or $\lim_{t \to T_{max}^-} \|u(t)\| = \infty$ if $T_{max} < \infty$.*

Proof Let

$$T_{max} = \sup\{T \in [0, \infty) : \exists \text{ unique local solution } u \text{ to problem (1.1)–(1.3)in } (0, T)\}.$$

Suppose that $T_{max} < \infty$ and there exists a positive constant $M < \infty$ such that $\|u(t)\| \le M$ for any $t \in [0, T_{max})$. Let $\{t_i\}_{i\in\mathbb{N}}$ be a sequence of $[0, T_{max})$ such that $t_i \to T_{max}^-$ as $i \to \infty$, we now consider the sequence $\{u(t_i)\} \in L^2(\Omega)$ and we will check that it is a Cauchy sequence in the space $L^2(\Omega)$. Setting $t_i > t_j$, we get

$$\begin{aligned} u(t_i) - u(t_j) =& \mathcal{S}_\beta(t_i)\phi - \mathcal{S}_\beta(t_j)\phi + \mathcal{R}_\beta(t_i)\phi - \mathcal{R}_\beta(t_j)\phi + \mathcal{P}_\beta(t_i)\psi - \mathcal{P}_\beta(t_j)\psi \\ &- \int_{t_j}^{t_i} \mathcal{R}'_\beta(t_i - s)u(s)ds - \int_0^{t_j} (\mathcal{R}'_\beta(t_i - s) - \mathcal{R}'_\beta(t_j - s))u(s)ds \\ &+ \int_{t_j}^{t_i} \mathcal{T}_\beta(t_i - s)f(u(s))ds + \int_0^{t_j} (\mathcal{T}_\beta(t_i - s) - \mathcal{T}_\beta(t_j - s))f(u(s))ds. \end{aligned}$$

Therefore, the same reasoning used as (3.8) in Theorem 3.1 and the similar proof in Theorem 4.2 ensure that

$$\|u(t_i) - u(t_j)\| \to 0, \quad \text{as } i, j \to \infty.$$

Hence, $\{u(t_i)\}_{i\in\mathbb{N}}$ is a Cauchy sequence and then there exists the limit

$$\lim_{i\to\infty} u(t_i) =: u(T_{max}) \in L^2(\Omega).$$

For above reasons, we may extend u over a large interval $[0, T_{max}]$. This shows a contradiction with the maximality of T_{max}. The proof is completed. □

4.3 Compactness method

In the sequel, we remove the Lipschitz condition or higher smoothness assumption of $f \in C^1(\mathbb{R})$, we also consider a more general condition. For this purpose, we need the following lemma.

Lemma 4.3 [30] *Let X be a Banach space and let $R(Q)$ be the range of operator Q. Assume that $Q : X \to X$ is linear.*

(i) *If the dimension of $R(Q)$ is finite, then Q is compact.*
(ii) *If $\{Q_n\}_{n\in\mathbb{N}}$ are a sequence of compact operators in $\mathcal{B}(X)$ that converge uniformly to Q, then Q is compact.*

Definition 4.2 Let X be a Banach space. An operator valued function $T(\cdot)$ defined on $\mathbb{R}_+$ is said to be

(i) *Uniformly continuous.* If the map $t \mapsto T(t)x$ from $\mathbb{R}_+$ to $\mathcal{B}(X)$ is continuous with respect to the operator topology;
(ii) *Strongly continuous.* If the map $t \mapsto T(t)x$ from $\mathbb{R}_+$ to X is continuous for every $x \in X$.

We get the following result which will be useful through this subsection.

Lemma 4.4 *Operator $\mathcal{T}_\beta(t)$ is compact for every $t \geq 0$ and is uniformly continuous on $L^2(\Omega)$ for all $t \geq 0$.*

Proof It is clear that $\mathcal{T}_\beta(0)$ is a zero operator, which is trivial involving with the compactness result. Let $t > 0$ be fixed, and let $\Omega_N = \text{span}\{e_1(x), \ldots, e_N(x)\}$, for every $N \in \mathbb{N}$. It is easy to see that $L^2(\Omega)$ can be expressed by $\text{span}\{e_1(x), \ldots, e_N(x), \ldots\}$. Obviously, Ω_N is a finite dimensional subspace of $L^2(\Omega)$. For all $N \in \mathbb{N}$, we denote operators $\mathcal{T}_\beta^N(t)\cdot : L^2(\Omega) \to \Omega_N$ by

$$\mathcal{T}_\beta^N(t)v = \sum_{n=1}^{N} t^{\beta-1} E_{\beta,\beta}(-\lambda_n t^\beta)(v, e_n)e_n(x).$$

Clearly, $\mathcal{T}_\beta^N(t)$ also is a linear finite dimensional operator. Applying (i) in Lemma 2.4 with respect to $\mu = 1 - \frac{1}{\beta}$, we have

$$\|\mathcal{T}_\beta^N(t)v\| = \left(\sum_{n=1}^{N} \lambda_n^{-\mu} \left(\lambda_n^{\mu} t^{\beta-1} E_{\beta,\beta}(-\lambda_n t^\beta)\right)^2 (v, e_n)^2\right)^{1/2} \lesssim \|v\|.$$

This yields that $\mathcal{T}_\beta^N(t)$ is well-defined in $\mathcal{B}(L^2(\Omega))$. Thus, $R(\mathcal{T}_\beta^N(t))$ is finite, we conclude from (i) in Lemma 4.1 that the operator $\mathcal{T}_\beta^N(t)$ is a compact operator for every $N \in \mathbb{N}$.

Now, we shall prove that $\mathcal{T}_\beta^N(t)$ converges uniformly to $\mathcal{T}_\beta(t)$ whenever N tends to infinite. By applying the above argument, it is notice that from the asymptotic property of the eigenvalues $\lambda_n \to \infty$ as $n \to \infty$, we have when $N \to \infty$,

$$\begin{aligned}\|\mathcal{T}_\beta(t)v - \mathcal{T}_\beta^N(t)v\| &\leq \left(\sum_{n=N+1}^{\infty} \left(t^{\beta-1} E_{\beta,\beta}(-\lambda_n t^\beta)\right)^2 (v, e_n)^2\right)^{1/2} \\ &\lesssim \lambda_{N+1}^{-\mu} \|v\| \to 0.\end{aligned}$$

It means from (ii) in Lemma 4.1 that the operator $\mathcal{T}_\beta(t)$ is a compact operator on $\mathcal{B}(L^2(\Omega))$ for every $t \geq 0$.

In addition, in view of Lemma 2.1 and (3.11), for any $v \in L^2(\Omega)$, for $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$, we find

$$\|\mathcal{T}_\beta(t_2)v - \mathcal{T}_\beta(t_1)v\| \lesssim (t_2 - t_1)^{\beta-1} \|v\| \to 0 \quad \text{as } t_2 \to t_1.$$

Therefore, we conclude that $\mathcal{T}_\beta(t)v$ is strong continuous for all $t \geq 0$. Combined with the compactness of $\mathcal{T}_\beta(t)$, this implies the desired result. The proof is completed. □

Remark 4.2 It is notice that, by using the method of finite dimensional approximation and the compact results Lemma 4.1, one easily prove that operators $\mathcal{S}_\beta(t)$,

$\mathcal{P}_\beta(t)$ and $\mathcal{R}_\beta(t)$ are compact for every $t \geq 0$ and are uniformly continuous for all $t \geq 0$. Additionally, $\mathcal{R}'_\beta(t)$ is also compact for every $t > 0$ since it may be unbounded at time $t = 0$ on $\mathcal{B}(L^2(\Omega))$ for $\beta - \alpha < 1$ and is uniformly continuous for all $t > 0$.

Theorem 4.4 *Let $(\phi, \psi) \in D(A^\gamma) \times L^2(\Omega)$ for $\gamma \in (0, 1)$. Assume that f is Lebesgue measurable with respect to t and is continuous with respect to u, and there exists a nonnegative nondecreasing function $W(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|f(u(t))\| \leq W(\|u(t)\|).$$

Assume further that

$$\liminf_{r\to\infty} \frac{W(r)}{r} = \varpi < \infty,$$

and the following inequality

$$\frac{T^{\beta-\alpha}}{(\beta-1)\Gamma(\beta-\alpha+1)} + \frac{\varpi T^\beta}{\Gamma(\beta+1)} \leq 1. \tag{4.10}$$

Then problem (1.1)–(1.3) possesses at least one mild solution on $C([0, T]; L^2(\Omega))$.

Proof For each $r > 0$, let us set

$$B_r = \{u \in C([0, T]; L^2(\Omega)) : \ \|u\|_C \leq r\}.$$

Then B_r is a bounded closed and convex subset of $C([0, T]; L^2(\Omega))$. Consequently, we need to show that the operator equation $u = \mathcal{Q}u$ has a solution where $\mathcal{Q}$ is defined in Theorem 4.1.

Let us first check that operator $\mathcal{Q}$ maps B_r into itself. In fact, if this is not true, then for each $r > 0$, there exists $u^r \in B_r$ such that $\|(\mathcal{Q}u^r)(t_*)\| > r$ for some $t_* \in [0, T]$. In view of Lemma 4.2 and (4.6), one finds

$$\begin{aligned}
r < \|(\mathcal{Q}u^r)(t_*)\| &\leq \|\mathcal{S}_\beta(t_*)\phi\| + \|\mathcal{R}'_\beta(t_*)\phi\| + \|\mathcal{P}_\beta(t_*)\psi\| \\
&\quad + \int_0^{t_*} \|\mathcal{R}'_\beta(t_*-s)u^r(s)\|ds + \int_0^{t_*} \|\mathcal{T}_\beta(t_*-s)f(u^r(s))\|ds \\
&\leq \lambda_1^{-\gamma} \|\phi\|_\gamma + t_*\|\psi\| + \sigma\ (\|\phi\| + r)\frac{t_*^{\beta-\alpha}}{\beta-\alpha} \\
&\quad + \frac{1}{\Gamma(\beta)}\int_0^{t_*} (t-s)^{\beta-1} W(\|u^r(s)\|)ds \\
&\leq \lambda_1^{-\gamma} \|\phi\|_\gamma + T\|\psi\| + \sigma \left(\lambda_1^{-\gamma}\|\phi\|_\gamma + r\right)\frac{T^{\beta-\alpha}}{\beta-\alpha} \\
&\quad + \frac{T^\beta}{\Gamma(\beta+1)} W(r),
\end{aligned}$$

where $\sigma = 1/((\beta-1)\Gamma(\beta-\alpha))$. Dividing both sides by r and taking the lower limit as $r \to \infty$, we obtain that

$$1 < \frac{\sigma}{\beta-\alpha} T^{\beta-\alpha} + \frac{\varpi}{\Gamma(\beta+1)} T^{\beta},$$

which contradicts (4.10). Therefore, one can selected r such that $\|\mathcal{Q}u\|_{\mathcal{C}} \le r$. This implies that $\mathcal{Q}(B_r) \subseteq B_r$.

We claim that operator $\mathcal{Q}$ is completely continuous. To prove this property, we will divide the proof into three steps. Firstly, we show that the set $\Theta = \{\mathcal{Q}u,\ u \in B_r\}$ is equicontinuous. Indeed, for $0 \le t_1 < t_2 \le T$, we have

$$\begin{aligned}
&\|(\mathcal{Q}u)(t_2) - (\mathcal{Q}u)(t_1)\| \\
&\quad \le \|\mathcal{S}_\beta(t_2)\phi - \mathcal{S}_\beta(t_1)\phi\| + \|\mathcal{R}_\beta(t_2)\phi - \mathcal{R}_\beta(t_1)\phi\| + \|\mathcal{P}_\beta(t_2)\psi - \mathcal{P}_\beta(t_1)\psi\| \\
&\qquad + \left\| \int_0^{t_2} \mathcal{R}'_\beta(t_2-s)u(s)ds - \int_0^{t_1} \mathcal{R}'_\beta(t_1-s)u(s)ds \right\| \\
&\qquad + \left\| \int_0^{t_2} \mathcal{T}_\beta(t_2-s)f(u(s))ds - \int_0^{t_1} \mathcal{T}_\beta(t_1-s)f(u(s))ds \right\| \\
&\quad := J_1 + J_2 + J_3.
\end{aligned}$$

Observe that, from Remark 4.2, we get

$$\begin{aligned}
J_1 = &\|\mathcal{S}_\beta(t_2)\phi - \mathcal{S}_\beta(t_1)\phi\| + \|\mathcal{R}_\beta(t_2)\phi - \mathcal{R}_\beta(t_1)\phi\| \\
&+ \|\mathcal{P}_\beta(t_2)\psi - \mathcal{P}_\beta(t_1)\psi\| \to 0, \text{ as } t_2 \to t_1.
\end{aligned}$$

As for J_2, for any $\varepsilon > 0$, we estimate

$$\begin{aligned}
J_2 \le &\int_0^{t_1-\varepsilon} \left\| \left(\mathcal{R}'_\beta(t_2-s) - \mathcal{R}'_\beta(t_1-s)\right) u(s) \right\| ds \\
&+ \int_{t_1-\varepsilon}^{t_1} \left\| \left(\mathcal{R}'_\beta(t_2-s) - \mathcal{R}'_\beta(t_1-s)\right) u(s) \right\| ds + \int_{t_1}^{t_2} \left\| \mathcal{R}'_\beta(t_2-s)u(s) \right\| ds \\
\le &\int_0^{t_1-\varepsilon} \|u(s)\| \, ds \sup_{s\in[0,t_1-\varepsilon]} \left\| \mathcal{R}'_\beta(t_2-s) - \mathcal{R}'_\beta(t_1-s) \right\|_{\mathcal{B}(L^2(\Omega))} \\
&+ \sigma \int_{t_1-\varepsilon}^{t_2} (t_2-s)^{\beta-\alpha-1} \|u(s)\| ds + \sigma \int_{t_1-\varepsilon}^{t_1} (t_1-s)^{\beta-\alpha-1} \|u(s)\| ds \\
\lesssim &r \sup_{s\in[0,t_1-\varepsilon]} \left\| \mathcal{R}'_\beta(t_2-s) - \mathcal{R}'_\beta(t_1-s) \right\|_{\mathcal{B}(L^2(\Omega))} + (\|\phi\| + r)((t_2 - t_1 + \varepsilon)^{\beta-\alpha} + \varepsilon^{\beta-\alpha}) \\
\to &0, \quad \text{as } t_2 \to t_1,\ \varepsilon \to 0.
\end{aligned}$$

Next, we estimate J_3. In fact, Lemma 4.2 shows that

$$\begin{aligned}
J_3 \le &\int_0^{t_1} W(\|u(s)\|)ds \sup_{s\in[0,t_1]} \left\| \mathcal{T}_\beta(t_2-s) - \mathcal{T}_\beta(t_1-s) \right\|_{\mathcal{B}(L^2(\Omega))} \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2-s)^{\beta-1} W(\|u(s)\|)ds
\end{aligned}$$

$$\lesssim W(r) \sup_{s\in[0,t_1]} \left\|\mathcal{T}_\beta(t_2-s)-\mathcal{T}_\beta(t_1-s)\right\|_{\mathcal{B}(L^2(\Omega))} + W(r)(t_2-t_1)^\beta$$
$$\to 0, \quad \text{as } t_2 \to t_1.$$

Hence, it follows that $\|(\mathcal{Q}u)(t_2)-(\mathcal{Q}u)(t_1)\|$ tends to zero as $t_2-t_1 \to 0$ independent of $u \in B_r$. Thus, we conclude that the set Θ is equicontinuous.

Secondly, we show that $\mathcal{Q}$ is continuous. For any $\{u_m\}_{m=1}^\infty \subset B_r$, $u \in B_r$ with $u_m \to u$ as $m \to \infty$. In view of the assumptions of f, one has

$$\lim_{m\to\infty} f(u_m(t)) = f(u(t)).$$

On the other hand, one has the inequalities

$$(t-s)^{\beta-\alpha-1}\|u_m(s)-u(s)\| \le 2(t-s)^{\beta-\alpha-1}r,$$

and

$$(t-s)^{\beta-1}\|f(u_m(s))-f(u(s))\| \le 2(t-s)^{\beta-1}W(r),$$

are integrable with respect to a.e. $s \in [0,t]$ and $t \in [0,T]$. Therefore, Lebesgue's dominated convergence theorem implies

$$\int_0^t (t-s)^{\beta-1}\|u_m(s)-u(s)\|ds \to 0, \quad \int_0^t (t-s)^{\beta-1}\|f(u_m(s))-f(u(s))\|ds \to 0,$$

as $m \to \infty$. Consequently, we get

$$\|(\mathcal{Q}u_m)(t)-(\mathcal{Q}u)(t)\| \le \sigma \int_0^t (t-s)^{\beta-\alpha-1}\|u_m(s)-u(s)\|ds$$
$$+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|f(u_m(s))-f(u(s))\|ds \to 0, \text{ as } m \to \infty.$$

This proves that $\mathcal{Q}u_m \to \mathcal{Q}u$ pointwise on $[0,T]$ as $m \to \infty$, which follows from the equicontinuity of Θ that $\mathcal{Q}u_m \to \mathcal{Q}u$ uniformly on $[0,T]$ as $m \to \infty$. Thus, $\mathcal{Q}$ is continuous.

Finally, we show that operator $\mathcal{Q}$ is compact. It is sufficient to prove that for any $t \in [0,T]$, $\Theta(t)$ is relatively compact in $L^2(\Omega)$. Obviously, for the case $t=0$, it is easy to see that $\Theta(0)$ is relatively compact. Let $t \in (0,T]$ be fixed, since $\mathcal{R}'_\beta(t)$ and $\mathcal{T}_\beta(t)$ are compact for every $t>0$ in view of Lemma 4.2 and Remark 4.2, we can structure a family of finite dimensional compact operators as the same way in Lemma 4.2 by

$$(\mathcal{Q}^N u)(t) = \mathcal{S}_\beta(t)\phi + \mathcal{R}_\beta(t)\phi + \mathcal{P}_\beta(t)\psi - \int_0^t \mathcal{R}_\beta'^N(t-s)u(s)ds$$
$$+\int_0^t \mathcal{T}_\beta^N(t-s)f(u(s))ds,$$

for every $N \in \mathbb{N}$, in which $\mathcal{T}_\beta^N(\cdot)$ is defined as in Lemma 4.2 and

$$\mathcal{R}_\beta'^N(t)u = \sum_{n=1}^{N} t^{\beta-\alpha-1} E_{\beta,\beta-\alpha}(-\lambda_n t^\beta)(u, e_n)e_n.$$

Obviously, one can repeat the proof process above that the relatively compactness of set $\Theta^N = \{\mathcal{Q}^N u : u \in B_r\}$ follows. On the other hand, by virtue of Lemma 2.2 and (i) in Lemma 2.4 with respect to $\mu \in \left(0, \frac{\beta-\alpha}{\beta}\right)$, it yields

$$\begin{aligned}
\|(\mathcal{Q}u)(t) - (\mathcal{Q}^N u)(t)\| &\le \int_0^t \|(\mathcal{R}_\beta'(t-s) - \mathcal{R}_\beta'^N(t-s))u(s)\|ds \\
&\quad + \int_0^t \|(\mathcal{T}_\beta(t-s) - \mathcal{T}_\beta^N(t-s))f(u(s))\|ds \\
&\lesssim \lambda_{N+1}^{-\mu} \int_0^t (t-s)^{\beta(1-\mu)-\alpha-1}\|u(s)\|ds \\
&\quad + \lambda_{N+1}^{-\mu} \int_0^t (t-s)^{\beta(1-\mu)-1}\|f(u(s))\|ds.
\end{aligned}$$

Hence, it is easy to show that

$$\|(\mathcal{Q}u)(t) - (\mathcal{Q}^N u)(t)\| \to 0, \quad \text{as } N \to \infty.$$

This means that there are relatively compact sets arbitrarily close to the set $\Theta(t)$. Therefore, $\Theta(t)$ is relatively compact in $L^2(\Omega)$, and we derive that $\mathcal{Q}$ is a compact operator.

Now, let's finish this proof. By above arguments and Ascoli-Arzelà theorem, we know that $\mathcal{Q}$ is completely continuous. Therefore, Schauder's fixed point theorem implies that $\mathcal{Q}$ has at least one fixed point, which means that there exists least one mild solution to problem (1.1)–(1.3). The proof is completed. □

Remark 4.3 Concerning the well-posedness in Theorem 4.1, it is indeed a local result corresponding existence interval $(0, T_0)$ sufficiently small such that (4.8) and (4.9) must satisfy. Despite all of this, it is a new result to some special nonlinear functions, such as $f(u) = |u|^{\vartheta-1}u$, $\vartheta \ge 1$. Besides, the existence interval of Theorem 4.4 is not needed to make sufficiently small since we can get the exact interval of time from the exact upper bounds of Mittag–Leffler functions by (4.10) and Lemma 4.2, and it means that there may appear multiple solutions. Thus this conclusion extends certain results in literatures.

5 An application

Let us take account of the following time fractional telegraph equation

$$\partial_t^{2\alpha} u(t,x) + \partial_t^\alpha u(t,x) = u_{xx}(t,x) + f(u(t,x)), \;\; 0 < x < 1, \; t > 0,$$

where $\partial_t^{2\alpha}$ and ∂_t^{α} are fractional derivatives in the sense of Caputo type with respect to t of order $1/2 < \alpha \leq 1$. Specially, the case $\alpha = 1$ is related to the well-known telegraph process, which describes the propagation process of electron in telegraph cable, and it can be regarded as an integral order wave equation with damped term $\partial_t u$.

In the sequel, let us consider the boundary conditions $u(0,t) = u(1,t) = 0$ and let $\lambda_n = n^2\pi^2$ and $e_n = \sin(n\pi x)$, $n \in \mathbb{N}$, $\Omega = [0,1]$, obviously, $\{-\lambda_n, e_n\}_{n=1}^{\infty}$ is the eigensystem in $L^2(\Omega)$ associated with operator $A = \frac{\partial^2}{\partial x^2}$. If $\phi(x) = \cos(x\pi/2)$, $\psi(x) = x$ and linear function $f(t,x) = t^2\sin(x\pi/2)$, then it is easy to check that $\phi \in D(A^\gamma)$ for $0 < \gamma < 1$, $\psi \in L^2(\Omega)$ and $f \in L^1(0,T;L^2(\Omega))$, then from [7], one can find a solution given by

$$u(t,x) = \sum_{n=1}^{\infty}\Bigg(\int_0^t \tau^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}\left(-\tau^{\alpha}, -\lambda_n\tau^{2\alpha}\right) f_n(t-\tau)d\tau + A_{1n}(0)B_1(t) + A_{2n}(0)B_2(t)\Bigg)e_n(x),$$

where $f_n(t) = 2\int_0^1 f(t,x)e_n(x)dx$, $A_{1n}(0) = 2\int_0^1 \phi(x)e_n(x)dx$, $A_{2n}(0) = 2\int_0^1 \psi(x)e_n(x)dx$,

$$B_1(t) = 1 - \lambda_n t^{2\alpha}E_{(\alpha,2\alpha),2\alpha+1}\left(-t^{\alpha}, -\lambda_n t^{2\alpha}\right),$$
$$B_2(t) = t - t^{\alpha+1}E_{(\alpha,2\alpha),2\alpha+2}\left(-\tau^{\alpha}, -\lambda_n\tau^{2\alpha}\right) - \lambda_n t^{2\alpha+1}E_{(\alpha,2\alpha),2\alpha+2}\left(-t^{\alpha}, -\lambda_n t^{2\alpha}\right),$$

and for $b > 0$, $a_i > 0$, $|z_i| < \infty$, $i = 1, 2$, the multivariate Mittag–Leffler function is defined as

$$E_{(\cdot),b}(\cdot) = E_{(a_1,a_2),b}(z_1,z_2) = \sum_{k=0}^{\infty}\sum_{\substack{l_1+l_2=k \\ l_1\geq 0, l_2\geq 0}} \frac{k!}{l_1!\times l_2!}\frac{z_1^{l_1}z_2^{l_2}}{\Gamma(b+a_1l_1+a_2l_2)}.$$

This ensures that the above solution also belongs to $C([0,T];L^2(\Omega))$ according to Theorem 3.1 and further the solution will satisfy (3.7). Nevertheless, if nonlinear function $f(u) = |u|^{\vartheta-1}u$ for $\vartheta \geq 1$ or $f(u) = \sin(u)$, then we cannot use the method of [7] to establish the existence of solution while there will exist a solution to the current problem by Theorem 4.1 as well as Theorem 4.4. Consequently, this extends the results in [7] and we will get a more general existence result for the case from the exponent $\beta = 2\alpha$ to $\beta \in (1,2]$, $\alpha \in (0,1]$.

Acknowledgements The authors are very grateful to the editor and reviewers for their valuable comments. Project supported by the Fundo para o Desenvolvimento das Ciências e da Tecnologia of Macau (Grant No. 0074/2019/A2) and National Natural Science Foundation of China (11671339) (12071396).

References

1. Affili, E., Valdinoci, E.: Decay estimates for evolution equations with classical and fractional time-derivatives. J. Differ. Equ. **266**(7), 4027–4060 (2019)
2. Alaimia, M.R., Tatar, N.-E.: Blow up for the wave equation with a fractional damping. J. Appl. Anal. **11**(1), 133–144 (2005)
3. Aloui, L., Ibrahim, S., Khenissi, M.: Energy decay for linear dissipative wave equations in exterior domains. J. Differ. Equ. **259**(5), 2061–2079 (2015)
4. Alvarez, E., Gal, C., Keyantuo, V., Warma, M.: Well-posedness results for a class of semi-linear super-diffusive equations. Nonlinear Anal. **181**, 24–61 (2019)
5. Beckers, S., Yamamoto, M.: Regularity and unique existence of solution to linear diffusion equation with multiple time-fractional derivatives. In: Bredies, K., Clason, C., Kunisch, K., von Winckel, G. (eds.) Control and Optimization with PDE Constraints. International Series of Numerical Mathematics, vol. 164. Birkhäuser, Basel (2013)
6. Chen, L.: Nonlinear stochastic time-fractional diffusion equations on $\mathbb{R}$: moments, Hölder regularity and intermittency. Tran. Am. Math. Soc. **369**(12), 8497–8535 (2017)
7. Chen, J., Liu, F., Anh, V.: Analytical solution for the time-fractional telegraph equation by the method of separating variables. J. Math. Anal. Appl. **338**(2), 1364–1377 (2008)
8. Djida, J.D., Fernandez, A., Area, I.: Well-posedness results for fractional semi-linear wave equations. Discrete Contin. Dyn. Syst. Ser. B **25**(2), 569–597 (2020)
9. Dong, H., Kim, D.: L_p-estimates for time fractional parabolic equations in divergence form with measurable coefficients. J. Funct. Anal. **278**, 108338 (2020)
10. Dong, H., Kim, D.: L_p-estimates for time fractional parabolic equations with coefficients measurable in time. Adv. Math. **345**, 289–345 (2019)
11. Fragnelli, G., Mugnai, D.: Stability of solutions for some classes of nonlinear damped wave equations. SIAM J. Control Optim. **47**(5), 2520–2539 (2008)
12. Giga, Y., Namba, T.: Well-posedness of Hamilton–Jacobi equations with Caputo's time fractional derivative. Commun. Partial Differ. Equ. **42**(7), 1088–1120 (2017)
13. Graham, I., Langer, U., Melenk, J., Sini, M. (eds.): Direct and Inverse Problems in Wave Propagation and Applications. Walter de Gruyter, Berlin (2013)
14. Ikehata, R., Todorova, G., Yordanov, B.: Wave equations with strong damping in Hilbert spaces. J. Differ. Equ. **254**(8), 3352–3368 (2013)
15. Kim, I., Kim, K.H., Lim, S.: An $L_q(L_p)$-theory for the time fractional evolution equations with variable coefficients. Adv. Math. **306**, 123–176 (2017)
16. Kian, Y., Yamamoto, M.: On existence and uniqueness of solutions for semilinear fractional wave equations. Fract. Calc. Appl. Anal. **20**(1), 117–138 (2017)
17. Li, L., Liu, J.G., Wang, L.: Cauchy problems for Keller–Segel type time-space fractional diffusion equation. J. Differ. Equ. **265**(3), 1044–1096 (2018)
18. Lin, C., Nakamura, G.: Unique continuation property for multi-terms time fractional diffusion equations. Math. Ann. **373**(3–4), 929–952 (2019)
19. Lin, C., Nakamura, G.: Unique continuation property for anomalous slow diffusion equation. Commun. Partial Differ. Equ. **41**(5), 749–758 (2016)
20. Luchko, Y.: Fractional wave equation and damped waves. J. Math. Phys. **54**(3), 031505 (2013)
21. Luchko, Y., Mainardi, F.: Fractional diffusion-wave phenomena. In: Tarasov, Vasily E. (ed.) Applications in Physics, Part B, pp. 71–98. De Gruyter, Berlin, Boston (2019)
22. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity, An Introduction to Mathematical Models. Imperial College Press, London (2010)
23. Orsingher, E., Beghin, L.: Time-fractional telegraph equations and telegraph process with Brownian time. Probab. Theory Related Fields **128**(1), 141–160 (2004)
24. Otárola, E., Salgado, A.J.: Regularity of solutions to space-time fractional wave equations: a PDE approach. Fract. Calc. Appl. Anal. **21**(5), 1262–1293 (2018)
25. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
26. Sakamoto, K., Yamamoto, M.: Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. J. Math. Anal. Appl. **382**(1), 426–447 (2011)
27. Schneider, W.R., Wyss, W.: Fractional diffusion and wave equations. J. Math. Phys. **30**, 134–144 (1989)
28. Stojanovic, M., Gorenflo, R.: Nonlinear two term time fractional diffusion wave problem. Nonlinear Anal. RWA **11**(5), 3512–3523 (2010)

29. Tatar, N.-E.: A blow up result for a fractionally damped wave equation. NoDEA Nonlinear Differ. Equ. Appl. **12**(2), 215–226 (2005)
30. Vivian, H., Pym, J., Cloud, M.: Applications of Functional Analysis and Operator Theory. Elsevier, Amsterdam (2005)
31. Ye, H., Gao, J., Ding, Y.: A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. **328**(2), 1075–1081 (2007)
32. Zacher, R.: A De Giorgi–Nash type theorem for time fractional diffusion equations. Math. Ann. **356**(1), 99–146 (2013)
33. Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)